

# Consumer Vulnerability and Behavioral Biases\*

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## Abstract

A wealth of evidence shows that individuals are biased and firms can often exploit consumers' behavioral biases in their contract designs. In this paper, we study how vulnerable biased individuals are to their own behavioral biases in market equilibrium, and focus on the role of risk aversion and intertemporal elasticity of substitution (IES). We measure *consumer vulnerability* by the percentage loss in a consumer's equilibrium certainty equivalent from a market with non-biased consumers to that with biased ones. We examine several important behavioral biases that have been extensively studied in the literature, including the impact of biased beliefs (either over- or under-confidence) in an insurance market, the impact of present bias and naïvete about present bias in a dynamic model of credit contract design, the impact of projection bias about habit formation, and the impact of expectation-based loss aversion on an investor's portfolio choice. We show that consumer vulnerability to all four commonly studied behavioral biases has a non-monotonic relationship with risk aversion or IES. This is in striking contrast to the deviations in the equilibrium *allocations* from the rational benchmark, which are often monotonic to the risk aversion or IES. We also consider a setting of biased agents with Epstein-Zin preferences to isolate the effect of risk aversion from that of IES.

**Keywords:** Behavioral Biases; Welfare; Vulnerability; Biased Belief; Present Bias; Projection Bias; Prospect Theory; Loss Aversion; Epstein-Zin Preferences.

**JEL Classification Codes:** D60, D81, D86, D91.

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# 1 Introduction

Behavioral economics is by now a well-established strand of literature where economists introduce a number of robust behavioral biases and heuristics first documented by psychologists into models of economic behavior. Numerous biases have been studied by economists, including present-biased discounting (Strotz, 1956; Laibson, 1997; O’Donoghue and Rabin, 1999; DellaVigna and Malmendier, 2006; Fang and Wang, 2015), biased-beliefs (Fang and Moscarini, 2005; Grubb, 2009, 2015), confirmatory bias (Rabin and Schrag, 1999), loss aversion (Genesove and Mayer, 2001), mental accounting (Barberis and Huang, 2001), and projection bias (Loewenstein, O’Donoghue, and Rabin, 2003), among others. In the theoretical models in behavioral economics, researchers typically introduce one of the documented biases and heuristics into stylized models with an aim of understanding how small, but psychologically relevant, deviations from the standard economic framework can influence *decisions or behavior* such as saving and consumption (Harris and Laibson, 2001; Kőszegi and Rabin, 2009; Pagel, 2017), investment (Barberis and Huang, 2001; Barberis and Xiong, 2009), labor supply (Fang and Silverman, 2004, 2006; Farber, 2008; Crawford and Meng, 2011), wage policies (Santos-Pinto, 2010, 2012); pricing strategies (Eliaz and Spiegler, 2008; Heidhues and Kőszegi, 2014; Martimort and Stole, 2020), advertising strategies (Karle and Schumacher, 2017; Karle and Möller, 2020); contracts (DellaVigna and Malmendier, 2004; Eliaz and Spiegler, 2006; Santos-Pinto, 2008; Grubb, 2009; Herweg, Müller, and Weinschenk, 2010; De la Rosa, 2011), etc. The focus of the analysis is to show that behavioral biases can lead to a departure from a “rational” benchmark in terms of *allocations*, whether it is contracts, consumptions and savings, hours of work, portfolio choices, etc.

The departures from the rational allocations due to behavioral biases typically lead to welfare losses,<sup>1</sup> and a burgeoning literature has been devoted to investigating how policy should respond to market outcomes in order to alleviate market inefficiency (e.g., Sandroni and Squintani, 2007; Bisin, Lizzeri, and Yariv, 2015; Fang and Wu, 2020; Heidhues and Kőszegi, 2010, 2017; Armstrong and Vickers, 2019). However, to the best of our knowledge, what determines the size of the welfare loss caused by the deviation from the rational benchmark allocations in the absence of market intervention remains largely unexplored. Our study aims to close this gap. Answers to this question can help us better evaluate the effectiveness of different policies that aim to protect consumers or improve efficiency. For instance, knowing that the welfare loss in a marketplace with biased consumers is insignificant without regulation would render consumer education campaigns or market interventions—which are often costly—less desirable.

In this paper, we investigate how welfare losses due to consumers’ behavioral biases de-

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<sup>1</sup>In Section 2.2, we provide an example in which a consumer benefits from his underestimation about his risks in a monopolistic insurance market.

pend on their characteristics, or more precisely, on the curvatures of their utility functions. We demonstrate that a large departure in terms of allocations from the rational benchmark does *not* necessarily lead to substantial welfare consequences. A simple example can illustrate the distinction between the magnitude of deviations in allocations and the size of welfare losses. Consider a potential insuree with a biased belief regarding his probability of loss. If the insuree is not very risk averse, a small bias in his subjective loss probability assessment will result in large deviations in his insurance coverage from the rational benchmark. However, because he has a low level of risk aversion, such a deviation is not likely to result in a large welfare loss. If the insuree is instead very risk averse, then a small bias in his subjective loss probability assessment will only result in a small deviation in his coverage from the rational benchmark. Again, the welfare loss from such a small deviation is likely to be small. Indeed, we will formally show in Section 2 that consumers with a moderate level of risk aversion are mostly likely to suffer the largest welfare loss from biased beliefs regarding their loss probabilities.

We systematically examine *consumer vulnerability* in models with behavioral biases. We measure consumer vulnerability by the ratio of the difference in the equilibrium certainty equivalent (CE) between the biased consumer and the non-biased consumer relative to that of a non-biased consumer with the same underlying instantaneous utility function. We then explore how consumer vulnerability is related to the *curvatures* of their utility functions. In particular, we focus on the role of risk aversion in static models and the role of intertemporal elasticity of substitution (IES) in dynamic models.

We examine several important behavioral biases that have been extensively studied in the literature. In Section 2, we consider a classical insurance market with overconfident or underconfident consumers. Assuming constant relative risk aversion (CRRA) preferences, we show that in a perfectly competitive market, consumers' vulnerability has an inverted-U relationship with their risk aversion. In contrast, in a monopolistic market, consumers with overconfidence—i.e., those who underestimate their loss probability—may even benefit from having biased beliefs, even though their insurance coverage deviates from the rational benchmark.

In Section 3, we study a dynamic model of competitive lenders' credit contract design—similar to that of Heidhues and Kőszegi (2010)—when borrowers with CRRA preferences have present bias in their intertemporal time preference and are naive about their present bias. We show that the consumers' vulnerability to their naïveté about present bias again has an inverted-U relationship with their IES.

Recognizing that under CRRA utility function, risk aversion and IES are by default inverse of each other, we consider in Section 4 a two-period model with the Epstein-Zin preference (Epstein and Zin, 1989, 1991) where risk aversion and IES are separately param-

eterized. The analysis under this general framework largely confirms the previous results. We show that when agents are naive about their time preference (or present bias), their vulnerability indeed has an inverted-U relationship with IES; and when agents have misperception on the second-period risks, their vulnerability has an inverted-U relationship with risk aversion.

In Section 5, we study another well-documented behavioral bias, the *projection bias* about the evolutions of his “habit stock” in a model of habit formation, as proposed by Loewenstein, O’Donoghue, and Rabin (2003). In contrast to the results obtained in the previous sections, we show that the consumer vulnerability in this case has a U-shaped relationship with respect to the IES. Thus, consumers with an intermediate degree of IES are less vulnerable than those with extreme values of IES. In Section 6, we consider the portfolio choice problem of investors who exhibit expectation-based reference-dependent preferences á la Kőszegi and Rabin (2006, 2007). We again show that consumers’ vulnerability to their loss aversion has an inverted-U relationship with risk aversion.

Our systematic analysis shows that consumer vulnerability to the commonly studied behavioral biases in the literature in general has *non-monotonic* relationship with risk aversion or IES. This is in striking contrast to the deviations in the allocations from the rational benchmark, which are often *monotonic* to the risk aversion or the IES. Our paper complements the existing literature in behavioral economics that focused almost exclusively on how behavioral bias leads to the deviations in allocations from the rational benchmark, summarized in Spiegel (2011), Kőszegi (2014), Grubb (2015), and Heidhues and Kőszegi (2018). As previously mentioned, most of these studies search for policy remedies that mitigate the welfare losses due to market participants’ behavioral biases under different economic contexts and identify conditions for such policies to be welfare-enhancing.

The remainder of the paper is structured as follows. In Section 2, we analyze a simple insurance model with over- or under-confident consumers, and show that consumer vulnerability is non-monotonic in their risk aversion; in Section 3, we consider a dynamic contracting model with present-biased consumers; in Section 4, we provide a unified framework that incorporates both risk aversion and IES to show how they interact to affect consumer’s vulnerability to biases; in Section 5, we analyze a model of habit formation with projection bias; in Section 6, we study a model of portfolio choice with investors that exhibit expectation-based reference-dependent preferences; and finally, in Section 7 we conclude. We collect all the proofs in the Appendix.

## 2 An Insurance Model with Over- or Under-confident Consumers

Consider a parsimonious model of insurance market with risk averse agents and two possible states. For simplicity, the agent's risk aversion is represented by a constant relative risk aversion (CRRA) utility function with *risk aversion* parameter  $\gamma \equiv 1/\rho > 0$ :

$$u(c) = \begin{cases} \frac{c^{1-\frac{1}{\rho}}-1}{1-\frac{1}{\rho}} & \text{if } \rho > 0 \text{ and } \rho \neq 1, \\ \ln(c) & \text{if } \rho = 1. \end{cases} \quad (1)$$

In dynamic models, which we will analyze in Section 3, the parameter  $\rho$  also represents the *intertemporal elasticity of substitution (IES)*.<sup>2</sup> The agent has initial wealth  $y > 0$  and is subject to a possible loss  $\omega \in (0, y)$  with *objective* probability  $\mu \in (0, 1)$ . The agent, however, *subjectively* believes that the loss probability is  $\hat{\mu} \in (0, 1)$ . When  $\hat{\mu} = \mu$ , the agent is rational. When  $\hat{\mu} < \mu$  ( $\hat{\mu} > \mu$ ), the agent underestimates (overestimates) his loss probability and is said to exhibit *overconfidence* (*underconfidence*).

In what follows, we analyze the equilibrium of the insurance market under perfect competition and that under monopoly, respectively.

### 2.1 Competitive Insurance Market

The consumer can purchase an insurance contract from a competitive insurance market at *premium*  $p$  to cover a fraction  $x$  of the loss if the loss occurs. We interpret  $x$  as the *degree of coverage*. Alternatively, the ratio  $p/x$  can be interpreted as the unit price of an insurance policy that covers the potential loss of  $\omega$ , and  $x$  is the quantity of insurance that a consumer purchases from the market.

The equilibrium contract, which we denote by  $(x^*, p^*)$ , maximizes consumer's *perceived* utility

$$\max_{\{x,p\}} \hat{\mu}u(y - (1-x)\omega - p) + (1-\hat{\mu})u(y - p), \quad (2)$$

subject to the zero-profit constraint

$$p = \mu x \omega. \quad (3)$$

Two remarks are in order. First, the objective function (2) in the above maximization problem is the agents' expected *perceived* utility using the subjective belief  $\hat{\mu}$  about the loss probability, instead of the utility based on the *objective* probability  $\mu$  of incurring a loss. Thus, the perceived expected utility in (2) is the *decision utility* in the terminology

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<sup>2</sup>In Section 4 we will analyze the case of Epstein-Zin preference which separates risk aversion and IES.

of Kahneman, Wakker, and Sarin (1997). As we will explain later, agents' expected utility according to the *correct*, or *objective*, probability of incurring a loss  $\mu$ , is used when we evaluate the consumer welfare, corresponding to the notion of *experienced utility* in Kahneman et al. (1997).

Second, we impose no restrictions on the contract space in the subsequent analysis, i.e., price can be either positive or negative in the equilibrium contract. When  $p > 0$ , the risk-averse agent is the consumer, and the risk-neutral agent is the insurance firm: the consumer pays the insurance firm and transfers his risk to the insurance firm. When  $p < 0$ , the risk-averse agent becomes a seller instead: he earns money through receiving more risks from the market. When  $p = 0$ , there is no trade on the market and the economy degenerates to an autarky.

The following first-order condition must hold in equilibrium:

$$\frac{u' [y - (1 - x^*)\omega - \mu x^*\omega]}{u'(y - \mu x^*\omega)} = \frac{1 - \hat{\mu}}{\hat{\mu}} \times \frac{\mu}{1 - \mu}.$$

Exploiting the CRRA form of  $u(\cdot)$  as described in (1), we obtain:

$$\frac{y - (1 - x^*)\omega - \mu x^*\omega}{y - \mu x^*\omega} = \left( \frac{1 - \hat{\mu}}{\hat{\mu}} \times \frac{\mu}{1 - \mu} \right)^{-\rho} =: A(\rho), \quad (4)$$

where  $A(\rho)$  is the ratio between the equilibrium consumption level when a loss occurs and that when no loss occurs and depends on  $\hat{\mu}$ ,  $\mu$ , and  $\rho$ .

Solving for  $(x^*, p^*)$  yields that:<sup>3</sup>

$$x^*(\rho; \mu, \hat{\mu}) = \frac{[A(\rho) - 1] y + \omega}{[A(\rho)\mu + (1 - \mu)] \omega}, \text{ and } p^*(\rho; \mu, \hat{\mu}) = \mu\omega x^*(\rho; \mu, \hat{\mu}). \quad (5)$$

The term  $A(\rho)$  captures the distortion caused by consumer's misperception regarding the loss probability. To see this, note that when consumers have unbiased beliefs, i.e., when  $\hat{\mu} = \mu$ , we have  $A(\rho) = 1$  and  $x^*(\rho; \mu, \hat{\mu}) = 1$  for all  $\rho > 0$ . In other words, the market provides full insurance independent of  $\rho$ . This reaffirms the classical result in the insurance theory. However, when  $\hat{\mu} < \mu$ , we have  $A(\rho) < 1$  and  $x^*(\rho; \mu, \hat{\mu}) < 1$ ; and when  $\hat{\mu} > \mu$ , we have  $A(\rho) > 1$  and  $x^*(\rho; \mu, \hat{\mu}) > 1$ . That is, an overconfident consumer will purchase less than full insurance; and an underconfident consumer is over insured in equilibrium.

Following the literature, we use the consumer's *experienced utility* with the *objective* loss

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<sup>3</sup>Interestingly, the formula for the insurance coverage choice in equilibrium (5) also clarifies that there is an identification problem between biased belief  $\hat{\mu}$  and the preference parameter  $\rho$  from just observing the insurance coverage choice of the consumer.

probability  $\mu$  to evaluate the consumer welfare.<sup>4</sup> Specifically, consumer's welfare measured by the equilibrium certainty equivalent, which we denote by  $CE^*(\rho; \mu, \hat{\mu})$ , is the solution to

$$u [CE^*(\rho; \mu, \hat{\mu})] = \mu u \left[ y - [1 - x^*(\rho; \mu, \hat{\mu})] \omega - p^*(\rho; \mu, \hat{\mu}) \right] + (1 - \mu) u \left[ y - p^*(\rho; \mu, \hat{\mu}) \right]. \quad (6)$$

Solving for  $CE^*(\rho; \mu, \hat{\mu})$  yields that

$$CE^*(\rho; \mu, \hat{\mu}) = \begin{cases} \left( \mu [y - (1 - x^*)\omega - p^*]^{1-\frac{1}{\rho}} + (1 - \mu)(y - p^*)^{1-\frac{1}{\rho}} \right)^{\frac{\rho}{\rho-1}} & \text{if } \rho > 0 \text{ and } \rho \neq 1, \\ [y - (1 - x^*)\omega - p^*]^\mu \cdot (y - p^*)^{1-\mu} & \text{if } \rho = 1. \end{cases} \quad (7)$$

A close look at (5) and (7) leads to the following.

**Remark 1** *Suppose that  $\hat{\mu} = \mu$ . Then  $(x^*, p^*) = (1, \mu\omega)$  and  $CE^*(\rho; \mu, \hat{\mu}) = y - \mu\omega$ .*

Recall that the shape of the concave utility function has no impact on the equilibrium contract when consumers are rational (i.e.,  $\hat{\mu} = \mu$ ): They always receive full insurance in equilibrium independent of  $\rho$  (i.e.,  $x^*(\rho; \mu, \mu) = 1$ ). Because full insurance is obtained and the insurance firms earn zero profits under the pressure of market competition, rational consumers' welfare is maximized and the corresponding certainty equivalent is  $y - \mu\omega$ , which is again independent of the utility function  $u(\cdot)$ . This desired independence property, however, does not hold when consumers misperceive the loss probability.

Given consumers' certainty equivalent described in (7), we can define a *loss function*  $\mathcal{L}(\rho; \mu, \hat{\mu})$  as follows:

$$\mathcal{L}(\rho; \mu, \hat{\mu}) := \frac{CE^*(\rho; \mu, \mu) - CE^*(\rho; \mu, \hat{\mu})}{CE^*(\rho; \mu, \mu)}.$$

In words,  $\mathcal{L}(\rho; \mu, \hat{\mu})$  is the percentage loss in consumers' certainty equivalent relative to the rational benchmark. We use it as the measure of the *vulnerability* of a biased consumer. The following observation follows immediately.

**Proposition 1** (*Competitive insurance market with biased consumers*) *Suppose that consumers have CRRA utility function (1) and  $\hat{\mu} \leq \mu$ . Then the following statements hold:*

(i) *The equilibrium coverage  $x^*(\rho; \mu, \hat{\mu}) \leq 1$ , and the deviation from full insurance coverage, i.e.,  $|x^*(\rho; \mu, \hat{\mu}) - 1|$ , is strictly decreasing in the risk aversion parameter  $\gamma \equiv 1/\rho$  for  $\gamma \in (0, \infty)$ .*

(ii) *The loss function  $\mathcal{L}(\rho; \mu, \hat{\mu})$  is an inverted U-shaped curve in  $\gamma \equiv 1/\rho$  for  $\gamma \in (0, \infty)$ .*

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<sup>4</sup>See Brunnermeier, Simsek, and Xiong (2014) for a welfare criterion for models with distorted beliefs.

The first part of Proposition 1 is intuitive: Consumers' misperception messes up consumption smoothing across states. When consumers underestimate their loss probability (i.e.,  $\hat{\mu} < \mu$ ), they put more weight on the no-loss state than they should, and thus underestimate their demand for insurance. The competitive insurers then respond by providing partial insurance. Similarly, when they overestimate their loss probability, they overestimate their insurance demand, and thus will be over insured in equilibrium.

Note that the deviation in insurance coverage from the unbiased belief benchmark is decreasing in the agent's risk aversion parameter  $\gamma$ . As we will show in the next sections, this observation holds in general in different behavioral models. Intuitively, when a consumer's risk aversion parameter is low, he has a weak propensity towards consumption smoothing. As a result, he is more easily exploited by the insurance firms than a consumer with a higher level of risk aversion. Define  $\tau := \left(\frac{1-\hat{\mu}}{\hat{\mu}} \times \frac{\mu}{1-\mu}\right)^{-1} > 0$ , and let  $\rho_1 := \frac{\ln(y-w)-\ln(y)}{\ln(\tau)} > 0$ . When the consumer exhibits overconfidence and  $\rho > \rho_1$ , he is even willing to increase his consumption volatility across the loss and no loss states by absorbing more risks from the market and become a seller.<sup>5</sup>

One may then draw the conclusion that a consumer with a lower risk aversion is more *vulnerable* because his equilibrium contract deviates from the first-best contract to a larger extent. However, this intuition is not accurate. In fact, part (ii) of Proposition 1 indicates that a consumer with a moderate IES is the most vulnerable. To explain the intuition most cleanly, it is useful to consider a small risk and employ Taylor expansion. More formally, we assume that the size of potential loss  $\omega$  is sufficiently small. The consumer's equilibrium certainty equivalent can then be approximated by:<sup>6</sup>

$$\begin{aligned} CE^*(\rho; \mu, \hat{\mu}) &\approx y - \mu\omega - \frac{\mu(1-\mu)\omega^2}{2(y-\mu\omega)} \times \frac{[1-x^*(\rho; \mu, \hat{\mu})]^2}{\rho} \\ &= y - \mu\omega - \frac{1}{2}\mu(1-\mu)(y-\mu\omega) \times \frac{1}{\rho \left[\mu + \frac{1}{A(\rho)-1}\right]^2}. \end{aligned}$$

Therefore, the loss function can be approximated by

$$\mathcal{L}(\rho; \mu, \hat{\mu}) \approx \frac{1}{2}\mu(1-\mu) \left[\frac{\omega}{y-\mu\omega}\right]^2 \times \underbrace{\frac{1}{\rho}}_{\text{curvature effect}} \times \underbrace{[1-x^*(\rho; \mu, \hat{\mu})]^2}_{\text{volatility effect}}. \quad (8)$$

Two important driving forces for the welfare analysis can be identified from the above ex-

<sup>5</sup>It is useful to point out that this observation continues to hold even if both sides are risk-averse.

<sup>6</sup>Suppose that a random variable  $X$  has finite mean  $\mathbb{E}(X)$  and a small variance  $\text{Var}(X)$ . The certainty equivalent of this gamble under utility function  $u(\cdot)$  can be approximated by  $\mathbb{E}(X) + \frac{1}{2} \frac{u''(\mathbb{E}(X))}{u'(\mathbb{E}(X))} \text{Var}(X)$ , which can be further simplified as  $\mathbb{E}(X) - \frac{\text{Var}(X)}{2\rho\mathbb{E}(X)}$ .

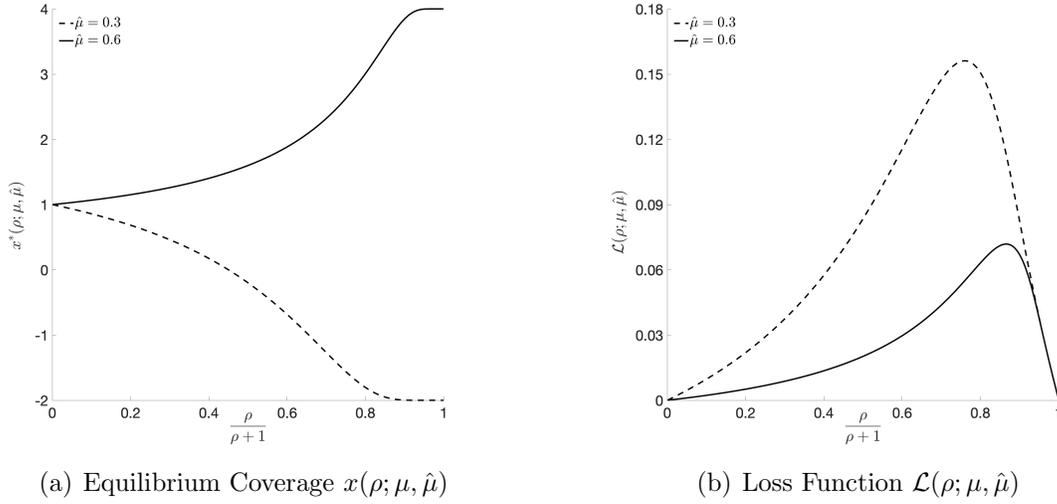


Figure 1: Competitive Insurance Market with Biased Consumers:  $(y, \omega, \mu) = (1, 0.5, 0.5)$ .

pression. By Equation (8), the size of the equilibrium welfare loss depends on (i) the shape of consumer's utility function, i.e.,  $\rho$ ; and (ii) the degree of deviation from the equilibrium coverage level under the rational benchmark, i.e.,  $[1 - x^*(\rho; \mu, \hat{\mu})]^2$ . On the one hand, the shape of the utility function affects consumer welfare, which we refer to as the *curvature effect*. Note that the utility function becomes less concave as  $\rho$  increases. This in turn implies that fixing a degree of deviation, the welfare loss is reduced with a utility function of a higher  $\rho$  (or equivalently, a lower  $\gamma$ ). On the other hand, when consumers are biased (either over- or under-confident), the magnitude of deviation from the equilibrium insurance coverage endogenously depends on the risk aversion of the utility function [see Equation (5) and Figure 1(a)]. As  $\rho$  increases, the equilibrium coverage deviates more from the unbiased benchmark, which increases the volatility of consumption across the two states, and thus contributes to welfare loss (the *volatility effect*). Because the welfare loss is decreasing in  $\rho$  by the curvature effect, and is increasing in  $\rho$  by the volatility effect, the welfare loss caused by the consumer's misperception (either overconfidence or underconfidence) is *non-monotone* in  $\rho$  as Figure 1(b) depicts.

## 2.2 Monopolistic Insurance Market

Next, we consider the monopolistic insurance market. Clearly, firms' market power would further distort the market outcome and thus result in inefficiency. A monopolistic insurance firm solves the following profit maximization problem:

$$\max_{\{x, p\}} p - \mu x \omega,$$

subject to the following individual rationality (IR) constraint:

$$\hat{\mu}u(y - (1 - x)\omega - p) + (1 - \hat{\mu})u(y - p) \geq \hat{\mu}u(y - \omega) + (1 - \hat{\mu})u(y). \quad (9)$$

It is evident that the IR constraint always binds in the optimal contract. Otherwise, the monopolistic insurance firm can increase premium without violating condition (9) and further increase the expected profits. Note that the value of a consumer's outside option depends on his utility function  $u(\cdot)$  as well as his subjective belief  $\hat{\mu}$ .

Denote the profit-maximizing contract by  $(x^m, p^m)$ , where we use the superscript  $m$  to indicate "monopoly." The first-order condition yields that

$$\frac{y - (1 - x^m)\omega - p^m}{y - p^m} = \left( \frac{1 - \hat{\mu}}{\hat{\mu}} \times \frac{\mu}{1 - \mu} \right)^{-\rho} \equiv A(\rho).$$

As in the case of perfect competition,  $A(\rho)$  appears and again measures the distortion of consumption across the two states caused by consumers' misperception. It is noteworthy that the ratio between consumption in the two states under the profit-maximizing contract  $(x^m, p^m)$  coincides with that under the competitive equilibrium contract  $(x^*, p^*)$ .

Solving for  $(x^m, p^m)$ , we obtain

$$x^m = 1 - \frac{1 - A}{\omega} \left[ \frac{\hat{\mu}(y - \omega)^{\frac{\rho-1}{\rho}} + (1 - \hat{\mu})y^{\frac{\rho-1}{\rho}}}{\hat{\mu}A^{\frac{\rho-1}{\rho}} + (1 - \hat{\mu})} \right]^{\frac{\rho}{\rho-1}},$$

and

$$p^m = y - \left[ \frac{\hat{\mu}(y - \omega)^{\frac{\rho-1}{\rho}} + (1 - \hat{\mu})y^{\frac{\rho-1}{\rho}}}{\hat{\mu}A^{\frac{\rho-1}{\rho}} + (1 - \hat{\mu})} \right]^{\frac{\rho}{\rho-1}}.$$

A close look at the above equations lead to the following.

**Remark 2** Suppose that  $\hat{\mu} = \mu$ . Then  $x^m = 1$ ,  $p^m = y - u^{-1}(\mu u(y - \omega) + (1 - \mu)u(y))$ .

When consumers are unbiased, they will be offered full insurance in a monopolistic market as in a competitive market; and the price will be higher than that under perfect competition and now depends on the utility function. The monopolistic insurance firm charges the premium such that consumers are indifferent between buying insurance and remaining uninsured. Note that when the risk aversion of  $u(\cdot)$  becomes sufficiently small, the profit-maximizing premium  $p^m$  is approaching the competitive premium  $p^*$ . In other words, a lower risk aversion limits the monopolistic insurance firm's market power.

Similar to the analysis under perfect competition, we can denote consumer's equilibrium certainty equivalent by  $CE^m(\rho; \mu, \hat{\mu})$  and the monopolist's maximum profit by  $\pi^m(\rho; \mu, \hat{\mu})$ ,

respectively.<sup>7</sup> Recall that  $\rho_1$  is defined as  $\rho_1 \equiv \frac{\ln(y-w)-\ln(y)}{\ln(\tau)} > 0$ , where  $\tau \equiv \left(\frac{1-\hat{\mu}}{\hat{\mu}} \times \frac{\mu}{1-\mu}\right)^{-1}$ . The following results can then be obtained.

**Proposition 2 (*Monopolistic insurance market with biased consumers*)** *The following statements hold under monopoly:*

- (i) *Suppose that consumers exhibit underconfidence, i.e.,  $\hat{\mu} > \mu$ . Then  $CE^m(\rho; \mu, \hat{\mu}) < CE^m(\rho; \mu, \mu)$  and  $\pi^m(\rho; \mu, \hat{\mu}) > \pi^m(\rho; \mu, \mu)$  for all  $\rho > 0$ .*
- (ii) *Suppose that consumers exhibit overconfidence, i.e.,  $\hat{\mu} < \mu$ . Then  $CE^m(\rho; \mu, \hat{\mu}) \geq CE^m(\rho; \mu, \mu)$  if  $\rho \leq \rho_1$ . Moreover, there exists a threshold  $\rho_2 \in (\rho_1, \infty)$  such that  $\pi^m(\rho; \mu, \hat{\mu}) \leq \pi^m(\rho; \mu, \mu)$  if  $\rho \leq \rho_2$ .*

Proposition 2 provides a stark contrast to Proposition 1, which states that consumer welfare is always reduced relative to the rational benchmark in a perfectly competitive market regardless of the type of consumer bias. By Proposition 2, a biased consumer is always worse off than an unbiased consumer if he exhibits underconfidence [see Figure 2(c)]. However, an overconfident consumer with a high risk aversion, i.e.,  $\gamma \equiv 1/\rho > 1/\rho_1$ , is better off than if he were unbiased.

The intuition is as follows. Consumer's biased belief has two effects under monopoly. On the one hand, it creates a *wedge* between the marginal utility across the two states, and leads the monopolistic insurance firm to design a contract that deviates from full insurance, which we refer to as the *distortion effect* [see Figure 2(a)]. This effect is the same as in the case of perfect competition, and reduces consumer welfare. On the other hand, it influences a consumer's *demand* for insurance—an effect that is not present in the case of perfect competition—as his expected utility without insurance depends on his subjective belief  $\hat{\mu}$ . If a consumer exhibits underconfidence, he *overestimates* his demand for insurance, which implies that fixing a coverage, the monopolist is able to extract more rents by charging the consumer a higher premium than the unbiased counterpart. This effect again tends to lower consumer welfare. Therefore, an underconfident consumer will always be worse off than a rational consumer.

In contrast, if a consumer exhibits overconfidence, on the one hand his incorrect belief results in him losing full insurance, which we refer to as the distortion effect; on the other hand, his incorrect belief also leads him to *underestimate* his demand for insurance. This effect decreases an overconfident consumer's perceived value from insurance, which in turn disciplines the monopolist in its pricing, and may thus protect the consumer. In particular,

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<sup>7</sup>We do not define the loss function under monopoly because a biased consumer's welfare can indeed be higher than that of a rational consumer under monopoly as Proposition 2 points out.

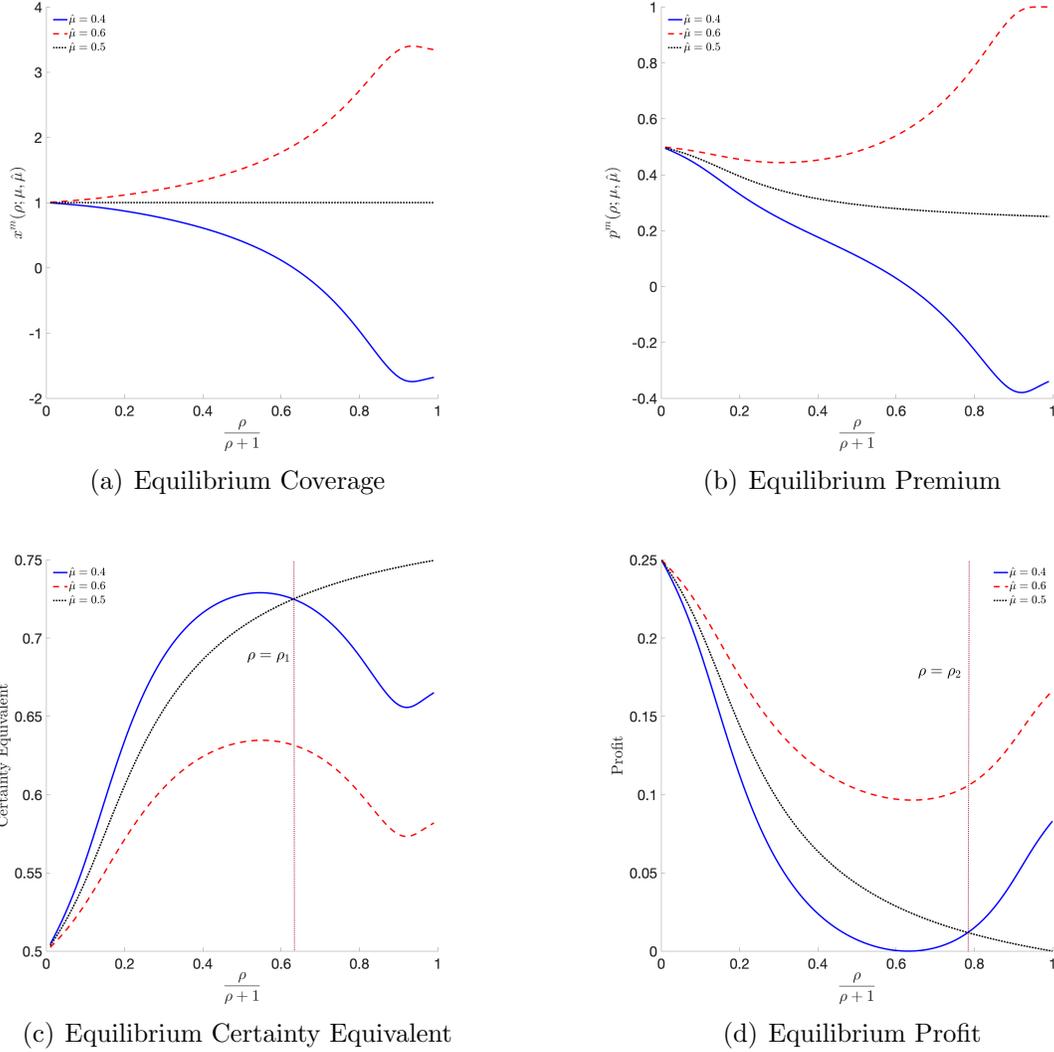


Figure 2: Monopolistic Insurance Market with Biased Consumers:  $(y, \omega, \mu) = (1, 0.5, 0.5)$ .

if the latter *disciplinary effect* outweighs the former *distortion effect*—which is the case for  $\rho < \rho_1$ —then a consumer’s welfare increases with the presence of the misperception.

Next, we consider the firm’s profit in the optimal contract. When consumers are unbiased (i.e.,  $\hat{\mu} = \mu$ ), firm’s profit strictly decreases with the IES  $\rho$  [see the middle curve in Figure 2(d)]. A monopolistic insurance firm provides full insurance and earns profits equal to consumers’ risk premium. As  $\rho$  increases, consumers become less risk averse, and thus firm’s profit decreases.<sup>8</sup> In contrast, when consumers exhibit overconfidence or underconfidence, it can be shown that the firm’s profit curve follows a U-shaped curve, as the upper and the lower curves in Figure 2(d) depict.<sup>9</sup> In addition to the aforementioned effect of  $\rho$

<sup>8</sup>See Pratt (1964, Theorem 1(b)) for a formal proof.

<sup>9</sup>A formal proof is provided for the case  $\hat{\mu} < \mu$  in Lemma 3; the proof for the case  $\hat{\mu} > \mu$  is similar.

on consumers’ risk premiums,  $\rho$  now also influences the equilibrium coverage in the optimal contract when consumers have biased beliefs. Intuitively, a consumer with a higher IES  $\rho$  (or equivalently, a lower risk aversion  $\gamma \equiv 1/\rho$ ) can be exploited by the firm more easily. This effect tends to increase firm’s profit as  $\rho$  increases, and thus reshapes the monotonically decreasing profit curve under the rational benchmark.

Proposition 2 also sheds some useful insight on market participants’ incentive to educate consumers. As Kőszegi (2014) points out: “...unshrouding [education] is often unprofitable because it turns profitable naive consumers into unprofitable sophisticated consumers. Hence, deceptive products or contracts can often survive in market.”<sup>10</sup> Proposition 2 confirms Kőszegi’s argument for *underconfident* consumers: A monopolistic insurance firm has no incentive to debias underconfident consumers [see Figure 2(d)]. However, it can be profitable for the monopolistic firm to educate *overconfident* consumers when their IES  $\rho$  falls below the threshold  $\rho_2$  in our model.

It is also worth noting that government intervention may not be necessary when consumers are overconfident. To see this more clearly, suppose that  $\rho \in (\rho_1, \rho_2)$ . In this case, the insurance firm becomes a buyer and the agents are sellers. It follows immediately from part (ii) of Proposition 2 that the monopolistic insurance firm has an incentive to educate consumers to correct their biased belief. Furthermore, such a move leads to Pareto improvements for both parties on the market [see Figures 2(c) and 2(d)].

### 3 Credit Market with Present-biased Borrowers

In Section 2, we studied the effects of biased beliefs in a static insurance market. In this section, we consider the effect of another common behavioral bias, namely, time-inconsistent present-biased preference, in a dynamic model of a competitive credit market proposed by Heidhues and Kőszegi (2010).

In order to explore the impact of consumer’s IES on the equilibrium contract, we slightly modify Heidhues and Kőszegi (2010) and assume that borrower’s utility function takes the CRRA form as described by (1) in all periods.<sup>11</sup> Consider a perfectly competitive credit market with three periods,  $t = 0, 1, 2$ . The consumer does not have any income in period 0, but receives an income of  $y$  in periods 1 and 2. The consumer’s intertemporal preference is subject to *present-bias* in the form of  $(\beta, \delta)$ -hyperbolic discounting (Strotz, 1956; Laibson, 1997; O’Donoghue and Rabin, 1999), where  $\beta \in (0, 1)$  is the present-bias factor and  $\delta \in (0, 1]$  is the standard exponential discounting factor. Without loss of generality, we assume that

<sup>10</sup>See Gabaix and Laibson (2006), Spiegel (2006), and Heidhues, Kőszegi, and Murooka (2017) for more details.

<sup>11</sup>Heidhues and Kőszegi (2010) postulate a preference in which the agent derives utility  $c$  from consumption  $c$ , and suffers a cost of  $k(q)$  from repayment of  $q$ .

$\delta = 1$ , and focus on the role of the present-bias factor  $\beta$ .

Period-0 self's utility is  $u(c) + u(y - r_1) + u(y - r_2)$ , where  $c \geq 0$  is the amount the consumer borrows in period 0, and  $r_1 \geq 0$  and  $r_2 \geq 0$  are the amounts he repays in period 1 and 2, respectively.<sup>12</sup> However, period-1 self maximizes  $u(y - r_1) + \beta u(y - r_2)$ , where  $\beta$  captures a borrower's time inconsistency and taste for immediate gratification. To capture borrower's naïvete, we assume that period-0 self *believes* that period-1 self will maximize  $u(y - r_1) + \hat{\beta} u(y - r_2)$ , where  $\hat{\beta} \geq \beta$  is a measure of sophistication. In particular,  $\hat{\beta} = \beta$  implies that a borrower is perfectly sophisticated, while  $\hat{\beta} = 1$  indicates that a borrower is completely naive.

**Sophisticated borrowers** We first consider the case of a *perfectly sophisticated* borrower (i.e.,  $\hat{\beta} = \beta$ ). Because the borrower correctly foresees his time inconsistency, it is without loss of generality to suppose that the menu of contracts consists of only one installment plan  $\langle c_s, (r_{1s}, r_{2s}) \rangle$ , where an amount  $c_s$  is lent to the borrower in period 0, and repayments of  $r_{1s}$  and  $r_{2s}$  are required in period 1 and 2 respectively.

A competitive lender's optimization problem is then given by

$$\begin{aligned} & \max_{\{c_s, (r_{1s}, r_{2s})\}} r_{1s} + r_{2s} - c_s, \\ \text{s.t.} \quad & u(c_s) + u(y - r_{1s}) + u(y - r_{2s}) \geq \underline{u}, \end{aligned}$$

where  $\underline{u}$  specifies the value of a borrower's outside option from alternative competitive firms and is endogenously determined by the zero-profit condition. The equilibrium contract, which we denote by  $\langle c_s^*, (r_{1s}^*, r_{2s}^*) \rangle$ , is determined by the following first-order condition

$$u'(y - r_{1s}^*) = u'(y - r_{2s}^*) = u'(c_s^*),$$

and the zero-profit condition

$$c_s^* = r_{1s}^* + r_{2s}^*.$$

We can show with simple algebra that the equilibrium contract is given by

$$\left\langle c_s^*, (r_{1s}^*, r_{2s}^*) \right\rangle = \left\langle \frac{2}{3}y, \left( \frac{1}{3}y, \frac{1}{3}y \right) \right\rangle. \quad (10)$$

To evaluate the welfare of the equilibrium contract  $\langle c_s^*, (r_{1s}^*, r_{2s}^*) \rangle$ , we follow the literature (e.g., DellaVigna and Malmendier, 2004; O'Donoghue and Rabin, 1999, 2001; Gottlieb and Zhang, 2020) and use the long-run self's utility as the welfare measure. Due to the timing

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<sup>12</sup>We follow Heidhues and Kőszegi (2010) and assume that a borrower signs the contract before period 0 starts. Therefore, from self 0's perspective,  $c$ ,  $r_1$ ,  $r_2$  all occur in the future, and thus his expected utility is  $\beta [u(c) + u(y - r_1) + u(y - r_2)]$ . It is evident that dropping the multiplier  $\beta$  will not influence our analysis.

assumption we adopted in this paper, the long-run self's utility is equivalent to period-0 self's utility. From the above analysis, the time-inconsistency problem is solved by the competitive credit market: Consumer welfare is maximized and consumption across all three periods is perfectly smoothed; the per period certainty equivalent is  $\frac{2}{3}y$ , which is independent of the curvature of the utility function  $u(\cdot)$ .

**Naive borrowers** Next, we consider the case of naive borrower (i.e.,  $\hat{\beta} > \beta$ ). By Heidhues and Köszegi (2010), the equilibrium contract is characterized by  $\langle c_n, (r_{1n}, r_{2n}), (\hat{r}_{1n}, \hat{r}_{2n}) \rangle$ , where  $c_n$  is borrower's consumption in period 0,  $(\hat{r}_{1n}, \hat{r}_{2n})$  is the installment plan that period-0 self thinks he will choose, and  $(r_{1n}, r_{2n})$  is the installment plan that period-1 self will actually follow. The competitive lender's maximization problem is given by

$$\begin{aligned} & \max_{\{c_n, (r_{1n}, r_{2n}), (\hat{r}_{1n}, \hat{r}_{2n})\}} r_{1n} + r_{2n} - c_n, \\ \text{s.t. } & u(c_n) + u(y - \hat{r}_{1n}) + u(y - \hat{r}_{2n}) \geq \underline{u}, \end{aligned} \quad (11)$$

$$u(y - \hat{r}_{1n}) + \hat{\beta}u(y - \hat{r}_{2n}) \geq u(y - r_{1n}) + \hat{\beta}u(y - r_{2n}), \quad (12)$$

$$u(y - r_{1n}) + \beta u(y - r_{2n}) \geq u(y - \hat{r}_{1n}) + \beta u(y - \hat{r}_{2n}), \quad (13)$$

where (11) is period-0 self's participation constraint when he signs the contracts in period 0; (12) is the perceived incentive constraint that period-0 self thinks his period-1 self will be facing; and (13) is the period-1 self's actual incentive constraint in period 1. Note that in condition (11), period-0 self, due to his naïvete about his own present bias, believes that  $(\hat{r}_{1n}, \hat{r}_{2n})$  are the relevant repayment schedules when he signs the contract in period 0; he believes, according to his perceived present bias factor  $\hat{\beta}$ , that his period-1 self will choose repayment plan  $(\hat{r}_{1n}, \hat{r}_{2n})$  over  $(r_{1n}, r_{2n})$  as guaranteed by (12). However, his period-1 self, using the actual present bias  $\beta$ , will choose repayment plan  $(r_{1n}, r_{2n})$  over  $(\hat{r}_{1n}, \hat{r}_{2n})$ , according to (13).

Following the arguments in Heidhues and Köszegi (2010), we can show that (11) and (13) must bind in the equilibrium contract. To see this, if (11) were not binding, then the firm can reduce  $c_n$  to increase the objective function; if (13) were not binding, then the firm can increase  $r_{1n}$  to increase the objective. In addition, since (13) must bind in the optimal contract, then the fact that  $\hat{\beta} > \beta$  implies that (12) will be satisfied as long as  $\hat{r}_{2n} < r_{2n}$ . In what follows, we assume that (12) does not bind in the maximization problem. We will confirm that it will indeed result in a solution that satisfies  $\hat{r}_{2n} < r_{2n}$ .

With the above reasoning, we consider the following relaxed problem:

$$\begin{aligned} & \max_{\{c_n, (r_{1n}, r_{2n}), (\hat{r}_{1n}, \hat{r}_{2n})\}} r_{1n} + r_{2n} - c_n, \\ \text{s.t. } & u(c_n) + u(y - \hat{r}_{1n}) + u(y - \hat{r}_{2n}) = \underline{u}, \end{aligned} \quad (14)$$

$$u(y - r_{1n}) + \beta u(y - r_{2n}) = u(y - \hat{r}_{1n}) + \beta u(y - \hat{r}_{2n}). \quad (15)$$

Denote the optimal contract by  $\langle c_n^*, (r_{1n}^*, r_{2n}^*), (\hat{r}_{1n}^*, \hat{r}_{2n}^*) \rangle$ . We can argue that  $\hat{r}_{2n}^* = 0$  in the solution to the above problem. To see this, if  $\hat{r}_{2n} > 0$ , then the firm can decrease  $\hat{r}_{2n}$  to 0 and increase  $\hat{r}_{1n}$  appropriately to leave (14) unaffected. This will create slackness in (15), which will allow the firm to increase profit through raising  $r_{1n}$ . Therefore, the equilibrium contract is governed by the following first-order condition

$$u'(c_n) = u'(y - r_{1n}) = \beta u'(y - r_{2n}), \quad (16)$$

and the zero-profit condition

$$c_n = r_{1n} + r_{2n}.$$

It follows immediately from Equation (16) that borrower's time inconsistency distorts the trade-off of resource allocation between period 2 (the last period) and the other periods, and thus generates inefficiency. When a borrower has time-inconsistent taste for immediate gratification and is naive about his present bias, the competitive firms will exploit his naïvete by designing a deceptive contract to induce excessive borrowing by including a high period-2 repayment (penalty),  $r_{2n}$ , which he thinks will never be triggered.

Exploiting the constant IES functional form of  $u(\cdot)$  as described in (1) and solving for the actual installment plan  $\langle c_n^*, (r_{1n}^*, r_{2n}^*) \rangle$  yields that:<sup>13</sup>

$$\left\langle c_n^*, (r_{1n}^*, r_{2n}^*) \right\rangle = \left\langle \frac{2}{2 + \beta^\rho} y, \left( \frac{\beta^\rho}{2 + \beta^\rho} y, \frac{2 - \beta^\rho}{2 + \beta^\rho} y \right) \right\rangle. \quad (17)$$

Comparing (10) with (17), we can obtain

$$\frac{c_n^*}{c_s^*} = \frac{3}{2 + \beta^\rho} > 1,$$

where the inequality follows from  $\beta \in (0, 1)$  and  $\rho > 0$ . In words, a naive consumer borrows more than a sophisticated consumer would and over-consumes in period 0 relative to the first-best benchmark. Furthermore, the ratio  $c_n^*/c_s^*$  increases with  $\rho$ , indicating that a consumer

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<sup>13</sup> $(\hat{r}_{1n}^*, \hat{r}_{2n}^*)$ , the installment plan offered by the firm that period-0 self thinks he will choose, is not consequential for the subsequent analysis, but it can be characterized as follows:  $\hat{r}_{2n}^* = 0$  as we already argued in the text; then  $\hat{r}_{1n}^*$  simply follows from (15) by substituting the solution  $(r_{1n}^*, r_{2n}^*)$  as characterized by (17).

with a higher IES is more vulnerable from his naïvete in terms of period-0 consumption.

Again, using the long-run self's utility from the contract  $\langle c_n^*, (r_{1n}^*, r_{2n}^*) \rangle$  as the welfare criterion, the average certainty equivalent of the naive borrower, denoted by  $CE_n^*(\rho)$ , solves

$$3u(CE_n^*(\rho)) = 2u\left(\frac{2}{2+\beta^\rho}y\right) + u\left(y - \frac{2-\beta^\rho}{2+\beta^\rho}y\right).$$

Simple algebra using the assumed CRRA utility function form (1) shows that:

$$CE_n^*(\rho) = \begin{cases} \frac{2}{2+\beta^\rho} \times \left(\frac{2+\beta^{\rho-1}}{3}\right)^{\frac{\rho}{\rho-1}} \times y & \text{if } \rho > 0 \text{ and } \rho \neq 1, \\ \frac{2}{2+\beta} \times \beta^{\frac{1}{3}} \times y & \text{if } \rho = 1. \end{cases}$$

A loss function relative to the perfectly sophisticated benchmark, denoted by  $\mathcal{L}(\rho)$  with slight abuse of notation—which we use to measure the consumer vulnerability from naïvete—can thus be defined as

$$\mathcal{L}(\rho) := \frac{\frac{2}{3}y - CE_n^*(\rho)}{\frac{2}{3}y}.$$

A proposition under the context of the credit market and present-biased consumers, which is similar to Proposition 1, can then be established as follows.

**Proposition 3** (*Credit market with present-biased borrowers with naïvete*) *The ratio of the first-period consumption between a naive borrower and a sophisticated borrower (i.e.,  $c_n^*/c_s^*$ ) is greater than one, and is strictly increasing in  $\rho$  for  $\rho \in (0, \infty)$ . However, the loss function  $\mathcal{L}(\rho)$  is an inverted U-shaped curve in  $\rho$ .*

## 4 Epstein-Zin Preferences: IES vs. Risk Aversion

Thus far, we have discussed static/dynamic models of different behavioral biases and market structures. The main insight is that the curvature of consumers' utility function is important to predict the deviation of their equilibrium behavior from the rational benchmark, and thus is a key determinant of welfare analysis. Note that consumer's utility is separable and additive for the first two examples, and CRRA utility functional form is employed. This implies instantly that the IES is the inverse of relative risk aversion, and thus it is impossible to isolate the effect of IES from that of risk aversion. Indeed, it is more appropriate to interpret our result using risk aversion in the first example (i.e., insurance market with biased consumers) due to its static nature, while it is more appropriate to use the interpretation of IES in the second example (i.e., credit market with present-biased borrowers) due to its dynamic nature.

In this section, we employ the recursive preferences proposed by Epstein and Zin (1989, 1991) and introduce behavioral biases into the model. This class of preferences has been applied broadly in asset pricing, portfolio choice, as well as macroeconomics, and allows us to break the link between risk aversion and IES.

**Epstein-Zin preferences** Consider a simple setting with two periods, in which an agent consumes  $c_1$  in the first period, and consumes  $c_2^i$  in state  $i \in \{1, 2\}$  in the second period, with  $\Pr(c_2 = c_2^i) = p^i \in [0, 1]$ . We follow most of the literature on Epstein-Zin preferences and assume that the agent's utility is given by:

$$F(c_1, z(c_2^1, c_2^2; p^1, p^2); \beta, \eta, \lambda) := \left( (1 - \beta)c_1^{1-\eta} + \beta [z(c_2^1, c_2^2; p^1, p^2)]^{1-\eta} \right)^{\frac{1}{1-\eta}}, \quad (18)$$

where  $c_1$  is agent's period-1 consumption and  $z(c_2^1, c_2^2; p^1, p^2)$  is the period-2 certainty equivalent using the following function:

$$G(c) = c^{1-\lambda}, \quad (19)$$

that is,

$$z(c_2^1, c_2^2; p^1, p^2) = G^{-1} \left( \mathbb{E}G((c_2)) \right) = \left[ p^1 (c_2^1)^{1-\lambda} + p^2 (c_2^2)^{1-\lambda} \right]^{\frac{1}{1-\lambda}}. \quad (20)$$

In the above recursive formulation,  $\beta \in (0, 1]$  is agent's time preference or discount factor,  $\eta > 0$  is the inverse of the IES for deterministic variations, and  $\lambda > 0$  represents the relative risk aversion coefficient for static gambles.

Two important observations follow immediately. First, when consumption is deterministic, the preference degenerates to the usual standard time-separable expected discounted utility with discount factor  $\beta$ , IES  $\rho = 1/\eta$ , and risk aversion  $\lambda = \eta$ . Second, when agent consumes  $c > 0$  in all states and periods, we have that  $F(c, z(c, c; p^1, p^2); \beta, \eta, \lambda) = c$ . This implies instantly that  $F(\cdot)$  is the average certainty equivalent of the agent.

**Income and contracts** The agent has income  $y_1$  in the first period, and income  $y_2^i$  in state  $i \in \{1, 2\}$  in the second period. For notational convenience, let us denote  $\Pr(y_2 = y_2^1)$  by  $\mu$  and agent's average income  $(1 - \beta)y_1 + \beta [\mu y_2^1 + (1 - \mu)y_2^2]$  by  $\bar{y}$ . Agent can go to a perfectly competitive financial market to smooth his consumption across different periods and states. A contract specifies the agent's consumption level in each period and state, and is in the form of  $(c_1, c_2^1, c_2^2)$ .

Before we proceed to the analysis, it is useful to point out that this simple model can be easily reinterpreted under different contexts. For instance, the model degenerates to our first static insurance example if  $\beta = 1$ . Similarly, the model shares exactly the same insight as our second example of credit market if  $\mu \in \{0, 1\}$  and agent holds incorrect belief regarding his time preference  $\beta$ .

**Rational Benchmark** We first consider the consumption allocation of a rational agent. Clearly, the equilibrium contract will result in a consumption bundle, denoted by  $(c_1^*, c_2^{1*}, c_2^{2*})$ , that solves the following maximization problem:

$$\begin{aligned} & \max_{\{c_1, c_2^1, c_2^2\}} F(c_1, z(c_2^1, c_2^2; p^1, p^2); \beta, \eta, \lambda), \\ \text{s.t. } & (1 - \beta)c_1 + \beta [\mu c_2^1 + (1 - \mu)c_2^2] = \bar{y}, \end{aligned} \quad (21)$$

where Equation (21) is the zero-profit condition. Simple algebra would verify that

$$c_1^* = c_2^{1*} = c_2^{2*} = \bar{y}, \text{ and thus } F(c_1^*, z(c_2^{1*}, c_2^{2*}; \mu, 1 - \mu); \beta, \eta, \lambda) = \bar{y}.$$

**Remark 3** *A rational agent with Epstein-Zin preferences receives a contract that perfectly smooths his consumption across all periods and states. In the competitive equilibrium, agent's average certainty equivalent is equal to his average income  $\bar{y}$ .*

Next, we introduce behavioral biases into the model by assuming that agent may misperceive either the time preferences  $\beta$  or the probability  $\mu$ .

## 4.1 Misperception on Time Preferences

Suppose the agent believes that his time preference parameter is  $\tilde{\beta} \neq \beta$ . The equilibrium competitive contract thus maximizes the agent's perceived utility

$$\max_{\{c_1, c_2^1, c_2^2\}} F(c_1, z(c_2^1, c_2^2; \mu, 1 - \mu); \tilde{\beta}, \eta, \lambda),$$

subject to the zero-profit condition (21). Note that  $\tilde{\beta}$  only enters into agent's perceived utility function and has no impact on the zero-profit constraint.

Solving for the equilibrium consumption bundle, which we denote by  $(c_{1\beta}, c_{2\beta}^1, c_{2\beta}^2)$ , yields that

$$c_{1\beta} = \frac{\bar{y}}{(1 - \beta) + \beta T(\eta)}, \text{ and } c_{2\beta}^1 = c_{2\beta}^2 = \frac{T(\eta)\bar{y}}{(1 - \beta) + \beta T(\eta)}, \quad (22)$$

where  $T(\eta)$  is a function of the IES parameter  $1/\eta$  and is given by

$$T(\eta) := \left( \frac{\beta}{1 - \beta} \times \frac{1 - \tilde{\beta}}{\tilde{\beta}} \right)^{-\frac{1}{\eta}}.$$

Two remarks are in order. First, the equilibrium consumption profile within the second period is not distorted by the agent's misperception of time preferences: The agent still

obtains full insurance in the second period because his belief regarding the period-2 risks is correct. Second and importantly, the equilibrium consumption bundle  $(c_{1\beta}, c_{2\beta}^1, c_{2\beta}^2)$  is independent of the risk aversion parameter  $\lambda$ ; the ratio between the period-1 and the period-2 consumption depends only on the IES parameter  $1/\eta$ .

Given the equilibrium consumption bundle specified in (22), we can calculate the equilibrium certainty equivalent of the agent using the true time preferences  $\beta$ , which we denote by  $F_\beta(\lambda, \eta)$ , as the following:

$$\begin{aligned} F_\beta(\lambda, \eta) &:= F\left(c_{1\beta}, z(c_{2\beta}^1, c_{2\beta}^2; \mu, 1 - \mu); \beta, \eta, \lambda\right) \\ &= \frac{\bar{y}}{(1 - \beta) + \beta T(\eta)} \left\{ (1 - \beta) + \beta [T(\eta)]^{1-\eta} \right\}^{\frac{1}{1-\eta}}. \end{aligned} \quad (23)$$

Recall that a rational agent's equilibrium certainty equivalent in a competitive equilibrium is  $\bar{y}$ . A loss function relative to the rational benchmark, denoted by  $\mathcal{L}_\beta(\lambda, \eta)$ , can be defined as

$$\mathcal{L}_\beta(\lambda, \eta) := \frac{\bar{y} - F_\beta(\lambda, \eta)}{\bar{y}}.$$

The following result can then be obtained:

**Proposition 4 (*Epstein-Zin preferences and misperception on time preferences*)**

*Suppose that agent has Epstein-Zin preferences as described in (18), and holds incorrect belief regarding his time preferences (i.e.,  $\tilde{\beta} \neq \beta$ ). Then the following statements hold:*

- (i) *The loss function  $\mathcal{L}_\beta(\lambda, \eta)$  is independent of the agent's risk aversion  $\lambda$ ;*
- (ii) *The loss function  $\mathcal{L}_\beta(\lambda, \eta)$  follows an inverted U-shaped curve in the agent's IES  $1/\eta$ .*

For part (i) of Proposition 4, recall that agent receives full insurance in the second period. This implies instantly that the period-2 consumption across states remain efficient. Therefore, agent's period-2 certainty equivalent will not be influenced by his risk aversion coefficient  $\lambda$ , and thus his average certainty equivalent  $F_\beta$  remains constant over  $\lambda$ .

The intuition for part (ii) of Proposition 4 is exactly the same as that for Proposition 3. Agent's biased belief about his time preference coefficient  $\beta$  distorts the dynamic trade-off between the period-1 consumption and period-2 consumption, which in turn leads to welfare inefficiency. In such a scenario, it is the IES  $1/\eta$ , rather than the risk aversion  $\lambda$ , that determines the agent's equilibrium certainty equivalent and the size of welfare losses.

## 4.2 Misperception on Second-period Risks

Next, suppose the agent believes that the probability that state- $y_2^1$  occurs in the second period is  $\tilde{\mu} \neq \mu$ . The equilibrium competitive contract maximizes agent's perceived utility

$$\max_{\{c_1, c_2^1, c_2^2\}} F(c_1, z(c_2^1, c_2^2; \tilde{\mu}, 1 - \tilde{\mu}); \beta, \eta, \lambda),$$

subject to the zero-profit condition (21). Similar to the previous analysis, agent's belief  $\tilde{\mu}$  only enters into his perceived objective function and has no impact on the zero-profit constraint.

Denote the equilibrium consumption profile by  $(c_{1\mu}, c_{2\mu}^1, c_{2\mu}^2)$ , where we use the subscript  $\mu$  to indicate that agent holds incorrect belief about distribution of the period-2 risks. The first-order condition implies that

$$\frac{c_{2\mu}^1}{c_{2\mu}^2} = \left[ \frac{\mu}{1 - \mu} \times \frac{1 - \tilde{\mu}}{\tilde{\mu}} \right]^{-\frac{1}{\lambda}} =: R(\lambda), \quad (24)$$

and

$$\frac{\mu c_{2\mu}^1 + (1 - \mu)c_{2\mu}^2}{c_{1\mu}} = \left( \frac{\left(1 - \tilde{\mu} + \tilde{\mu} [R(\lambda)]^{1-\lambda}\right)^{\frac{1}{1-\lambda}}}{1 - \mu + \mu R(\lambda)} \right)^{\frac{1-\eta}{\eta}} =: Q(\lambda, \eta). \quad (25)$$

Equations (24) and (25) unveil two important sources of inefficiency. The term  $R(\lambda)$  defined in Equation (24) measures the impact of the agent's misperception of  $\mu$  on the period-2 consumption profile, and hence captures the *period-2 distortion*. Without loss of generality, suppose that  $\tilde{\mu} > \mu$ , i.e., agent assigns more weights on state- $y_2^1$ . This implies instantly that  $R(\lambda) > 1$ , and thus the agent ends up with more consumption in state- $y_2^1$  in the competitive equilibrium and results in welfare loss.

Equation (25) indicates that agent's biased belief regarding the period-2 risks also varies the intertemporal tradeoff between the period-1 consumption (i.e.,  $c_{1\mu}$ ) and the average period-2 consumption (i.e.,  $\mu c_{2\mu}^1 + (1 - \mu)c_{2\mu}^2$ ). This *dynamic distortion* is captured by the term  $Q(\lambda, \eta)$ , which clearly depends on both  $\lambda$  and  $\eta$ . Therefore, agent's welfare depends on both the risk aversion coefficient  $\lambda$  and the IES parameter  $1/\eta$  when he incorrectly estimates his period-2 risks.

Solving for the equilibrium consumption profile  $(c_{1\mu}, c_{2\mu}^1, c_{2\mu}^2)$  yields that

$$\begin{aligned} c_{1\mu} &= \frac{1}{1 - \beta + \beta Q(\lambda, \eta)} \bar{y}, \\ c_{2\mu}^1 &= \frac{Q(\lambda, \eta)}{1 - \beta + \beta Q(\lambda, \eta)} \times \frac{R(\lambda)}{1 - \mu + \mu R(\lambda)} \bar{y}, \\ c_{2\mu}^2 &= \frac{Q(\lambda, \eta)}{1 - \beta + \beta Q(\lambda, \eta)} \times \frac{1}{1 - \mu + \mu R(\lambda)} \bar{y}. \end{aligned}$$

Given the equilibrium consumption profile specified above, we can derive agent's *experienced* welfare, which we denote by  $F_\eta(\eta, \lambda)$ , as the following:

$$\begin{aligned} F_\eta(\lambda, \eta) &:= F(c_{1\mu}, z(c_{2\mu}^1, c_{2\mu}^2; \mu, 1 - \mu); \beta, \eta, \lambda) \\ &:= \frac{\bar{y}}{[1 - \beta + \beta Q(\eta, \lambda)] \times [1 - \mu + \mu R(\lambda)]} \\ &\quad \times \left[ (1 - \beta) [1 - \mu + \mu R(\lambda)]^{1-\eta} + \beta [Q(\lambda, \eta)]^{1-\eta} \left( 1 - \mu + \mu [R(\lambda)]^{1-\lambda} \right)^{\frac{1-\eta}{1-\lambda}} \right]^{\frac{1}{1-\eta}}. \end{aligned}$$

Similar to the previous analysis, we can define a loss function, which we denote by  $\mathcal{L}_\eta(\lambda, \eta)$ , as the following:

$$\mathcal{L}_\eta(\lambda, \eta) := \frac{\bar{y} - F_\eta(\lambda, \eta)}{\bar{y}}.$$

The following result can then be obtained:

**Proposition 5 (*Epstein-Zin preferences and misperception on period-2 risks*)**

Suppose that agent has Epstein-Zin preferences as described in (18), and holds incorrect belief regarding the period-2 risks (i.e.,  $\tilde{\mu} \neq \mu$ ). Then the following statements hold:

- (i)  $\lim_{\lambda \searrow 0} \mathcal{L}_\eta(\lambda, \eta) > \lim_{\lambda \nearrow \infty} \mathcal{L}_\eta(\lambda, \eta) = 0$ . As a result,  $\frac{\partial \mathcal{L}_\eta(\lambda, \eta)}{\partial \lambda}$  is negative as  $\lambda$  becomes sufficiently large. Moreover, there exists a threshold  $\bar{\beta} \in [0, 1)$  such that for  $\beta > \bar{\beta}$ ,  $\frac{\partial \mathcal{L}_\eta(\lambda, \eta)}{\partial \lambda}$  is positive as  $\lambda$  becomes sufficiently small.<sup>14</sup>
- (ii)  $\lim_{\eta \nearrow \infty} \mathcal{L}_\eta(\lambda, \eta) > \lim_{\eta \searrow 0} \mathcal{L}_\eta(\lambda, \eta) > \lim_{\eta \rightarrow 1} \mathcal{L}_\eta(\lambda, 1) > 0$ . Moreover,  $\frac{\partial \mathcal{L}_\eta(\lambda, \eta)}{\partial \eta}$  is positive as  $\eta$  becomes sufficiently small or sufficiently large.

Part (i) of Proposition 5 indicates that the loss function  $\mathcal{L}_\eta(\lambda, \eta)$  follows an inverted U-shaped curve in agent's relative risk aversion coefficient  $\lambda$  as Figure 3(a) illustrates. The intuition is similar to that in part (ii) of Proposition 4. Fixing the IES parameter  $\eta$ , although a change in the risk aversion coefficient  $\lambda$  distorts agent's intertemporal tradeoff  $Q(\lambda, \eta)$  and

<sup>14</sup>It can be verified from the proof of Proposition 5 that  $\bar{\beta} = 0$  if  $(\max\{\frac{1-\tilde{\mu}}{1-\mu}, \frac{\tilde{\mu}}{\mu}\})^{1-\eta} \geq 1 - \eta$ .

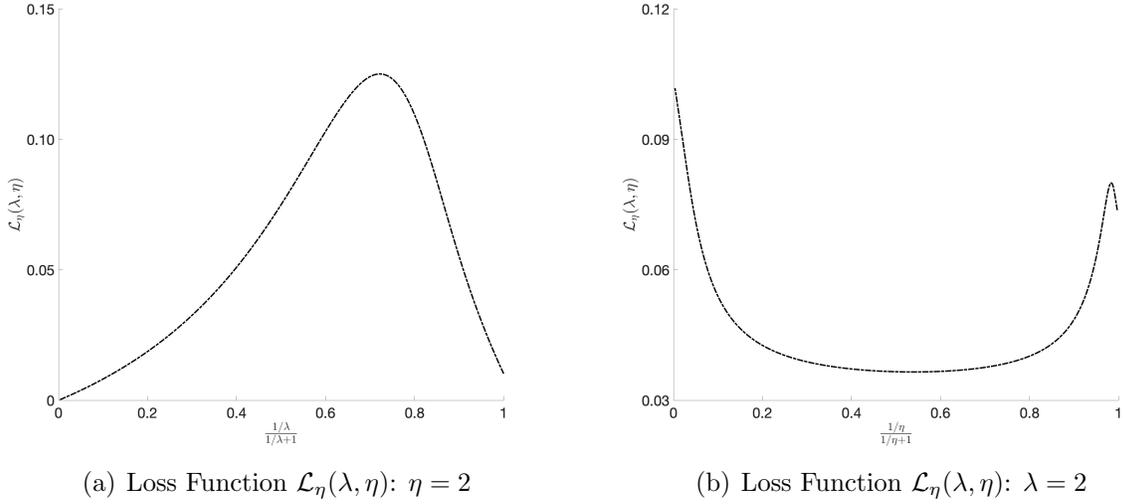


Figure 3: Epstein-Zin Preferences with Biased Belief on Period-2 Risks:  $(\bar{y}, \mu, \tilde{\mu}, \beta) = (1, 0.5, 0.25, 0.5)$ .

thus the welfare analysis is substantially complicated, the main insight as established in Proposition 1 remains qualitatively unchanged.

In contrast, by part (ii) of Proposition 5, the welfare analysis with respect to agent's IES  $1/\eta$  is less than explicit. It is somewhat surprising that Proposition 5 indicates that agent's average certainty equivalent needs to change the its monotonicity at least twice as Figure 3(b) depicts; and more importantly, that agents with extreme values of IES are now more vulnerable. The reason why agents with a moderate degree of IES are less vulnerable stems from the observation that a moderate IES limits the dynamic distortion caused by the misperception on period-2 risks. To see this more clearly, it is useful to consider the extreme case where  $1/\eta$  approaches one. It is straightforward to verify that Equation (25) degenerates to  $[\mu c_{2\mu}^1 + (1 - \mu)c_{2\mu}^2]/c_{1\mu} = 1$ , which in turn implies that  $c_{1\mu} = \bar{y} = c_1^*$ . In other words, the equilibrium period-1 consumption is equal to the average period-2 consumption, and coincides with the first-best period-1 consumption level despite the fact that agent is biased. In such a scenario, inefficiency is solely triggered by the unbalanced period-2 consumptions across state- $y_2^1$  and state- $y_2^2$ , and the dynamic distortion caused by the biased belief about  $\mu$  vanishes in the limit. As a result, the agent with a moderate level of IES  $1/\eta$  has the maximum welfare.

## 5 Projection Bias and Habit Formation

Next, we consider projection bias and investigate the impact of IES on consumer vulnerability in a simple decision-theoretic model. In particular, we consider a habit formation

model with projection biased consumers proposed by Loewenstein, O’Donoghue, and Rabin (2003).

A consumer has total income of  $y$  and allocate consumption  $c_1$  and  $c_2$  between two periods. Consumer’s instantaneous utility in period  $t = 1, 2$  is  $v(c_t, s_t) := u(c_t - s_t)$ , where  $c_t$  and  $s_t$  are respectively his consumption and the habit stock in period  $t$ . We follow Loewenstein et al. (2003) and assume that habit stock evolves according to

$$s_t = (1 - \phi)s_{t-1} + \phi c_{t-1}, \phi \in (0, 1],$$

with an initial habit stock normalized to zero (i.e.,  $s_1 = 0$ ). For simplicity, we assume that there is no discounting across periods.

Now we introduce projection bias into the model. When a consumer exhibits simple projection bias as in Loewenstein et al. (2003), his perceived utility is

$$u(c_1 - s_1) + (1 - \alpha)u(c_2 - s_2) + \alpha u(c_2 - s_1),$$

where  $\alpha \in [0, 1]$  captures the degree of projection bias in an intuitive way. Note that  $\alpha = 0$  corresponds to case of “rational habits” wherein a consumer fully takes into consideration the impact of his current assumption on his future well-being. Similarly,  $\alpha = 1$  corresponds to the case of “myopic habits” wherein a consumer does not account at all for how his current consumption influences his future habit stock.

A consumer’s decision on the first-period consumption, which we denote by  $c_1^*(\rho, \alpha)$ , is the solution to the following first-order condition:

$$u'(c_1^*(\rho, \alpha)) - (1 - \alpha)(1 + \phi)u'(y - (1 + \phi)c_1^*(\rho, \alpha)) - \alpha u'(y - c_1^*(\rho, \alpha)) = 0. \quad (26)$$

It can be shown that  $c_1^*(\rho, \alpha) > c_1^*(\rho, 0)$  for all  $\alpha \in (0, 1]$ , implying that a consumer with projection bias plans a consumption profile that consumes his income more quickly relative to a rational consumer. This result is intuitive: projection bias leads a consumer to *under-appreciate* how his current consumption will reduce his utility from future consumption, and thus leads one to consume too much.

Given the derived consumption profile  $(c_1^*(\rho, \alpha), c_2^*(\rho, \alpha))$ , we use the utility function  $u(\cdot)$  to calculate the average certainty equivalent, which we denote by  $CE^*(\rho, \alpha)$ , as follows:

$$2u [CE^*(\rho, \alpha)] = u [c_1^*(\rho, \alpha) - s_1] + u [c_2^*(\rho, \alpha) - s_2]. \quad (27)$$

A loss function, which is similar to that in Section 2.1, can be defined as

$$\mathcal{L}(\rho, \alpha) := \frac{CE^*(\rho, 0) - CE^*(\rho, \alpha)}{CE^*(\rho, 0)}. \quad (28)$$

Note that a closed-form solution to Equation (26) cannot be obtained in general even with the CRRA utility functional form. For ease of presentation, we consider the two aforementioned polar cases  $\alpha = 0$  and  $\alpha = 1$ . A discussion for the case of intermediate degree of projection bias (i.e.,  $0 < \alpha < 1$ ) will be presented after Proposition 6.

**Rational habits:  $\alpha = 0$ .** A rational consumer correctly anticipates his changes in habit stock, and chooses a consumption profile  $(c_1, c_2)$  to maximize his total utility across the two periods

$$u(c_1 - s_1) + u(c_2 - s_2), \quad (29)$$

subject to  $s_2 = (1 - \phi)s_1 + \phi c_1$ ,  $s_1 = 0$ , and  $c_1 + c_2 = y$ . The first-order condition (26) can be simplified as

$$\frac{u'(c_1 - s_1)}{u'(c_2 - s_2)} = 1 + \phi. \quad (30)$$

When  $\phi = 0$ , the model degenerates to the one without habit formation and  $u'(c_1) = u'(c_2)$ , in which case the consumer perfectly smooths his consumption across the two periods. When  $\phi > 0$ , an increase in the period-1 consumption influences the period-2 habit stock and thus decreases the period-2 utility, holding fixed the period-2 consumption level, a fact that a rational consumer would take into consideration when deciding his consumption.

Solving for the optimal consumption profile yields that

$$(c_1^*(\rho, 0), c_2^*(\rho, 0)) = \left( \frac{1}{(1 + \phi) + (1 + \phi)^\rho} y, \frac{\phi + (1 + \phi)^\rho}{(1 + \phi) + (1 + \phi)^\rho} y \right).$$

It follows immediately from above expression that  $c_1^* < c_2^*$ . This confirms the increasing consumption pattern typically emphasized in the literature on habit formation. As  $\rho$  increases, or equivalent, as a consumer becomes more risk neutral, he cares more about his total net consumption level [i.e.,  $(c_1 - s_1) + (c_2 - s_2)$ ] and incurs less disutility from an unbalanced consumption path. As a result, he optimally postpones his consumption to the second period so as to avoid the welfare loss caused by the period-1 habit stock. In the extreme case that  $\rho$  approaches infinity, his optimal plan is to consume nothing in the first period and all his income in the second period.

It is noteworthy that the rational benchmark now depends on the shape of the utility function due to the presence of habit formation, and the per period certainty equivalent

evaluated using the utility function  $u(\cdot)$  is given by

$$CE^*(\rho, 0) = \frac{y}{(1 + \phi) + (1 + \phi)^\rho} \left[ \frac{1}{2} + \frac{1}{2}(1 + \phi)^{\rho-1} \right]^{\frac{\rho}{\rho-1}},$$

which is a function of the IES coefficient  $\rho$ .

**Myopic habits:  $\alpha = 1$ .** Next, we consider the consumption behavior of a completely myopic consumer. Plugging  $\alpha = 1$  into the first-order condition (26) yields that

$$\frac{u'(c_1 - s_1)}{u'(c_2 - s_1)} = 1; \quad (31)$$

together with  $c_1 + c_2 = y$ , the optimal consumption profile can be solved as

$$c_1^*(\rho, 1) = c_2^*(\rho, 1) = \frac{y}{2}.$$

When a consumer is fully myopic, he perfectly smooths his consumption across the two periods. Although the consumption path achieve the first-best under the standard expected utility theory in the absence of habit formation (i.e.,  $\phi = 0$ ), it is suboptimal in a model of a habit formation because a consumer's period-1 consumption lowers his period-2 utility. Given the above consumption profile and Equation (27), we can derive the consumer's average certainty equivalent as follows:

$$CE^*(\rho, 1) = \frac{y}{2} \left[ \frac{1}{2} + \frac{1}{2}(1 - \phi)^{\frac{\rho-1}{\rho}} \right]^{\frac{\rho}{\rho-1}}.$$

Comparing the first-order conditions (30) and (31), we can see that, similar to the previous examples in Sections 2 to 4, projection bias creates a wedge between the marginal utility of consumption across the two periods and generates inefficiency. The following result can be obtained:

**Proposition 6 (*Implications of projection bias over habit formation*)** *Consider a model of habit formation with projection bias. The following statements hold:*

- (i) *The ratio of the period-1 consumption between a myopic consumer and a rational consumer, i.e.,  $c_1^*(\rho, 1)/c_1^*(\rho, 0)$ , is greater than one, and strictly increases with  $\rho$  for  $\rho \in (0, \infty)$ .*
- (ii) *The loss function  $\mathcal{L}(\rho, 1)$ , as defined by (28), of a consumer with myopic habits strictly decreases (increases, respectively) with  $\rho$  as  $\rho$  becomes sufficiently small (large, respectively).*

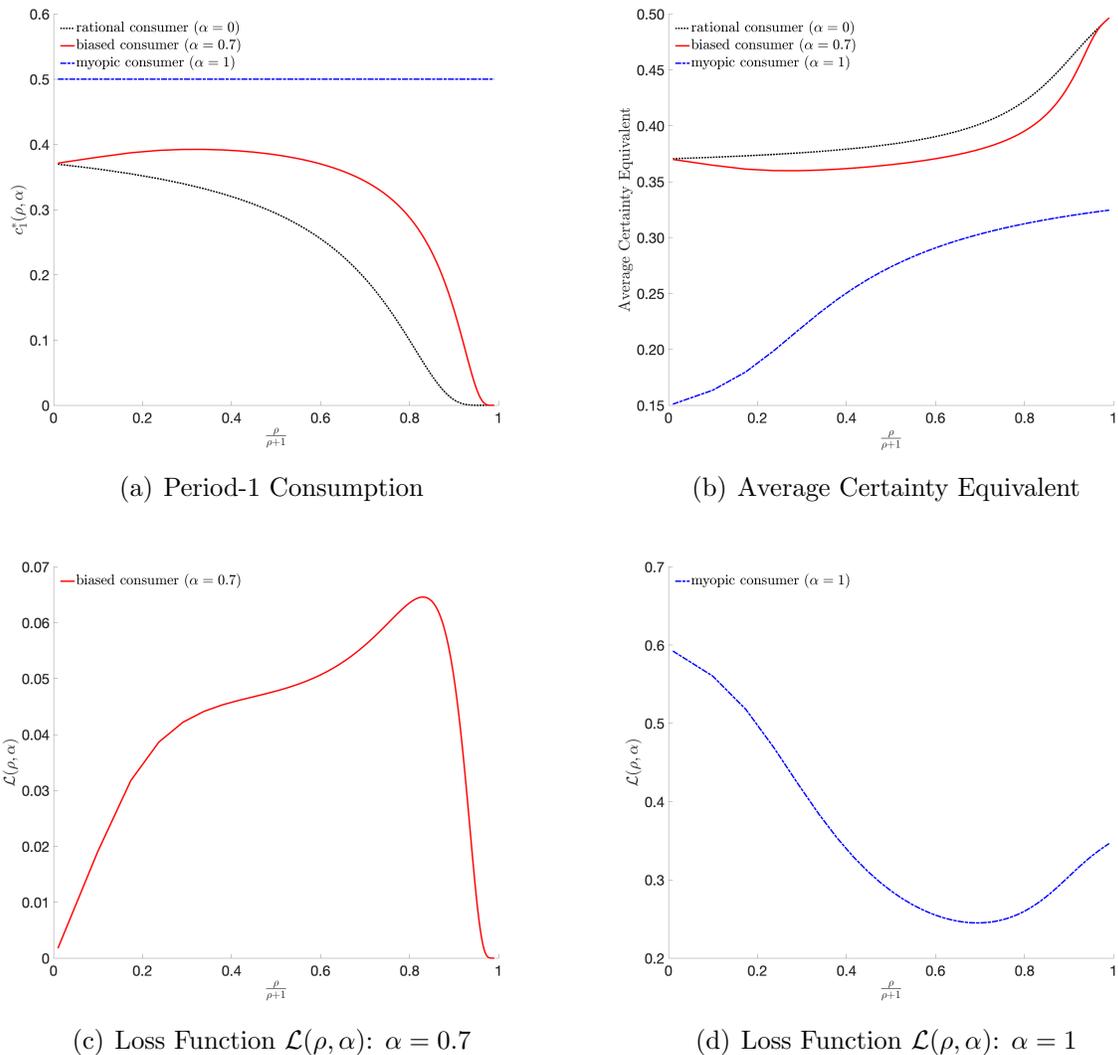


Figure 4: Habit Formation with Project Bias:  $(y, \phi) = (1, 0.7)$ .

The first part of Proposition 6 confirms the intuition obtained in the previous two examples: When the consumer exhibits extreme projection bias (i.e.,  $\alpha = 1$ ), the curvature of the utility function is closely related to the deviation of their consumption behavior from that of a consumer with rational habits. Again, a utility function with a higher degree of IES leads to more deviations from the first-best consumption path, as Figure 4(a) depicts.

Interestingly, although the loss function  $\mathcal{L}(\rho, 1)$  is non-monotone in the IES coefficient  $\rho$ , Proposition 6 indicates that it follows a U-shaped curve rather than an inverted U-shaped curve [see Figure 4(d)]. This in turn implies that consumers with an intermediate degree of IES are less vulnerable than those with extreme values of IES, and thus contrasts to the welfare results stated in Propositions 1 and 3. To understand the result, recall that under the context of habit formation, period-1 consumption causes a loss in total income of size

$s_2 = \phi c_1^*$  from Equation (29), and decreases consumer welfare, which we refer to as the *income loss effect*. The size of this income loss effect depends evidently on the period-1 consumption level  $c_1^*(\rho, \alpha)$ , and looms large when consumers' degree of projection bias (i.e.,  $\alpha$ ) increases [see Figure 4(a)]. In the extreme case that a consumer completely ignores the evolution of his habit stock (i.e.,  $\alpha = 1$ ) and  $\rho$  approaches infinity, he consumes  $y/2$  in the first period, whereas a rational consumer consumes zero. This causes a huge welfare loss to a myopic consumer, and thus the loss function exhibits an opposite pattern from those in Propositions 1 and 3 due to such income loss effect.

**Intermediate projection bias:  $\alpha \in (0, 1)$ .** Next, we provide some numerical results for intermediate degree of consumer projection bias. As Figure 4(a) illustrates, period-1 consumption level of a moderately biased consumer (the middle curve) approximates the benchmark case of rational habits (the bottom curve) for extremes values of  $\rho$ . Furthermore, it can be verified that the ratio  $c_1^*(\rho, \alpha)/c_1^*(\rho, 0)$  is strictly increasing in  $\rho$  for all  $\alpha \in (0, 1)$ , indicating that the first part of Proposition 6 is robust to intermediate projection bias.

However, the welfare analysis is different. Because  $c_1(\rho, \alpha)$  approaches  $c_1(\rho, 0)$  as  $\rho \nearrow \infty$  and  $\rho \searrow 0$ , the income loss effect vanishes in the limit; this indicates that the average certainty equivalent of a biased consumer coincides with that of a rational consumer [see Figure 4(b)]. By Figure 4(c), the loss function instead follows an inverted U-shaped curve. Under such a scenario, consumers with a moderate level of IES are the most vulnerable.

## 6 Reference-Dependent Preferences and Portfolio Choice

Since the seminal contribution by Kahneman and Tversky (1979), prospect theory is perhaps one of the most influential and well-cited framework in behavioral economics. A central component of prospect theory is that outcomes are not evaluated on an absolute scale, but rather are evaluated relative to some point of reference. In other words, agents' preferences are reference-dependent. Moreover, economic agents are loss averse in the sense that a loss relative to the reference point outweighs a gain of equal size.<sup>15</sup>

A robust literature has applied prospect theory to different subfields of economics such as finance, industrial organization, and public economics. For instance, Barberis and Xiong (2009) develop a dynamic model with loss-averse investors to investigate whether prospect theory preferences can predict a disposition effect, the tendency of retail investors to be more prone to sell their winners than their losers.<sup>16</sup> In this section, we present a simple

<sup>15</sup>The other two components are diminishing sensitivity and probability weighting.

<sup>16</sup>In Barberis and Xiong (2009), the reference point is exogenously given. Meng and Weng (2018) revisit the question and assume that the reference point is endogenously determined.

static portfolio choice setting with expectation-based loss-averse investors. Because our focus is to analyze the welfare consequences of reference-dependent preferences rather than to predict the disposition effect, we abstract from the dynamic structure and consider a simple static setting. Although there exists a plethora of studies that take a positive approach and focus on whether reference-dependent preferences can explain observed behavior, somewhat surprisingly, the welfare implications of reference-dependent preferences remain largely unexplored (O’Donoghue and Sprenger, 2018).

Consider a static model of asset allocation between a risk-free asset and a stock. There are two assets: a risk-free asset, which earns a gross return of  $R_f = 1$ ,<sup>17</sup> and a risky asset/stock. Specifically, the gross return of the asset is  $R_u > R_f$  with probability  $\kappa \in (0, 1)$ , and is  $R_d < R_f$  with probability  $1 - \kappa$ . To create a tradeoff between the risk-free asset and the stock, we assume that the expected stock return exceeds the risk-free rate, i.e.,  $\kappa R_u + (1 - \kappa) R_d > R_f$ . The price for each share of the stock is set to be unity.

The investor has initial wealth  $y$  and must decide how to split his wealth between the risk-free and risky assets. Denote the number of shares of the risky asset an investor purchases by  $\theta$ . For simplicity, we assume that short-selling is forbidden (i.e.,  $\theta \geq 0$ ), and the investor can borrow to invest in the risky asset (i.e.,  $\theta$  may exceed  $y$ ). In addition, we assume that an investor’s realized wealth in any state must be nonnegative.<sup>18</sup>

**Investor preferences** Investors have reference-dependent preferences and are expectation-based loss averse á la Kőszegi and Rabin (2006, 2007). To put it formally, when an investor’s consumption is  $c$  and his reference point is  $r$ , her *perceived* utility is given by

$$\tilde{u}(c | r) = \begin{cases} u(c) + \eta [u(c) - u(r)] & \text{if } c \geq r, \\ u(c) + \eta\lambda [u(c) - u(r)] & \text{if } c < r, \end{cases}$$

where  $u(c)$  is an investor’s *intrinsic* utilities from consumption;  $\eta \geq 0$  is the weight on gain-loss *psychological* utility; and  $\lambda \geq 1$  captures the idea that a loss looms larger than a gain of equal size and thus represents the degree of the agent’s loss aversion.

We follow Kőszegi and Rabin (2006, 2007) and assume that investor’s reference point is set to equal to his rational expectations as defined by his full probabilistic beliefs. Furthermore, at the time the investor decides on his asset composition, the investor anticipates that the lottery he faces regarding the terminal income depends on his investment decision. This assumption is incorporated into the solution concept “choice-acclimating personal equilibrium” (CPE) (Kőszegi and Rabin, 2007). More formally, fixing  $\theta \geq 0$ , investor’s reference point is the following lottery: With probability  $\kappa$ , his wealth becomes  $y - \theta + \theta R_u$ , with

<sup>17</sup>Normalizing  $R_f$  to one is without loss of generality.

<sup>18</sup>With CRRA specification, this constraint never binds due to Inada condition (i.e.,  $\lim_{c \searrow 0} u'(c) = \infty$ ).

probability  $1 - \kappa$ , he ends up with a wealth level of  $y - \theta + \theta R_d$ . Therefore, the expected *perceived* utility of an expectation-based loss-averse investor is given by

$$\begin{aligned} \tilde{U}(\theta) := & \left[ \kappa u(y - \theta + \theta R_u) + (1 - \kappa)u(y - \theta + \theta R_d) \right] \\ & + \eta \left\{ \begin{array}{l} \kappa [\kappa \cdot 0 + (1 - \kappa)[u(y - \theta + \theta R_u) - u(y - \theta + \theta R_d)]] \\ + (1 - \kappa) [\kappa \cdot \lambda [u(y - \theta + \theta R_d) - u(y - \theta + \theta R_u)] + (1 - \kappa) \cdot 0] \end{array} \right\}. \end{aligned}$$

The above expression can be further simplified as

$$\tilde{U}(\theta) = \underbrace{\left[ \kappa u(y - \theta + \theta R_u) + (1 - \kappa)u(y - \theta + \theta R_d) \right]}_{\text{intrinsic material utility}} - \underbrace{\Lambda \kappa (1 - \kappa) [u(y - \theta + \theta R_u) - u(y - \theta + \theta R_d)]}_{\text{gain-loss utility}}, \quad (32)$$

where  $\Lambda := \eta(\lambda - 1)$ . Our setup degenerates to a model with standard preferences if the second term vanishes, i.e., if  $\Lambda = 1$ .

Two remarks are in order. First, our model can be interpreted more broadly to incorporate alternative behavioral biases. To see this, note that the utility function  $\tilde{U}(\theta)$  coincides with that of disappointment aversion of Bell (1985) and Loomes and Sugden (1986) in this simple binary-lottery environment. Second, it follows from Equation (32) that the gain-loss utility is always negative for risky outcomes, and its size depends on the spread of the realized intrinsic utility. To maximize the psychological gain-loss utility, the investor would minimize dispersion in his realized outcome, which can be done by choosing  $\theta = 0$ .

**Optimal share holdings of risky assets** The optimal investment level, which we denote by  $\theta^*(\Lambda)$ , solves the following first-order condition:

$$\frac{d\tilde{U}(\theta)}{d\theta} = 0 \Leftrightarrow \frac{u'(y - \theta + \theta R_d)}{u'(y - \theta + \theta R_u)} = \frac{\kappa(R_u - 1)}{(1 - \kappa)(1 - R_d)} \times \frac{1 - (1 - \kappa)\Lambda}{1 + \kappa\Lambda}.$$

The term  $[1 - (1 - \kappa)\Lambda]/(1 + \kappa\Lambda)$ , which is less than one, captures the influence of an investor's loss aversion. Analogous to analyses for other behavioral biases in previous sections, loss aversion again creates a *wedge* between the marginal utility of consumption across the two states, which varies the investor's incentive.

Exploiting the constant IES functional form of  $u(\cdot)$  as described in (1), we obtain:

$$\frac{y - \theta + \theta R_d}{y - \theta + \theta R_u} = \left[ \frac{\kappa(R_u - 1)}{(1 - \kappa)(1 - R_d)} \times \frac{1 - (1 - \kappa)\Lambda}{1 + \kappa\Lambda} \right]^{-\rho} =: \mathcal{T}(\rho),$$

from which the optimal investment level, denoted by  $\theta^*(\rho; \Lambda)$ , can be solved as

$$\theta^*(\rho; \Lambda) = \frac{1 - \mathcal{T}(\rho; \Lambda)}{(R_u - 1)\mathcal{T}(\rho; \Lambda) + (1 - R_d)}y.$$

**Welfare with reference-dependent utility** Next, we define investor’s welfare. Welfare implications of reference-dependent preferences is an open question in the literature. A central issue to the question is that whether reference dependence and loss aversion are a manifestation of real experienced utility, or are more of a mistake. Unfortunately, no consensus has been reached on this question thus far. In the former scenario, gain-loss utility represents true feeling of “pleasure” and “pain,” and it seems natural that the psychological gain-loss component should be given a normative weight when we calculate welfare. However, experimental evidence suggests that gains and losses can be recoded through relatively innocuous changes in experimental procedures. In this case, incorporating gain-loss utility into welfare analysis seems inappropriate.<sup>19</sup>

In what follows, we take the second viewpoint and interpret the investors’ tendency to avoid losses as a “mistake.” Consequently, consumer’s welfare is measured by the certainty equivalent of his intrinsic material utility in expression (32) under the optimal portfolio choices. Given the optimal investment decision  $\theta^*(\rho; \Lambda)$ , an investor’s certainty equivalent of the intrinsic material utility, which we denote by  $CE^*(\rho; \Lambda)$ , can be derived as

$$CE^*(\rho; \Lambda) = \begin{cases} \left( \kappa [y + (R_u - 1)\theta^*]^{\frac{\rho-1}{\rho}} + (1 - \kappa) [y - (1 - R_d)\theta^*]^{\frac{\rho-1}{\rho}} \right)^{\frac{\rho}{\rho-1}} & \text{if } \rho > 0 \text{ and } \rho \neq 1, \\ [y + (R_u - 1)\theta^*]^{\kappa} \cdot [y - (1 - R_d)\theta^*]^{1-\kappa} & \text{if } \rho = 1. \end{cases} \quad (33)$$

A loss function can then be defined as

$$\mathcal{L}(\rho; \Lambda) := \frac{CE^*(\rho; 0) - CE^*(\rho; \Lambda)}{CE^*(\rho; 0)}.$$

**Main Results** We are now ready to present our main result. For notational convenience, let us define

$$\bar{\Lambda} := \frac{\kappa R_u + (1 - \kappa)R_d - 1}{\kappa(1 - \kappa)(R_u - R_d)}.$$

The following proposition can be established:

**Proposition 7 (Impact of expectation-based loss aversion on investors’ portfolio choice and welfare)** *Suppose that investors are expectation-based loss averse and  $\Lambda < \bar{\Lambda}$ .<sup>20</sup> Then the following statements hold:*

<sup>19</sup>See O’Donoghue and Sprenger (2018, p.69) for more discussions.

<sup>20</sup>When  $\Lambda$  exceeds  $\bar{\Lambda}$ , short-selling arises as the optimal trading strategy. Similar condition is commonly

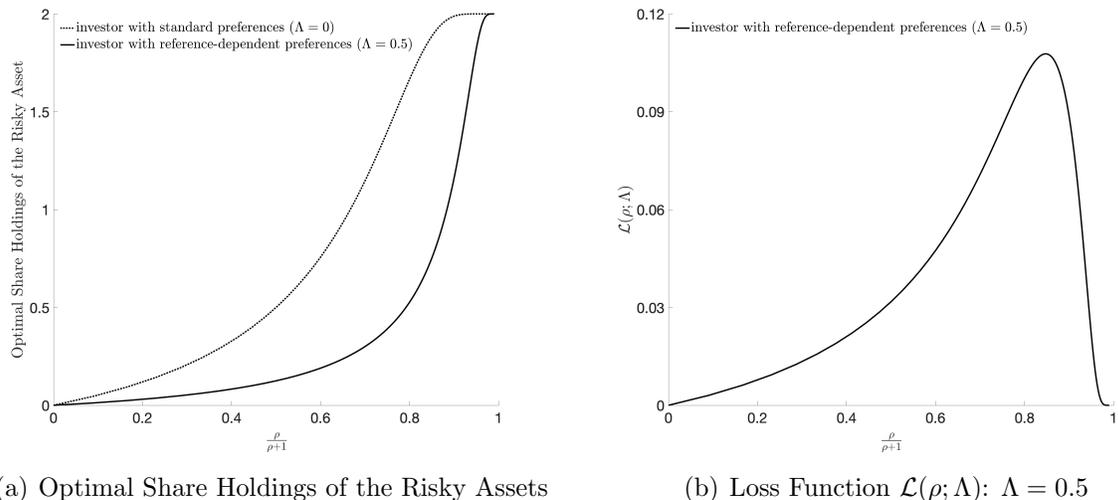


Figure 5: Portfolio Choice with Reference-dependent Preferences:  $(y, \kappa, R_u, R_d, R_f) = (1, 0.5, 2, 0.5, 1)$ .

- (i) Fix  $\Lambda > 0$ , then  $\theta^*(\rho; \Lambda) < \theta^*(\rho; 0)$ . Moreover,  $\lim_{\rho \searrow 0} \theta^*(\rho; \Lambda) = \lim_{\rho \searrow 0} \theta^*(\rho; 0) = 0$  and  $\lim_{\rho \nearrow \infty} \theta^*(\rho; \Lambda) = \lim_{\rho \nearrow \infty} \theta^*(\rho; 0) = y/(1 - R_d)$ .
- (ii)  $\lim_{\rho \searrow 0} \mathcal{L}(\rho; \Lambda) = \lim_{\rho \nearrow \infty} \mathcal{L}(\rho; \Lambda) = 0$ , and hence the loss function  $\mathcal{L}(\rho; \Lambda)$  is strictly increasing (decreasing, respectively) in  $\rho$  for sufficiently small (large, respectively)  $\rho$ .

Part (i) of Proposition 7 states that an expectation-based loss-averse investor will buy fewer shares than an investor with standard preferences. Recall from expression (32) that the investor strongly dislikes uncertainty in the sense that dispersion in realized outcome generates psychological disutility. As a result, in addition to the traditional consumption-smoothing motive caused by the concave intrinsic utility material function  $u(\cdot)$ , an expectation-based loss-averse investor has further incentive to reduce the spread in intrinsic material utilities through a more conservative investment strategy compared to an investor with standard preferences.

The strength of the aforementioned uncertainty-reducing effect vanishes as investors become sufficiently risk averse (i.e.,  $\rho \searrow 0$ ), in which case the investor with standard preferences would purchase no risky assets in the limit. Meanwhile, as investor becomes sufficiently risk neutral (i.e.,  $\rho \nearrow \infty$ ), he greatly values the additional expected payoff from holding the risky assets compared to risk-free assets; and thus both the consumption-smoothing and the uncertainty-reducing incentives become negligible.<sup>21</sup> Therefore, the investor would purchase

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assumed in the literature (e.g., Herweg, Müller, and Weinschenk, 2010) to avoid violations of first-order stochastic dominance.

<sup>21</sup>In the extreme case that  $\Lambda = \bar{\Lambda}$ , the incentive to reduce the spread of intrinsic utility is sufficiently high such that the investor would not buy any stock in the optimum and his consumption is perfectly smoothed.

the maximum amount of assets in the limit.<sup>22</sup> To summarize, the optimal portfolio choice of investors with reference-dependent preferences are the same as that of investors with standard preferences in these two extreme scenarios [see Figure 5(a)], and thus no welfare losses will be triggered in the limit. This confirms the prediction in part (ii) of Proposition 7, which in turn indicates that the most vulnerable investors are those with a moderate level of  $\rho$ , as depicted in Figure 5(b).

## 7 Concluding Remarks

In this paper, we systematically examine *consumer vulnerability* in models with behavioral biases. We measure consumer vulnerability by the percentage loss in a consumer’s equilibrium certainty equivalent from a market with non-biased consumers to that with biased ones with *the same* underlying instantaneous utility function. We consider several important behavioral biases that have been extensively studied in the literature—including the impact of biased beliefs (either over- or under-confidence) in an insurance market, the impact of present bias and naïveté about present bias in a dynamic model of credit contract design, the impact of projection bias about habit formation, and reference-dependent preferences with loss aversion—and investigate how consumer vulnerability is related to the *curvatures* of their utility functions.

We focus on the role of risk aversion in static models and the role of intertemporal elasticity of substitution (IES) in dynamic models. Although insight or effect specific to certain behavioral biases is identified, a robust pattern on welfare is observed: Consumer vulnerability to the commonly studied behavioral biases in the literature, as measured by the *welfare loss* relative to a rational benchmark, has in general a non-monotonic relationship with risk aversion or IES. This is in stark contrast to the deviations in the *allocations* from the rational benchmark, which are often monotonic to the risk aversion or IES. Our paper complements the existing literature in behavioral economics that focused almost exclusively on how behavioral bias leads to the deviations in allocations from the rational benchmark and the potential solutions to the associated market inefficiency.

Our study leaves large room for future extension. This paper assumes homogeneous agents. In practice, consumers often possess valuable private information (e.g., risks, income, risk preferences) to firms. It would be promising to introduce consumer heterogeneity and private information into the stylized modeling frameworks we considered in this paper, and investigate how the externality rational consumers exert on biased consumers, or vice versa, depends on risk aversion or IES. Another possible avenue for future research

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<sup>22</sup>Recall that the nonnegativity of wealth constraint requires that  $y - \theta + \theta R_d \geq 0$ , implying that  $\theta \leq y/(1 - R_d)$ .

is to extend our analysis to allow for regulations and/or policies, and examine how their effectiveness depends on risk aversion or IES under different economic contexts. We leave the exploration of these possibilities to future research.

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# Appendix: Proof of Propositions

## Proof of Proposition 1

**Proof.** Note that  $CE^*(\rho; \mu, \mu)$  is a constant. Therefore, it suffices to show that  $CE^*(\rho; \mu, \hat{\mu})$  is U-shaped with respect to  $\rho \in (0, \infty)$  for  $\hat{\mu} \neq \mu$ . Recall that  $\tau \equiv \left(\frac{1-\hat{\mu}}{\hat{\mu}} \times \frac{\mu}{1-\mu}\right)^{-1} > 0$ . It follows from Equation (7) that

$$\begin{aligned} CE^*(\rho; \mu, \hat{\mu}) &= \left( \mu [y - (1 - x^*)\omega - p^*]^{\frac{\rho-1}{\rho}} + (1 - \mu)(y - p^*)^{\frac{\rho-1}{\rho}} \right)^{\frac{\rho}{\rho-1}} \\ &= (y - p^*) \times \left\{ 1 - \mu + \mu \left[ \frac{y - (1 - x^*)\omega - p^*}{y - p^*} \right]^{\frac{\rho-1}{\rho}} \right\}^{\frac{\rho}{\rho-1}} \\ &= \frac{y - \mu\omega}{1 - \mu + \mu\tau^\rho} \times (1 - \mu + \mu\tau^{\rho-1})^{\frac{\rho}{\rho-1}}, \end{aligned}$$

where the last equality follows from (4) and (5). It is useful to prove the following intermediate result.

**Lemma 1** *Suppose that  $\mu \in (0, 1)$ ,  $\tau > 0$ , and  $\tau \neq 1$ . Then*

$$f(\rho; \mu, \tau) = \frac{1}{1 - \mu + \mu\tau^\rho} (1 - \mu + \mu\tau^{\rho-1})^{\frac{\rho}{\rho-1}}$$

*is U-shaped in  $\rho$  for  $\rho > 0$ .*

**Proof.** Note that

$$f(\rho; \mu, \tau) = f\left(\rho; 1 - \mu, \frac{1}{\tau}\right).$$

Therefore, it suffices to show that  $f(\rho; \mu, \tau)$  is U-shaped only for the case  $\tau > 1$ . Let

$$U(\rho) := \ln(f(\rho; \mu, \tau)) = \frac{\rho}{\rho-1} \ln(1 - \mu + \mu\tau^{\rho-1}) - \ln(1 - \mu + \mu\tau^\rho).$$

It follows immediately that

$$\frac{\partial f(\rho; \mu, \tau)}{\partial \rho} = f(\rho; \mu, \tau) \times U'(\rho),$$

and

$$U'(\rho) = \frac{\mu\rho\tau^{\rho-1} \ln(\tau)}{(\rho-1)(\mu\tau^{\rho-1} - \mu + 1)} + \frac{\ln(\mu\tau^{\rho-1} - \mu + 1)}{\rho-1} - \frac{\rho \ln(\mu\tau^{\rho-1} - \mu + 1)}{(\rho-1)^2} - \frac{\mu\tau^\rho \ln(\tau)}{\mu\tau^\rho - \mu + 1}.$$

It is straightforward to verify that

$$\lim_{\rho \searrow 0} U'(\rho) = - \left[ \ln \left( 1 - \mu + \frac{\mu}{\tau} \right) + \mu \ln(\tau) \right] < 0;$$

together with the fact that  $f(\rho) > 0$  for all  $\rho \in (0, \infty)$ , we must have that  $f'(\rho) < 0$  when  $\rho$  is sufficiently small. Therefore, to prove the lemma, it suffices to show that there exists a unique solution to  $U'(\rho) = 0$  for  $\rho \in (0, \infty)$ .

Let  $z := \tau^{\rho-1} > \frac{1}{\tau}$ . Then  $\rho = \frac{\ln(z)}{\ln(\tau)} + 1$ . Moreover,  $U'(\rho) > 0$  is equivalent to

$$\begin{aligned} \Delta(z) &:= \frac{(1 - \mu + \mu z)(1 - \mu + \mu \tau z) \ln^2(z)}{\ln(\tau)} U' \left( \frac{\ln(z)}{\ln(\tau)} + 1 \right) \\ &= -\mu(1 - \mu)(\tau - 1)z [\ln(z)]^2 + \mu(1 - \mu + \mu \tau z) \ln(\tau)z \ln(z) \\ &\quad - \ln(\tau)(1 - \mu + \mu z)(1 - \mu + \mu \tau z) \ln(1 - \mu + \mu z) > 0. \end{aligned} \quad (\text{A1})$$

Clearly,  $z = 1$  is one solution to  $\Delta(z) = 0$ . In order to prove that there exists a unique solution to  $U'(\rho) = 0$  for  $\rho \in (0, \infty)$ , it suffices to show that (i) there exists exactly one more solution to  $\Delta(z) = 0$  for  $z \in (\frac{1}{\tau}, \infty)$ , and  $\lim_{\rho \rightarrow 1} U'(\rho) \neq 0$ ; or (ii)  $z = 1$  is the unique solution to  $\Delta(z) = 0$  and  $\lim_{\rho \rightarrow 1} U'(\rho) = 0$ .

Carrying out the algebra, we have that

$$\begin{aligned} \Delta'(z) &= \mu \left\{ -(1 - \mu)(\tau - 1) \ln^2(z) + [-2(1 - \mu)(\tau - 1) + \ln(\tau) + \mu \ln(\tau)(2\tau z - 1)] \ln(z) \right. \\ &\quad \left. + [-\tau + \mu(\tau - 2\tau z + 1) - 1] \ln(\tau) \ln(1 - \mu + \mu z) \right\}, \\ \Delta''(z) &= 2\mu^2 \tau \ln(\tau) [\ln(z) - \ln(1 - \mu + \mu z)] - 2\mu(1 - \mu)(\tau - 1) \frac{\ln(z) + 1}{z} \\ &\quad + \mu \ln(\tau) \left( \frac{1}{z} - \frac{\mu}{1 - \mu + \mu z} \right) (1 - \mu + \mu \tau z), \\ \Delta'''(z) &= \mu(1 - \mu) \left\{ \frac{2(\tau - 1) \ln(z)}{z^2} + \frac{\ln(\tau) [3\mu \tau z(1 - \mu + \mu z) - (1 - \mu + 2\mu z)(1 - \mu + \mu \tau z)]}{z^2(1 - \mu + \mu z)^2} \right\}. \end{aligned}$$

We first start from the third derivative of  $\Delta(z)$ . It is straightforward to verify that  $\Delta'''(z) > 0$  is equivalent to

$$\mathcal{G}(z) := \frac{2(\tau - 1) \ln(z)}{\ln(\tau)} - \frac{(1 - \mu + 2\mu z)(1 - \mu + \mu \tau z) - 3\mu \tau z(1 - \mu + \mu z)}{(1 - \mu + \mu z)^2} > 0.$$

Note that

$$\mathcal{G}'(z) = \frac{2(\tau - 1)}{z \ln(\tau)} + \frac{2\mu(1 - \mu) [(1 - \mu)\tau + \mu z]}{(1 - \mu + \mu z)^3} > 0.$$

Therefore,  $\mathcal{G}(z)$  is strictly increasing in  $z$ , which in turn implies that there exists at most one solution to  $\Delta'''(z) = 0$  for  $z > \frac{1}{\tau}$ . Note that  $\lim_{z \nearrow \infty} \mathcal{G}(z) = \infty$ . Therefore,  $\Delta'''(z)$  must fall into one of the following two possibilities:

- (i)  $\Delta'''(z) > 0$  for all  $z > \frac{1}{\tau}$ .
- (ii) There exists  $z^* \in (\frac{1}{\tau}, \infty)$  such that  $\Delta'''(z^*) = 0$ ; moreover,  $\Delta'''(z) \geq 0$  for  $z \geq z^*$ .

The above observation, together with the fact that  $\lim_{z \nearrow \infty} \Delta''(z) = -2\mu^2\tau \ln(\mu) \ln(\tau) > 0$ , indicates that  $\Delta''(z)$  must fall into one of the following three possibilities:

- (a)  $\Delta''(z) > 0$  for all  $z > \frac{1}{\tau}$ .
- (b) There exists  $z^{**} > \frac{1}{\tau}$  such that  $\Delta''(z^{**}) = 0$ ; moreover,  $\Delta''(z) \geq 0$  for  $z \geq z^{**}$ .
- (c) There exist  $z_1$  and  $z_2$ , with  $z_2 > z_1 > \frac{1}{\tau}$ , such that  $\Delta''(z_1) = \Delta''(z_2) = 0$ ; moreover,  $\Delta''(z) < 0$  for  $z \in (z_1, z_2)$  and  $\Delta''(z) > 0$  for  $z \in (\frac{1}{\tau}, z_1) \cup (z_2, \infty)$ .

The above observation implies instantly that there are three possibilities regarding the convexity/concavity of  $\Delta(z)$ .

**Case (a)** We show that this case is impossible. Recall that  $\Delta(1) = 0$ . It is straightforward to verify that  $\Delta'(1) = 0$ . Moreover,

$$\lim_{z \searrow \frac{1}{\tau}} \Delta(z) = \left(1 - \mu + \frac{\mu}{\tau}\right) \ln(\tau) U'(0) < 0,$$

and

$$\lim_{z \nearrow \infty} \Delta(z) = \infty.$$

Therefore,  $z = 1$  is the unique solution to  $\Delta(z) = 0$  and  $\Delta'(1) > 0$  from the convexity of  $\Delta(\cdot)$ , which is a contradiction against  $\Delta'(1) = 0$ .

**Case (b)** First, it can be verified that

$$\Delta''(1) = 0 \Leftrightarrow \lim_{\rho \rightarrow 1} U'(\rho) = 0.$$

We consider the following three subcases:

1.  $z^{**} < 1$ . Recall that  $\Delta(1) = 0$ ,  $\Delta'(1) = 0$ ,  $\lim_{z \searrow \frac{1}{\tau}} \Delta(z) < 0$ , and  $\lim_{z \nearrow \infty} \Delta(z) > 0$ . Therefore, in addition to  $z = 1$ , there exists exactly one more solution to  $\Delta(z) = 0$ , which we denote by  $z^*$ . Moreover, we must have that  $z^* \in (\frac{1}{\tau}, z^{**})$  and thus  $\Delta''(z^*) < 0$ , which in turn implies that  $\lim_{\rho \rightarrow 1} U'(\rho) \neq 0$ .

2.  $z^{**} = 1$ . In this case,  $z = 1$  is the unique solution to  $\Delta(z) = 0$ . Moreover, we have that  $\Delta''(1) = 0$ .
3.  $z^{**} > 1$ . Similar to subcase 1, there exists exactly one more solution to  $\Delta(z) = 0$ , which we denote by  $\hat{z}^*$ . Moreover, we must have that  $\hat{z}^* > z^{**}$  and thus  $\Delta''(\hat{z}^*) > 0$ , which in turn implies that  $\lim_{\rho \rightarrow 1} U'(\rho) \neq 0$ .

**Case (c)** In this case,  $\Delta(z)$  is convex in  $z$  for  $z \in (\frac{1}{\tau}, z_1)$ , concave for  $z \in (z_1, z_2)$ , and then convex again for  $z \in (z_2, \infty)$ . The analysis is similar to Case b and is omitted for brevity.

To summarize,  $\Delta(z) = 0$  has one or two solutions on the interval  $(\frac{1}{\tau}, \infty)$ . When there exists a unique solution, it must be  $z = 1$  and  $\lim_{\rho \rightarrow 1} U'(\rho) = 0$ . When there exists another solution in addition to  $z = 1$ , we must have  $\lim_{\rho \rightarrow 1} U'(\rho) \neq 0$ . This in turn implies that there exists a unique solution to  $U'(\rho) = 0$  for  $\rho > 0$ , and thus  $f(\rho; \mu, \tau)$  is U-shaped in  $\rho$ . ■

Proposition 1 follows directly from Lemma 1. This completes the proof. ■

## Proof of Proposition 2

**Proof.** We first consider consumer's certainty equivalent, which can be written as

$$CE^m(\rho; \mu, \hat{\mu}) = [B(\rho)]^{\frac{\rho}{\rho-1}} \times \left[ (1 - \mu) + \mu [A(\rho)]^{\frac{\rho-1}{\rho}} \right]^{\frac{\rho}{\rho-1}},$$

where

$$A(\rho) = \tau^\rho, \text{ and } B(\rho) := \frac{\hat{\mu}(y - \omega)^{\frac{\rho-1}{\rho}} + (1 - \hat{\mu})y^{\frac{\rho-1}{\rho}}}{\hat{\mu}A^{\frac{\rho-1}{\rho}} + (1 - \hat{\mu})}.$$

**Case I:  $\rho < 1$ .** Carrying out the algebra,  $CE^m(\rho; \mu, \hat{\mu}) < CE^m(\rho; \mu, \mu)$  is equivalent to

$$(1 - \mu)B(\rho) + \mu B(\rho)\tau^{\rho-1} > \mu(y - \omega)^{\frac{\rho-1}{\rho}} + (1 - \mu)y^{\frac{\rho-1}{\rho}},$$

which can be further simplified as

$$[\hat{\mu} - \mu] \times \left[ \left(1 - \frac{\omega}{y}\right)^{\frac{\rho-1}{\rho}} - \tau^{\rho-1} \right] > 0. \quad (\text{A2})$$

For  $\hat{\mu} > \mu$ , condition (A2) always holds. For  $\hat{\mu} < \mu$ , condition (A2) holds for  $\rho > \frac{\ln(y-\omega) - \ln(y)}{\ln(\tau)}$ .

**Case II:  $\rho > 1$ .** It can be verified that  $CE^m(\rho; \mu, \hat{\mu}) < CE^m(\rho; \mu, \mu)$  is equivalent to

$$(1 - \mu)B(\rho) + \mu B(\rho)\tau^{\rho-1} < \mu(y - \omega)^{\frac{\rho-1}{\rho}} + (1 - \mu)y^{\frac{\rho-1}{\rho}},$$

which can be further simplified as

$$[\hat{\mu} - \mu] \times \left[ \left(1 - \frac{\omega}{y}\right)^{\frac{\rho-1}{\rho}} - \tau^{\rho-1} \right] < 0. \quad (\text{A3})$$

For  $\hat{\mu} > \mu$ , condition (A3) always holds. For  $\hat{\mu} < \mu$ , condition (A3) holds for  $\rho > \frac{\ln(y-\omega) - \ln(y)}{\ln(\tau)}$ . This completes the proof for certainty equivalent.

Next, we consider firm's profit. Carrying out the algebra, the monopolistic firm's profit can be written as

$$\pi^m(\rho; \mu, \hat{\mu}) = y - \mu\omega + [\mu(1 - \tau^\rho) - 1] \times [B(\rho)]^{\frac{\rho}{\rho-1}}.$$

For  $\hat{\mu} > \mu$ , it is straightforward to see that the optimal contract to rational consumers is also a feasible contract to biased consumers without violating the IR constraint. Therefore, we must have  $\pi^m(\rho; \mu, \hat{\mu}) > \pi^m(\rho; \mu, \mu)$  for all  $\rho > 0$  in this case; and it remains to prove the result for the case  $\hat{\mu} < \mu$ .

It is useful to prove several intermediate results.

**Lemma 2** *Suppose that  $\hat{\mu} < \mu$ . Then  $\pi^m(\rho; \mu, \hat{\mu}) < \pi^m(\rho; \mu, \mu)$  for all  $\rho \leq \rho_1$ .*

**Proof.** Carrying out algebra,  $\pi^m(\rho; \mu, \hat{\mu}) < \pi^m(\rho; \mu, \mu)$  is equivalent to

$$\frac{1 - \mu}{1 - \hat{\mu}} \times (1 - \hat{\mu} + \hat{\mu}\tau^{\rho-1})^{-\frac{1}{\rho-1}} > \left( \frac{\mu \left(1 - \frac{\omega}{y}\right)^{\frac{\rho-1}{\rho}} + 1 - \mu}{\hat{\mu} \left(1 - \frac{\omega}{y}\right)^{\frac{\rho-1}{\rho}} + 1 - \hat{\mu}} \right)^{\frac{\rho}{\rho-1}}. \quad (\text{A4})$$

It suffices to show that the above inequality holds for all  $\rho \leq \rho_1 \equiv \frac{\ln(y-\omega) - \ln(y)}{\ln(\tau)}$ . Note that the left-hand side of the above inequality is independent of  $\omega$ . For notational convenience, let us denote the left-hand side and the right-hand side of (A4) by  $\Psi_L(\rho; \mu, \hat{\mu})$  and  $\Psi_R(\rho, \omega; \mu, \hat{\mu})$  respectively. Simple algebra yields that

$$\frac{\partial \ln \Psi_R(\rho, \omega; \mu, \hat{\mu})}{\partial \omega} = -\frac{\left(1 - \frac{\omega}{y}\right)^{-\frac{1}{\rho}}}{y} \times \left( \frac{1}{\frac{1-\mu}{\mu} + \left(1 - \frac{\omega}{y}\right)^{\frac{\rho-1}{\rho}}} - \frac{1}{\frac{1-\hat{\mu}}{\hat{\mu}} + \left(1 - \frac{\omega}{y}\right)^{\frac{\rho-1}{\rho}}} \right) < 0,$$

where the strict inequality follows from  $\hat{\mu} < \mu$ . Therefore,  $\Psi_R(\rho, \omega; \mu, \hat{\mu})$  is strictly decreasing in  $\omega$  for  $\omega \in (0, y)$ . Recall that  $\tau$  is defined as  $\tau \equiv \left(\frac{1-\hat{\mu}}{\hat{\mu}} \times \frac{\mu}{1-\mu}\right)^{-1}$ ; together with the assumption  $\hat{\mu} < \mu$ , we have  $\tau < 1$  and

$$\rho \leq \rho_1 \Leftrightarrow 1 - \tau^\rho \leq 1 - \tau^{\rho_1} \equiv \frac{\omega}{y} \Leftrightarrow \omega \geq (1 - \tau^\rho)y.$$

This in turn implies that

$$\Psi_R(\rho, \omega; \mu, \hat{\mu}) \leq \Psi_R(\rho, (1 - \tau^\rho)y; \mu, \hat{\mu}) = \left(\frac{\mu\tau^{\rho-1} + 1 - \mu}{\hat{\mu}\tau^{\rho-1} + 1 - \hat{\mu}}\right)^{\frac{\rho}{\rho-1}}. \quad (\text{A5})$$

Next, we show that

$$\left(\frac{\mu\tau^{\rho-1} + 1 - \mu}{\hat{\mu}\tau^{\rho-1} + 1 - \hat{\mu}}\right)^{\frac{\rho}{\rho-1}} < \frac{1 - \mu}{1 - \hat{\mu}} \times (1 - \hat{\mu} + \hat{\mu}\tau^{\rho-1})^{-\frac{1}{\rho-1}}. \quad (\text{A6})$$

Exploiting the fact that  $\tau \equiv \left(\frac{1-\hat{\mu}}{\hat{\mu}} \times \frac{\mu}{1-\mu}\right)^{-1}$ , the above inequality can be simplified as

$$(\mu\tau^{\rho-1} + 1 - \mu)^{\frac{1}{\rho-1}} < (\mu\tau^\rho + 1 - \mu)^{\frac{1}{\rho}},$$

which holds due to the well-known result that the general mean (power mean) function  $M(\rho) = \left(\sum_{i=1}^n f_i x_i^\rho\right)^{\frac{1}{\rho}}$  is strictly increasing in  $\rho$  for  $\rho \in (-\infty, \infty)$ , holding fixed  $\mathbf{x} \equiv (x_1, \dots, x_n) > (0, \dots, 0)$  and  $\mathbf{f} \equiv (f_1, \dots, f_n) > (0, \dots, 0)$ . Therefore, the strict inequality (A6) holds.

Combining (A5) and (A6), we can obtain that

$$\Psi_R(\rho, \omega; \mu, \hat{\mu}) \leq \Psi_R(\rho, (1 - \tau^\rho)y; \mu, \hat{\mu}) < \frac{1 - \mu}{1 - \hat{\mu}} \times (1 - \hat{\mu} + \hat{\mu}\tau^{\rho-1})^{-\frac{1}{\rho-1}} \equiv \Psi_L(\rho; \mu, \hat{\mu}).$$

This completes the proof. ■

**Lemma 3** *Suppose that  $\hat{\mu} < \mu$ . Then  $\pi^m(\rho; \mu, \hat{\mu})$  is strictly decreasing in  $\rho$  for  $\rho < \rho_1$ , and is strictly increasing in  $\rho$  for  $\rho > \rho_1$ .*

**Proof.** Note that

$$\begin{aligned} & \ln(y - \mu\omega - \pi^m(\rho; \mu, \hat{\mu})) \\ &= \ln\left(\frac{1 - \mu}{1 - \hat{\mu}}y\right) + \frac{\rho}{\rho - 1} \ln\left(1 - \hat{\mu} + \hat{\mu}\left(1 - \frac{\omega}{y}\right)^{\frac{\rho-1}{\rho}}\right) - \frac{1}{\rho - 1} \ln(1 - \hat{\mu} + \hat{\mu}\tau^{\rho-1}), \end{aligned}$$

and thus

$$\frac{\partial \ln (y - \mu\omega - \pi^m(\rho; \mu, \hat{\mu}))}{\partial \rho} = G \left( \left(1 - \frac{\omega}{y}\right)^{\frac{1}{\rho}} \right) - G(\tau),$$

where  $G(x)$  is defined as

$$G(x) := -\frac{1}{(\rho-1)^2} \ln(1 - \hat{\mu} + \hat{\mu}x^{\rho-1}) + \frac{1}{\rho-1} \times \frac{\hat{\mu}x^{\rho-1} \ln x}{1 - \hat{\mu} + \hat{\mu}x^{\rho-1}}.$$

Therefore, we have that

$$\frac{\partial \pi^m(\rho; \mu, \hat{\mu})}{\partial \rho} \geq 0 \Leftrightarrow G \left( \left(1 - \frac{\omega}{y}\right)^{\frac{1}{\rho}} \right) \leq G(\tau).$$

Note that  $\rho \geq \rho_1$  is equivalent to  $\tau \leq \left(1 - \frac{\omega}{y}\right)^{\frac{1}{\rho}}$ . Therefore, to prove the lemma, it suffices to show that  $G(x)$  is strictly decreasing in  $x$  for  $x \in (0, 1)$ . Carrying out the algebra, we can obtain that

$$G'(x) = \frac{\hat{\mu}(1 - \hat{\mu})x^{\rho-2} \ln x}{(1 - \hat{\mu} + \hat{\mu}x^{\rho-1})^2},$$

which is clearly strictly negative. This concludes the proof. ■

We are now ready to prove the proposition. Lemma 2 states that  $\pi^m(\rho; \mu, \hat{\mu}) < \pi^m(\rho; \mu, \mu)$  for all  $\rho \in (0, \rho_1]$ . Lemma 3 states that  $\pi^m(\rho; \mu, \hat{\mu})$  is strictly increasing in  $\rho$ . Moreover, it can be verified that  $\pi^m(\rho; \mu, \mu)$  is strictly decreasing in  $\rho$  for all  $\rho \in (0, +\infty)$ .

Next, note that  $\lim_{\rho \nearrow \infty} \tau^\rho = 0$  and  $\lim_{\rho \nearrow \infty} B(\rho) = \frac{y - \hat{\mu}\omega}{1 - \hat{\mu}}$ . Therefore,

$$\lim_{\rho \nearrow \infty} \pi^m(\rho; \mu, \mu) = y - \mu\omega - \lim_{\rho \nearrow \infty} \left[ \mu(y - \omega)^{\frac{\rho-1}{\rho}} + (1 - \mu)y^{\frac{\rho-1}{\rho}} \right]^{\frac{\rho}{\rho-1}} = 0,$$

and

$$\lim_{\rho \nearrow \infty} \pi^m(\rho; \mu, \hat{\mu}) = (y - \mu\omega) - \frac{1 - \mu}{1 - \hat{\mu}}(y - \hat{\mu}\omega) > 0.$$

It follows immediately from the above equations that  $\lim_{\rho \nearrow \infty} \pi^m(\rho; \mu, \hat{\mu}) > \lim_{\rho \nearrow \infty} \pi^m(\rho; \mu, \mu)$ . Moreover, we have that  $\pi^m(\rho_1; \mu, \hat{\mu}) < \pi^m(\rho_1; \mu, \mu)$  by Lemma 3. These facts, together with the monotonicity of  $\pi^m(\rho; \mu, \hat{\mu})$  and  $\pi^m(\rho; \mu, \mu)$ , imply instantly the existence of a unique threshold  $\rho_2 \in (\rho_1, \infty)$  such that  $\pi^m(\rho; \mu, \hat{\mu}) < \pi^m(\rho; \mu, \mu)$  for  $\rho < \rho_2$  and  $\pi^m(\rho; \mu, \hat{\mu}) > \pi^m(\rho; \mu, \mu)$  for  $\rho > \rho_2$ . This completes the proof. ■

### Proof of Proposition 3

**Proof.** It is straightforward to see that  $c_n/c_s = 3/(2 + \beta^\rho) > 1$  is strictly increasing in  $\rho$ . Note that

$$\mathcal{L}(\rho) = 1 - f\left(\rho; \frac{1}{3}, \beta\right),$$

where the function  $f(\cdot)$  is defined in Lemma 1. Recall that  $f(\rho; \frac{1}{3}, \beta)$  follows an U-shaped curve in  $\rho$  from Lemma 1. Therefore,  $\mathcal{L}(\rho)$  follows an inverted U-shaped curve in  $\rho$ . This completes the proof. ■

### Proof of Proposition 4

**Proof.** The first part of Proposition 4 is obvious, and the second part follows immediately from Lemma 1. This completes the proof. ■

### Proof of Proposition 5

**Proof.** We first prove part (i) of the proposition. By the expression of the loss function  $\mathcal{L}_\eta(\lambda, \eta)$ , it suffices to show that  $\lim_{\lambda \searrow 0} F_\eta(\lambda, \eta) < \lim_{\lambda \nearrow \infty} F_\eta(\lambda, \eta) = \bar{y}$  and  $\frac{\partial F_\eta(\lambda, \eta)}{\partial \lambda}$  is negative (positive, respectively) as  $\lambda$  becomes sufficiently small (large, respectively).

For notational convenience, let us define  $Q_0(\eta)$  as

$$Q_0(\eta) := \left( \max \left\{ \frac{1 - \tilde{\mu}}{1 - \mu}, \frac{\tilde{\mu}}{\mu} \right\} \right)^{\frac{1-\eta}{\eta}}.$$

Carrying out the algebra, it can be verified that

$$\lim_{\lambda \searrow 0} [R(\lambda)]^{1-\lambda} = \lim_{\lambda \searrow 0} R(\lambda) = \begin{cases} 0 & \text{if } \tilde{\mu} < \mu, \\ \infty & \text{if } \tilde{\mu} > \mu, \end{cases}$$

which in turn implies that

$$\lim_{\lambda \searrow 0} Q(\lambda, \eta) = \lim_{\lambda \searrow 0} \left( \frac{[1 - \tilde{\mu} + \tilde{\mu} [R(\lambda)]^{1-\lambda}]^{\frac{1}{1-\lambda}}}{1 - \mu + \mu R(\lambda)} \right)^{\frac{1-\eta}{\eta}} = \left( \max \left\{ \frac{1 - \tilde{\mu}}{1 - \mu}, \frac{\tilde{\mu}}{\mu} \right\} \right)^{\frac{1-\eta}{\eta}} \equiv Q_0(\eta),$$

and

$$\lim_{\lambda \searrow 0} \frac{[1 - \mu + \mu [R(\lambda)]^{1-\lambda}]^{\frac{1}{1-\lambda}}}{1 - \mu + \mu R(\lambda)} = 1.$$

Therefore, we have that

$$\begin{aligned} \lim_{\lambda \searrow 0} F_\eta(\lambda, \eta) &= \frac{\bar{y}}{1 - \beta + \beta \lim_{\lambda \searrow 0} Q(\lambda, \eta)} \\ &\quad \times \left[ (1 - \beta) + \beta \lim_{\lambda \searrow 0} [Q(\lambda, \eta)]^{1-\eta} \left( \lim_{\lambda \searrow 0} \frac{[1 - \mu + \mu [R(\lambda)]^{1-\lambda}]^{\frac{1}{1-\lambda}}}{1 - \mu + \mu R(\lambda)} \right)^{1-\eta} \right]^{\frac{1}{1-\eta}} \\ &= \frac{\bar{y}}{1 - \beta + \beta Q_0(\eta)} \times [(1 - \beta) + \beta [Q_0(\eta)]^{1-\eta}]^{\frac{1}{1-\eta}} < \bar{y}. \end{aligned}$$

Next, note that

$$\lim_{\lambda \nearrow \infty} R(\lambda) = \lim_{\lambda \nearrow \infty} \left[ \frac{\mu}{1 - \mu} \times \frac{1 - \tilde{\mu}}{\tilde{\mu}} \right]^{-\frac{1}{\lambda}} = 1.$$

It follows immediately that

$$\lim_{\lambda \nearrow \infty} Q(\eta, \lambda) = \lim_{\lambda \nearrow \infty} \left( \frac{[1 - \tilde{\mu} + \tilde{\mu} [R(\lambda)]^{1-\lambda}]^{\frac{1}{1-\lambda}}}{1 - \mu + \mu R(\lambda)} \right)^{\frac{1-\eta}{\eta}} = 1.$$

Therefore, we have that

$$\begin{aligned} \lim_{\lambda \nearrow \infty} F_\eta(\lambda, \eta) &= \frac{\bar{y}}{1 - \beta + \beta \lim_{\lambda \nearrow \infty} Q(\eta, \lambda)} \\ &\quad \times \left[ (1 - \beta) + \beta \lim_{\lambda \nearrow \infty} [Q(\eta, \lambda)]^{1-\eta} \times \lim_{\lambda \nearrow \infty} \left( \frac{[1 - \mu + \mu [R(\lambda)]^{1-\lambda}]^{\frac{1}{1-\lambda}}}{1 - \mu + \mu R(\lambda)} \right)^{1-\eta} \right]^{\frac{1}{1-\eta}} \\ &= \frac{\bar{y}}{1 - \beta + \beta} \times [(1 - \beta) + \beta]^{\frac{1}{1-\eta}} = \bar{y}. \end{aligned}$$

Note that  $F(\lambda, \eta) < \bar{y}$  for all  $\lambda > 0$  when  $\tilde{\mu} \neq \mu$ , we must have that  $\frac{\partial F_\eta(\lambda, \eta)}{\partial \lambda}$  is positive as  $\lambda$  becomes sufficiently large. Therefore, it remains to show that  $\frac{\partial F_\eta(\lambda, \eta)}{\partial \lambda}$  is negative as  $\lambda$  becomes sufficiently small; it suffices to show that  $\frac{\partial \ln F_\eta(\lambda, \eta)}{\partial \lambda}$  is negative as  $\lambda$  becomes sufficiently small.

With slight abuse of notation, we follow the notation in Section 2.1 and denote  $\frac{1-\mu}{\mu} \times \frac{\tilde{\mu}}{1-\tilde{\mu}}$

by  $\tau$ . Define  $q(\lambda)$  and  $\ell(\lambda)$  as follows:

$$q(\lambda) := \frac{\left[1 - \tilde{\mu} + \tilde{\mu} [R(\lambda)]^{1-\lambda}\right]^{\frac{1}{1-\lambda}}}{1 - \mu + \mu R(\lambda)} = \frac{\left(1 - \tilde{\mu} + \tilde{\mu} \tau^{\frac{1-\lambda}{\lambda}}\right)^{\frac{1}{1-\lambda}}}{1 - \mu + \mu \tau^{\frac{1}{\lambda}}}, \quad (\text{A7})$$

$$\ell(\lambda) := \frac{\left[1 - \mu + \mu [R(\lambda)]^{1-\lambda}\right]^{\frac{1}{1-\lambda}}}{1 - \mu + \mu R(\lambda)} = \frac{\left(1 - \mu + \mu \tau^{\frac{1-\lambda}{\lambda}}\right)^{\frac{1}{1-\lambda}}}{1 - \mu + \mu \tau^{\frac{1}{\lambda}}}. \quad (\text{A8})$$

It follows immediately from the previous analysis that

$$\lim_{\lambda \searrow 0} q(\lambda) = \max \left\{ \frac{1 - \tilde{\mu}}{1 - \mu}, \frac{\tilde{\mu}}{\mu} \right\} =: q_0, \text{ and } \lim_{\lambda \searrow 0} \ell(\lambda) = 1,$$

which in turn implies that

$$\begin{aligned} \lim_{\lambda \searrow 0} \frac{\partial q(\lambda)}{\partial \lambda} &= \lim_{\lambda \searrow 0} q(\lambda) \times \lim_{\lambda \searrow 0} \frac{\partial \ln q(\lambda)}{\partial \lambda} \\ &= q_0 \lim_{\lambda \searrow 0} \left[ \frac{1}{(1-\lambda)^2} \ln \left( 1 - \tilde{\mu} + \tilde{\mu} \tau^{\frac{1}{\lambda}-1} \right) - \frac{1}{\lambda^2(1-\lambda)} \frac{\tilde{\mu} \tau^{\frac{1}{\lambda}-1} \ln \tau}{1 - \tilde{\mu} + \tilde{\mu} \tau^{\frac{1}{\lambda}-1}} + \frac{1}{\lambda^2} \frac{\mu \tau^{\frac{1}{\lambda}} \ln \tau}{1 - \mu + \mu \tau^{\frac{1}{\lambda}}} \right], \end{aligned} \quad (\text{A9})$$

and

$$\begin{aligned} \lim_{\lambda \searrow 0} \frac{\partial \ell(\lambda)}{\partial \lambda} &= \lim_{\lambda \searrow 0} \ell(\lambda) \times \lim_{\lambda \searrow 0} \frac{\partial \ln \ell(\lambda)}{\partial \lambda} \\ &= \lim_{\lambda \searrow 0} \left[ \frac{1}{(1-\lambda)^2} \ln \left( 1 - \mu + \mu \tau^{\frac{1}{\lambda}-1} \right) - \frac{1}{\lambda^2(1-\lambda)} \frac{\mu \tau^{\frac{1}{\lambda}-1} \ln \tau}{1 - \mu + \mu \tau^{\frac{1}{\lambda}-1}} + \frac{1}{\lambda^2} \frac{\mu \tau^{\frac{1}{\lambda}} \ln \tau}{1 - \mu + \mu \tau^{\frac{1}{\lambda}}} \right]. \end{aligned} \quad (\text{A10})$$

The partial derivative of  $\ln F_\eta(\lambda, \eta)$  with respect to  $\lambda$  can be rewritten as

$$\begin{aligned} &\frac{\partial \ln F_\eta(\lambda, \eta)}{\partial \lambda} \\ &= \frac{1}{1-\eta} \frac{\beta(1-\eta) [\ell(\lambda)]^{-\eta} [q(\lambda)]^{\frac{(1-\eta)^2}{\eta}} \frac{\partial \ell(\lambda)}{\partial \lambda} + \beta [\ell(\lambda)]^{1-\eta} \frac{(1-\eta)^2}{\eta} [q(\lambda)]^{\frac{(1-\eta)^2}{\eta}-1} \frac{\partial q(\lambda)}{\partial \lambda}}{1 - \beta + \beta [\ell(\lambda)]^{1-\eta} [q(\lambda)]^{\frac{(1-\eta)^2}{\eta}}} \\ &\quad - \frac{\beta \frac{1-\eta}{\eta} [q(\lambda)]^{\frac{1}{\eta}-2} \frac{\partial q(\lambda)}{\partial \lambda}}{1 - \beta + \beta [q(\lambda)]^{\frac{1-\eta}{\eta}}}. \end{aligned}$$

We consider the following two cases depending on  $\tau$  relative to one.

**Case I:  $\tau < 1$ .** Note that

$$\frac{\tau^{\frac{1}{\lambda}}}{\lambda^2} = o(1), \text{ for } \lambda \searrow 0.$$

The above equation, together with (A9) and (A10), implies that

$$\lim_{\lambda \searrow 0} \frac{\partial q(\lambda)}{\partial \lambda} = q_0 \ln(1 - \tilde{\mu}),$$

and

$$\lim_{\lambda \searrow 0} \frac{\partial \ell(\lambda)}{\partial \lambda} = \ln(1 - \mu).$$

Further, note that  $\tau < 1$  is equivalent to  $\tilde{\mu} < \mu$ , which implies that  $q_0 = \max\{\frac{1-\tilde{\mu}}{1-\mu}, \frac{\tilde{\mu}}{\mu}\} = \frac{1-\tilde{\mu}}{1-\mu}$ . Therefore, we have that

$$\begin{aligned} & \lim_{\lambda \searrow 0} \frac{\partial \ln F_\eta(\lambda, \eta)}{\partial \lambda} \\ &= \frac{1}{1-\eta} \lim_{\lambda \searrow 0} \left[ \frac{\beta(1-\eta) [\ell(\lambda)]^{-\eta} [q(\lambda)]^{\frac{(1-\eta)^2}{\eta}} \frac{\partial \ell(\lambda)}{\partial \lambda} + \beta [\ell(\lambda)]^{1-\eta} \frac{(1-\eta)^2}{\eta} [q(\lambda)]^{\frac{(1-\eta)^2}{\eta}-1} \frac{\partial q(\lambda)}{\partial \lambda}}{1 - \beta + \beta [\ell(\lambda)]^{1-\eta} [q(\lambda)]^{\frac{(1-\eta)^2}{\eta}}} \right] \\ & \quad - \lim_{\lambda \searrow 0} \left[ \frac{\beta \frac{1-\eta}{\eta} [q(\lambda)]^{\frac{1}{\eta}-2} \frac{\partial q(\lambda)}{\partial \lambda}}{1 - \beta + \beta [q(\lambda)]^{\frac{1-\eta}{\eta}}} \right] \\ &= \frac{1}{1-\eta} \frac{\beta(1-\eta) q_0^{\frac{(1-\eta)^2}{\eta}} \ln(1-\mu) + \beta \frac{(1-\eta)^2}{\eta} q_0^{\frac{(1-\eta)^2}{\eta}} \ln(1-\tilde{\mu})}{1 - \beta + \beta q_0^{\frac{(1-\eta)^2}{\eta}}} - \frac{\beta \frac{1-\eta}{\eta} q_0^{\frac{1-\eta}{\eta}} \ln(1-\tilde{\mu})}{1 - \beta + \beta q_0^{\frac{1-\eta}{\eta}}} \\ &= \frac{\beta q_0^{\frac{(1-\eta)^2}{\eta}}}{1 - \beta + \beta q_0^{\frac{(1-\eta)^2}{\eta}}} \ln(1-\mu) + \frac{1-\eta}{\eta} \times \left[ \frac{\beta q_0^{\frac{(1-\eta)^2}{\eta}}}{1 - \beta + \beta q_0^{\frac{(1-\eta)^2}{\eta}}} - \frac{\beta q_0^{\frac{1-\eta}{\eta}}}{1 - \beta + \beta q_0^{\frac{1-\eta}{\eta}}} \right] \ln(1-\tilde{\mu}) \\ &= \frac{\beta q_0^{\frac{(1-\eta)^2}{\eta}}}{1 - \beta + \beta q_0^{\frac{(1-\eta)^2}{\eta}}} \ln(1-\mu) + \frac{1-\eta}{\eta} \times \left[ \frac{\beta q_0^{\frac{(1-\eta)^2}{\eta}}}{1 - \beta + \beta q_0^{\frac{(1-\eta)^2}{\eta}}} - \frac{\beta q_0^{\frac{1-\eta}{\eta}}}{1 - \beta + \beta q_0^{\frac{1-\eta}{\eta}}} \right] [\ln(1-\mu) + \ln q_0] \\ &= \underbrace{\frac{1}{\eta}}_{>0} \times \underbrace{\left[ \frac{q_0^{\frac{(1-\eta)^2}{\eta}}}{1 - \beta + \beta q_0^{\frac{(1-\eta)^2}{\eta}}} \times \frac{1 - \beta + \beta q_0^{\frac{1-\eta}{\eta}}}{q_0^{\frac{1-\eta}{\eta}}} - (1-\eta) \right]}_{\mathcal{M}(\beta)} \times \underbrace{\frac{\beta q_0^{\frac{1-\eta}{\eta}}}{1 - \beta + \beta q_0^{\frac{1-\eta}{\eta}}}}_{>0} \times \underbrace{\ln(1-\mu)}_{<0} \\ & \quad + \underbrace{\frac{1-\eta}{\eta}}_{\leq 0 \text{ if } \eta \geq 1} \times \underbrace{\left[ \frac{\beta q_0^{\frac{(1-\eta)^2}{\eta}}}{1 - \beta + \beta q_0^{\frac{(1-\eta)^2}{\eta}}} - \frac{\beta q_0^{\frac{1-\eta}{\eta}}}{1 - \beta + \beta q_0^{\frac{1-\eta}{\eta}}} \right]}_{\geq 0 \text{ if } \eta \geq 1} \times \underbrace{\ln q_0}_{>0}. \end{aligned}$$

Evidently,  $\mathcal{M}(\beta) > 1 - \eta$  for  $\eta \geq 1$ . For  $\eta < 1$ , it is straightforward to verify that

$$\begin{aligned} \frac{d \log (\mathcal{M}(\beta))}{d \beta} &= \frac{q_0^{\frac{1-\eta}{\eta}} - 1}{1 - \beta + \beta q_0^{\frac{1-\eta}{\eta}}} - \frac{q_0^{\frac{(1-\eta)^2}{\eta}} - 1}{1 - \beta + \beta q_0^{\frac{(1-\eta)^2}{\eta}}} \\ &= \frac{q_0^{\frac{1-\eta}{\eta}} - q_0^{\frac{(1-\eta)^2}{\eta}}}{\left(1 - \beta + \beta q_0^{\frac{1-\eta}{\eta}}\right) \left(1 - \beta + \beta q_0^{\frac{(1-\eta)^2}{\eta}}\right)} > 0. \end{aligned}$$

Therefore,  $\mathcal{M}(\beta)$  is strictly increasing in  $\beta$ . Note that  $\mathcal{M}(0) = q_0^{-(1-\eta)}$  and  $\mathcal{M}(1) = 1 > 1 - \eta$ . Define  $\bar{\beta}$  as

$$\bar{\beta} := \begin{cases} 0, & \text{if } q_0^{-(1-\eta)} \geq 1 - \eta, \\ \text{the unique solution to } \mathcal{M}(\beta) = 1 - \eta, & \text{otherwise.} \end{cases}$$

It follows immediately that  $\mathcal{M}(\beta) - (1 - \eta) > 0$  for  $\beta > \bar{\beta}$ , and thus  $\lim_{\lambda \searrow 0} \frac{\partial \ln F_\eta(\lambda, \eta)}{\partial \lambda} < 0$ .

**Case II:  $\tau > 1$ .** Note that

$$\frac{\tilde{\mu} \tau^{\frac{1}{\lambda}-1}}{1 - \tilde{\mu} + \tilde{\mu} \tau^{\frac{1}{\lambda}-1}}, \frac{\mu \tau^{\frac{1}{\lambda}-1}}{1 - \mu + \mu \tau^{\frac{1}{\lambda}-1}}, \frac{\mu \tau^{\frac{1}{\lambda}}}{1 - \mu + \mu \tau^{\frac{1}{\lambda}}} = 1 + o(\lambda^2), \text{ for } \lambda \searrow 0,$$

where the equality follows from  $\tau^{-\frac{1}{\lambda}} = o(\lambda^2)$  for  $\lambda \searrow 0$ .

Moreover,

$$\lim_{\lambda \searrow 0} \frac{1}{(1 - \lambda)^2} \ln \left( 1 - \tilde{\mu} + \tilde{\mu} \tau^{\frac{1}{\lambda}-1} \right) = \frac{\ln \tau}{\lambda(1 - \lambda)} + \ln \tilde{\mu} + o(1).$$

The above equation, together with (A9) and (A10), implies that

$$\lim_{\lambda \searrow 0} \frac{\partial q(\lambda)}{\partial \lambda} = q_0 \ln \tilde{\mu},$$

and

$$\lim_{\lambda \searrow 0} \frac{\partial \ell(\lambda)}{\partial \lambda} = \ln \mu.$$

Further, note that  $\tau > 1$  is equivalent to  $\tilde{\mu} > \mu$ , which implies that  $q_0 = \max\left\{\frac{1-\tilde{\mu}}{1-\mu}, \frac{\tilde{\mu}}{\mu}\right\} =$

$\tilde{\mu}$ . Therefore, we have that

$$\begin{aligned}
& \lim_{\lambda \searrow 0} \frac{\partial \ln F_\eta(\lambda, \eta)}{\partial \lambda} \\
&= \frac{1}{1-\eta} \lim_{\lambda \searrow 0} \left[ \frac{\beta(1-\eta) [\ell(\lambda)]^{-\eta} [q(\lambda)]^{\frac{(1-\eta)^2}{\eta}} \frac{\partial \ell(\lambda)}{\partial \lambda} + \beta [\ell(\lambda)]^{1-\eta} \frac{(1-\eta)^2}{\eta} [q(\lambda)]^{\frac{(1-\eta)^2}{\eta}-1} \frac{\partial q(\lambda)}{\partial \lambda}}{1-\beta + \beta [\ell(\lambda)]^{1-\eta} [q(\lambda)]^{\frac{(1-\eta)^2}{\eta}}} \right] \\
&\quad - \lim_{\lambda \searrow 0} \left[ \frac{\beta \frac{1-\eta}{\eta} [q(\lambda)]^{\frac{1}{\eta}-2} \frac{\partial q(\lambda)}{\partial \lambda}}{1-\beta + \beta [q(\lambda)]^{\frac{1-\eta}{\eta}}} \right] \\
&= \frac{1}{1-\eta} \frac{\beta(1-\eta) q_0^{\frac{(1-\eta)^2}{\eta}} \ln \mu + \beta \frac{(1-\eta)^2}{\eta} q_0^{\frac{(1-\eta)^2}{\eta}} \ln \tilde{\mu}}{1-\beta + \beta q_0^{\frac{(1-\eta)^2}{\eta}}} - \frac{\beta \frac{1-\eta}{\eta} q_0^{\frac{1-\eta}{\eta}} \ln \tilde{\mu}}{1-\beta + \beta q_0^{\frac{1-\eta}{\eta}}} \\
&= \frac{\beta q_0^{\frac{(1-\eta)^2}{\eta}}}{1-\beta + \beta q_0^{\frac{(1-\eta)^2}{\eta}}} \ln \mu + \frac{1-\eta}{\eta} \times \left[ \frac{\beta q_0^{\frac{(1-\eta)^2}{\eta}}}{1-\beta + \beta q_0^{\frac{(1-\eta)^2}{\eta}}} - \frac{\beta q_0^{\frac{1-\eta}{\eta}}}{1-\beta + \beta q_0^{\frac{1-\eta}{\eta}}} \right] \ln \tilde{\mu} \\
&= \frac{\beta q_0^{\frac{(1-\eta)^2}{\eta}}}{1-\beta + \beta q_0^{\frac{(1-\eta)^2}{\eta}}} \ln \mu + \frac{1-\eta}{\eta} \times \left[ \frac{\beta q_0^{\frac{(1-\eta)^2}{\eta}}}{1-\beta + \beta q_0^{\frac{(1-\eta)^2}{\eta}}} - \frac{\beta q_0^{\frac{1-\eta}{\eta}}}{1-\beta + \beta q_0^{\frac{1-\eta}{\eta}}} \right] \times (\ln \mu + \ln q_0) \\
&= \underbrace{\frac{1}{\eta}}_{>0} \times \underbrace{\left[ \frac{q_0^{\frac{(1-\eta)^2}{\eta}}}{1-\beta + \beta q_0^{\frac{(1-\eta)^2}{\eta}}} \times \frac{1-\beta + \beta q_0^{\frac{1-\eta}{\eta}}}{q_0^{\frac{1-\eta}{\eta}}} - (1-\eta) \right]}_{\mathcal{M}(\beta)} \times \underbrace{\frac{\beta q_0^{\frac{1-\eta}{\eta}}}{1-\beta + \beta q_0^{\frac{1-\eta}{\eta}}}}_{>0} \times \underbrace{\ln \mu}_{<0} \\
&\quad + \underbrace{\frac{1-\eta}{\eta}}_{\leq 0 \text{ if } \eta \geq 1} \times \underbrace{\left[ \frac{\beta q_0^{\frac{(1-\eta)^2}{\eta}}}{1-\beta + \beta q_0^{\frac{(1-\eta)^2}{\eta}}} - \frac{\beta q_0^{\frac{1-\eta}{\eta}}}{1-\beta + \beta q_0^{\frac{1-\eta}{\eta}}} \right]}_{\geq 0 \text{ if } \eta \geq 1} \times \underbrace{\ln q_0}_{>0}.
\end{aligned}$$

By the same argument as in Case I, we can show that  $\lim_{\lambda \searrow 0} \frac{\partial \ln F_\eta(\lambda, \eta)}{\partial \lambda} < 0$  for  $\beta > \bar{\beta}$ . This concludes the proof of part (i) of the proposition.

Next, we prove part (ii). Again, by the expression of  $\mathcal{L}_\eta(\lambda, \eta)$ , it suffices to show that  $\lim_{\eta \nearrow \infty} F_\eta(\lambda, \eta) < \lim_{\eta \searrow 0} F_\eta(\lambda, \eta) < \lim_{\eta \rightarrow 1} F_\eta(\lambda, 1) < \bar{y}$  and  $\frac{\partial F_\eta(\lambda, \eta)}{\partial \eta}$  is negative as  $\eta$  becomes sufficiently small or sufficiently large. A closer look at the expression of  $q(\lambda)$  and  $\ell(\lambda)$  in Equations (A7) and (A8) yields that

$$q(\lambda) > 1, \text{ and } \ell(\lambda) < 1.$$

Therefore, we have that

$$\begin{aligned}
\lim_{\eta \searrow 0} F_\eta(\lambda, \eta) &= \lim_{\eta \searrow 0} \frac{\bar{y}}{1 - \beta + \beta [q(\lambda)]^{\frac{1-\eta}{\eta}}} \times \left[ (1 - \beta) + \beta [\ell(\lambda)]^{1-\eta} [q(\lambda)]^{\frac{(1-\eta)^2}{\eta}} \right]^{\frac{1}{1-\eta}} \\
&= \lim_{\eta \searrow 0} \frac{\bar{y}}{\beta [q(\lambda)]^{\frac{1-\eta}{\eta}}} \times \left[ \beta [\ell(\lambda)]^{1-\eta} [q(\lambda)]^{\frac{(1-\eta)^2}{\eta}} \right]^{\frac{1}{1-\eta}} \\
&= \ell(\lambda) \bar{y},
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\eta \nearrow \infty} F_\eta(\lambda, \eta) &= \lim_{\eta \nearrow \infty} \frac{\bar{y}}{1 - \beta + \beta [q(\lambda)]^{\frac{1-\eta}{\eta}}} \times \left[ (1 - \beta) + \beta [\ell(\lambda)]^{1-\eta} [q(\lambda)]^{\frac{(1-\eta)^2}{\eta}} \right]^{\frac{1}{1-\eta}} \\
&= \frac{\bar{y}}{1 - \beta + \beta [q(\lambda)]^{-1}} \times \frac{\ell(\lambda)}{q(\lambda)}.
\end{aligned}$$

Because  $q(\lambda) > 1$  and  $0 < \beta < 1$ , we have that

$$1 - \beta + \beta [q(\lambda)]^{-1} > [q(\lambda)]^{-1},$$

which in turn implies that

$$\lim_{\eta \nearrow \infty} F_\eta(\lambda, \eta) < \lim_{\eta \searrow 0} F_\eta(\lambda, \eta).$$

Next, note that

$$\begin{aligned}
\lim_{\eta \rightarrow 1} F_\eta(\lambda, \eta) &= \lim_{\eta \rightarrow 1} \frac{\bar{y}}{1 - \beta + \beta [q(\lambda)]^{\frac{1-\eta}{\eta}}} \times \left[ (1 - \beta) + \beta [\ell(\lambda)]^{1-\eta} [q(\lambda)]^{\frac{(1-\eta)^2}{\eta}} \right]^{\frac{1}{1-\eta}} \\
&= \bar{y} \lim_{\eta \rightarrow 1} \left[ (1 - \beta) + \beta [\ell(\lambda)]^{1-\eta} [q(\lambda)]^{\frac{(1-\eta)^2}{\eta}} \right]^{\frac{1}{1-\eta}} \\
&= [\ell(\lambda)]^\beta \bar{y}.
\end{aligned}$$

Therefore, we have that

$$\lim_{\eta \nearrow \infty} F_\eta(\lambda, \eta) < \lim_{\eta \searrow 0} F_\eta(\lambda, 1) = [\ell(\lambda)] \bar{y} < [\ell(\lambda)]^\beta \bar{y} = \lim_{\eta \rightarrow 1} F_\eta(\lambda, \eta) < \bar{y}.$$

Next, we show that  $\frac{\partial F_\eta(\lambda, \eta)}{\partial \eta}$  is negative as  $\eta$  becomes sufficiently small or sufficiently large. It is equivalent to show that  $\ln F_\eta(\lambda, \eta)$  is strictly increasing in  $\eta$  when  $\eta$  is sufficiently small

or sufficiently large. Carrying out the algebra, we have that

$$\begin{aligned}
\frac{\partial \ln F_\eta(\lambda, \eta)}{\partial \eta} &= \frac{1}{(1-\eta)^2} \ln \left( 1 - \beta + \beta [\ell(\lambda)]^{1-\eta} [q(\lambda)]^{\frac{(1-\eta)^2}{\eta}} \right) \\
&+ \frac{1}{1-\eta} \frac{\beta [\ell(\lambda)]^{1-\eta} [q(\lambda)]^{\frac{(1-\eta)^2}{\eta}} \left[ -\ln [\ell(\lambda)] + \left( 1 - \frac{1}{\eta^2} \right) \ln [q(\lambda)] \right]}{1 - \beta + \beta [\ell(\lambda)]^{1-\eta} [q(\lambda)]^{\frac{(1-\eta)^2}{\eta}}} \\
&+ \frac{\beta [q(\lambda)]^{\frac{1-\eta}{\eta}} \frac{1}{\eta^2} \ln [q(\lambda)]}{1 - \beta + \beta [q(\lambda)]^{\frac{1-\eta}{\eta}}}. \tag{A11}
\end{aligned}$$

We consider the following two cases.

**Case I:  $\eta \searrow 0$ .** In this case, we have that

$$\frac{\beta [\ell(\lambda)]^{1-\eta} [q(\lambda)]^{\frac{(1-\eta)^2}{\eta}}}{1 - \beta + \beta [\ell(\lambda)]^{1-\eta} [q(\lambda)]^{\frac{(1-\eta)^2}{\eta}}} = 1 + o(\eta^2), \tag{A12}$$

and

$$\frac{\beta [q(\lambda)]^{\frac{1-\eta}{\eta}}}{1 - \beta + \beta [q(\lambda)]^{\frac{1-\eta}{\eta}}} = 1 + o(\eta^2). \tag{A13}$$

Moreover, when  $\eta \searrow 0$ , we have that

$$\ln \left( 1 - \beta + \beta [\ell(\lambda)]^{1-\eta} [q(\lambda)]^{\frac{(1-\eta)^2}{\eta}} \right) = \ln \beta + (1-\eta) \ln [\ell(\lambda)] + \frac{(1-\eta)^2}{\eta} \ln [q(\lambda)] + o(1). \tag{A14}$$

Combining (A11), (A12), (A13), and (A14), we can obtain that

$$\begin{aligned}
\frac{\partial \ln F_\eta(\lambda, \eta)}{\partial \eta} &= \frac{1}{(1-\eta)^2} \left[ \ln \beta + (1-\eta) \ln [\ell(\lambda)] + \frac{(1-\eta)^2}{\eta} \ln [q(\lambda)] + o(1) \right] \\
&+ \frac{1}{1-\eta} \left[ -\ln [\ell(\lambda)] + \left( 1 - \frac{1}{\eta^2} \right) \ln [q(\lambda)] + o(1) \right] + \frac{1}{\eta^2} \ln [q(\lambda)] + o(1) \\
&= \frac{1}{(1-\eta)^2} \ln \beta + o(1).
\end{aligned}$$

Therefore,  $\frac{\partial \ln F_\eta(\lambda, \eta)}{\partial \eta} < 0$  when  $\eta$  is sufficiently small.

**Case II:  $\eta \nearrow \infty$ .** In this case, we have that

$$\frac{\beta [\ell(\lambda)]^{1-\eta} [q(\lambda)]^{\frac{(1-\eta)^2}{\eta}}}{1 - \beta + \beta [\ell(\lambda)]^{1-\eta} [q(\lambda)]^{\frac{(1-\eta)^2}{\eta}}} = 1 + o(\eta^{-5}). \quad (\text{A15})$$

Because  $q(\lambda) > 1$  and  $\lim_{\eta \nearrow \infty} \frac{1-\eta}{\eta} = -1$ , we have that

$$\frac{\beta [q(\lambda)]^{\frac{1-\eta}{\eta}}}{1 - \beta + \beta [q(\lambda)]^{\frac{1-\eta}{\eta}}} = \frac{\beta [q(\lambda)]^{-1}}{1 - \beta + \beta [q(\lambda)]^{-1}} + o(1). \quad (\text{A16})$$

Similarly, we can obtain that

$$\ln \left( 1 - \beta + \beta [\ell(\lambda)]^{1-\eta} [q(\lambda)]^{\frac{(1-\eta)^2}{\eta}} \right) = \ln \beta + (1-\eta) \ln [\ell(\lambda)] + \frac{(1-\eta)^2}{\eta} \ln [q(\lambda)] + o(1). \quad (\text{A17})$$

Combining (A11), (A15), (A16), and (A17), we have that

$$\begin{aligned} \frac{\partial \ln F_\eta(\lambda, \eta)}{\partial \eta} &= \frac{1}{(1-\eta)^2} \times \left[ \ln \beta + (1-\eta) \ln [\ell(\lambda)] + \frac{(1-\eta)^2}{\eta} \ln [q(\lambda)] + o(1) \right] \\ &\quad + \frac{1}{1-\eta} \times \left[ -\ln [\ell(\lambda)] + \left( 1 - \frac{1}{\eta^2} \right) \ln [q(\lambda)] + o(\eta^{-5}) \right] \\ &\quad + \frac{\ln [q(\lambda)]}{\eta^2} \times \left[ \frac{\beta [q(\lambda)]^{-1}}{1 - \beta + \beta [q(\lambda)]^{-1}} + o(1) \right] \\ &= \frac{\ln \beta}{\eta^2} - \frac{\ln [q(\lambda)]}{\eta^2} \times \frac{1 - \beta}{1 - \beta + \beta [q(\lambda)]^{-1}} + o(\eta^{-2}). \end{aligned}$$

Therefore,  $\frac{\partial \ln F_\eta(\lambda, \eta)}{\partial \eta} < 0$  when  $\eta$  is sufficiently large. This concludes the proof.

■

## Proof of Proposition 6

**Proof.** The monotonicity of  $c_1^*(\rho, 1)/c_1^*(\rho, 0)$  is obvious, and it remains to prove the monotonicity of  $\mathcal{L}(\rho, 1)$  for extreme values of  $\rho$ . Define  $\mathcal{D}(\rho)$  as

$$\begin{aligned}\mathcal{D}(\rho) &:= \ln [CE^*(\rho, 1)] - \ln [CE^*(\rho, 0)] \\ &= -\ln 2 + \ln [(1 + \phi) + (1 + \phi)^\rho] \\ &\quad + \frac{\rho}{\rho - 1} \times \left\{ \ln \left[ \frac{1}{2} + \frac{1}{2}(1 - \phi)^{\frac{\rho-1}{\rho}} \right] - \ln \left[ \frac{1}{2} + \frac{1}{2}(1 + \phi)^{\rho-1} \right] \right\}.\end{aligned}$$

It is equivalent to show that  $\mathcal{D}'(\rho) > 0$  as  $\rho$  becomes sufficiently small, and  $\mathcal{D}'(\rho) < 0$  as  $\rho$  becomes sufficiently large. Carrying out the algebra, we have that

$$\begin{aligned}\mathcal{D}'(\rho) &= \frac{(1 + \phi)^\rho \ln(1 + \phi)}{(1 + \phi) + (1 + \phi)^\rho} \\ &\quad - \frac{1}{(\rho - 1)^2} \times \left\{ \ln \left[ \frac{1}{2} + \frac{1}{2}(1 - \phi)^{\frac{\rho-1}{\rho}} \right] - \ln \left[ \frac{1}{2} + \frac{1}{2}(1 + \phi)^{\rho-1} \right] \right\} \\ &\quad + \frac{1}{\rho(\rho - 1)} \frac{\frac{1}{2}(1 - \phi)^{\frac{\rho-1}{\rho}} \ln(1 - \phi)}{\frac{1}{2} + \frac{1}{2}(1 - \phi)^{\frac{\rho-1}{\rho}}} - \frac{\rho}{\rho - 1} \frac{\frac{1}{2}(1 + \phi)^{\rho-1} \ln(1 + \phi)}{\frac{1}{2} + \frac{1}{2}(1 + \phi)^{\rho-1}}.\end{aligned}\tag{A18}$$

**Case I:  $\rho \searrow 0$ .** It can be verified that the following equations hold as  $\rho \searrow 0$ :

$$\frac{(1 + \phi)^\rho \ln(1 + \phi)}{(1 + \phi) + (1 + \phi)^\rho} = \frac{\ln(1 + \phi)}{2 + \phi} + o(1),\tag{A19}$$

$$\ln \left[ \frac{1}{2} + \frac{1}{2}(1 - \phi)^{\frac{\rho-1}{\rho}} \right] - \ln \left[ \frac{1}{2} + \frac{1}{2}(1 + \phi)^{\rho-1} \right] = \frac{\rho - 1}{\rho} \ln(1 - \phi) - \ln \left( \frac{2 + \phi}{1 + \phi} \right) + o(1),\tag{A20}$$

$$\frac{\frac{1}{2}(1 - \phi)^{\frac{\rho-1}{\rho}} \ln(1 - \phi)}{\frac{1}{2} + \frac{1}{2}(1 - \phi)^{\frac{\rho-1}{\rho}}} = \ln(1 - \phi) + o(\rho),\tag{A21}$$

and

$$\frac{\frac{1}{2}(1 + \phi)^{\rho-1} \ln(1 + \phi)}{\frac{1}{2} + \frac{1}{2}(1 + \phi)^{\rho-1}} = \frac{1}{2 + \phi} \ln(1 + \phi) + o(1).\tag{A22}$$

Combining (A18), (A19), (A20), (A21), and (A22), we obtain that

$$\mathcal{D}'(\rho) = \frac{1}{1 - \rho} \frac{\ln(1 + \phi)}{2 + \phi} + \frac{1}{(\rho - 1)^2} \ln \left( \frac{2 + \phi}{1 + \phi} \right) + o(1).$$

Clearly,  $\mathcal{D}'(\rho) > 0$  as  $\rho$  becomes sufficiently small.

**Case II:  $\rho \nearrow \infty$ .** Similarly, we can obtain the following equations as  $\rho \nearrow \infty$ :

$$\frac{(1 + \phi)^\rho \ln(1 + \phi)}{(1 + \phi) + (1 + \phi)^\rho} = \ln(1 + \phi) + o(\rho^{-2}), \quad (\text{A23})$$

$$\ln \left[ \frac{1}{2} + \frac{1}{2}(1 - \phi)^{\frac{\rho-1}{\rho}} \right] - \ln \left[ \frac{1}{2} + \frac{1}{2}(1 + \phi)^{\rho-1} \right] = \ln(2 - \phi) - (\rho - 1) \ln(1 + \phi) + o(1), \quad (\text{A24})$$

$$\frac{\frac{1}{2}(1 - \phi)^{\frac{\rho-1}{\rho}} \ln(1 - \phi)}{\frac{1}{2} + \frac{1}{2}(1 - \phi)^{\frac{\rho-1}{\rho}}} = \frac{1 - \phi}{2 - \phi} \ln(1 - \phi) + o(1), \quad (\text{A25})$$

and

$$\frac{\frac{1}{2}(1 + \phi)^{\rho-1} \ln(1 + \phi)}{\frac{1}{2} + \frac{1}{2}(1 + \phi)^{\rho-1}} = \ln(1 + \phi) + o(\rho^{-2}). \quad (\text{A26})$$

Combining (A18), (A23), (A24), (A25), and (A26), we obtain that

$$\mathcal{D}'(\rho) = -\frac{1}{(\rho - 1)^2} \ln(2 - \phi) + \frac{1}{\rho(\rho - 1)} \frac{1 - \phi}{2 - \phi} \ln(1 - \phi) + o(\rho^{-2}).$$

Clearly,  $\mathcal{D}'(\rho) < 0$  as  $\rho$  becomes sufficiently large. This concludes the proof.

■

## Proof of Proposition 7

**Proof.** The first part of the proposition is obvious:  $\theta^*(\rho; \Lambda) < \theta^*(\rho; 0)$  follows immediately from the facts that  $\partial \mathcal{T} / \partial \Lambda > 0$  and  $\partial \theta^* / \partial \mathcal{T} < 0$ ; and the optimal investment decision in the limit follows from  $\lim_{\rho \searrow 0} \mathcal{T}(\rho) = 1$  and  $\lim_{\rho \nearrow \infty} \mathcal{T}(\rho) = 0$ .

Now we prove the second part. For notational convenience, define

$$\psi := \frac{\kappa(R_u - 1)}{(1 - \kappa)(1 - R_d)} \times \frac{1 - (1 - \kappa)\Lambda}{1 + \kappa\Lambda}.$$

Note that  $\Lambda < \bar{\Lambda}$  implies instantly that  $\psi > 1$ . Moreover, the investor's certainty equivalent can be rewritten as:

$$CE^*(\rho; \Lambda) = \frac{(R_u - R_d)y}{(R_u - 1) + (1 - R_d)\psi^\rho} \times [\kappa\psi^{\rho-1} + (1 - \kappa)]^{\frac{\rho}{\rho-1}}.$$

Carrying out the algebra, we have that

$$\lim_{\rho \searrow 0} CE^*(\rho; \Lambda) = y \times \lim_{\rho \searrow 0} \left\{ [\kappa\psi^{\rho-1} + (1 - \kappa)]^{\frac{1}{\rho-1}} \right\}^\rho = y,$$

and

$$\lim_{\rho \nearrow \infty} CE^*(\rho; \Lambda) = (R_u - R_d)y \times \lim_{\rho \nearrow \infty} \frac{[\kappa\psi^{\rho-1} + (1 - \kappa)]^{\frac{\rho}{\rho-1}}}{(R_u - 1) + (1 - R_d)\psi^\rho} = \frac{R_u - R_d}{1 - R_d} \kappa y.$$

Therefore,  $\lim_{\rho \searrow 0} \mathcal{L}(\rho; \Lambda) = \lim_{\rho \nearrow \infty} \mathcal{L}(\rho; \Lambda) = 0$ . This completes the proof. ■