This note concerns Lemma 9.10 in the Supplemental Material of Andrews and Cheng (2012). This is an argmax theorem as in Theorem 3.2.2 in van der Vaart and Wellner (1996) except that the parameter space A_n is allowed to be indexed by sample size n, i.e., if (i) a stochastic process $M_n(h)$ weakly converges to another stochastic process $M(h)$, and (ii) a set A_n converges to another set A_0 in the Hausdorff metric, then $\text{argmax}_{h \in A_n} M_n(h) \to_d \text{argmax}_{h \in A} M(h)$ under the standard regularities conditions given in the Lemma.

Cox (2022) points out that although the conclusion is correct as long as the parameter space is separable, the proof requires a modification. Specifically, equation (9.98) in the proof assumes that $F \cap A_n$ converges to $F \cap A_0$ for every closed set F if $A_n \to A$. Cox (2022) points out that this argument does not hold for some closed sets F[1] and show that the proof can go through by replacing the requirement $A_n \cap F$ converging to $A_0 \cap F$ with the requirement $A_n \cap F$ converging to a *subset* of $A_0 \cap F$ along a subsequence. The rest of the proof follows the same steps as in the original argmax theorem in Theorem 3.2.2 in van der Vaart and Wellner (1996).

References

Andrews, D. W. K. and Cheng X. (2012). Estimation and inference with weak, semi-strong, and strong identification. *Econometrica.* 80:2153-2211

Cox, G. (2022). A Generalized Argmax Theorem with Applications. arXiv: 2209: 08793

van der Vaart, A. adn Wellner , J. (1996). Weak Convergence and Empirical Processes.

[1]For example, suppose $A_n = [1 + 1/n, 2]$, $A_0 = [1, 2]$, and $F = [0, 1] \cup \{2\}$. In this case, $A_n \cap F = \{2\} \subset A_0 \cap F = \{1, 2\}$.