The Eigenvalue Problem of Singular Ergodic Control

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Abstract

We consider the problem of finding a real number λ and a function u satisfying the PDE

$$\max\{\lambda - \Delta u - f,\ |Du| - 1\} = 0, \quad x \in \mathbb{R}^n.$$

Here f is a convex, superlinear function. We prove that there is a unique λ^* such that the above PDE has a viscosity solution u satisfying $\lim_{|x|\to\infty} u(x)/|x| = 1$. Moreover, we show that associated to λ^* is a convex solution u^* with $D^2u^* \in L^\infty(\mathbb{R}^n)$ and give two min-max formulae for λ^*. λ^* has a probabilistic interpretation as being the least, long-time averaged (ergodic) cost for a singular control problem involving f. © 2011 Wiley Periodicals, Inc.

1 Introduction

In this paper, we address the following problem: find $\lambda \in \mathbb{R}$ such that the PDE

$$\max\{\lambda - \Delta u - f,\ |Du| - 1\} = 0, \quad x \in \mathbb{R}^n, \tag{1.1}$$

has a solution $u : \mathbb{R}^n \to \mathbb{R}$. Since we seek the pair (λ, u), we consider this to be an *eigenvalue problem* and therefore any such λ will be called an *eigenvalue*. Here it is assumed that $f \in C^\infty(\mathbb{R}^n)$ is convex and satisfies the growth condition

$$\lim_{|x|\to\infty} \frac{f(x)}{|x|} = +\infty.$$

These assumptions imply that f is bounded from below and without any loss of generality it will also be assumed that f is nonnegative.

Our main result is the following:

THEOREM 1.1. *There is a unique $\lambda^* \in \mathbb{R}$ such that* (1.1) *has a viscosity solution $u \in C(\mathbb{R}^n)$ satisfying*

$$\lim_{|x|\to\infty} \frac{u(x)}{|x|} = 1. \tag{1.2}$$

Moreover, associated to this eigenvalue $\lambda^* \in \mathbb{R}$ is a convex solution u^* of (1.1) *satisfying* (1.2) *such that $D^2u^* \in L^\infty(\mathbb{R}^n)$.*

Our approach to proving this theorem is as follows. We use an idea inspired from the maximum principle for solutions of second-order elliptic PDEs to establish a

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comparison principle among eigenvalues. Then we present a simple technique for approximating the value of an eigenvalue. It is based on studying the related elliptic PDE

$$\max\{\delta u - \Delta u - f,\ |Du| - 1\} = 0, \quad x \in \mathbb{R}^n, \tag{1.3}$$

for δ small and positive. In particular, we prove the following:

THEOREM 1.2.

(i) *For each $\delta > 0$, there is a unique viscosity solution u_δ of* (1.3) *satisfying* (1.2).

(ii) *There is a universal constant L such that*

$$0 \le D^2 u_\delta(x) \le L, \quad \text{a.e. } x \in \mathbb{R}^n,$$

for $0 < \delta \le 1$.

(iii) *There is a sequence $(\delta_k)_{k\in\mathbb{N}}$ of positive numbers tending to* 0 *such that as* $k \to \infty$

$$\delta_k u_{\delta_k} \to \lambda^*$$

locally uniformly on $\mathbb{R}^n$, and for each $x_0 \in \mathbb{R}^n$, $(u_{\delta_k} - u_{\delta_k}(x_0))_{k\in\mathbb{N}}$ converges to a solution u^* of (1.1) *in $C^1_{\text{loc}}(\mathbb{R}^n)$ that satisfies* (1.2).

Along the way to proving Theorem 1.1, we will make a few interesting observations. The first, which was already known, is that when f is rotational, u_δ and thus u^* are also rotational and in particular of class $C^2(\mathbb{R}^n)$. This result was first proved in [14]. However, our argument is elegant and does not require an involved, explicit construction of the solution.

THEOREM 1.3. *Suppose that f is rotationally symmetric, i.e., $f(x) = f_0(|x|)$ for some nondecreasing, convex, superlinear function $f_0 : [0,\infty) \to \mathbb{R}$. Then u_δ is rotationally symmetric and belongs to the space $C^2(\mathbb{R}^n)$.*

Our second observation, which is original, is that the eigenvalue λ^* can be characterized via min-max formulae.

THEOREM 1.4. *Define*

$$\lambda_- = \sup\Big\{\inf_{x\in\mathbb{R}^n}\{\Delta\phi(x) + f(x)\} : \phi \in C^2(\mathbb{R}^n), |D\phi| \le 1\Big\}$$

and

$$\lambda_+ = \inf\Big\{\sup_{|D\psi(x)|<1}\{\Delta\psi(x) + f(x)\} : \psi \in C^2(\mathbb{R}^n), \liminf_{|x|\to\infty}\tfrac{\psi(x)}{|x|} \ge 1\Big\}.$$

Then

$$\lambda_- = \lambda^* = \lambda_+.$$

As far as we know, this work is the first to consider the eigenvalue problem as posed above. However, a big part of our motivation was the work of Menaldi et al. [16], who studied a closely related problem arising in stochastic control theory. With regards to our framework, they used probabilistic arguments to build an eigenvalue. In this paper, we establish the existence of a *unique* eigenvalue and obtain a better regularity result than what was obtained in [16] because it does not require any special assumptions on f; we only require convexity and superlinear growth. Moreover, we have employed methods that are entirely analytic and use nothing from probability theory. Before carrying out our analysis let us see how this eigenvalue problem arises in the theory of singular stochastic control.

1.1 Probabilistic Interpretation of the Eigenvalue

Let $(\Omega, \mathcal{F}, \mathbb{P})$ represent a probability space with n-dimensional Brownian motion $(W(t), t \geq 0)$. Also set

$$X^\nu(t) := \sqrt{2}\, W(t) + \nu(t), \quad t \geq 0,$$

where ν is an $\mathbb{R}^n$-valued control process (adapted to the filtration generated by W) that satisfies

$$\begin{cases} \nu(0) = 0 \text{ a.s.} \\ t \mapsto \nu(t) \text{ is left-continuous a.s.} \\ |\nu|(t) := TV_\nu[0,t) < \infty \text{ for all } t > 0 \text{ a.s.}^1 \end{cases}$$

We say ν is a *singular control* because it may have sample paths that may not be absolutely continuous with respect to Lebesgue measure on $[0, \infty)$.

An optimization problem of interest is to find a singular control ν that minimizes the quantity

$$\limsup_{t\to\infty} \frac{1}{t}\, \mathbb{E}\left\{ \int_0^t f(X^\nu(s))ds + |\nu|(t) \right\}. \tag{1.4}$$

Since (1.4) is a "long-time" average, we interpret this problem as one of *singular ergodic control*.

To see how (1.1) is related to the control problem described above, we suppose that there is $\lambda \in \mathbb{R}$ such that (1.1) has a convex solution $u \in C^2(\mathbb{R}^n)$. Let ν be a singular control process. According to Ito's rule for semimartingales [12, theorem 9.5],

$$\begin{aligned} \mathbb{E}u(X^\nu(t)) = u(x) + \mathbb{E}\int_0^t \Delta u(X^\nu(s))ds + \mathbb{E}\int_0^t Du(X^\nu(s)) \cdot d\nu(s) \\ + \sum_{0\leq s<t} \mathbb{E}\int_0^t \big[u(X^\nu(s+)) - u(X^\nu(s)) \\ - Du(X^\nu(s)) \cdot (X^\nu(s+) - X^\nu(s))\big] \geq \end{aligned}$$

[1] $TV_g I$ denotes the total variation of g on the interval $I \subset \mathbb{R}$.

$$\geq u(x) + t\lambda - \mathbb{E}\int_0^t f(X^\nu(s))ds - \mathbb{E}\int_0^t |Du(X^\nu(s))|\, d|\nu|(s)$$

$$\geq u(x) + t\lambda - \mathbb{E}\int_0^t f(X^\nu(s))ds - \mathbb{E}|\nu|(t).$$

Thus

$$\lambda \leq \frac{1}{t}\,\mathbb{E}\left\{\int_0^t f(X^\nu(s))ds + |\nu|(t)\right\} + \frac{\mathbb{E}u(X^\nu(t)) - u(x)}{t}, \quad t > 0. \tag{1.5}$$

We would like to conclude that

$$\lambda \leq \limsup_{t\to\infty} \frac{1}{t}\,\mathbb{E}\left\{\int_0^t f(X^\nu(s))ds + |\nu|(t)\right\}. \tag{1.6}$$

Suppose that the right-hand side of inequality (1.6) is finite or else (1.6) clearly holds. In this case,

$$\limsup_{t\to\infty} \frac{1}{t}\,\mathbb{E}\left\{\int_0^t f(X^\nu(s))ds\right\} < \infty,$$

which implies that there is a sequence of positive numbers $t_k \to \infty$ as $k \to \infty$ such that

$$\limsup_{k\to\infty} \mathbb{E} f(X^\nu(t_k)) < \infty.$$

Since f grows superlinearly and u grows at most as fast as $|x|$, as $|x| \to \infty$,

$$\limsup_{k\to\infty} \mathbb{E}u(X^\nu(t_k)) < \infty.$$

Choosing $t = t_k$ in (1.5) and sending $k \to \infty$ establishes (1.6). In particular,

$$\lambda \leq \lambda^* := \inf_\nu \limsup_{t\to\infty} \frac{1}{t}\,\mathbb{E}\left\{\int_0^t f(X^\nu(s))ds + |\nu|(t)\right\}. \tag{1.7}$$

If there is a control ν^* such that

$$\lambda - \Delta u(X^{\nu^*}(t)) - f(X^{\nu^*}(t)) = 0, \tag{1.8}$$

$$\int_0^t Du(X^{\nu^*}(s)) \cdot d\nu^*(s) = -\nu^*(t), \tag{1.9}$$

and

$$u(X^{\nu^*}(t+)) - u(X^{\nu^*}(t)) - Du(X^{\nu^*}(t)) \cdot (X^{\nu^*}(t+) - X^{\nu^*}(t)) = 0 \tag{1.10}$$

for $t \in (0, \infty)$ almost surely, then equality holds in (1.7). In this case, we have λ^* as a probabilistic formula for the eigenvalue appearing in (1.1).

Remark 1.5. ν^* satisfying (1.8), (1.9), and (1.10) is a good candidate for an optimal control. Designing such an optimal control can be done via reflected diffusions if enough regularity is assumed on u and on the boundary of the set of points x such that $|Du(x)| < 1$. This procedure is discussed in detail in [16, sec. 5] and in [17, sec. 12].

2 Comparison of Eigenvalues

The purpose of this section is to prove that there can be at most one eigenvalue for which the PDE (1.1) has a viscosity solution satisfying (1.2). We will not motivate viscosity solutions except to say that it is a weak notion of solutions of PDE and that it provides a very flexible approach to establishing comparison principles for sub- and supersolutions of scalar, nonlinear, elliptic, and parabolic PDEs. We refer the interested reader to a few of the well-known references on the theory of viscosity solutions such as [6, 10]; in this paper, we use the notation of [6].

DEFINITION 2.1. $u \in USC(\mathbb{R}^n)$ is a *viscosity subsolution* of (1.1) with *eigenvalue* $\lambda \in \mathbb{R}$ if for each $x_0 \in \mathbb{R}^n$,

$$\max\{\lambda - \Delta\varphi(x_0) - f(x_0), |D\varphi(x_0)| - 1\} \le 0$$

whenever $u - \varphi$ has a local maximum at x_0 and $\varphi \in C^2(\mathbb{R}^n)$. $v \in LSC(\mathbb{R}^n)$ is a *viscosity supersolution* of (1.1) with *eigenvalue* $\mu \in \mathbb{R}$ if for each $y_0 \in \mathbb{R}^n$,

$$\max\{\mu - \Delta\psi(y_0) - f(y_0), |D\psi(y_0)| - 1\} \ge 0$$

whenever $v - \psi$ has a local minimum at y_0 and $\psi \in C^2(\mathbb{R}^n)$. $u \in C(\mathbb{R}^n)$ is a *viscosity solution* of (1.1) with *eigenvalue* $\lambda \in \mathbb{R}$ if it is both a viscosity sub- and supersolution of (1.1) with eigenvalue λ.

Towards establishing a uniqueness result, we first establish a comparison principle for eigenvalues with sub- and supersolutions. As we shall do several times in this paper, we will first present a formal (heuristic) argument followed by a rigorous proof. The purpose of doing this is to convey the motivating ideas.

PROPOSITION 2.2 (Comparison Principle). *Suppose u is a subsolution of* (1.1) *with eigenvalue λ and that v is a supersolution of* (1.1) *with eigenvalue μ. If in addition*

$$\limsup_{|x|\to\infty} \frac{u(x)}{|x|} \le 1 \le \liminf_{|x|\to\infty} \frac{v(x)}{|x|}, \tag{2.1}$$

then $\lambda \le \mu$.

FORMAL PROOF OF COMPARISON PRINCIPLE. Here we assume that $u, v \in C^2(\mathbb{R}^n)$. Fix $0 < \tau < 1$ and set

$$w^\tau(x) = \tau u(x) - v(x), \quad x \in \mathbb{R}^n.$$

By (2.1), we have $\lim_{|x|\to\infty} w^\tau(x) = -\infty$, so there is $x_\tau \in \mathbb{R}^n$ such that

$$w^\tau(x_\tau) = \sup_{x\in\mathbb{R}^n} w^\tau(x).$$

Basic calculus gives

$$\begin{cases} 0 = Dw^\tau(x_\tau) = \tau Du(x_\tau) - Dv(x_\tau), \\ 0 \ge \Delta w^\tau(x_\tau) = \tau\Delta u(x_\tau) - \Delta v(x_\tau). \end{cases}$$

Note in particular that

$$|Dv(x_\tau)| = \tau|Du(x_\tau)| \le \tau < 1,$$

and since v is a supersolution of (1.1) with eigenvalue μ,

$$0 \le \mu - \Delta v(x_\tau) - f(x_\tau).$$

Because u is a subsolution of (1.1) with eigenvalue λ,

$$\begin{aligned}\tau\lambda - \mu &\le \tau\Delta u(x_\tau) - \Delta v(x_\tau) - (1-\tau) f(x_\tau)\\ &\le -(1-\tau) f(x_\tau)\\ &\le 0.\end{aligned}$$

Here we have used that f is nonnegative. Letting $\tau \to 1^-$ gives $\lambda \le \mu$. □

We now make this rigorous by using a "doubling-the-variables" type of argument. This is a fairly standard approach in the theory of viscosity solutions. What makes this problem different is the fact that the domain is the whole space $\mathbb{R}^n$. However, we have the growth condition (1.2), which plays the role of a boundary condition.

PROOF OF PROPOSITION 2.2.

(1) Fix $0 < \tau < 1$ and set

$$w^\tau(x, y) = \tau u(x) - v(y), \quad x, y \in \mathbb{R}^n.$$

For $\delta > 0$, we also set

$$\varphi_\delta(x, y) = \frac{1}{2\delta}|x-y|^2, \quad x, y \in \mathbb{R}^n.$$

The inequality

$$\begin{aligned}w^\tau(x, y) - \varphi_\delta(x, y) &= \tau(u(x) - u(y)) - \frac{1}{2\delta}|x-y|^2 + \tau u(y) - v(y)\\ &\le \left(|x-y| - \frac{1}{2\delta}|x-y|^2\right) + \tau u(y) - v(y)\end{aligned}$$

implies

$$\lim_{|(x,y)|\to\infty}\{w^\tau(x, y) - \varphi_\delta(x, y)\} = -\infty.$$

Therefore, $w^\tau - \varphi_\delta$ achieves a global maximum at a point $(x_\delta, y_\delta) \in \mathbb{R}^n \times \mathbb{R}^n$.

(2) According to theorem 3.2 in [6], for each $\rho > 0$, there are $X, Y \in \mathcal{S}(n)$ such that

$$\begin{aligned}\left(\frac{x_\delta - y_\delta}{\delta}, X\right) &= (D_x\varphi_\delta(x_\delta, y_\delta), X) \in \overline{J}^{2,+}(\tau u)(x_\delta),\\ \left(\frac{x_\delta - y_\delta}{\delta}, Y\right) &= (-D_y\varphi_\delta(x_\delta, y_\delta), Y) \in \overline{J}^{2,-}v(y_\delta),\end{aligned} \tag{2.2}$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le A + \rho A^2. \tag{2.3}$$

Here

$$A = D^2\varphi_\delta(x_\delta, y_\delta) = \frac{1}{\delta}\begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix}.$$

Applying both sides of the matrix inequality (2.3) to the vector $(\xi, \xi)^\mathsf{T} \in \mathbb{R}^{2n}$ and then taking the dot product with $(\xi, \xi)^\mathsf{T}$ yields

$$X\xi \cdot \xi - Y\xi \cdot \xi \le 0.$$

Because $\xi \in \mathbb{R}^n$ is arbitrary, $X \le Y$.

(3) We also have

$$\frac{1}{\tau}\frac{x_\delta - y_\delta}{\delta} \in \bar{J}^{1,+}u(x_\delta),$$

and since $|Du| \le 1$ (in the sense of viscosity solutions),

$$\left|\frac{x_\delta - y_\delta}{\delta}\right| \le \tau < 1.$$

Since v is a viscosity supersolution of (1.1) with eigenvalue μ, we have

$$0 \le \mu - \operatorname{tr} Y - f(y_\delta)$$

by (2.2). Since u is a viscosity subsolution of (1.1) with eigenvalue λ,

$$\lambda - \frac{\operatorname{tr} X}{\tau} - f(x_\delta) \le 0.$$

Therefore,

$$\tau\lambda - \mu \le \operatorname{tr}[X - Y] + \tau f(x_\delta) - f(y_\delta) \le f(x_\delta) - f(y_\delta). \tag{2.4}$$

(4) We now claim that $x_\delta \in \mathbb{R}^n$ is bounded for all small enough $\delta > 0$. If not, then there is a sequence of $\delta \to 0$ for which $(w^\tau - \varphi_\delta)(x_\delta, y_\delta)$ tends to $-\infty$ as this sequence of δ tends to 0. Indeed,

$$\begin{aligned}(w^\tau - \varphi_\delta)(x_\delta, y_\delta) &= (\tau u(x_\delta) - v(x_\delta)) + v(x_\delta) - v(y_\delta) - \frac{|x_\delta - y_\delta|^2}{2\delta} \\ &\le (\tau u(x_\delta) - v(x_\delta)) + |x_\delta - y_\delta| - \frac{|x_\delta - y_\delta|^2}{2\delta} \\ &\le \tau u(x_\delta) - v(x_\delta) + \frac{\delta}{2},\end{aligned}$$

which tends to $-\infty$ as $\delta \to 0$ provided $\lim_{\delta\to 0^+} |x_\delta| = +\infty$. This would be the case for some sequence of $\delta \to 0$ provided x_δ is unbounded.

However,

$$\begin{aligned}(w^\tau - \varphi_\delta)(x_\delta, y_\delta) &= \max_{x,y\in\mathbb{R}^n}\left\{\tau u(x) - v(y) - \frac{|x-y|^2}{2\delta}\right\}\\ &\geq \tau u(0) - v(0)\\ &> -\infty,\end{aligned}$$

and thus x_δ lies in a bounded subset of $\mathbb{R}^n$. Likewise, we conclude that y_δ is also bounded for all small $\delta > 0$. It then follows from lemma 3.1 in [6] that

$$\lim_{\delta\to 0^+} \frac{|x_\delta - y_\delta|^2}{2\delta} = 0,$$

and therefore the sequence $((x_\delta, y_\delta))_{\delta>0}$ has a cluster point (x_τ, x_τ) for some sequence of $\delta \to 0^+$. Passing to the limit along such a sequence of δ tending to 0 gives

$$\tau\lambda - \mu \leq 0.$$

We conclude by letting $\tau \to 1^-$.

□

Uniqueness of eigenvalues with solutions having the appropriate growth for large values of $|x|$ is now immediate.

COROLLARY 2.3. *There can be at most one* $\lambda \in \mathbb{R}$ *such that* (1.1) *has a viscosity solution* u *satisfying the growth condition* (1.2).

3 Approximation Scheme

Another interesting corollary of Proposition 2.2 is the following:

COROLLARY 3.1. *Suppose there exists an eigenvalue* λ^* *as described in Theorem* 1.1. *Then*

$$\lambda^* = \sup\Big\{\lambda \in \mathbb{R} : \text{there exists a subsolution } u \text{ of (1.1) with eigenvalue } \lambda \text{ satisfying } \limsup_{|x|\to\infty} \tfrac{u(x)}{|x|} \leq 1\Big\} \tag{3.1}$$

and

$$\lambda^* = \inf\Big\{\mu \in \mathbb{R} : \text{there exists a supersolution } v \text{ of (1.1) with eigenvalue } \mu, \text{ satisfying } \liminf_{|x|\to\infty} \tfrac{v(x)}{|x|} \geq 1\Big\}. \tag{3.2}$$

It would be of great interest to show that both expressions on the right-hand sides of (3.1) and (3.2) are equal and that this number is the unique eigenvalue of equation (1.1). Such a procedure for producing eigenvalues would be reminiscent of Perron's method for exhibiting viscosity solutions of PDEs enjoying a comparison principle. Unfortunately, this method does not work so directly in our context because it is not clear that if, say, the right-hand side of (3.1) is not an eigenvalue we can find a strictly bigger number with a corresponding u that is a subsolution of (1.1). Therefore, we are led to an alternative procedure of approximating an eigenvalue.

The method we propose is a PDE version of the probabilistic approach used by Menaldi et al. [16]. This approach essentially amounts to replacing λ in (1.1) with δu and studying the resulting PDE for $\delta > 0$ and small. If this resulting PDE has a unique solution u_δ, the hope is that there is a sequence of δ tending to 0 such that δu_δ tends to λ. However, we believe the earliest application of this method appears in periodic homogenization of viscosity solutions of PDE in [7, 8, 15]; for this approach applied to general ergodic control problems, see reference [4].

To this end, we will study solutions of the PDE (1.3)

$$\max\{\delta u - \Delta u - f,\ |Du| - 1\} = 0, \quad x \in \mathbb{R}^n.$$

In particular, we seek a viscosity solution u satisfying (1.2)

$$\lim_{|x|\to\infty} \frac{u(x)}{|x|} = 1.$$

PROPOSITION 3.2. *Suppose u is a subsolution of* (1.3) *and that v is a supersolution of* (1.3). *If in addition*

$$\limsup_{|x|\to\infty} \frac{u(x)}{|x|} \le 1 \le \liminf_{|x|\to\infty} \frac{v(x)}{|x|},$$

then $u \le v$.

PROOF. We omit the proof since it is almost identical to the proof of Proposition 2.2. □

COROLLARY 3.3. *There can be at most one viscosity solution u* (1.3) *satisfying the growth condition* (1.2).

With a comparison principle in hand, we can now employ a routine application of Perron's method to obtain existence of solutions once we have appropriate sub- and supersolutions.

LEMMA 3.4. *Fix $0 < \delta \le 1$.*

(i) *There is a universal constant $K > 0$ such that*

$$\underline{u}(x) = (|x| - K)^+, \quad x \in \mathbb{R}^n, \tag{3.3}$$

is a viscosity subsolution of (1.3) *satisfying the growth condition* (1.2).

(ii) *There is a universal constant $K > 0$ such that*

$$\overline{u}(x) = \frac{K}{\delta} + \begin{cases} \frac{1}{2}|x|^2, & |x| \le 1, \\ |x| - \frac{1}{2}, & |x| \ge 1, \end{cases} \quad x \in \mathbb{R}^n, \tag{3.4}$$

is a viscosity supersolution of (1.3) *satisfying the growth condition* (1.2).

PROOF.

(i) Choose $K > 0$ such that

$$\underline{u}(x) \leq f(x), \quad x \in \mathbb{R}^n.$$

Such a K can be chosen due to the superlinear growth of f.

Since $\underline{u}$ is convex and $\mathrm{Lip}(\underline{u}) = 1$, if $(p, X) \in J^{2,+}\underline{u}(x_0)$,

$$|p| \leq 1 \quad \text{and} \quad X \geq 0.$$

Hence

$$\begin{aligned} \max\{\delta\underline{u}(x_0) - \operatorname{tr} X - f(x_0), |p| - 1\} &\leq \max\{\underline{u}(x_0) - f(x_0), |p| - 1\} \\ &\leq 0. \end{aligned}$$

Thus $\underline{u}$ is a viscosity subsolution.

(ii) Choose

$$K := n + \max_{|x|\leq 1} f(x),$$

and assume that $(p, X) \in J^{2,-}\underline{u}(x_0)$. If $|x_0| < 1$, $\overline{u}$ is smooth in a neighborhood of x_0 and

$$\begin{cases} \overline{u}(x_0) = \frac{K}{\delta} + \frac{|x_0|^2}{2}, \\ D\overline{u}(x_0) = x_0 = p, \\ \Delta\overline{u}(x_0) = n = \operatorname{tr} X. \end{cases}$$

Therefore,

$$\delta\overline{u}(x_0) - \Delta\overline{u}(x_0) - f(x_0) \geq K - n - f(x_0) \geq 0,$$

which implies

$$\max\{\delta\overline{u}(x_0) - \Delta\overline{u}(x_0) - f(x_0),\ |D\overline{u}(x_0)| - 1\} \geq 0. \tag{3.5}$$

Now suppose $|x_0| \geq 1$. $\overline{u} \in C^1(\mathbb{R}^n)$, so $p = D\overline{u}(x_0) = x_0/|x_0|$, and in particular $|D\overline{u}(x_0)| = 1$. Thus (3.5) still holds, and consequently, $\overline{u}$ is a viscosity supersolution. □

The following proposition, which implies Theorem 1.2(ii), follows directly from theorem 4.1 in [6] using $\underline{u}$ and $\overline{u}$ above. Because the proof is a routine application of Perron's method, we omit it.

PROPOSITION 3.5. *Fix* $0 < \delta \leq 1$. *There is a unique viscosity solution* $u = u_\delta$ *of* (1.3) *satisfying* (1.2).

3.1 Basic Estimates

Before we attempt to pass to the limit as $\delta \to 0$, we will obtain some better estimates on u_δ that will help us build an eigenvalue λ^* and establish estimates on a solution u^* of (1.1) corresponding to this eigenvalue. So far we have shown that

(1.3) has a unique solution u_δ that satisfies the growth condition (1.2). Moreover, from the sub- and supersolutions (3.3) and (3.4) above, we have for each $0 < \delta \le 1$

$$(|x| - K)^+ \le u_\delta(x) \le \frac{K}{\delta} + |x|, \quad x \in \mathbb{R}^n,$$

and

$$|u_\delta(x) - u_\delta(y)| \le |x - y|, \quad x, y \in \mathbb{R}^n.$$

Our goal now is to obtain *second-derivative estimates* on u_δ. We first prove the following:

PROPOSITION 3.6. *There is a constant $C > 0$ such that for all $0 < \delta \le 1$ and (Lebesgue) almost every $x \in \mathbb{R}^n$, the following estimate holds*:

$$0 \le D^2 u_\delta(x) \le \frac{1}{\delta} \max_{|y| \le C} |D^2 f(y)|. \tag{3.6}$$

The above proposition follows from the following two lemmas. In the first lemma, we show that u_δ is convex by adapting the classical "convexity maximum principle" argument of Korevaar [13]; in the second lemma, we estimate the second-order difference quotient of u_δ from above to obtain the upper bound in (3.6).

LEMMA 3.7. *u_δ is convex.*

PROOF.

(1) We first assume $u \in C^2(\mathbb{R}^n)$ and for ease of notation, we write u for u_δ. Fix $0 < \tau < 1$ and set

$$\mathcal{C}^\tau(x, y) = \tau u\left(\frac{x+y}{2}\right) - \frac{u(x) + u(y)}{2}, \quad x, y \in \mathbb{R}^n.$$

We aim to bound $\mathcal{C}^\tau$ from above and later send $\tau \to 1^-$.

We first claim that $\mathcal{C}^\tau$ achieves its maximum value at some point $(x_\tau, y_\tau) \in \mathbb{R}^n \times \mathbb{R}^n$; it suffices to show

$$\lim_{|(x,y)| \to \infty} \mathcal{C}^\tau(x, y) = -\infty. \tag{3.7}$$

Let $(x_k, y_k) \in \mathbb{R}^n \times \mathbb{R}^n$ be such that

$$|x_k| + |y_k| \to \infty$$

as $k \to \infty$. Let N be large enough so that $|x_k| + |y_k| > 0$ for $k \ge N$.

Note that for $k \ge N$

$$\begin{aligned}\frac{\mathcal{C}^\tau(x_k, y_k)}{|x_k| + |y_k|} &= \tau \frac{u\left(\frac{x_k + y_k}{2}\right)}{|x_k| + |y_k|} \\ &\quad - \frac{1}{2}\left\{\left(\frac{|x_k|}{|x_k| + |y_k|}\right)\frac{u(x_k)}{|x_k|} + \left(\frac{|y_k|}{|x_k| + |y_k|}\right)\frac{u(y_k)}{|y_k|}\right\}.\end{aligned}$$

If $|x_k + y_k|$ happens to be bounded, then

$$\limsup_{k\to\infty} \frac{\mathcal{C}^\tau(x_k, y_k)}{|x_k| + |y_k|} \le -\frac{1}{2} < 0.$$

While if $|x_k + y_k| \to \infty$ as $k \to \infty$, we have for large k

$$\frac{\mathcal{C}^\tau(x_k, y_k)}{|x_k| + |y_k|} \le \frac{\tau}{2} \frac{u\left(\frac{x_k+y_k}{2}\right)}{\left|\frac{x_k+y_k}{2}\right|} - \frac{1}{2}\left\{\left(\frac{|x_k|}{|x_k| + |y_k|}\right)\frac{u(x_k)}{|x_k|} + \left(\frac{|y_k|}{|x_k| + |y_k|}\right)\frac{u(y_k)}{|y_k|}\right\};$$

and thus,

$$\limsup_{k\to\infty} \frac{\mathcal{C}^\tau(x_k, y_k)}{|x_k| + |y_k|} \le \frac{(\tau - 1)}{2} < 0.$$

Consequently, $\limsup_{k\to\infty} \mathcal{C}^\tau(x_k, y_k) = -\infty$. The claim (3.7) follows since (x_k, y_k) was an arbitrary unbounded sequence.

(2) Because (x_τ, y_τ) is a maximizing point for $\mathcal{C}^\tau$,

$$0 = D_x\mathcal{C}^\tau(x_\tau, y_\tau) = \frac{\tau}{2} Du\left(\frac{x_\tau + y_\tau}{2}\right) - \frac{1}{2} Du(x_\tau)$$

and

$$0 = D_y\mathcal{C}^\tau(x_\tau, y_\tau) = \frac{\tau}{2} Du\left(\frac{x_\tau + y_\tau}{2}\right) - \frac{1}{2} Du(y_\tau).$$

Thus,

$$\tau Du\left(\frac{x_\tau + y_\tau}{2}\right) = Du(x_\tau) = Du(y_\tau).$$

The function $v \mapsto \mathcal{C}^\tau(x_\tau + v, y_\tau + v)$ has a maximum at $v = 0$, which implies

$$0 \ge \tau\Delta u\left(\frac{x_\tau + y_\tau}{2}\right) - \frac{\Delta u(x_\tau) + \Delta u(y_\tau)}{2}.$$

Since

$$|Du(x_\tau)| = |Du(y_\tau)| = \tau\left|Du\left(\frac{x_\tau + y_\tau}{2}\right)\right| \le \tau < 1,$$

we have

$$\delta u(z) - \Delta u(z) - f(z) = 0, \quad z = x_\tau, y_\tau.$$

Combining the above inequalities gives

$$\begin{aligned}
\delta\mathcal{C}^{\tau}(x,y) &\le \delta\mathcal{C}^{\tau}(x_\tau,y_\tau)\\
&= \tau\delta u\left(\frac{x_\tau+y_\tau}{2}\right)-\frac{\delta u(x_\tau)+\delta u(y_\tau)}{2}\\
&\le \tau\Delta u\left(\frac{x_\tau+y_\tau}{2}\right)-\frac{\Delta u(x_\tau)+\Delta u(y_\tau)}{2}\\
&\quad + \tau f\left(\frac{x_\tau+y_\tau}{2}\right)-\frac{f(x_\tau)+f(y_\tau)}{2}\\
&\le f\left(\frac{x_\tau+y_\tau}{2}\right)-\frac{f(x_\tau)+f(y_\tau)}{2}\\
&\le 0
\end{aligned}$$

by the convexity of f for each $x, y \in \mathbb{R}^n$. Sending $\tau \to 1^-$, we conclude that u is convex.

(3) To make this formal argument rigorous, we employ a doubling-the-variables type of argument. Specifically, we substitute for the variable $(x+y)/2$ in $\mathcal{C}^\tau$ and study the related function

$$w^\tau(x,y,z)=\tau u(z)-\frac{u(x)+u(y)}{2},\quad x,y,z\in\mathbb{R}^n.$$

As in our proof of Proposition 3.2, it will also be useful for us to employ the penalty function

$$\varphi_\eta(x,y,z)=\frac{1}{2\eta}\left|\frac{x+y}{2}-z\right|^2,\quad x,y,z\in\mathbb{R}^n,$$

for small $\eta > 0$. Notice that

$$\begin{aligned}
(w^\tau-\varphi_\eta)(x,y,z) &= \tau\left\{u(z)-u\left(\frac{x+y}{2}\right)\right\}-\frac{1}{2\eta}\left|\frac{x+y}{2}-z\right|^2\\
&\quad + \tau u\left(\frac{x+y}{2}\right)-\frac{u(x)+u(y)}{2}\\
&\le \left(\tau\left|\frac{x+y}{2}-z\right|-\frac{1}{2\eta}\left|\frac{x+y}{2}-z\right|^2\right)\\
&\quad + \tau u\left(\frac{x+y}{2}\right)-\frac{u(x)+u(y)}{2}.
\end{aligned}\tag{3.8}$$

From part 1 of this proof, it follows that

$$\lim_{|(x,y,z)|\to\infty}(w^\tau-\varphi_\eta)(x,y,z)=-\infty$$

and, in particular, that there is $(x_\eta, y_\eta, z_\eta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ maximizing $w^\tau - \varphi_\eta$. By theorem 3.2 in [6], for each $\rho > 0$ there are $X, Y, Z \in \mathcal{S}(n)$ such that

$$\begin{cases} (-2D_x\varphi_\eta(x_\eta, y_\eta, z_\eta), X) \in \overline{J}^{2,-}u(x_\eta), \\ (-2D_y\varphi_\eta(x_\eta, y_\eta, z_\eta), Y) \in \overline{J}^{2,-}u(y_\eta), \\ (\frac{1}{\tau}D_z\varphi_\eta(x_\eta, y_\eta, z_\eta), Z) \in \overline{J}^{2,+}u(z_\eta), \end{cases} \tag{3.9}$$

$$\begin{pmatrix} -\frac{1}{2}X & 0 & 0 \\ 0 & -\frac{1}{2}Y & 0 \\ 0 & 0 & \tau Z \end{pmatrix} \le A + \rho A^2. \tag{3.10}$$

Here

$$A := D^2\varphi_\eta(x_\eta, y_\eta, z_\eta) = \frac{1}{\eta}\begin{pmatrix} \frac{1}{4}I_n & \frac{1}{4}I_n & -\frac{1}{2}I_n \\ \frac{1}{4}I_n & \frac{1}{4}I_n & -\frac{1}{2}I_n \\ -\frac{1}{2}I_n & -\frac{1}{2}I_n & I_n \end{pmatrix}.$$

Note that the matrix inequality (3.10) immediately implies

$$\tau Z - \frac{1}{2}(X + Y) \le 0. \tag{3.11}$$

Set

$$p_\eta := \frac{1}{\eta}\left(z_\eta - \frac{x_\eta + y_\eta}{2}\right),$$

and note

$$p_\eta = -2D_x\varphi_\eta(x_\eta, y_\eta, z_\eta) = -2D_y\varphi_\eta(x_\eta, y_\eta, z_\eta) = D_z\varphi_\eta(x_\eta, y_\eta, z_\eta).$$

By the third inclusion in (3.9),

$$\max\{\delta u(z_\eta) - \operatorname{tr} Z - f(z_\eta), |p_\eta/\tau| - 1\} \le 0.$$

Hence, $|p_\eta| \le \tau < 1$ and by the top two inclusions in (3.9),

$$\begin{cases} \delta u(x_\eta) - \operatorname{tr} X - f(x_\eta) \ge 0, \\ \delta u(y_\eta) - \operatorname{tr} Y - f(y_\eta) \ge 0. \end{cases}$$

Combining these inequalities with (3.11) gives

$$\begin{aligned} \delta w^\tau(x, y, z) &\le \delta w(x_\eta, y_\eta, z_\eta) \\ &= \tau\delta u(z_\eta) - \frac{\delta u(x_\eta) + \delta u(y_\eta)}{2} \\ &\le \tau(\operatorname{tr} Z + f(z_\eta)) - \frac{(\operatorname{tr} X + f(x_\eta)) + (\operatorname{tr} Y + f(y_\eta))}{2} \\ &= \operatorname{tr}\left[\tau Z - \frac{X + Y}{2}\right] + \tau f(z_\eta) - \frac{f(x_\eta) + f(y_\eta)}{2} \\ &\le f(z_\eta) - \frac{f(x_\eta) + f(y_\eta)}{2} \end{aligned} \tag{3.12}$$

for each $(x, y, z) \in \mathbb{R}^n$.

(4) Another simple estimate for $w^\tau - \varphi_\eta$ that follows from (3.8) is

$$(w^\tau - \varphi_\eta)(x, y, z) \le \tau u\left(\frac{x+y}{2}\right) - \frac{u(x)+u(y)}{2} + \frac{\tau^2\eta}{2}. \tag{3.13}$$

This estimate implies that $(x_\eta, y_\eta)_{\eta>0}$ is a bounded sequence, for otherwise $(w^\tau - \varphi_\eta)(x_\eta, y_\eta, z_\eta)$ tends to $-\infty$ while

$$\begin{aligned}(w^\tau - \varphi_\eta)(x_\eta, y_\eta, z_\eta) &= \max_{x,y,z}(w^\tau - \varphi_\eta)(x, y, z)\\ &\ge (w^\tau - \varphi_\eta)(0, 0, 0)\\ &= (\tau - 1)u(0)\\ &> -\infty\end{aligned}$$

for each $\eta > 0$. Likewise, $(z_\eta)_{\eta>0}$ is a bounded sequence. By lemma 3.1 in [6], there is a cluster point $(x_\tau, y_\tau, (x_\tau + y_\tau)/2)$ of the sequence $((x_\eta, y_\eta, z_\eta))_{\eta>0}$ through some sequence of $\eta \to 0$ that maximizes the function

$$(x, y) \mapsto \tau u\left(\frac{x+y}{2}\right) - \frac{u(x)+u(y)}{2}.$$

Passing to the limit through this sequence of η tending to 0 in (3.12) gives for any $x, y \in \mathbb{R}^n$

$$\tau u\left(\frac{x+y}{2}\right) - \frac{u(x)+u(y)}{2} \le f\left(\frac{x_\tau + y_\tau}{2}\right) - \frac{f(x_\tau)+f(y_\tau)}{2} \le 0$$

due to the convexity of f. Sending $\tau \to 1^-$ establishes the claim. □

Aleksandrov's theorem [9, sec. 6.4] now implies the following corollary:

COROLLARY 3.8. *u_δ is twice differentiable at (Lebesgue) almost every point in $\mathbb{R}^n$.*

Since u_δ is convex and f is superlinear, we expect

$$\delta u_\delta(x) - \Delta u_\delta(x) - f(x) \le \delta u_\delta(x) - f(x) \le K + |x| - f(x) < 0$$

for all x large enough and all $0 < \delta \le 1$. Here K is the constant in (3.4). In other words, if $|Du_\delta(x)| < 1$, then $|x| \le C$ for some C independent of $0 < \delta \le 1$. We give a precise statement of this in terms of jets.

COROLLARY 3.9. *There is a constant $C > 0$, independent of $0 < \delta \le 1$, such that if $|x| \ge C$ and $p \in J^{1,-}u_\delta(x)$, then $|p| \ge 1$.*

PROOF. Let K be the constant in (3.4) and choose C so large that

$$K + |z| < f(z), \quad |z| \ge C.$$

Recall that $J^{1,-}u_\delta(x) = \underline{\partial} u_\delta(x)$ by the convexity of u_δ (see proposition 4.7 in [1]). Here

$$\underline{\partial} u(x) = \{p \in \mathbb{R}^n : u(y) \ge u(x) + p \cdot (y - x) \text{ for all } y \in \mathbb{R}^n\}$$

is the *subdifferential* of u at the point x. Moreover, $(p, 0) \in J^{2,-}u_\delta(x)$, and so

$$\max\{\delta u_\delta(x) - f(x), |p| - 1\} \geq 0.$$

Since $\delta u_\delta(x) - f(x) \leq K + |x| - f(x) < 0$, $|p| \geq 1$. □

LEMMA 3.10. *Let $C_1 > C$, where C is the constant in the previous corollary. For almost every $x \in \mathbb{R}^n$,*

$$D^2 u_\delta(x) \leq \frac{1}{\delta} \max_{|y| \leq C_1} |D^2 f(y)|$$

for all $0 < \delta < 1$.

PROOF.

(1) Fix $0 < \tau < 1$, $0 < |z| < C_1 - C$, and set

$$\mathcal{C}(x) := \tau u_\delta(x + z) - 2u_\delta(x) + \tau u_\delta(x - z), \quad x \in \mathbb{R}^n.$$

As in previous arguments, we will give a formal proof (i.e., assuming $u \in C^2(\mathbb{R}^n)$) first and then later describe how to our justify arguments. For ease of notation, we write u for u_δ.

Since $\lim_{|x|\to\infty} u(x)/|x| = 1$,

$$\lim_{|x|\to\infty} \mathcal{C}(x) = -\infty.$$

Thus, there is $\hat{x} \in \mathbb{R}^n$ such that

$$\mathcal{C}(\hat{x}) = \max_{x \in \mathbb{R}^n} \mathcal{C}(x).$$

At $\hat{x}$, we have

$$\begin{cases} 0 = D\mathcal{C}(\hat{x}) = \tau Du(\hat{x} + z) - 2Du(\hat{x}) + \tau Du(\hat{x} - z), \\ 0 \geq \Delta\mathcal{C}(\hat{x}) = \tau \Delta u(\hat{x} + z) - 2\Delta u(\hat{x}) + \tau \Delta u(\hat{x} - z). \end{cases}$$

Thus,

$$|Du(\hat{x})| = \frac{\tau}{2}|Du(\hat{x} + z) + Du(\hat{x} - z)| \leq \tau < 1,$$

and in particular

$$\begin{cases} |\hat{x}| \leq C \text{ (from the previous corollary)}, \\ \delta u(\hat{x}) - \Delta u(\hat{x}) - f(\hat{x}) = 0. \end{cases}$$

Hence, for $x \in \mathbb{R}^n$,

$$\begin{aligned} \delta\mathcal{C}(x) &\leq \delta\mathcal{C}(\hat{x}) \\ &= \tau\delta u(\hat{x} + z) - 2\delta u(\hat{x}) + \tau\delta u(\hat{x} - z) \\ &\leq \tau\Delta u(\hat{x} + z) - 2\Delta u(\hat{x}) + \tau\Delta u(\hat{x} - z) \\ &\quad + \tau(f(\hat{x} + z) + f(\hat{x} - z)) - 2f(\hat{x}) \\ &\leq f(\hat{x} + z) - 2f(\hat{x}) + f(\hat{x} - z) \leq \end{aligned}$$

$$\leq \max_{-1\leq \xi\leq 1} D^2 f(\hat{x}+\xi z)z\cdot z$$
$$\leq \max_{|y|\leq C_1} |D^2 f(y)||z|^2.$$

Since the last expression is independent of τ, we send $\tau \to 1^-$ and arrive at the inequality

$$\frac{u(x+z)-2u(x)+u(x-z)}{|z|^2} \leq \frac{1}{\delta}\max_{|y|\leq C_1} |D^2 f(y)|, \quad 0<|z|<C_1-C.$$

The claim now follows because D^2u exists a.e. in $\mathbb{R}^n$.

(2) Similar to previous proofs, we will "triple the variables." Again we fix $0<\tau<1$ and $0<|z|<C_1-C$. Set

$$\begin{cases} w(x_1,x_2,x_3) := \tau(u(x_1+z)+u(x_2-z))-2u(x_3), \\ \varphi_\eta(x_1,x_2,x_3) := \frac{1}{2\eta}\{|x_1-x_3|^2+|x_2-x_3|^2\}, \end{cases}$$

for $x_1,x_2,x_3\in\mathbb{R}^n$ and $\eta>0$. Notice that

$$\begin{aligned}(w-\varphi_\eta)(x_1,x_2,x_3) &= \tau(u(x_1+z)-u(x_3-z)) \\ &\quad + \tau(u(x_2+z)-u(x_3-z)) + \mathcal{C}(x_3) \\ &\quad - \frac{1}{2\eta}\{|x_1-x_3|^2+|x_2-x_3|^2\} \\ &\leq \left(\tau|x_1-x_3| - \frac{1}{2\eta}|x_1-x_3|^3\right) \\ &\quad + \left(\tau|x_2-x_3| - \frac{1}{2\eta}|x_2-x_3|^3\right) + \mathcal{C}(x_3),\end{aligned}$$

which immediately implies

$$\lim_{|(x_1,x_2,x_3)|\to\infty} (w-\varphi_\eta)(x_1,x_2,x_3) = -\infty.$$

In particular, there is $(x_1^\eta, x_2^\eta, x_3^\eta)$ globally maximizing $w-\varphi_\eta$. Now we can invoke theorem 3.2 in [6] and argue very similarly to how we did in the proof of the convexity of solutions of (1.3). We leave the details to the reader. □

COROLLARY 3.11. *We have the following*:

(i) $u_\delta \in C^{1,1}(\mathbb{R}^n)$.
(ii) $\Omega_\delta := \{x\in\mathbb{R}^n : |Du_\delta(x)|<1\}$ *is open and bounded independently of all* $0<\delta\leq 1$.
(iii) $u_\delta\in C^\infty(\Omega_\delta)$.
(iv) *There is* L (*independent of* $0<\delta\leq 1$) *such that*

$$D^2u_\delta(x)\leq L, \quad x\in\Omega_\delta.$$

PROOF. As usual, we write u for u_δ. (i) is immediate from Proposition 3.6. (ii) follows from Corollary 3.9 and (i), since $x \mapsto |Du(x)|$ is continuous on $\mathbb{R}^n$. (iii) follows from basic elliptic regularity theory since u satisfies the linear elliptic PDE

$$\delta u(x) - \Delta u(x) = f(x), \quad x \in \Omega_\delta,$$

and $f \in C^\infty(\mathbb{R}^n)$ (see theorem 6.17 [11]). As for (iv), we have by convexity that if $x \in \Omega_\delta$ and $|\xi| = 1$,

$$D^2u(x)\xi \cdot \xi \le \Delta u(x) = \delta u(x) - h(x) \le K + \delta|x| \le K + C =: L.$$

□

We conclude this subsection with a statement that u_δ is a Lipschitz extension of its values in $\overline{\Omega_\delta}$.

PROPOSITION 3.12.

$$u_\delta(x) = \min_{y\in\overline{\Omega_\delta}}\{u_\delta(y) + |x - y|\}, \quad x \in \mathbb{R}^n. \tag{3.14}$$

PROOF. It is simple to check that, since $\mathrm{Lip}[u_\delta] \le 1$, the formula above holds for $x \in \overline{\Omega_\delta}$. We now proceed to show that the formula above also holds in the complement of $\overline{\Omega_\delta}$.

An easy argument using the convexity of u_δ establishes that the minimum in (3.14) is achieved on $\partial\Omega_\delta$ for $x \notin \Omega_\delta$. So we are left to show

$$u_\delta(x) = \min_{y\in\partial\Omega_\delta}\{u_\delta(y) + |x - y|\}, \quad x \notin \Omega_\delta. \tag{3.15}$$

To this end, we first notice that u_δ satisfies the eikonal equation

$$\begin{cases} |Dv| = 1, & x \in \overline{\Omega_\delta}^{\mathsf{c}} \\ v = u_\delta, & x \in \partial\Omega_\delta, \end{cases} \tag{3.16}$$

and we claim this PDE has a unique solution given by the right-hand side (RHS) of (3.15). It is not hard to see that the RHS above is a solution of (3.16). The RHS clearly defines a function with Lipschitz constant at most 1 and hence is a subsolution of (3.16). The RHS also dominates *every* subsolution of the eikonal equation that is equal to u_δ on $\partial\Omega_\delta$ and therefore is a supersolution of (3.16) by lemma 4.4 in [6]. The proof of uniqueness is a straightforward adaptation of the proof of the comparison of sub- and supersolutions of (1.3) (see also theorem 5.9 of [1]). □

COROLLARY 3.13. *There is a universal constant $C > 0$ such that the estimate*

$$D^2u_\delta(x) \le \frac{1}{|x| - C} \quad \text{a.e. } |x| > C$$

holds for all $0 < \delta \le 1$.

PROOF. Recall that Ω_δ is bounded independently of $0 < \delta \le 1$; let C be chosen so large that if $x \in \Omega_\delta$, then $|x| \le C$. Also recall that $x \mapsto |x|$ is smooth on $\mathbb{R}^n \setminus \{0\}$ and that

$$D^2|x| = \frac{1}{|x|}\left(I_n - \frac{x \otimes x}{|x|^2}\right) \le \frac{1}{|x|} I_n, \quad x \neq 0.$$

Let $x \in \mathbb{R}^n$ with $|x| > C$. From (3.14), there is $y \in \partial\Omega_\delta$ such that $u_\delta(x) = u_\delta(y) + |x-y|$; moreover, $|y| \le C$. Also, from (3.14) and the above computation, we have that as $|z| \to 0$

$$\begin{aligned}&\frac{u_\delta(x+z) - 2u_\delta(x) + u_\delta(x-z)}{|z|^2}\\ &\quad\le \frac{|x+z-y| - 2|x-y| + |x-z-y|}{|z|^2}\\ &\quad\le \frac{1}{|x-y|} + o(1) \le \frac{1}{|x|-|y|} + o(1) \le \frac{1}{|x|-C} + o(1).\end{aligned}$$

The corollary now follows because u_δ is twice differentiable almost everywhere in $\mathbb{R}^n$. □

3.2 C^2 Regularity of Solutions for Rotational f

We take a brief break from our analysis of u_δ to prove Theorem 1.3, which asserts that when f is rotational, $u_\delta \in C^2(\mathbb{R}^n)$. We assume that

$$f(x) := f_0(|x|), \quad x \in \mathbb{R}^n,$$

where $f_0 : [0,\infty) \to \mathbb{R}$ is convex, nondecreasing, and superlinear. Using the uniqueness of solutions of (1.3), it is straightforward to show that u_δ will also be radially symmetric. That is,

$$u_\delta(x) = \phi(|x|), \quad x \in \mathbb{R}^n,$$

for some ϕ satisfying

$$\begin{cases}\phi \in C^{1,1}(0,\infty),\\ \phi' \ge 0,\\ \phi'' \ge 0,\\ \lim_{r\to\infty} \frac{\phi(r)}{r} = 1,\end{cases}$$

and

$$\max\left\{\delta\phi - \frac{n-1}{r}\phi' - \phi'' - f_0(r),\ \phi' - 1\right\} = 0, \quad r > 0, \tag{3.17}$$

in the sense of viscosity solutions. We claim that ϕ and therefore u_δ is necessarily C^2 (and not just $C^{1,1}$).

Since ϕ is nondecreasing, there is an $r_0 > 0$ such that

$$\Omega_\delta = B_{r_0}(0).$$

Therefore, it suffices to show that $\phi''(r)$ exists at $r = r_0$ and equals 0. Of course, $\phi'(r) = 1$ for $r \geq r_0$, so

$$\phi''(r_0+) = 0.$$

Thus, we focus on establishing that

$$\rho := \phi''(r_0-) = 0.$$

By the convexity of ϕ, ρ exists and is nonnegative. By (3.17),

$$\delta\phi - \frac{n-1}{r}\phi' - \phi'' - f_0(r) = 0, \quad 0 < r < r_0,$$

and so letting $r \to r_0^-$

$$\delta\phi(r_0) - \frac{n-1}{r_0}\phi'(r_0) - \rho - f_0(r_0) = 0. \tag{3.18}$$

However, we always have

$$\delta\phi - \frac{n-1}{r}\phi' - \phi'' - f_0(r) \leq 0, \quad r > 0,$$

and so letting $r \to r_0^+$ gives

$$\delta\phi(r_0) - \frac{n-1}{r}\phi'(r_0) - f_0(r_0) \leq 0. \tag{3.19}$$

Comparing (3.18) and (3.19), we have

$$\rho = \delta\phi(r_0) - \frac{n-1}{r_0}\phi'(r_0) - f_0(r_0) \leq 0.$$

Hence $\rho = 0$ as desired, which proves Theorem 1.3.

Remark 3.14. Many singular control problems involving one state variable admit value functions that are twice continuously differentiable. See the original work of Beneš, Shepp, and Witsenhausen identifying this phenomenon [2]; for a more modern treatment, consult [10, chap. VIII].

4 A Uniform Second-Derivative Estimate

According to Corollary 3.13, D^2u_δ is bounded from above for all x large enough independently of all δ positive and small. However, the upper bound we have in the whole space

$$\frac{1}{\delta}\max_{|y|\leq C_1}|D^2 f(y)|$$

blows up as $\delta \to 0^+$. Our aim in this section is to obtain an estimate on the second derivative of u_δ that is *uniform* in all small $\delta > 0$. In fact, we prove the following:

PROPOSITION 4.1. *For each ball $B \subset \mathbb{R}^n$, there is a constant $C = C(B)$ such that*

$$|Du_\delta(x) - Du_\delta(y)| \leq C|x - y|, \quad x, y \in B,$$

for each $0 < \delta \leq 1$.

Having established the above proposition, we would immediately have from Corollary 3.13 the following uniform second-derivative estimate. This bound implies Theorem 1.2(ii).

COROLLARY 4.2. *There is a universal constant L such that*

$$0 \le D^2 u_\delta(x) \le L, \quad a.e.\ x \in \mathbb{R}^n, \tag{4.1}$$

for all $0 < \delta \le 1$.

4.1 Penalty Method

Towards proving Lemma 4.1, we fix $0 < \delta \le 1$, and for ϵ positive and small study solutions of the PDE

$$\begin{cases} \delta v - \Delta v + \beta_\epsilon(|Dv|^2 - 1) = f, & x \in B, \\ v = u_\delta, & x \in \partial B. \end{cases} \tag{4.2}$$

Here $(\beta_\epsilon)_{\epsilon>0}$ is a family of functions satisfying

$$\begin{cases} \beta_\epsilon \in C^\infty(\mathbb{R}), \\ \beta_\epsilon = 0, & z \le 0, \\ \beta_\epsilon > 0, & z > 0, \\ \beta'_\epsilon \ge 0, \\ \beta''_\epsilon \ge 0, \\ \beta_\epsilon(z) = \frac{z-\epsilon}{\epsilon}, & z \ge 2\epsilon. \end{cases} \tag{4.3}$$

For each $\epsilon > 0$, we think of β_ϵ as a smoothing of $z \mapsto (z/\epsilon)^+$. We consider this PDE a "penalization" of equation (1.3) because the values of $\beta_\epsilon(|Dv|^2 - 1)$ will be large for small ϵ if $|Dv|^2 \ge 1$. This approach is well-known in control theory and can be found, for instance, in the classic book of Bensoussan and Lions [3].

Since (4.2) is a uniformly elliptic, semilinear PDE, it has a unique classical solution $v_\epsilon \in C^\infty(B) \cap C^1(\overline{B})$ for each $\epsilon > 0$ (by a straightforward variant of [11, theorem 15.10]). Our goal is to deduce a pointwise bound $D^2 v_\epsilon$ that is independent of all ϵ (and δ) positive and small. With such an estimate we would be in a good position to pass to the limit and show $v_\epsilon \to u_\delta$ in $C^1(B)$ and in particular that $u_\delta \in W^{2,\infty}(B)$.[2]

This method was introduced by L. C. Evans to study elliptic equations with gradient constraints [7]. Although the results of [7] apply to this problem, we derive our own estimates below because we need to understand how the estimates depend on δ. Our principal result related to this penalization scheme is the following:

THEOREM 4.3. *Let $B' \Subset B$ be open. Then there is a constant $C = C(B')$ such that*

$$|D^2 v_\epsilon(x)| \le C, \quad x \in B',$$

for all $0 < \epsilon < 1$.

[2] We will actually do this on a slightly smaller ball than B.

We prove the above theorem by establishing several lemmas. We acknowledge that v_ϵ depends on δ and the overall goal is to obtain an estimate on $D^2 v_\epsilon$ that is also independent of δ. For now, we consider $\delta > 0$ to be fixed and later see how to establish such a uniform estimate while proving Theorem 4.3 (see Corollary 4.8).

LEMMA 4.4. *There is a constant C, independent of $\epsilon > 0$, such that*

$$|v_\epsilon(x)| \le C, \quad x \in \overline{B}.$$

PROOF. Equation (4.2) enjoys a comparison principle for viscosity sub- and supersolutions. It is easily verified that u_δ is a viscosity subsolution of (4.2) and therefore

$$u_\delta \le v_\epsilon.$$

Likewise, the same comparison principle establishes

$$v_\epsilon \le w$$

where w is the unique solution of the PDE

$$\begin{cases} \delta w - \Delta w = f, & x \in B, \\ w = u_\delta, & x \in \partial B. \end{cases}$$

□

COROLLARY 4.5. *There is a constant C, independent of $\epsilon > 0$, such that*

$$|Dv_\epsilon(x)| \le C, \quad x \in \partial B.$$

PROOF. We have from the proof of the previous lemma that $u_\delta \le v_\epsilon \le w$ and equality holds on ∂B. Hence,

$$\frac{\partial w(x)}{\partial \nu} \le \frac{\partial v_\epsilon(x)}{\partial \nu} \le \frac{\partial u_\delta(x)}{\partial \nu}, \quad x \in \partial B,$$

where ν is the outer unit normal on ∂B. □

LEMMA 4.6. *There is a constant C, independent of $0 < \epsilon < 1$, such that*

$$|Dv_\epsilon(x)| \le C, \quad x \in \overline{B}.$$

PROOF. For ease of notation we write v for v_ϵ. Set

$$\phi(x) := |Dv(x)|^2 - \lambda v(x), \quad x \in \overline{B},$$

for $\lambda > 0$, which will be chosen below. To prove the lemma, it suffices to bound ϕ from above.

Direct computation gives

$$\begin{cases} D\phi = 2D^2 v Dv - \lambda Dv, \\ \Delta\phi \ge -(|Dv|^2 + C) + \beta'(|Dv|^2 - 1)(2Dv \cdot D\varphi + \lambda |Dv|^2), \end{cases} \tag{4.4}$$

for some constant C independent of ϵ. Choose x_0 that maximizes ϕ. If $x_0 \in \partial B$, the bound on ϕ follows from the previous lemma. If

$$\beta'(|Dv(x_0)|^2 - 1) < 1,$$

then

$$\beta'(|Dv(x_0)|^2 - 1) < \frac{1}{\epsilon},$$

which implies that $\beta(|Dv(x_0)|^2 - 1) \le 1$, and in particular (4.3) implies that $|Dv(x_0)|^2 - 1 \le 2\epsilon \le 2$. Therefore, if $x_0 \in \partial B$ or $\beta'(|Dv(x_0)|^2 - 1) < 1$, then ϕ is bounded above.

Alternatively, if

$$x_0 \in B \quad \text{and} \quad \beta'(|Dv(x_0)|^2 - 1) \ge 1,$$

we have from computation (4.4)

$$0 \ge -|Dv(x_0)|^2 + C + \lambda |Dv(x_0)|^2.$$

For λ chosen large enough and independently of $0 < \epsilon < 1$, the above inequality implies a uniform bound on $|Dv(x_0)|^2$ and in turn on ϕ. □

LEMMA 4.7. *For each open $B' \Subset B$, there is a constant $C = C(B')$, independent of $0 < \epsilon < 1$, such that*

$$\beta_\epsilon(|Dv_\epsilon(x)|^2 - 1) \le C, \quad x \in B'.$$

PROOF.

(1) It suffices to bound

$$\phi_\epsilon(x) = \xi(x)\beta_\epsilon(|Dv_\epsilon(x)|^2 - 1), \quad x \in B,$$

for each $\xi \in C_c^\infty(B)$, $0 \le \xi \le 1$. For ease of notation we will omit ϵ subscripts, omit the argument of functions, and write β for $\beta_\epsilon(|Dv_\epsilon|^2 - 1)$. Using

$$\beta = \Delta v - \delta v + f \le C\{1 + |D^2 v|\}$$

and the fact that β is convex gives

$$\begin{aligned}
D\phi &= \beta D\xi + \xi D\beta \\
\Delta\phi &= \Delta\xi\beta + 2D\xi \cdot D\beta + \xi\Delta\beta \\
&= \Delta\xi\beta + 4\beta' D\xi \cdot D^2 v Dv \\
&\quad + \xi\big(\beta''|2D^2 v Dv|^2 + 2\beta'(|D^2 v|^2 + Dv \cdot D\Delta v)\big) \\
&\ge -C(1 + |D^2 v|) - C\beta'|D^2 v| \\
&\quad + 2\beta'\big(\xi|D^2 v|^2 + Dv \cdot \xi D\beta + \xi Dv \cdot D(\delta v - f)\big) \\
&\ge -C(1 + |D^2 v|) \\
&\quad + 2\beta'\{\xi|D^2 v|^2 + D\phi \cdot Dv - \beta D\xi \cdot Dv - C|D^2 v| - C\} \\
&\ge -C(1 + |D^2 v|) + 2\beta'\{\xi|D^2 v|^2 + D\phi \cdot Dv - C|D^2 v| - C\}
\end{aligned}$$

for various constants C independent of $0 < \epsilon < 1$ (although dependent on ξ).

(2) Choose a maximizing point x_0 for ϕ. If $x_0 \in \partial B$, $\phi \le \phi(x_0) = 0$. Now suppose that $x_0 \in B$. Necessarily

$$D\phi(x_0) = 0 \quad \text{and} \quad 0 \ge \Delta\phi(x_0),$$

and the above computations imply that at the point x_0

$$0 \ge -C(1 + |D^2 v|) + 2\beta'\{\xi |D^2 v|^2 - C|D^2 v| - C\}. \tag{4.5}$$

If $\beta' \le 1 < 1/\epsilon$, then $\beta \le 1$ and thus v is bounded uniformly from above. If $\beta' \ge 1$, (4.5) gives

$$0 \ge \beta'\{\xi |D^2 v|^2 - C|D^2 v| - C\}.$$

Because $\beta' > 0$,

$$0 \ge \xi |D^2 v|^2 - C|D^2 v| - C,$$

which implies a bound on $\xi(x_0)|D^2 v(x_0)|$ that is independent of $0 < \epsilon < 1$. We conclude the proof:

$$\phi \le \phi(x_0) = \xi(x_0)\beta(|Dv(x_0)|^2 - 1) \le C(\xi(x_0)|D^2 v(x_0)| + 1) \le C.$$

□

PROOF OF THEOREM 4.3.

(1) It suffices to bound the quantity

$$M_\epsilon := \max_{x \in \overline{B}}\{\eta(x)|D^2 v_\epsilon(x)|\}$$

uniformly in all small enough $\epsilon > 0$, for each $\eta \in C_c^\infty(B)$, $0 \le \eta \le 1$. To this end, we bound from the above the quantity

$$\begin{aligned}\phi_\epsilon(x) = {} & \frac{1}{2}\eta(x)^2 |D^2 v_\epsilon(x)|^2 \\ & + \eta(x)\lambda\beta_\epsilon(|Dv_\epsilon(x)|^2 - 1) + \frac{\mu}{2}|Dv_\epsilon(x)|^2, \quad x \in \overline{B},\end{aligned}$$

where λ and μ are positive constants that will be chosen below. For ease of notation, we will omit ϵ-dependence, omit function arguments, and write β for $\beta_\epsilon(|Dv_\epsilon|^2 - 1)$. We follow the arguments of Wiegner [18] closely here.

(2) As in previous arguments, we perform various computations that will help us study ϕ near its maximum value.

$$\begin{cases} \phi_{x_i} = \eta\eta_{x_i}|D^2 v|^2 + \eta^2 D^2 v \cdot D^2 v_{x_i} + \lambda(\eta_{x_i}\beta + 2\eta\beta' Dv \cdot Dv_{x_i}) \\ \qquad + \mu Dv \cdot Dv_{x_i}, \quad i = 1, \dots, n, \\ \Delta\phi = (\Delta\eta + |D\eta|^2)|D^2 v|^2 + 4\eta^2 \sum_{i=1}^n \eta_{x_i} D^2 v \cdot D^2 v_{x_i} \\ \qquad + \eta^2(|D^3 v|^2 + D^2 v \cdot D^2 \Delta v) \\ \qquad + \lambda\big[\Delta\eta\beta + 4\beta' D^2 v Dv \cdot D\eta \\ \qquad\quad + \eta\{\beta''|2D^2 v Dv|^2 + 2\beta'(|D^2 v|^2 + Dv \cdot D\Delta v)\}\big] \\ \qquad + \mu(|D^2 v|^2 + Dv \cdot D\Delta v). \end{cases}$$

Since

$$\Delta v = \beta + \delta v - f,$$

we have for $i, j = 1, \dots, n$,

$$\begin{cases} (\Delta v)_{x_i} = 2\beta' Dv \cdot Dv_{x_i} + \partial_{x_i}(\delta v - f) \\ (\Delta v)_{x_i x_j} = 4\beta''(Dv \cdot Dv_{x_i})(Dv \cdot Dv_{x_j}) \\ \qquad\qquad + 2\beta'(Dv_{x_i} \cdot Dv_{x_j} + Dv \cdot Dv_{x_i x_j}) + \partial^2_{x_i x_j}(\delta v - f). \end{cases}$$

Substituting these values into the expression above that we have for Δv gives

$$\begin{aligned} \Delta\phi = {} & (\Delta\eta + |D\eta|^2)|D^2 v|^2 + 4\eta^2 \sum_{i=1}^n \eta_{x_i} D^2 v \cdot D^2 v_{x_i} \\ & + \eta^2 \Big[|D^3 v|^2 + 4\beta'' D^2 v (D^2 v D v)(D^2 v D v) \\ & \qquad + 2\beta' \Big(D^2 v \cdot (D^2 v)^2 + \sum_{i=1}^n v_{x_i x_j} Dv \cdot Dv_{x_i x_j} \Big) \\ & \qquad + D^2 v \cdot D^2(\delta v - f) \Big] \\ & + \lambda \big[\Delta\eta\beta + 4\beta' D^2 v D v \cdot D\eta \\ & \qquad + \eta \big\{ \beta'' |2 D^2 v D v|^2 \\ & \qquad\qquad + 2\beta'(|D^2 v|^2 + 2\beta' D^2 v D v \cdot Dv + Dv \cdot D(\delta v - f)) \big\} \big] \\ & + \mu(|D^2 v|^2 + 2\beta' Dv \cdot D^2 v D v + Dv \cdot D(\delta v - f)). \end{aligned}$$

Moreover, using our computation of ϕ_{x_i} gives our final expression for $\Delta\phi$:

$$\begin{aligned} \Delta\phi = {} & (\Delta\eta + |D\eta|^2)|D^2 v|^2 + 4\eta \sum_{i=1}^n \eta_{x_i} D^2 v \cdot \eta D^2 v_{x_i} + \eta^2 |D^3 v|^2 \\ & + \lambda \Delta\eta\beta + 4\beta''\eta\{ D^2 v (D^2 v D v)(D^2 v D v) + \lambda |D^2 v D v|^2 \} \\ & + \eta^2 D^2 v \cdot D^2(\delta v - f) + 2\beta' Dv \cdot D\phi \\ & + \beta' \big[\lambda \big(4 D^2 v D v \cdot D\eta + 2\eta |D^2 v|^2 - 2\beta Dv \cdot D\eta \\ & \qquad + \eta Dv \cdot D(\delta v - f) \big) - 2\eta |D^2 v|^2 D\eta \cdot Dv \big] \\ & + \mu(|D^2 v|^2 + Dv \cdot D(\delta v - f)). \end{aligned} \tag{4.6}$$

(3) Choose a maximizing point x_0 for ϕ. If $x_0 \in \partial B$, we would be able to conclude the proof as we already have a uniform gradient bound for v. Now

suppose that $x_0 \in B$. By calculus,

$$D\phi(x_0) = 0 \quad \text{and} \quad \Delta\phi(x_0) \le 0;$$

and from equation (4.6), we have at the point x_0

$$\begin{aligned} 0 \ge {} & -C(|D^2 v|^2 + 1) + 4\beta''|D^2 v Dv|^2 \eta\{\lambda - \eta|D^2 v|\} \\ & + 2\beta'\big[\lambda(\eta|D^2 v|^2 - C - C|D^2 v|) - C\eta|D^2 v|^2\big] + \mu(|D^2 v|^2 - C) \end{aligned} \tag{4.7}$$

for various constants C independent of $\epsilon \in (0,1)$. In deriving the above inequality, we used the Cauchy-Schwarz inequality several times, the uniform bounds on v and $|Dv|$, and also the fact that

$$\beta \le (|Dv|^2 - 1)\beta' \le C\beta'$$

to simplify the term $\lambda\Delta\eta\beta$. The inequality above is due to the uniform gradient bounds on $|Dv|$ and the inequality $\beta(z) \le z\beta'(z)$, which is immediate from the convexity of β and $\beta(0) = 0$.

(4) Now choose

$$\lambda = \lambda_\epsilon := 2M_\epsilon \ge 2\eta(x_0)|D^2 v(x_0)|$$

so that (4.7) becomes

$$\begin{aligned} 0 \ge {} & -C(|D^2 v|^2 + 1) + 2\beta'\big[\lambda(\eta|D^2 v|^2 - C - C|D^2 v|) - C\eta|D^2 v|^2\big] \\ & + \mu(|D^2 v|^2 - C). \end{aligned} \tag{4.8}$$

If for this choice of λ

$$\lambda(\eta|D^2 v|^2 - C - C|D^2 v|) - C\eta|D^2 v|^2 \le 0,$$

we would have a bound on $\eta(x_0)|D^2 v(x_0)|$ independent of $\epsilon \in (0,1)$. If the above inequality does not hold, then from (4.8) we infer

$$0 \ge -C(|D^2 v|^2 + 1) + \mu(|D^2 v|^2 - C).$$

Clearly, there is a choice of $\mu > 0$ so that $\eta(x_0)|D^2 v(x_0)|$ is bounded independently of $\epsilon \in (0,1)$.

(5) Finally, we observe that from our bounds on $\eta(x_0)|D^2 v(x_0)|$

$$\begin{aligned} M_\epsilon^2 &\le \max_{\bar{B}} \phi_\epsilon(x) \\ &= \frac{1}{2}\eta(x_0)^2 |D^2 v_\epsilon(x_0)|^2 + \eta(x_0)\lambda_\epsilon \beta_\epsilon(|Dv_\epsilon(x_0)|^2 - 1) + \frac{\mu}{2}|Dv_\epsilon(x_0)|^2 \\ &\le \frac{1}{2}\eta(x_0)^2 |D^2 v_\epsilon(x_0)|^2 + CM_\epsilon + C \\ &\le C(M_\epsilon + 1) \end{aligned}$$

for some constant C, independent of $\epsilon \in (0,1)$. It is now plain that M_ϵ is bounded independently of $\epsilon \in (0,1)$. □

A close inspection of the above proof of Theorem 4.3 reveals that we actually have the following crucial inequality:

COROLLARY 4.8. *Let $B' \Subset B$ be open. Then there is a constant $C = C(B')$ such that*

$$|D^2 v_\epsilon(x)| \le C(1 + |\delta v_\epsilon|_{L^\infty(B)} + |Dv_\epsilon|^2_{L^\infty(B)}), \quad x \in B', \tag{4.9}$$

for all $0 < \epsilon < 1$.

The above estimates will now be used to establish Proposition 4.1. To this end, we first show here that there is a subsequence of $\epsilon \to 0^+$ such that $v_\epsilon \to u_\delta$ in $C^1_{\text{loc}}(B)$, where u_δ is the solution of (1.3). By the local, pointwise estimates that we have established for $D^2 v_\epsilon$, this convergence would imply $u_\delta \in W^{2,\infty}_{\text{loc}}(B)$. Then we use (4.9) to establish inequality (4.1).

PROOF OF PROPOSITION 4.1.

(1) So far we have established that there is a constant $C > 0$ such that

$$|v_\epsilon|_{W^{1,\infty}(B)} \le C, \quad \epsilon \in (0,1),$$

and for each open $B' \Subset B$, there is a constant C' such that

$$|v_\epsilon|_{W^{2,\infty}(B')} \le C', \quad \epsilon \in (0,1).$$

Standard compactness arguments can be used to show that there is a function $v \in W^{1,\infty}(B) \cap W^{2,\infty}_{\text{loc}}(B)$ and a sequence of $\epsilon_k \to 0$ such that

$$\begin{cases} v_{\epsilon_k} \to v & \text{uniformly in } \overline{B}, \\ v_{\epsilon_k} \to v & \text{in } C^1_{\text{loc}}(B), \end{cases} \tag{4.10}$$

as $k \to \infty$.[3]

(2) We now claim that v is a viscosity solution of (1.3) and therefore has to coincide with u_δ, the unique viscosity solution of the PDE

$$\begin{cases} \max\{\delta v - \Delta v - f, |Dv| - 1\} = 0, & x \in B, \\ v = u_\delta, & x \in \partial B. \end{cases}$$

Suppose that $v - \varphi$ has a local maximum at $x_0 \in B$ and that $\varphi \in C^2(B)$. We must show

$$\max\{\delta v(x_0) - \Delta\varphi(x_0) - f(x_0), |D\varphi(x_0)| - 1\} \le 0. \tag{4.11}$$

By adding $x \mapsto \frac{\rho}{2}|x - x_0|^2$ to φ and later sending $\rho \to 0$, we may assume that $v - \varphi$ has a *strict* local maximum. Since v_{ϵ_k} converges to v uniformly as $k \to \infty$, there is a sequence of x_k such that

$$\begin{cases} x_k \to x_0 \text{ as } k \to \infty, \\ v_{\epsilon_k} - \varphi \text{ has a local maximum at } x_k. \end{cases}$$

[3] That is, $v_{\epsilon_k} \to v$ uniformly in $\overline{B}$ and $v_{\epsilon_k} \to v$ in $C^1(B')$ for each $B' \Subset B$ as $k \to \infty$.

Since v_ϵ is a smooth solution of (4.2), we have

$$\delta v_{\epsilon_k}(x_k) - \delta\varphi(x_k) + \beta_{\epsilon_k}(|D\varphi(x_k)|^2 - 1) \le f(x_k).$$

Since $\beta_\epsilon \ge 0$, we can send $k \to \infty$ to arrive at

$$\delta v(x_0) - \Delta\varphi(x_0) \le f(x_0).$$

By Theorem 4.3,

$$0 \le \beta_{\epsilon_k}(|D\varphi(x_k)|^2 - 1) = \beta_{\epsilon_k}(|Dv_{\epsilon_k}(x_k)|^2 - 1) \le C,$$

which implies that when $k \to \infty$

$$|D\varphi(x_0)|^2 - 1 \le 0.$$

Thus, (4.11) holds.

Now suppose that $v - \psi$ has a (strict) local minimum at $x_0 \in B$ and that $\psi \in C^2(B)$. We must show

$$\max\{\delta v(x_0) - \Delta\psi(x_0) - f(x_0), |D\psi(x_0)| - 1\} \ge 0. \tag{4.12}$$

Arguing as above, we discover there is a sequence $\epsilon_k \to 0$ as $k \to \infty$ and x_k such that

$$\begin{cases} x_k \to x_0 \text{ as } k \to \infty, \\ v_{\epsilon_k} - \psi \text{ has a local minimum at } x_k. \end{cases}$$

If

$$|Dv(x_0)|^2 \ge 1,$$

then (4.12) holds. Suppose now that

$$|Dv(x_0)|^2 < 1.$$

Since v_ϵ is a smooth solution of (4.2), we have

$$\delta v_{\epsilon_k}(x_k) - \Delta\psi(x_k) + \beta_{\epsilon_k}(|D\psi(x_k)|^2 - 1) - f(x_k) \ge 0. \tag{4.13}$$

By the convergence established in the first part of this proof, $|Dv_{\epsilon_k}(x_k)| < 1$ for all k sufficiently large. Hence,

$$\lim_{k\to\infty} \beta_{\epsilon_k}(|D\psi(x_k)|^2 - 1) = 0.$$

With the above limit, we can pass to the limit (4.13) to get

$$\delta v(x_0) - \Delta\psi(x_0) - f(x_0) \ge 0,$$

and thus (4.12) holds in this case, too.

(3) From (4.9), we have that for $x, y \in B' \Subset B$, there is a constant C' such that

$$|Dv_{\epsilon_k}(x) - Dv_{\epsilon_k}(y)| \le C'(1 + |\delta v_{\epsilon_k}|_{L^\infty(B)} + |Dv_{\epsilon_k}|^2_{L^\infty(B)})|x - y|$$

for all k sufficiently large. As $v_{\epsilon_k} \to u_\delta$ in $C^1_{\text{loc}}(B)$ and as

$$|\delta u_\delta|_{L^\infty(B)} + |Du_\delta|^2_{L^\infty(B)} \text{ is bounded for } 0 < \delta \le 1,$$

we let $k \to \infty$ to discover that there is a constant $L = L(B')$ such that

$$|Du_\delta(x) - Du_\delta(y)| \le L|x - y|, \quad x, y \in B'. \qquad \square$$

4.2 Passing to the Limit

We now have the following estimates on u_δ $(0 < \delta \le 1)$:

$$\begin{cases} (|x| - K)^+ \le u_\delta(x) \le \frac{K}{\delta} + |x|, & x \in \mathbb{R}^n, \\ |Du_\delta(x)| \le 1, & x \in \mathbb{R}^n, \\ |Du_\delta(x) - Du_\delta(y)| \le L|x - y|, & x, y \in \mathbb{R}^n. \end{cases}$$

Our aim is to pass to the limit as $\delta \to 0^+$ and prove there is an eigenvalue λ^* as stated in Theorem 1.1. To this end, we define

$$\begin{cases} \lambda_\delta := \delta u_\delta(x_\delta), \\ v_\delta(x) := u_\delta(x) - u_\delta(x_\delta), \end{cases}$$

where x_δ is a global minimizer of u_δ. Of course, $Du_\delta(x_\delta) = 0$, and in particular $x_\delta \in \Omega_\delta$. Moreover, Corollary 3.9 asserts that $|x_\delta| \le C$ for some constant C independent of all $0 < \delta \le 1$.

For this constant C, we have that

$$0 \le \lambda_\delta \le K + C$$

and that v_δ satisfies

$$\begin{cases} |v_\delta(x)| \le |x| + C, \\ |Dv_\delta(x)| \le 1, \\ |Dv_\delta(x) - Dv_\delta(y)| \le L|x - y|, \end{cases}$$

for all $x, y \in \mathbb{R}^n$, $0 < \delta \le 1$. We will now use the above estimates to prove the following lemma:

LEMMA 4.9. *There is a sequence* $\delta_k > 0$ *tending to* 0 *as* $k \to \infty$, $\lambda^* \in \mathbb{R}$, *and* $u^* \in C^{1,1}(\mathbb{R}^n)$ *such that*

$$\begin{cases} \lambda^* = \lim_{k\to\infty} \lambda_{\delta_k}, \\ v_{\delta_k} \to u^* \text{ in } C^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ as } k \to \infty. \end{cases} \tag{4.14}$$

Moreover, u^* *is a solution of* (1.1) *satisfying the growth condition* (1.2) *with eigenvalue* λ^* *and*

$$0 \le D^2 u^* \le L \quad \text{a.e. } x \in \mathbb{R}^n. \tag{4.15}$$

PROOF. Routine compactness and diagonalization arguments establish the convergence (4.14). The estimate (4.15) is also immediate from this convergence.

It follows directly from the definition that viscosity solutions pass to the limit under local uniform convergence. Consequently, u^* satisfies the PDE

$$\max\{\lambda^* - \Delta u^* - f, |Du^*| - 1\} = 0, \quad x \in \mathbb{R}^n,$$

in the sense of viscosity solutions. Since $|u^*(x)| \le |x| + C$ for all $x \in \mathbb{R}^n$,

$$\limsup_{|x|\to\infty} \frac{u^*(x)}{|x|} \le 1.$$

By the Lipschitz extension formula (3.14) and Corollary 3.9, we also have for all $|x|$ sufficiently large,

$$v_\delta(x) = u_\delta(x) - u_\delta(x_\delta) \ge |x| - C$$

for some C independent of $0 < \delta \le 1$. Thus,

$$\liminf_{|x|\to\infty} \frac{u^*(x)}{|x|} \ge 1,$$

and so u^* satisfies the growth rate (1.2). □

Note that for any $x \in \mathbb{R}^n$, with say $|x| \le R$,

$$\delta u_\delta(x) = \delta u_\delta(x_\delta) + \delta(u_\delta(x) - u_\delta(x_\delta)) = \delta u_\delta(x_\delta) + O(\delta)$$

as $\delta \to 0^+$. This follows since $|x_\delta|$ is bounded uniformly for all small $\delta > 0$, and thus

$$|\delta(u_\delta(x) - u_\delta(x_\delta))| \le \delta|x - x_\delta| \le C\delta, \quad 0 < \delta \le 1.$$

Therefore, Lemma 4.9 implies Theorem 1.2(iii), and this result along with the comparison principle for eigenvalues (Proposition 2.2) completes the proof of Theorem 1.1. □

Remark 4.10. We emphasize that the main point to establishing a uniform second-derivative estimate on u_δ was to show that $u_\delta - u_\delta(x_0)$ converges to u^* in $C^1_{\text{loc}}(\mathbb{R}^n)$ through some sequence of δ tending to 0. Uniform convergence would have followed without this estimate since we have a uniform Lipschitz estimate on u_δ.

Remark 4.11. As we established for u_δ, u^* is its own Lipschitz extension

$$u^*(x) = \min_{y\in\overline{\Omega_0}} \{u^*(y) + |x - y|\}, \quad x \in \mathbb{R}^n,$$

where $\Omega_0 = \{x \in \mathbb{R}^n : |Du^*(x)| < 1\}$. Therefore, it suffices only to know u^* within Ω_0 to know it everywhere in space.

5 Min-Max Formulae

We conclude this work by proving Theorem 1.4, which asserts alternative, min-max characterizations of the eigenvalue λ^*. To this end, we recall formula (3.1),

$$\lambda^* = \sup\big\{\lambda \in \mathbb{R} : \text{there exists a subsolution } u \text{ of (1.1) with eigenvalue } \lambda, \text{ satisfying } \limsup_{|x|\to\infty} \tfrac{u(x)}{|x|} \le 1\big\},$$

and formula (3.2),

$$\lambda^* = \inf\big\{\mu \in \mathbb{R} : \text{there exists a supersolution } v \text{ of (1.1) with eigenvalue } \mu, \text{ satisfying } \liminf_{|x|\to\infty} \tfrac{v(x)}{|x|} \ge 1\big\},$$

which are now consequences of Theorem 1.1 and the comparison principle established in Proposition 2.2. Our goal is to use the above equalities to show

$$\lambda_- = \lambda^* = \lambda_+,$$

where

$$\lambda_- := \sup\Big\{\inf_{x\in\mathbb{R}^n}\{\Delta\phi(x) + f(x)\} : \phi \in C^2(\mathbb{R}^n), |D\phi| \le 1\Big\}$$

and

$$\lambda_+ := \inf\left\{\sup_{|D\psi(x)|<1}\{\Delta\psi(x) + f(x)\} : \psi \in C^2(\mathbb{R}^n), \liminf_{|x|\to\infty}\frac{\psi(x)}{|x|} \ge 1\right\}.$$

PROOF OF THEOREM 1.4.

(1) First we prove $\lambda^* = \lambda_-$. For $\phi \in C^2(\mathbb{R}^n)$ with $|D\phi| \le 1$, set

$$\mu^\phi := \inf_{x\in\mathbb{R}^n}\{\Delta\phi(x) + f(x)\}.$$

If $\mu^\phi = -\infty$, then $\mu^\phi \le \lambda^*$. If $\mu^\phi > -\infty$, then

$$\max\{\mu^\phi - \Delta\phi(x) - f(x), |D\phi(x)| - 1\} \le 0, \quad x \in \mathbb{R}^n.$$

Equality (3.1) implies $\mu^\phi \le \lambda^*$. Consequently, $\lambda_- = \sup \mu^\phi \le \lambda^*$.

(2) Now let u^* be as in Lemma 4.9 and $u^\epsilon := \eta^\epsilon * u^*$ be the standard mollification of u^* for $\epsilon > 0$. Note that since $|Du^*| \le 1$ and $0 \le D^2u^* \le L$, we have

$$|Du^\epsilon| \le 1 \quad \text{and} \quad 0 \le D^2u^\epsilon \le L \quad \text{for all } \epsilon > 0.$$

Also note that since $u^* \in C^{1,1}(\mathbb{R}^n)$,

$$\Delta u^\epsilon = \eta^\epsilon * \Delta u^* \ge \lambda^* - f^\epsilon,$$

where f^ϵ is the standard mollification of f.

Since f grows superlinear and D^2u^* is bounded, there is $R > 0$ such that $x \mapsto \Delta u^\epsilon(x) + f(x)$ achieves its minimum value for an $x \in B_R$ for all $\epsilon > 0$. Hence, as $\epsilon \to 0^+$,

$$\begin{aligned}\lambda^* \le \inf_{|x|\le R}\{\Delta u^\epsilon(x) + f^\epsilon(x)\} &\le \inf_{|x|\le R}\{\Delta u^\epsilon(x) + f(x)\} + o(1)\\ &= \inf_{x\in\mathbb{R}^n}\{\Delta u^\epsilon(x) + f(x)\} + o(1)\\ &\le \lambda_- + o(1).\end{aligned}$$

(3) To prove $\lambda^* = \lambda_+$, assume that $\psi \in C^2(\mathbb{R}^n)$ and that

$$\liminf_{|x|\to\infty}\frac{\psi(x)}{|x|} \ge 1.$$

Similar to our argument above, we set

$$\tau^\psi := \sup_{|D\psi(x)|<1}\{\Delta\psi(x) + f(x)\}.$$

If $\tau^\psi = +\infty$, then $\lambda^* \le \tau^\psi$. If $\tau^\psi < \infty$, then

$$\max\{\tau^\psi - \Delta\psi(x) - f(x), |D\psi(x)| - 1\} \ge 0, \quad x \in \mathbb{R}^n.$$

Equality (3.2) implies $\lambda^* \le \tau^\psi$. Consequently, $\lambda^* \le \inf \tau^\psi = \lambda_+$.

(4) Next, we claim that $\lambda^* \ge \lambda_+$. To prove this, we mollify u^* and scale u^* by a factor larger than 1:

$$u^{\epsilon,\delta} := (1+\delta)u^\epsilon = (1+\delta)\eta^\epsilon * u^*,$$

where $\delta > 0$ and $\epsilon > 0$. Notice that

$$|Du^{\epsilon,\delta}(x)| < 1 \Leftrightarrow |Du^\epsilon(x)| < \frac{1}{1+\delta}.$$

Select $\epsilon_0 = \epsilon_0(\delta) > 0$ such that

$$\gamma := \frac{1}{1+\delta} + \epsilon_0 |D^2 u^*|_{L^\infty(\mathbb{R}^n)} < 1.$$

Such an ϵ_0 exists because $1/(1+\delta) < 1$. Note that when $|Du^{\epsilon,\delta}(x)| < 1$ and $0 < \epsilon < \epsilon_0$,

$$|Du^*(x)| \le |Du^\epsilon(x)| + \epsilon |D^2 u^*|_{L^\infty(\mathbb{R}^n)} < \frac{1}{1+\delta} + \epsilon_0 |D^2 u^*|_{L^\infty(\mathbb{R}^n)} = \gamma.$$

Moreover, there is also $\epsilon_1 = \epsilon_1(\delta)$ such that

$$\{x \in \mathbb{R}^n : |Du^*(x)| < \gamma\} \subset \Omega^\epsilon := \{x \in \Omega_0 : \operatorname{dist}(x, \partial\Omega_0) > \epsilon\}$$

for $0 < \epsilon < \epsilon_1$, where $\Omega_0 = \{|Du^*| < 1\}$. This follows since $\{|Du^*| < \gamma\}$ is an open subset of Ω_0.

Hence for $0 < \epsilon < \min\{\epsilon_0, \epsilon_1\}$, we have

$$|Du^{\epsilon,\delta}(x)| < 1 \Rightarrow x \in \Omega^\epsilon.$$

In particular, if $|Du^{\epsilon,\delta}(x)| < 1$, then

$$\lambda^* - \Delta u^\epsilon(x) - f(x) = o(1)$$

as $\epsilon \to 0^+$, since $\lambda^* - \Delta u^* - f = 0$ on Ω_0.

With the above computations, the fact that Du^* and $D^2 u^*$ are bounded, and Ω^ϵ is a uniformly bounded set, we have

$$\begin{aligned}
\lambda_+ &\le \sup_{|Du^{\epsilon,\delta}(x)|<1} \{\Delta u^{\epsilon,\delta}(x) + f(x)\} \\
&= \sup_{|Du^{\epsilon,\delta}(x)|<1} \{(1+\delta)\Delta u^\epsilon(x) + f(x)\} \\
&= \sup_{|Du^{\epsilon,\delta}(x)|<1} \{\Delta u^\epsilon(x) + f(x)\} + O(\delta) \\
&\le \sup_{x\in\Omega^\epsilon} \{\Delta u^\epsilon(x) + f(x)\} + O(\delta) \\
&\le \lambda^* + o(1) + O(\delta)
\end{aligned}$$

(here $o(1)$ tends to 0 as $\epsilon \to 0^+$). We conclude the proof by first sending $\epsilon \to 0^+$ and then $\delta \to 0^+$. $\square$

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