Online Supplementary Material for "Identifying the Volatility Risk Price Through the Leverage Effect"

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This Version: April 23, 2024

Abstract

This document provides proofs of some technical results in the main paper.

Proof of Proposition 3. By the Law of Iterated Expectations,

$$\begin{aligned}
\mathcal{L}_t(\zeta_r, \zeta) &= E[\exp\left\{-\zeta\mu_{t+1}\right\}\mathcal{L}_{(r|\sigma),t}(\zeta_r)\,|I(t)] \\
&= B\left(\zeta_r, I(t)\right) E[\exp\left\{-\zeta\mu_{t+1}\right\} A\left(\zeta_r, \mu_{t+1}\right)|I(t)], \\
\mathcal{L}_t(\zeta_r - 1, \zeta) &= E[\exp\left\{-\zeta\mu_{t+1}\right\}\mathcal{L}_{(r|\sigma),t}(\zeta_r - 1)\,|I(t)] \\
&= B\left(\zeta_r - 1, I(t)\right) E[\exp\left\{-\zeta\mu_{t+1}\right\} A\left(\zeta_r - 1, \mu_{t+1}\right)|I(t)].
\end{aligned}$$

From the condition in (**??**), we have $A\left(\zeta_r, \mu_{t+1}\right) = A\left(\zeta_r - 1, \mu_{t+1}\right)$. Therefore, $\mathcal{L}_t(\zeta_r, \zeta) = \mathcal{L}_t(\zeta_r - 1, \zeta) \Longleftrightarrow B\left(\zeta_r, I(t)\right) = B\left(\zeta_r - 1, I(t)\right)$. ■

The proof of Proposition 4 uses Lemma A.1 below. The proof of Lemma A.1 is given at the end of this appendix.

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Lemma A.1 *For all* (u, v) ,

$$\mathcal{L}_t^*(u,v) = E^*[\exp\left(-v\mu_{t+1}\right) g_t(u\left|\mu_{t+1}\right) |I(t)] \,, \text{ where}$$
$$g_t(u\left|\mu_{t+1}\right) = \exp\left[-\alpha^*\left(u\right)\mu_{t+1} - \beta^*\left(u\right)\mu_t - \gamma^*\left(u\right)\right]$$

with functions $\alpha^*(\cdot)\,, \beta^*(\cdot)\,, \gamma^*(\cdot)$ *defined by Proposition 3.*

Proof of Proposition 4. We first consider the risk-neutral distribution of the volatility factor. The risk-neutral conditional Laplace transform of μ_{t+1} given $I(t)$ is

$$\mathcal{L}_{\sigma,t}^*(v) = \frac{\mathcal{L}_t(\zeta_r, v+\zeta_\sigma)}{\mathcal{L}_t(\zeta_r, \zeta_\sigma)}. \tag{A.1}$$

The numerator of (A.1) can be made explicit with the conditional Laplace transforms defined in (**??**) and (**??**):

$$\begin{aligned}
\mathcal{L}_t(\zeta_r, v+\zeta_\sigma) &= E[\exp\left(-\zeta_r r_{t+1} - (v+\zeta_\sigma)\mu_{t+1}\right) |I(t)] \\
&= E[\exp\left\{-(v+\zeta_\sigma)\mu_{t+1}\right\} E[\exp\left\{-\zeta_r r_{t+1}\right\} |I^\sigma(t)] \,|I(t)] \\
&= E[\exp\left\{-(v+\zeta_\sigma)\mu_{t+1}\right\} E[\exp\left\{-\alpha(\zeta_r)\mu_{t+1} - \beta(\zeta_r)\mu_t - \gamma(\zeta_r)\right\} |I(t)] \\
&= \exp\left\{-\beta(\zeta_r)\mu_t - \gamma(\zeta_r)\right\} \exp\left\{-a\left[v+\zeta_\sigma+\alpha(\zeta_r)\right]\mu_t - b\left[v+\zeta_\sigma+\alpha(\zeta_r)\right]\right\}.
\end{aligned}$$

We get the denominator of (A.1) by setting $v = 0$ and simplify the ratio to obtain

$$\begin{aligned}
\mathcal{L}_{\sigma,t}^*(v) &= \exp\left\{-a\left[v+\zeta_\sigma+\alpha(\zeta_r)\right]\mu_t + a\left[\zeta_\sigma+\alpha(\zeta_r)\right]\mu_t\right\} \\
&\quad \exp\left\{-b\left[v+\zeta_\sigma+\alpha(\zeta_r)\right] + b\left[\zeta_\sigma+\alpha(\zeta_r)\right]\right\}.
\end{aligned}$$

Analogous to (**??**), the risk-neutral conditional Laplace transform of the volatility factor can be written as $E^*[\exp\left(-v\mu_{t+1}\right)|I(t)] = \exp\left\{-a^*(v)\mu_t - b^*(v)\right\}$, where

$$a^*(v) = a\left[v+\zeta_\sigma+\alpha(\zeta_r)\right] - a\left[\zeta_\sigma+\alpha(\zeta_r)\right], \quad b^*(v) = b\left[v+\zeta_\sigma+\alpha(\zeta_r)\right] - b\left[\zeta_\sigma+\alpha(\zeta_r)\right].$$

Next, we consider the risk-neutral distribution of the return conditional on the volatility factor. The conditional risk neutral distribution of the return given the volatility factor is defined by the decomposition (from the Law of Iterated Expectations, see (**??**) for the historical analog):

$$\mathcal{L}_t^*(u,v) = E^*[\exp\left\{-v\mu_{t+1}\right\} \mathcal{L}_{(r|\sigma),t}^*(u)\,|I(t)]\,. \tag{A.2}$$

Moreover (see e.g., argument in Bierens, 1982), this decomposition is unique. Hence, the result of Lemma A.1 allows us to conclude that $\mathcal{L}_{(r|\sigma),t}^*(u) = g_t(u\left|\mu_{t+1}\right) =$

$\exp\left[-\alpha^*(u)\mu_{t+1} - \beta^*(u)\mu_t - \gamma^*(u)\right]$. ■

Proof of Proposition 5. From (**??**) and (**??**), we have

$$\begin{aligned}\mathcal{L}_t(u,v) &= E[\exp\left\{-v\mu_{t+1}\right\}\exp\left\{-\alpha(u)\mu_{t+1} - \beta(u)\mu_t - \gamma(u)\right\}|I(t)] \\ &= \exp\{-\beta(u)\mu_t - \gamma(u)\}\exp\{-a[v+\alpha(u)]\mu_t\}\exp\{-b[v+\alpha(u)]\}.\end{aligned}$$

Therefore, the condition $\mathcal{L}_t(\zeta_r,\zeta_\sigma) = \mathcal{L}_t(\zeta_r - 1,\zeta_\sigma)$ is equivalent to the conjunction of two conditions

$$\begin{aligned}a[\zeta_\sigma + \alpha(\zeta_r)] + \beta(\zeta_r) &= a[\zeta_\sigma + \alpha(\zeta_r - 1)] + \beta(\zeta_r - 1), \\ b[\zeta_\sigma + \alpha(\zeta_r)] + \gamma(\zeta_r) &= b[\zeta_\sigma + \alpha(\zeta_r - 1)] + \gamma(\zeta_r - 1).\end{aligned}$$

■

Proof of Proposition 6. From the CAR model for the historical distribution of r_{t+1} given $I(t)$ (see section 3.1), the constraint $\mu_t = Var[r_{t+1}\,|I(t)]$ is characterized as

$$\begin{aligned}\mu_t &= E\{Var[r_{t+1}\,|I^\sigma(t)]\,|I(t)]\} + Var\{E[r_{t+1}\,|I^\sigma(t)]\,|I(t)]\}, \text{ where} \\ E[r_{t+1}\,|I^\sigma(t)] &= \alpha'(0)\mu_{t+1} + \beta'(0)\mu_t + \gamma'(0), \\ Var[r_{t+1}\,|I^\sigma(t)] &= -\alpha''(0)\mu_{t+1} - \beta''(0)\mu_t - \gamma''(0),\end{aligned}$$

such that the constraint is equivalent to

$$\mu_t = -\alpha''(0)E[\mu_{t+1}\,|I(t)] - \beta''(0)\mu_t - \gamma''(0) + [\alpha'(0)]^2 Var[\mu_{t+1}\,|I(t)].$$

Moreover, from the historical distribution of μ_{t+1} given $I(t)$, we have

$$E[\mu_{t+1}\,|I(t)] = a'(0)\mu_t + b'(0),\ Var[\mu_{t+1}\,|I(t)] = -a''(0)\mu_t - b''(0).$$

As such, the constraint above is equivalent to

$$\mu_t + \alpha''(0)[a'(0)\mu_t + b'(0)] + \beta''(0)\mu_t + \gamma''(0) = [\alpha'(0)]^2[-a''(0)\mu_t - b''(0)].$$

Both the coefficients of μ_t and the constant term must be identically zero, implying that the constraint is equivalent to the conjunction of the following two restrictions:

$$\begin{aligned}1 + \alpha''(0)a'(0) + \beta''(0) + [\alpha'(0)]^2 a''(0) &= 0, \\ \alpha''(0)b'(0) + \gamma''(0) + [\alpha'(0)]^2 b''(0) &= 0.\end{aligned}$$

Proposition 6 follows directly with the definition of ψ and ξ. ■

Proof of Theorem 1. We obtain this result by applying Theorem 1 of Andrews and Mikusheva (2016). We now verify Assumptions 1–3 in Andrews and Mikusheva (2016). To show weak convergence of $\eta_T(\cdot)$ to $\eta(\cdot)$ uniformly over $\mathcal{P}$, note that by a second-order Taylor expansion,

$$\begin{aligned}\eta_T(\lambda) &\coloneqq T^{1/2}\left[\widehat{g}(\lambda) - g_0(\lambda)\right] = G_0(\lambda)\Omega^{1/2}\xi_T + \delta_T, \text{ where} \\ \xi_T &= \Omega^{-1/2}T^{1/2}\left(\widehat{\omega} - \omega_0\right), \text{ and } \delta_T = \left(G(\lambda, \widetilde{\omega}) - G(\lambda, \omega_0)\right)T^{1/2}(\widehat{\omega} - \omega_0)\end{aligned} \tag{A.3}$$

and $\widetilde{\omega}$ is between $\widehat{\omega}$ and ω_0. Because $\|G(\lambda, \widetilde{\omega}) - G(\lambda, \omega_0)\| \leq C\|\widetilde{\omega} - \omega_0\|$, $\delta_T = o_p(1)$ uniformly over $\mathcal{P}$ following Assumption R. To show $G_0(\lambda)\Omega^{1/2}\xi_T$ weakly converges to $\eta(\cdot)$, it is sufficient to show (i) the pointwise convergence

$$\begin{pmatrix} G_0(\lambda_1)\Omega^{1/2}\xi_T \\ G_0(\lambda_2)\Omega^{1/2}\xi_T \end{pmatrix} \to_d \begin{pmatrix} \eta(\lambda_1) \\ \eta(\lambda_2) \end{pmatrix}, \tag{A.4}$$

which follows from Assumption R, and (ii) the stochastic equicontinuity condition, i.e., for every $\varepsilon > 0$ and $\xi > 0$, there exists a $\delta > 0$ such that

$$\limsup_{T\to\infty} \Pr\left(\sup_{P\in\mathcal{P}} \sup_{\|\lambda_1 - \lambda_2\| \leq \delta} \left\| G_0(\lambda_1)\Omega^{1/2}\xi_T - G_0(\lambda_2)\Omega^{1/2}\xi_T \right\| > \varepsilon \right) < \xi. \tag{A.5}$$

For some $C < \infty$, we have $\|G_0(\lambda_1) - G_0(\lambda_2)\| \leq C\|\lambda_1 - \lambda_2\|$ under a uniform bound for the derivative in **??**, and we have $\|\Omega^{1/2}\| \leq C$ under **??**. Thus,

$$\begin{aligned}&\limsup_{T\to\infty} \Pr\left(\sup_{P\in\mathcal{P}} \sup_{\|\lambda_1 - \lambda_2\| \leq \delta} \left\| G_0(\lambda_1)\Omega^{1/2}\xi_T - G_0(\lambda_2)\Omega^{1/2}\xi_T \right\| > \varepsilon \right) \\ &\leq \limsup_{T\to\infty} \Pr\left(C^2 \sup_{P\in\mathcal{P}} \|\xi_T\| > \frac{\varepsilon}{\delta} \right).\end{aligned} \tag{A.6}$$

Because $\xi_T = O_p(1)$ uniformly over $P \in \mathcal{P}$, there exists δ such that ε/δ is large enough to make the right hand side of the inequality in (A.6) smaller than ξ. Assumptions 2 and 3 of Andrews and Mikusheva (2016) follow directly from **??** and **??**. ■

Proof of Lemma A.1. We know from (**??**) that

$$\mathcal{L}_t^*(u, v) = \frac{\mathcal{L}_t(u + \zeta_r, v + \zeta_\sigma)}{\mathcal{L}_t(\zeta_r, \zeta_\sigma)} = \frac{N_t}{D_t}.$$

The numerator can be written as

$$
\begin{aligned}
N_t &= E[\exp\left\{-(u+\zeta_r)r_{t+1}-(v+\zeta_\sigma)\mu_{t+1}\right\}|I(t)] \\
&= E[\exp\left\{-(v+\zeta_\sigma)\mu_{t+1}\right\} E[\exp\left\{-(u+\zeta_r)r_{t+1}\right\}|I^\sigma(t)]\,|I(t)] \\
&= \exp\left\{-\gamma(u+\zeta_r)-\beta(u+\zeta_r)\mu_t\right\} E[\exp\left\{-\left[v+\zeta_\sigma+\alpha\left(u+\zeta_r\right)\right]\mu_{t+1}\right\}|I(t)] \,.
\end{aligned}
$$

Similarly, the denominator can be written as

$$
D_t = \exp\left\{-\gamma(\zeta_r)-\beta(\zeta_r)\mu_t\right\} E[\exp\left\{-\left[\zeta_\sigma+\alpha\left(\zeta_r\right)\right]\mu_{t+1}\right\}|I(t)] \,.
$$

By computing the ratio N_t/D_t, we get $\mathcal{L}_t^*(u,v)=\exp\left\{-\gamma^*(u)-\beta^*(u)\mu_t\right\} B_t/C_t$, where

$$
B_t = E[\exp\left\{-\left[v+\zeta_\sigma+\alpha\left(u+\zeta_r\right)\right]\mu_{t+1}\right\}|I(t)] \,, C_t \;= E[\exp\left\{-\left[\zeta_\sigma+\alpha\left(\zeta_r\right)\right]\mu_{t+1}\right\}|I(t)] \,.
$$

Using the definition of $\alpha^*(\cdot)$, we can write

$$
\begin{aligned}
B_t &= E[\exp\left\{-\left[v+\zeta_\sigma+\alpha^*(u)+\alpha(\zeta_r)\right]\mu_{t+1}\right\}|I(t)] \\
&= \exp\left\{-a\left[v+\zeta_\sigma+\alpha^*(u)+\alpha(\zeta_r)\right]\mu_t\right\}\exp\left\{-b\left[v+\zeta_\sigma+\alpha^*(u)+\alpha(\zeta_r)\right]\right\}.
\end{aligned}
$$

By definition of functions $a^*(\cdot)$ and $b^*(\cdot)$,

$$
\begin{aligned}
a\left[v+\zeta_\sigma+\alpha^*(u)+\alpha(\zeta_r)\right] &= a^*\left[v+\alpha^*(u)\right]+a\left[\zeta_\sigma+\alpha(\zeta_r)\right], \\
b\left[v+\zeta_\sigma+\alpha^*(u)+\alpha(\zeta_r)\right] &= b^*\left[v+\alpha^*(u)\right]+b\left[\zeta_\sigma+\alpha(\zeta_r)\right].
\end{aligned}
$$

Hence,

$$
B_t = \exp\left\{-a^*\left[v+\alpha^*(u)\right]\mu_t-a\left[\zeta_\sigma+\alpha(\zeta_r)\right]\mu_t\right\}\exp\left\{-b^*\left[v+\alpha^*(u)\right]-b\left[\zeta_\sigma+\alpha(\zeta_r)\right]\right\}.
$$

For $u=v=0$, we get $C_t=\exp\left\{-a\left[\zeta_\sigma+\alpha(\zeta_r)\right]\mu_t-b\left[\zeta_\sigma+\alpha(\zeta_r)\right]\right\}$. By computing the ratio, we obtain $B_t/C_t=\exp\left\{-a^*\left[v+\alpha^*(u)\right]\mu_t\right\}\exp\left\{-b^*\left[v+\alpha^*(u)\right]\right\}$. Therefore, we have shown that

$$
\begin{aligned}
\mathcal{L}_t^*(u,v) &= \exp\left\{-\gamma^*(u)-\beta^*(u)\mu_t\right\}\exp\left\{-a^*\left[v+\alpha^*(u)\right]\mu_t\right\}\exp\left\{-b^*\left[v+\alpha^*(u)\right]\right\} \\
&= E^*[\exp\left(-v\mu_{t+1}\right) g_t(u\left|\mu_{t+1}\right)|I(t)]\,,\ \text{where,} \\
g_t(u\left|\mu_{t+1}\right) &= \exp\left[-\alpha^*\left(u\right)\mu_{t+1}-\beta^*\left(u\right)\mu_t-\gamma^*\left(u\right)\right].
\end{aligned}
$$

This proves the desired result in the lemma. ■

References

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