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PARTIAL REGULARITY OF WEAK SOLUTIONS OF THE VISCOELASTIC NAVIER–STOKES EQUATIONS WITH DAMPING*

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Abstract. We prove an analogue of the Caffarelli–Kohn–Nirenberg theorem for weak solutions of a system of PDEs that model a viscoelastic fluid in the presence of an energy damping mechanism. The system was recently introduced as a possible method of establishing the global-in-time existence of weak solutions of the well-known Oldroyd system.

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1. Introduction.

The Oldroyd model for an incompressible, viscoelastic fluid is governed by the following system of equations:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u = \Delta u - \nabla p + \nabla \cdot FF^t, \\ \partial_t F + (u \cdot \nabla)F = \nabla u F, \\ \nabla \cdot u = 0. \end{cases} \tag{1.1}$$

Informally, we refer to (1.1) as the *viscoelastic Navier–Stokes equations*. This system of PDEs is always assumed to be satisfied in an open subset of spacetime $\mathbb{R}^3 \times \mathbb{R}$. At a point (x, t) in spacetime, $u = u(x, t) \in \mathbb{R}^3$ represents the fluid's velocity, $p = p(x, t) \in \mathbb{R}$ represents the fluid's pressure, and $F = F(x, t) \in \mathbb{R}^{3\times3}$ represents the local deformation of the fluid. The associated energy law for any smooth solution (u, p, F) on $\mathbb{R}^3 \times (0, \infty)$ that vanishes rapidly enough as $|x| \to \infty$ is

$$\frac{d}{dt}\int_{\mathbb{R}^3} \left\{ \frac{|u(x,t)|^2}{2} + \frac{|F(x,t)|^2}{2} \right\} dx = -\int_{\mathbb{R}^3} |\nabla u(x,t)|^2 dx$$

for $t > 0$.

Solutions of the initial value problem associated with (1.1) have been studied extensively. For instance, the short time existence of a smooth solution and the global existence of a smooth solution that is initially small (in an appropriate norm) has been established in various settings [6, 5]. However, it is not known if solutions with smooth initial and boundary data develop singularities or even if some meaningful type of weak solutions exist globally in time.

In pursuing the former problem, the authors of [5] introduced the following system as a way of approximating solutions of (1.1):

$$\begin{cases} \partial_t u + (u \cdot \nabla)u = \Delta u - \nabla p + \nabla \cdot FF^t, \\ \partial_t F + (u \cdot \nabla)F = \mu\Delta F + \nabla u F, \\ \nabla \cdot u = 0 \end{cases} \tag{1.2}$$

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for a parameter $\mu > 0$. The associated energy law for any smooth solution (u, p, F) of (1.2) on $\mathbb{R}^3 \times (0, \infty)$, which vanishes rapidly enough as $|x| \to \infty$, is

$$\frac{d}{dt} \int_{\mathbb{R}^3} \left\{ \frac{|u(x,t)|^2}{2} + \frac{|F(x,t)|^2}{2} \right\} dx = -\int_{\mathbb{R}^3} \left(\mu |\nabla F(x,t)|^2 + |\nabla u(x,t)|^2 \right) dx$$

for $t > 0$. Therefore, the presence of μ acts to create energy dissipation, and so we interpret μ as a *damping parameter* and call (1.2) the *viscoelastic Navier–Stokes equations with damping.*

It is not difficult to establish the existence of a global-in-time weak solution of (1.2) analogous to that of the Leray–Hopf solutions of the incompressible Navier–Stokes equations. Unfortunately, standard weak convergence methods do not allow one to pass to the limit as $\mu \to 0^+$ to generate weak solutions of (1.1); see the end of section 2 in [5] for more on this. Nevertheless, the system (1.2) is itself of interest and is the topic of study in our work.

A fundamental simplification that we will make in our analysis is that we consider solutions (u, p, F) of (1.2) that additionally satisfy the equation

$$\nabla \cdot F^t = 0. \tag{1.3}$$

That is, a standing assumption that we shall make is that the deformation F has divergence-free columns. This assumption is motivated by taking the divergence of the second equation in (1.2), which yields the following transport equation:

$$\partial_t (\nabla \cdot F^t) + (u \cdot \nabla)(\nabla \cdot F^t) = \mu \Delta (\nabla \cdot F^t).$$

The above PDE formally implies that if (1.3) holds at some instance of time, then it will hold at all later times. Therefore, we believe that our results below will pertain to solutions of (1.2) that initially satisfy (1.3).

As our results do not change qualitatively as $\mu > 0$ is varied, we set

$$\mu = 1$$

in our analysis of solutions of (1.2). Our main result is the analogue for (1.2) of the Caffarelli–Kohn–Nirenberg theorem for the incompressible Navier–Stokes equations [1]. The statement of this theorem involves the concept of a weak solution, which we will define in the next section, and the concept of the singular set of a solution (u, p, F). The *singular set* corresponding to a solution (u, p, F) is defined as the set of points (x, t) in the domain of (u, p, F) for which either u or F is not Hölder continuous in any neighborhood of (x, t). Any point not belonging to the singular set of (u, p, F) is a *regular point.*

THEOREM 1.1. *There is a universal constant $\epsilon > 0$ such that if (u, p, F) is a weak solution of* (1.2) *and if*

$$\limsup_{r \to 0^+} \frac{1}{r} \iint_{Q_r(x,t)} \left\{ |\nabla u|^2 + |\nabla F|^2 \right\} dyds < \epsilon, \tag{1.4}$$

then u and F are Hölder continuous on some neighborhood of (x, t). In particular, $(x, t) \notin S$.

In the limit above (1.4), and in this work,

$$Q_r(x,t) := B_r(x) \times (t - r^2/2, t + r^2/2) \subset \mathbb{R}^3 \times \mathbb{R}$$

is a parabolic cylinder of radius r centered at (x, t). Using standard covering arguments (see in particular section 6 of [1]), we have the following corollary.

COROLLARY 1.2. *Assume that* (u, p, F) *is a suitable weak solution of* (1.2) *on an open subset of* $\mathbb{R}^3 \times \mathbb{R}$ *and let* S *be the singular set of* (u, p, F). *Then* $\mathcal{P}^1(S) = 0$, *where* $\mathcal{P}^1$ *denotes one-dimensional parabolic Hausdorff measure on* $\mathbb{R}^3 \times \mathbb{R}$.

In proving Theorem 1.1, we employ the compactness method introduced by Lin [4] that leads to a relatively simple proof of the Caffarelli–Kohn–Nirenberg theorem. We also use ideas from the very clear account on these topics given by Ladyzhenskaya and Seregin [3]. The organization of this paper is as follows. In section 2, we define suitable weak solutions and establish an important compactness result for these solutions; a fundamental corollary of this compactness is a "decay" or "blow-up" lemma. In section 3, we use the decay lemma to deduce a condition that, if satisfied, implies solutions are locally Hölder continuous. Finally, in section 4, we show that (1.4) implies that our regularity condition is satisfied, which in turn furnishes a proof of Theorem 1.1.

2. Weak solutions. As mentioned above, we set $\mu = 1$ in (1.2) and study the following system of PDEs:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = \Delta u - \nabla p + \nabla \cdot F F^t, \\ \partial_t F + (u \cdot \nabla) F = \Delta F + \nabla u F, \\ \nabla \cdot u = 0. \end{cases} \tag{2.1}$$

Sometimes it will be beneficial for us to write (2.1) in terms of the columns of the matrix valued mapping F. Setting $F_j := F e_j$ for $j = 1, 2, 3$, we have from the assumption (1.3)

$$\nabla \cdot F F^t = \sum_{k=1}^{3} (F_k \cdot \nabla) F_k.$$

In particular, (2.1) and (1.3) can be rewritten together as

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = \Delta u - \nabla p + \sum_{k=1}^{3} (F_k \cdot \nabla) F_k, \\ \partial_t F_j + (u \cdot \nabla) F_j = \Delta F_j + (F_j \cdot \nabla) u, \quad j = 1, 2, 3, \\ \nabla \cdot u = \nabla \cdot F_j = 0. \end{cases} \tag{2.2}$$

As Theorem 1.1 is local, we only consider solutions on the unit cylinder $Q_1 := Q_1(0, 0)$. Before pursuing the analysis of solutions, let us make some basic observations that will motivate the definition of suitable weak solutions and other ideas that follow.

Scale invariance. If (u, p, F) is a solution of (2.1) on $Q_r(x_0, t_0)$, then $(u^\lambda, p^\lambda, F^\lambda)$ is a solution on $Q_{r/\lambda}(0, 0)$ for $\lambda > 0$, where

$$\begin{cases} u^\lambda(x, t) = \lambda u(x_0 + \lambda x, t_0 + \lambda^2 t), \\ p^\lambda(x, t) = \lambda^2 p(x_0 + \lambda x, t_0 + \lambda^2 t), \\ F^\lambda(x, t) = \lambda F(x_0 + \lambda x, t_0 + \lambda^2 t). \end{cases} \tag{2.3}$$

Local energy identity. If (u,p,F) is a smooth solution of (2.1) on Q_1 and $\phi \in C_c^\infty(Q_1)$, then

$$\frac{d}{dt}\int_{B_1}\phi\left(\frac{|u|^2}{2}+\frac{|F|^2}{2}\right)dx+\int_{B_1}\phi\left(|\nabla u|^2+|\nabla F|^2\right)dx$$
$$=\int_{B_1}\left\{(\phi_t+\Delta\phi)\left(\frac{|u|^2}{2}+\frac{|F|^2}{2}\right)-u\otimes\nabla\phi\cdot FF^t+\left(\frac{|u|^2}{2}+\frac{|F|^2}{2}+p\right)u\cdot\nabla\phi\right\}dx$$

for $t\in(-1/2,1/2)$.

A spacetime $L^{10/3}$ bound on u, F. From the local energy identity above, we expect solutions (u,p,F) to satisfy

$$\sup_{-1/2\le t\le 1/2}\int_{B_1}|u|^2+|F|^2dx+\iint_{Q_1}|\nabla u|^2+|\nabla F|^2dxdt\le C. \tag{2.4}$$

An application of the interpolation inequality

$$|v|_{L^r(\Omega)}\le C\left\{|v|^{\alpha}_{L^2(\Omega)}|\nabla v|^{1-\alpha}_{L^2(\Omega)}+\frac{|v|_{L^2(\Omega)}}{|\Omega|^{1/2-1/r}}\right\} \tag{2.5}$$

for $v\in H^1(\Omega;\mathbb{R}^3)$, where

$$\alpha=\frac{3}{r}-\frac{1}{2},\quad 2\le r\le 6,$$

provides the bound

$$\iint_{Q_1}|u|^{10/3}+|F|^{10/3}dxdt\le C_1. \tag{2.6}$$

Here C_1 depends only on the constant C in (2.4).

A spacetime $L^{5/3}$ bound on p. Taking the divergence of (2.1) gives

$$-\Delta p=\nabla\cdot\left[(u\cdot\nabla)u-\sum_{k=1}^{3}(F_k\cdot\nabla)F_k\right]. \tag{2.7}$$

Note that

$$|(u\cdot\nabla)u|_{L^{15/14}(B_1)}\le|\nabla u|_{L^2(B_1)}|u|_{L^{30/13}(B_1)}. \tag{2.8}$$

By the interpolation inequality (2.5) with $r=30/13$ and $\alpha=4/5$

$$|u|_{L^{30/13}(B_1)}\le C\left\{|\nabla u|^{1/5}_{L^2(B_1)}+1\right\}. \tag{2.9}$$

With (2.8) and (2.9), we have

$$|(u\cdot\nabla)u|^{5/3}_{L^{15/14}(B_1)}\le C\left\{|\nabla u|^2_{L^2(B_1)}+1\right\},$$

and likewise

$$|(F_j\cdot\nabla)F_j|^{5/3}_{L^{15/14}(B_1)}\le C\left\{|\nabla F_j|^2_{L^2(B_1)}+1\right\},\quad j=1,2,3.$$

As a result, standard elliptic estimates following from (2.7) give

$$\begin{aligned}\int_{-\theta^2/2}^{\theta^2/2} |\nabla p|^{5/3}_{L^{15/14}(B_\theta)} dt &\leq C \int_{-1/2}^{1/2} \left(|(u\cdot\nabla)u|^{5/3}_{L^{15/14}(B_1)} + \sum_{j=1}^{3} |(F_j\cdot\nabla)F_j|^{5/3}_{L^{15/14}(B_1)} + 1 \right) dt \\ &\leq C \iint_{Q_1} \left(|\nabla u|^2 + \sum_{j=1}^{3} |\nabla F_j|^2 + 1 \right) dxdt \qquad (2.10)\end{aligned}$$

for any $\theta \in (0,1)$. Since the Sobolev conjugate of 15/14 is 5/3,

$$\left(\frac{15}{14}\right)^* = \frac{5}{3},$$

$p \in L^{5/3}_{\mathrm{loc}}(Q_1)$.

The above observations motivate the following definition of weak solutions of the system (2.1). This definition is of course also consistent with the notion of (suitable) weak solutions presented in the original work of Caffarelli, Kohn, and Nirenberg [1].

Definition 2.1. *(u,p,F) is a weak solution of (2.1) on Q_1, provided*

(i) *$u, F \in L^3(Q_1)$ and $p \in L^{3/2}(Q_1)$,*

(ii) *equation (2.1) holds in the sense of distributions on Q_1, and*

(iii) *for each $\phi \in C_c^\infty(Q_1)$, with $\phi \geq 0$,*

$$\begin{aligned}&\int_{B_1\times\{t\}} \phi\left(\frac{|u|^2}{2} + \frac{|F|^2}{2}\right) dx + \int_{-1/2}^{t}\int_{B_1} \phi\left(|\nabla u|^2 + |\nabla F|^2\right) dxds \\ &\leq \int_{-1/2}^{t}\int_{B_1} \Bigg\{ (\phi_t + \Delta\phi)\left(\frac{|u|^2}{2} + \frac{|F|^2}{2}\right) \\ &\quad - u\otimes\nabla\phi\cdot FF^t + \left(\frac{|u|^2}{2} + \frac{|F|^2}{2} + p\right) u\cdot\nabla\phi \Bigg\} dxds, \qquad (2.11)\end{aligned}$$

$t \in [-1/2, 1/2]$.

Our first task is to establish that a bounded sequence of weak solutions has a convergent subsequence whose limit is again a weak solution. We will not use this result directly although the ideas that go into proving this result will be essential to our proof of Theorem 1.1. To this end, we start by quoting a standard compactness lemma.

Lemma 2.2. *Let X_0, X_1, X_2 denote Banach spaces, with X_0 and X_2 reflexive, that satisfy*

$$X_0 \subset X_1 \subset X_2.$$

Also suppose that the embedding of X_0 into X_1 is compact and the embedding of X_1 into X_2 is continuous. Let $p, q \in (1,\infty)$ and assume that $(u_k)_{k\in\mathbb{N}} \in L^p([0,1];X_0)$ is a bounded sequence such that each u_k has a weak derivative $\partial_t u_k$ and the sequence

$$(\partial_t u_k)_{k\in\mathbb{N}} \in L^q([0,1];X_2)$$

is also bounded. Then there is a subsequence of u_k converging strongly in $L^p([0,1],X_1)$.

Remark 1. We can replace $[0,1]$ in the statement above with any compact interval of $\mathbb{R}$.

We refer the reader to Theorem 2.1, section III of [8] for a detailed proof of the above lemma. Its primary application is the following theorem.

THEOREM 2.3. *Suppose that $(u^k, p^k, F^k)_{k\in\mathbb{N}}$ is a sequence of weak solutions of* (2.1) *on Q_1 satisfying*

$$\iint_{Q_1} \left\{|u^k|^3 + |p^k|^{3/2} + |F^k|^3\right\} dxdt \le C \tag{2.12}$$

for all $k \ge 1$ and some constant $C \ge 0$. Then there is a subsequence $(u^{k_j}, p^{k_j}, F^{k_j})_{j\in\mathbb{N}}$ and (u, P, F), such that

$$\begin{cases} u^{k_j} \to u \quad in \quad L^q_{loc}(Q_1), \\ p^{k_j} \rightharpoonup P \quad in \quad L^{3/2}(Q_1), \\ F^{k_j} \to F \quad in \quad L^q_{loc}(Q_1) \end{cases}$$

for $1 \le q < 10/3$. Moreover, (u, P, F) is a weak solution of (2.1) *on Q_1.*

Remark 2. It is straightforward to adapt the proof below to build globally defined weak solutions of a large class of initial value problems associated with (2.1). One may use a Galerkin-type approximation, for example.

Proof. First, we select the weak limit P of the sequence $(p^k)_{k\in\mathbb{N}}$ and without any loss of generality assume $p^k \rightharpoonup P$ in $L^{3/2}(Q_1)$ as $k \to \infty$. Next, we observe that by (2.2), we have that

$$\partial_t u^k = \nabla \cdot \left(-u^k \otimes u^k + \nabla u^k - p^k I + F^k (F^k)^t\right)$$

and

$$\partial_t F^k_j = \nabla \cdot \left(-F^k_j \otimes u^k + u^k \otimes F^k_j + \nabla F^k_j\right), \quad j = 1, 2, 3,$$

belong to the space

$$L^{3/2}((-1/2, 1/2); W^{-1,3}(B_1)).$$

By (2.12), $\partial_t u^k$ and $\partial_t F^k$ are in fact bounded in this space for all $k \ge 1$.

For a fixed $\theta \in (0, 1)$, we note that as (u^k, p^k, F^k) satisfies the local energy estimate (2.11)

$$\iint_{Q_\theta} |\nabla u^k|^2 + |\nabla F^k|^2 dxdt \le C \tag{2.13}$$

for some $C = C(\theta)$. These observations lead to the choice of exponents

$$p := 2 \quad \text{and} \quad q := 3/2$$

and Banach spaces

$$\begin{cases} X_0 := H^1(B_\theta), \\ X_1 := L^2(B_\theta), \\ X_2 := W^{-1,3}(B_\theta). \end{cases}$$

By our observations above, the hypotheses of the previous lemma are satisfied with the sequences $(u^k)_{k\in\mathbb{N}}$ and $(F^k)_{k\in\mathbb{N}}$ (and exponents and Banach spaces as indicated above). Hence, some subsequence of $(u^k)_{k\in\mathbb{N}}$, $(F^k)_{k\in\mathbb{N}}$ converges in $L^2(Q_\theta)$ to some u_θ, F_θ. By the bound (2.13), u^k, F^k are bounded in $L^{10/3}(Q_\theta)$ (recall the estimate (2.6)). Consequently, the interpolation of Lebesgue spaces implies $(u^k, F^k) \to (u_\theta, F_\theta)$ as $k \to$ in $L^q(Q_\theta)$ for $1 \le q < 10/3$. As $\theta \in (0,1)$ was arbitrary, we can employ a routine diagonalization argument to construct a $u, F \in L^{10/3}_{\text{loc}}(Q_1)$ such that a subsequence of u^k and F^k converge, respectively, to u and F in $L^q_{\text{loc}}(Q_1)$ for $1 \le q < 10/3$. It is now immediate to pass to the limit as $k \to \infty$ and show that (u, P, F) is a solution of (2.1) in the distributional sense and that (2.11) holds. Therefore, (u, P, F) is a weak solution of (2.1) on Q_1. □

Another application of the compactness lemma is the following "blow-up" or "decay" lemma; these terms come from the fact that the lemma's proof uses a rescaling and blow-up argument, while the conclusion involves a type of decay. This is arguably the most important step in the proof of Theorem 1.1. These ideas originated with the groundbreaking work of Lin [4]. However, the specific approach we use here follows closely the reinterpretation by Ladyzhenskaya and Seregin [3]. We make use of the notation for integral averages: for $Q_r = Q_r(x,t)$ and $B_r = B_r(x)$,

$$v_{Q_r} = \fint\!\!\fint_{Q_r} v(y,s)dyds := \frac{1}{|Q_r|}\iint_{Q_r} v(y,s)dyds$$

and

$$q_{B_r}(t) = \fint_{B_r} q(y,t)dy := \frac{1}{|B_r|}\int_{B_r} q(y,t)dy.$$

Lemma 2.4. *Let (u,p,F) be a weak solution on Q_1 and set*

$$E(x,t,r) := \left(\fint\!\!\fint_{Q_r} |u-u_{Q_r}|^3 dyds\right)^{1/3} + r\left(\fint\!\!\fint_{Q_r} |p-p_{B_r}(t)|^{3/2} dyds\right)^{2/3} + \left(\fint\!\!\fint_{Q_r} |F-F_{Q_r}|^3 dyds\right)^{1/3}$$

for $Q_r = Q_r(x,t) \subset Q_1$ and $B_r = B_r(x)$. For each

$$0 < \theta < 1/2 \quad \text{and} \quad M > 0$$

there are positive numbers ϵ_1, R_1, c_1 such that if

$$\begin{cases} \text{(i)} \quad Q_r(x,t) \subset Q_1,\ r \le R_1, \\ \text{(ii)} \quad |ru_{Q_r(x,t)}| + |rF_{Q_r(x,t)}| \le M, \\ \text{(iii)} \quad E(x,t,r) \le \epsilon_1, \end{cases}$$

then

$$E(x,t,\theta r) \le c_1 \theta^{2/3} E(x,t,r).$$

Moreover, c_1 can be chosen independent of θ.

Remark 3. In particular we can arrange $c_1\theta^{2/3}$ to be small, so this is a type of decay lemma.

Proof. 1. We argue by contradiction. Suppose the statement of the lemma is false. Then there are sequences

$$\begin{cases}(x_k,t_k)\in Q_1,\\ \epsilon_k\to 0,\\ r_k\to 0,\\ c_k\equiv c_1 \text{ (which will be chosen sufficiently large below)}\end{cases}$$

with

$$\begin{cases}\text{(i)}\quad Q_{r_k}(x_k,t_k)\subset Q_1,\\ \text{(ii)}\quad |r_k u_{Q_{r_k}(x_k,t_k)}|+|r_k F_{Q_{r_k}(x_k,t_k)}|\le M_0,\\ \text{(iii)}\quad E(x_k,t_k,r_k)=\epsilon_k\end{cases} \tag{2.14}$$

and

$$E(x_k,t_k,\theta_0 r_k)>c_1\theta_0^{2/3}E(x_k,t_k,r_k). \tag{2.15}$$

We set

$$\begin{cases}u^k(y,s):=\epsilon_k^{-1}\left(u(x_k+r_ky,t_k+r_k^2s)-u_{Q_{r_k}(x_k,t_k)}\right),\\ p^k(y,s):=r_k\epsilon_k^{-1}\left(p(x_k+r_ky,t_k+r_k^2s)-p_{B_{r_k}(x_k)}(t_k+r_k^2s)\right),\qquad (y,s)\in Q_1,\\ F^k(y,s):=\epsilon_k^{-1}\left(F(x_k+r_ky,t_k+r_k^2s)-F_{Q_{r_k}(x_k,t_k)}\right)\end{cases}$$

and observe that (u^k,p^k,F^k) is a weak solution (defined analogously as in Definition 2.1) of the PDE

$$\begin{cases}\partial_t u^k+((a^k+r_k\epsilon_k u^k)\cdot\nabla)u^k\\ \qquad =\Delta u^k-\nabla p^k+\sum_{j=1}^3((b_j^k+r_k\epsilon_kF_j^k)\cdot\nabla)F_j^k,\\ \partial_tF_j^k+((a^k+r_k\epsilon_ku^k)\cdot\nabla)F_j^k=\Delta F_j^k+((b_j^k+r_k\epsilon_kF_j^k)\cdot\nabla)u^k,\\ \qquad \nabla\cdot u^k=\nabla\cdot F_j^k=0\end{cases}\quad (y,s)\in Q_1. \tag{2.16}$$

for $j=1,2,3$. Here

$$a^k:=r_ku_{Q_{r_k}(x_k,t_k)}\quad\text{and}\quad b_j^k:=r_k(F_j)_{Q_{r_k}(x_k,t_k)}$$

are bounded sequences in $\mathbb{R}^3$ for each $j=1,2,3$ by (2.14). Moreover, the sequence (u^k,p^k,F^k) satisfies the integral bounds

$$\left(⨍⨍_{Q_1}|u^k|^3\right)^{1/3}+\left(⨍⨍_{Q_1}|p^k|^{3/2}\right)^{2/3}+\left(⨍⨍_{Q_1}|F^k|^3\right)^{1/3}=\frac{E(x_k,t_k,r_k)}{\epsilon_k}=1 \tag{2.17}$$

and the generalized energy inequality

$$\begin{aligned}\int_{B_1\times\{s\}}\phi\left(\frac{|u^k|^2}{2}+\frac{|F^k|^2}{2}\right)dx+\int_{-1/2}^s\int_{B_1}\phi\left(|\nabla u^k|^2+|\nabla F^k|^2\right)dxds\\ \le\int_{-1/2}^s\int_{B_1}\left[\left(\frac{|u^k|^2}{2}+\frac{|F^k|^2}{2}\right)(\phi_t+\Delta\phi)\right.\\ \left.+\left\{-\sum_{j=1}^3u^k\cdot F_j^k(b_j^k+r_k\epsilon_kF_j^k)+\left(\frac{|u^k|^2}{2}+\frac{|F^k|^2}{2}\right)(a^k+r_k\epsilon_ku^k)+p^ku^k\right\}\cdot\nabla\phi\right]dxds\end{aligned} \tag{2.18}$$

for $\phi \in C_c^\infty(B), \phi \geq 0$ and $s \in (-1/2, 1/2)$.

2. Arguing as we did in the proof of Theorem 2.3 one checks that the sequence $(u^k)_{k\in\mathbb{N}}$ and $(F^k)_{k\in\mathbb{N}}$ satisfies the hypotheses of Lemma 2.2. Moreover, we conclude that there is $u \in L^3_{\text{loc}}(Q_1)$, $p \in L^{3/2}_{\text{loc}}(Q_1)$, and $F \in L^3_{\text{loc}}(Q_1)$ and subsequences of $(u^k)_{k\in\mathbb{N}}$, $(p^k)_{k\in\mathbb{N}}$, and $(F^k)_{k\in\mathbb{N}}$ (again labeled u^k, p^k, F^k) such that

$$\begin{cases} u^k \to u \text{ in } L^3_{\text{loc}}(Q_1), \\ p^k \rightharpoonup p \text{ in } L^{3/2}(Q_1), \\ F^k \to F \text{ in } L^3_{\text{loc}}(Q_1) \end{cases}$$

as $k \to \infty$. From (2.14), we may assume without any loss of generality that

$$\begin{cases} a^k \to a, \\ b_j^k \to b_j, \end{cases} \qquad j = 1, 2, 3,$$

in $\mathbb{R}^3$ as $k \to \infty$.

It is follows from this convergence that (u, p, F) are weak solutions of the linear PDE

$$\begin{cases} \partial_t u + (a \cdot \nabla)u = \Delta u - \nabla p + \sum_{j=1}^3 (b_j \cdot \nabla)F_j, \\ \partial_t F_j + (a \cdot \nabla)F_j = \Delta F_j + (b_j \cdot \nabla)u, \\ \nabla \cdot u = \nabla \cdot F_j = 0 \end{cases} \tag{2.19}$$

and satisfy the energy inequality

$$\begin{aligned} &\int_{B_1\times\{s\}} \phi \left(\frac{|u|^2}{2} + \frac{|F|^2}{2}\right) dx + \int_{-1/2}^{s}\int_{B_1} \phi \left(|\nabla u|^2 + |\nabla F|^2\right) dxds \\ &\leq \int_{-1/2}^{s}\int_{B_1} \left[\left(\frac{|u|^2}{2} + \frac{|F|^2}{2}\right)(\phi_t + \Delta\phi) \right. \\ &\qquad \left. + \left\{ -\sum_{j=1}^{3} (u \cdot F_j) b_j + \left(\frac{|u|^2}{2} + \frac{|F|^2}{2}\right) a + pu \right\} \cdot \nabla\phi \right] dxds \end{aligned} \tag{2.20}$$

for $\phi \in C_c^\infty(B), \phi \geq 0$ and $s \in (-1/2, 1/2)$.

3. We claim that there is a constant $C_1 = C_1(M_0)$ such that

$$\begin{cases} |u(x,t) - u(y,s)| \leq C_1(|x-y| + |t-s|^{1/3}), \\ |F(x,t) - F(y,s)| \leq C_1(|x-y| + |t-s|), \end{cases} \tag{2.21}$$

for $(x,t), (y,s) \in Q_{1/2}$. In particular, we are asserting that u is Hölder continuous and F is Lipschitz continuous on $Q_{1/2}$.

To verify this claim, we first take the divergence of the first equation in (2.19) to get, for almost every $t \in (-1/2, 1/2)$,

$$-\Delta p(x,t) = 0, \quad x \in B_1.$$

Thus $p(t) \in C^\infty(B_1)$ for almost every $t \in (-1/2, 1/2)$. Recall the following estimate for harmonic functions: $-\Delta h = 0$ in $B_1 \subset \mathbb{R}^n$; then for each multi-index α, $q \geq 1$, and $\theta \in (0,1)$, there is a $C(q,\alpha)$ such that

$$|\partial^\alpha h|_{L^\infty(B_\theta)} \leq \frac{C(q,\alpha)}{(1-\theta)^{|\alpha|+n}} |h|_{L^q(B_1)} \tag{2.22}$$

(see Theorem 7, page 29 of [2]). As $p \in L^{3/2}(Q_1)$, we conclude that

$$\partial_x^\alpha p \in L^{3/2}((-1/2,1/2), L^\infty_{\mathrm{loc}}(B_1))$$

for each multi-index α.

Next, we take the curl of the first two equations in (2.19) to arrive at

$$\begin{cases} \partial_t w + (a\cdot\nabla) w = \Delta w + \sum_{j=1}^3 (b_j\cdot\nabla) r_j, \\ \partial_t r_j + (a\cdot\nabla) r_j = \Delta r_j + (b_j \cdot \nabla) w, \end{cases} \tag{2.23}$$

where of course

$$w := \nabla \times u \quad \text{and} \quad r_j := \nabla \times F_j.$$

From the local energy estimate for (u,p,F) (2.20), we have $w, r_j \in L^2_{\mathrm{loc}}(Q_1)$. As

$$\begin{cases} \partial_t w - \Delta w = \nabla \cdot \left(\sum_{j=1}^3 r_j \otimes b_j - w\otimes a\right) \in L^2((-1/2,1/2), H^{-1}(B_1)), \\ \partial_t r_j - \Delta r_j = \nabla\cdot (w\otimes b_j - r_j \otimes a) \in L^2((-1/2,1/2), H^{-1}(B_1)), \end{cases}$$

it follows from standard energy estimates for the parabolic system (2.23) that

$$w, r_j \in L^\infty((-1/2,1/2), L^2(B_1)) \cap L^2((-1/2,1/2), H^1(B_1)).$$

As (2.23) is linear, and in particular each of the derivatives $\partial_x^\alpha w, \partial_x^\alpha r_j$ also satisfies (2.23), we conclude by induction that for each multi-index α

$$\partial_x^\alpha w, \partial_x^\alpha r_j \in L^\infty((-1/2,1/2), L^2(B_1)) \cap L^2((-1/2,1/2), H^1(B_1)).$$

Therefore, each space variable of both w and r_j is locally bounded on Q_1.

The condition $\nabla \cdot u = 0$ implies

$$-\Delta u = \nabla \times w. \tag{2.24}$$

In particular, the Biot–Savart law reads, for each compact $G \subset B_1$,

$$u(x,t) = \int_G \nabla\Phi(x-y)\times w(y,t)dy + A(x,t), \quad (x,t) \in G\times(-1/2,1/2),$$

where $G \ni x \mapsto A(x,t)$ is harmonic for almost each $t \in (-1/2,1/2)$ (see Lemma 2 of [7] for a proof) and $\Phi(z) = (4\pi|z|)^{-1}$. From the local energy estimate for the solution (u,p,F) (2.20), we know that the $L^2(G)$ norm of u is bounded independently of t belonging to compact subintervals of $(-1/2,1/2)$. From our estimates on w, we conclude that the $L^2(G)$ norm of A is also bounded independently of such t. As A is harmonic, it is immediate from the mean value property that A is locally bounded on $G \times I$ for each interval $I \subset (-1/2, 1/2)$. Hence u is locally bounded on Q_1; differentiating (2.24) and recalling our estimates on w, we also see that $\partial_x^\alpha u$ is locally bounded on Q_1.

Arguing in the same manner, we have that $\partial_x^\alpha F_j$ is locally bounded on Q_1 for each multi-index α and $j = 1, 2, 3$. It is now immediate from (2.19) that

$$\partial_t u = -(a \cdot \nabla)u + \Delta u - \nabla p + \sum_{j=1}^{3} (b_j \cdot \nabla) F_j \in L^{3/2}_{\mathrm{loc}}((-1/2, 1/2), L^\infty_{\mathrm{loc}}(B_1))$$

and

$$\partial_t F_j = -(a \cdot \nabla) F_j + \Delta F_j + (b_j \cdot \nabla) u \in L^\infty_{\mathrm{loc}}((-1/2, 1/2), L^\infty_{\mathrm{loc}}(B_1))$$

from which the claimed estimates (2.21) readily follow.

4. Notice that the established continuity of u and F (2.21) implies that there is a constant $C_2 = C_2(M_0)$ such that

$$\left(\fint\!\!\fint_{Q_{\theta_0}} |u - u_{Q_{\theta_0}}|^3 dyds \right)^{1/3} + \left(\fint\!\!\fint_{Q_{\theta_0}} |F - F_{Q_{\theta_0}}|^3 dyds \right)^{1/3} \leq C_2 \theta_0^{2/3}.$$

By the convergence in $L^3_{\mathrm{loc}}(Q_1)$ of (u^k, F^k) to (u, F) as $k \to \infty$,

$$\left(\fint\!\!\fint_{Q_{\theta_0}} |u^k - u^k_{Q_{\theta_0}}|^3 dyds \right)^{1/3} + \left(\fint\!\!\fint_{Q_{\theta_0}} |F^k - F^k_{Q_{\theta_0}}|^3 dyds \right)^{1/3} \leq 2C_2 \theta_0^{2/3} \tag{2.25}$$

for all $k \in \mathbb{N}$ sufficiently large.

Let us now derive an estimate on the average of the p^k. Direct computation from (2.16) gives

$$-\Delta p^k = \epsilon_k \nabla \cdot \left[(u^k \cdot \nabla) u^k - \sum_{j=1}^{3} (F^k_j \cdot \nabla) F^k_j \right], \quad x \in B_1,$$

for almost every $s \in (-1/2, 1/2)$. For each $s \in (-1/2, 1/2)$ that this equation holds weakly on B_1, let $g^k(s)$ denote the unique solution of the PDE

$$\begin{cases} -\Delta g^k = \epsilon_k \nabla \cdot [(u^k(s) \cdot \nabla) u^k(s) - \sum_{j=1}^{3} (F^k_j(s) \cdot \nabla) F^k_j(s)], \quad x \in B_{3/4}, \\ \quad g^k = 0, \quad x \in \partial B_{3/4}, \end{cases}$$

and set

$$h^k(s) := p^k(s) - g^k(s).$$

Clearly, $h^k(s)$ is harmonic on $B_{3/4}$. Virtually the same argument that led us to the bound (2.10) will provide the estimate

$$\iint_{Q_{1/2}} |g^k|^{3/2} dyds \leq C\epsilon_k,$$

where C depends (say) only on an upper bound for the integrals

$$\iint_{Q_{5/6}} \left(|\nabla u^k|^2 + |\nabla F^k|^2 \right) dyds.$$

The local energy estimate for the solution (u^k, p^k, F^k) (2.18) combined with the $L^3(Q_1)$ bounds (2.17) for (u^k, F^k) assures us that these integrals are uniformly bounded above. Therefore, by these remarks and the estimate (2.22)

$$\begin{aligned}
\theta_0 \left(\fint\!\!\fint_{Q_{\theta_0}} |p^k - p^k_{B_{\theta_0}}|^{3/2} \right)^{2/3} &\le \theta_0 \left(\fint\!\!\fint_{Q_{\theta_0}} |g^k - g^k_{B_{\theta_0}}|^{3/2} \right)^{2/3} + \theta_0 \left(\fint\!\!\fint_{Q_{\theta_0}} |h^k - h^k_{B_{\theta_0}}|^{3/2} \right)^{2/3} \\
&\le 2\theta_0 \left(\fint\!\!\fint_{Q_{\theta_0}} |g^k|^{3/2} \right)^{2/3} + C\theta_0 \left(\frac{1}{\theta_0^5} \int_{-\theta_0^2/2}^{\theta_0^2/2} \int_{B_{\theta_0}} |h - h_{B_{\theta_0}}|^{3/2} dy ds \right)^{2/3} \\
&\le \frac{2\theta_0}{\theta_0^{10/3}} \left(\iint_{Q_{\theta_0}} |g^k|^{3/2} \right)^{2/3} \\
&+ C\theta_0 \left(\frac{1}{\theta_0^5} \int_{-\theta_0^2/2}^{\theta_0^2/2} \int_{B_{\theta_0}} (\theta_0 |\nabla h^k(s)|_{L^\infty(B_{\theta_0})})^{3/2} ds \right)^{2/3} \\
&\le \frac{2\theta_0}{\theta_0^{10/3}} \left(\iint_{Q_{1/2}} |g^k|^{3/2} \right)^{2/3} + C\theta_0^{2/3} \left(\int_{-\theta_0^2/2}^{\theta_0^2/2} |\nabla h^k(s)|^{3/2}_{L^\infty(B_{\theta_0})} ds \right)^{2/3} \\
&\le \frac{C\theta_0 \epsilon_k}{\theta_0^{10/3}} + C\theta_0^{2/3} \left(\int_{-\theta_0^2/2}^{\theta_0^2/2} |h^k(s)|^{3/2}_{L^\infty(B_{5/8})} ds \right)^{2/3} \\
&\le \frac{C\epsilon_k}{\theta_0^{7/3}} + C\theta_0^{2/3} \left(\iint_{Q_{5/8}} |h^k|^{3/2} dy ds \right)^{2/3} \\
&\le \frac{C\epsilon_k}{\theta_0^{7/3}} + C\theta_0^{2/3} \left(\iint_{Q_{5/8}} (|g^k|^{3/2} + |p^k|^{3/2}) dy ds \right)^{2/3} \\
&\le \frac{C\epsilon_k}{\theta_0^{7/3}} + C\theta_0^{2/3}.
\end{aligned}$$

Consequently, for k sufficiently large

$$\theta_0 \left(\fint\!\!\fint_{Q_{\theta_0}} |p^k - p^k_{B_{\theta_0}}|^{3/2} dy ds \right)^{2/3} \le C_3 \theta_0^{2/3} \tag{2.26}$$

for some universal constant C_3 (since for all large k we have $\epsilon_k \le \theta_0^3$). Combining (2.25) and (2.26), it is easily checked that a contradiction to (2.15) is obtained, for all k sufficiently large, by choosing

$$c_1(M_0) := 2(2C_2(M_0) + C_3), \quad M_0 > 0.$$

Moreover, our choice c_1 is independent of θ. □

3. A local criterion for Hölder continuity. In view of Lemma 2.4, we are now in a position to iterate its conclusion, which is the major step in establishing a local criterion for Hölder continuity for weak solutions of the system of PDEs (2.1). As a corollary of this Hölder continuity criterion, we prove Theorem 3.5, which is reminiscent of Serrin's higher regularity result for weak solutions of the incompressible Navier–Stokes equation [7].

LEMMA 3.1. *Assume (u,p,F) is weak solution on Q_1 and fix*

$$0<\theta<1/2,\quad M>0,\quad 0<\beta<2/3, \tag{3.1}$$

so that

$$c_1(M)\theta^{1/3-\beta/2}\le 1.$$

There is $\epsilon_2>0$ such that if

$$\begin{cases}\text{(i)}\quad Q_r(x,t)\subset Q_1,\ r\le R_1,\\ \text{(ii)}\quad |ru_{Q_r(x,t)}|+|rF_{Q_r(x,t)}|<\frac{1}{2}M,\\ \text{(iii)}\quad E(x,t,r)\le\epsilon_2,\end{cases}$$

then for any $k=0,1,2,3,\dots$

$$\begin{cases}\text{(i)}\quad |\theta^k rF_{Q_{\theta^k r}(x,t)}|+|(\theta^k r)u_{Q_{\theta^k r}(x,t)}|\le M,\\ \text{(ii)}\quad E(x,t,\theta^k r)\le\epsilon_1,\\ \text{(iii)}\quad E(x,t,\theta^{k+1}r)\le\theta^{(k+1)\beta}(1-\theta^{1/3-\beta/2})^{-1}E(x,t,r).\end{cases}$$

Proof. 1. First set

$$I(x,t,r):=\left(\iint\!\!\!\!\!\!-_{Q_r}|u-u_{Q_r}|^3dyds\right)^{1/3}+\left(\iint\!\!\!\!\!\!-_{Q_r}|F-F_{Q_r}|^3dyds\right)^{1/3}$$

for $Q_r=Q_r(x,t)$, and notice that

$$|u_{Q_r(x,t)}-u_{Q_{\theta r}(x,t)}|+|F_{Q_r(x,t)}-F_{Q_{\theta r}(x,t)}|\le\frac{I(x,t,r)}{\theta^{5/3}}.$$

The above estimate follows directly from Hölder's inequality, and using the triangle inequality we obtain

$$|(\theta^k r)u_{Q_{\theta^k r}(x,t)}|+|\theta^k rF_{Q_{\theta^k r}(x,t)}|\le\frac{\theta^k r}{\theta^{5/3}}\sum_{j=0}^{k-1}I(x,t,\theta^j r)+\theta^k(|ru_{Q_r(x,t)}|+|rF_{Q_r(x,t)}|).$$

We define

$$\epsilon_2:=(1-\theta^{1/3-\beta/2})\min\left\{\frac{1}{2}\epsilon_1,\frac{\theta^{5/3}(1-\theta^\beta)}{R_1}\frac{M}{2}\right\}$$

as our candidate for ϵ_2 described in the statement of the lemma. We argue by induction below.

2. We establish the claim for $k=0$. By assumption and our choice of ϵ_2,

$$E(x,t,r)\le\epsilon_2\le(1-\theta^{1/3-\beta/2})\frac{\epsilon_1}{2}\le\frac{\epsilon_1}{2}\le\epsilon_1.$$

It is also easy to verify that the conditions of Lemma 2.4 are satisfied, and thus

$$\begin{aligned}E(x,t,\theta r)&\le c_1\theta^{2/3}E(x,t,r)\\&\le c_1\theta^{1/3-\beta/2}\theta^{1/3+\beta/2}E(x,t,r)\\&\le 1\cdot\theta^{1/3+\beta/2}E(x,t,r)\\&\le\theta^\beta(1-\theta^{1/3-\beta/2})^{-1}E(x,t,r)\end{aligned}$$

as $\theta^{1/3+\beta/2}\le\theta^\beta(1-\theta^{1/3-\beta/2})^{-1}$ (which happens if and only if $\theta^{1/3-\beta/2}\le 2$).

3. Assume for $s = 0, 1, \ldots, k$

$$\begin{cases} \text{(i)}_s \quad |\theta^s r u_{Q_{\theta^s r}(x,t)}| + |\theta^s r F_{Q_{\theta^s r}(x,t)}| \le M, \\ \text{(ii)}_s \quad E(x,t,\theta^s r) \le \epsilon_1, \\ \text{(iii)}_s \quad E(x,t,\theta^{s+1} r) \le \theta^{(s+1)\beta}(1-\theta^{1/3-\beta/2})^{-1} E(x,t,r). \end{cases}$$

We show that these assumptions imply that the above bounds hold in turn for $s = k+1$. For simplicity, we suppress the (x,t) dependence in our arguments below.

For (i)_{k+1}:

$$\begin{aligned} |(r\theta^{k+1}) u_{r\theta^{k+1}}| + |r\theta^{k+1} F_{r\theta^{k+1}}| &\le \frac{R_1}{\theta^{5/3}} \sum_{j=0}^{k} I(\theta^j r) + |r u_{Q_r}| + |r F_{Q_r}| \\ &< \frac{R_1}{\theta^{5/3}} \sum_{j=0}^{k} E(\theta^j r) + \frac{M}{2} \\ &\le \frac{R_1}{\theta^{5/3}} (1-\theta^{1/3-\beta/2})^{-1} E(r) \sum_{j=0}^{k} \theta^{\beta j} + \frac{M}{2} \\ &\le \frac{R_1}{\theta^{5/3}} \frac{E(r)}{(1-\theta^{1/3-\beta/2})(1-\theta^\beta)} + \frac{M}{2} \\ &\le \frac{R_1 \epsilon_2}{\theta^{5/3}(1-\theta^{1/3-\beta/2})(1-\theta^\beta)} + \frac{M}{2} \\ &\le \frac{M}{2} + \frac{M}{2} \\ &= M. \end{aligned}$$

For (ii)_{k+1}: By assumption (iii)_k, we have

$$\begin{aligned} E(\theta^{k+1} r) &\le \theta^{(k+1)\beta}(1-\theta^{1/3-\beta/2})^{-1} E(r) \\ &\le \theta^{(k+1)\beta}(1-\theta^{1/3-\beta/2})^{-1} \epsilon_2 \\ &\le \theta^{(k+1)\beta} \frac{\epsilon_1}{2} \\ &< \epsilon_1. \end{aligned}$$

For (iii)_{k+1}: Observe that the hypotheses of Lemma 2.4 are satisfied with $r \mapsto \theta^k r$. Moreover, using (3.1)

$$\begin{aligned} E(\theta^{k+2} r) &= E(\theta\theta^{k+1} r) \\ &\le c_1 \theta^{2/3} E(\theta^{k+1} r) \\ &\le c_1 \theta^{2/3} \theta^{(k+1)\beta}(1-\theta^{1/3-\beta/2})^{-1} E(r) \\ &= c_1 \theta^{1/3-\beta/2} \theta^{1/3+\beta/2} \theta^{(k+1)\beta}(1-\theta^{1/3-\beta/2})^{-1} E(r) \\ &\le 1 \cdot \theta^{1/3+\beta/2} \theta^{(k+1)\beta}(1-\theta^{1/3-\beta/2})^{-1} E(r) \\ &\le \theta^{(k+2)\beta}(1-\theta^{1/3-\beta/2})^{-1} E(r). \end{aligned}$$

The last inequality follows since $\beta/2 + 1/3 \ge \beta$ and $\theta \in (0,1)$, which trivially implies $\theta^{1/3+\beta/2} \le \theta^\beta$. ∎

The main use of the decay and iteration lemmas is the following proposition. We omit the proof as it follows from the above lemma and a relatively standard application of Campanato's criterion for Hölder continuity. See Proposition 2.8 in [3] for a related result, concerning the incompressible Navier–Stokes equations, and its proof, which is easily adapted to our framework.

PROPOSITION 3.2. *Assume that* (u, p, F) *is a weak solution of* (2.1). *There are universal constants* ϵ_3, R_2 *such that if*

$$\begin{cases} Q_r = Q_r(x,t) \subset Q_1, \quad r \leq R_2, \\ \overline{E}(x,t,r) < \epsilon_3, \end{cases}$$

then u *and* F *are Hölder continuous in some neighborhood of* (x, t). *Here*

$$\overline{E}(x,t,r) := \left(⨍\!\!\!\!\!\!\iint_{Q_r} |u|^3 \right)^{1/3} + r \left(⨍\!\!\!\!\!\!\iint_{Q_r} |p|^{3/2} \right)^{2/3} + \left(⨍\!\!\!\!\!\!\iint_{Q_r} |F|^3 \right)^{1/3}.$$

The scaling invariance properties of the system (2.1) listed in (2.3) imply the following improvement of Proposition 3.2. This simple observation is as important as any that we make in this work.

COROLLARY 3.3. *Assume* (u, p, F) *is a weak solution on* Q_1 *and let* ϵ_3, R_2 *be as in Proposition* 3.2. *Further suppose that*

$$\begin{cases} Q_r(x,t) \subset Q_1, \quad r \leq R_2, \\ \frac{r}{R_2}\overline{E}(x,t,r) < \epsilon_3. \end{cases}$$

Then (x, t) *is a regular point for* (u, p, F).

Proof. Set $\lambda = r/R_2$ and define

$$\begin{cases} u^\lambda(y,s) := \lambda u(x + \lambda y, t + \lambda^2 s), \\ p^\lambda(y,s) := \lambda^2 p(x + \lambda y, t + \lambda^2 s), \quad (y,s) \in Q_{R_2}(0,0), \\ F^\lambda(y,s) := \lambda F(x + \lambda y, t + \lambda^2 s). \end{cases}$$

It is immediate that $(u^\lambda, p^\lambda, F^\lambda)$ is a weak solution on $Q_{R_2}(0, 0)$, and direct computation yields

$$\begin{aligned} \overline{E}(0,0,R_2; u^\lambda, p^\lambda, F^\lambda) &= \lambda \overline{E}(t,x,r;u,p,F) \\ &= \frac{r}{R_2}\overline{E}(t,x,r;u,p,F) \\ &< \epsilon_3. \end{aligned}$$

By Proposition 3.2, u^λ and F^λ are Hölder continuous in a neighborhood of $(0, 0)$. Consequently, u and F are Hölder continuous in a neighborhood of (x, t). ∎

COROLLARY 3.4. *Assume* (u, p, F) *is a weak solution on* Q_1 *and let* ϵ_3, R_2 *be as in Proposition* 3.2. *If*

$$\liminf_{r \to 0^+} r\overline{E}(x,t,r) < \frac{1}{2}\epsilon_3 R_2,$$

then (x, t) *is a regular point for* (u, p, F). *In particular, there is* $\epsilon_4 > 0$ *such that the same conclusion holds, provided*

$$\liminf_{r \to 0^+} \frac{1}{r^2} \iint_{Q_r(x,t)} \left\{ |u|^3 + |p|^{3/2} + |F|^3 \right\} dyds < \epsilon_4. \tag{3.2}$$

Proof. If

$$\liminf_{r\to 0^+} r\overline{E}(x,t,r) < \frac{1}{2}\epsilon_3 R_2,$$

then $\inf_{0<r<\delta} r\overline{E}(x,t,r) < \frac{1}{2}\epsilon_3 R_2$ for any $\delta > 0$. Moreover, there is $r < R_2$ such that $\frac{r}{R_2}E(x,t,r) < \epsilon_3$ and $Q_r(x,t) \subset Q_1$. The previous corollary then applies. The second assertion follows immediately from the first. ∎

The following claim is in the spirit of Serrin's regularity criterion for weak solutions of the incompressible Navier–Stokes equations. Note, however, that our result requires integrability of the pressure as well as integrability of the velocity and the deformation.

THEOREM 3.5. *Assume that (u,p,F) is a weak solution on Q_1 such that*

$$\int_{-1/2}^{1/2}\left(\int_{B_1}\left\{|u|^s + |p|^{s/2} + |F|^s\right\}dx\right)^{s'/s} dt < \infty \tag{3.3}$$

for some $s \le s' < \infty$ satisfying

$$\frac{3}{s} + \frac{2}{s'} < 1.$$

Then u, F is regular on Q_1.

Proof. For each point $(x,t) \in Q_1$, the bound (3.3) implies

$$\lim_{r\to 0}\frac{1}{r^2}\iint_{Q_r(x,t)}\left\{|u|^3 + |p|^{3/2} + |F|^3\right\}dyds = 0,$$

which in turn implies (3.2). The result thus follows from the previous corollary. ∎

4. "A-B-C-D" estimates. In the previous section, we deduced that there is an $\epsilon_5 > 0$ such that if

$$\liminf_{r\to 0^+} r\overline{E}(x,t,r) < \epsilon_5, \tag{4.1}$$

then u and F are both Hölder continuous near (x,t); for instance, we can take $\epsilon_5 := \frac{1}{2}\epsilon_3 R_2$. We claim that the central hypothesis of Theorem 1.1 implies the above limit. More precisely, we assert the following fundamental proposition.

PROPOSITION 4.1. *Let (u,p,F) be a weak solution on Q_1. There is a universal constant $\epsilon > 0$ such that inequality* (4.1) *holds whenever*

$$\limsup_{r\to 0^+}\frac{1}{r}\iint_{Q_r(x,t)}\left\{|\nabla u|^2 + |\nabla F|^2\right\}dyds < \epsilon.$$

In view of Corollary 3.4, a proof of Proposition 4.1 establishes Theorem 1.1. To this end, we shall need three estimates involving the following integral quantities. For $(x,t) \in Q_1$, and $r > 0$ so small that $Q_r(x,t) \subset Q_1$, we define

$$\begin{cases} A(x,t,r) := \sup_{|t-s|\le r^2/2}\frac{1}{r}\int_{B_r(x)}\{|u(y,s)|^2 + |F(y,s)|^2\}dy, \\ B(x,t,r) := \frac{1}{r}\iint_{Q_r(x,t)}\left\{|\nabla u(y,s)|^2 + |\nabla F(y,s)|^2\right\}dyds, \\ C(x,t,r) := \frac{1}{r^2}\iint_{Q_r(x,t)}\{|u(y,s)|^3 + |F(y,s)|^3\}dyds, \\ D(x,t,r) := \frac{1}{r^2}\iint_{Q_r(x,t)}|p(y,s)|^{3/2}dyds. \end{cases}$$

Our arguments below are independent of (x,t), so without any loss of generality we establish our results for $(x,t)=(0,0)$. For ease of notation, we also write $A(r):=A(0,0,r), B(r):=B(0,0,r), C(r):=C(0,0,r)$, and $D(r):=D(0,0,r)$ for $0<r\le 1$.

We will need three estimates before undertaking the proof of Proposition 4.1. They are very similar to the sequence of lemmas needed in [4], and as in the previous section we follow the path of [3] closely. The statement of the lemmas in fact are nearly identical to the ones used to prove the version of Proposition 4.1 in [3], but unfortunately the proofs had to be modified. Nevertheless, we would like to emphasize that the work of [4] and [3] served as a guide for this section.

LEMMA 4.2. *There is a universal constant $c>0$ such that*

$$C(r)\le c\left\{\left(\frac{r}{\rho}\right)^3 A^{3/2}(\rho)+\left(\frac{\rho}{r}\right)^3 A^{3/4}(\rho)B^{3/4}(\rho)\right\}$$

for $0<r\le\rho$.

Proof. This assertion follows directly from following the well-known inequality established in [4] (Lemma 2.1). There is a universal constant $c>0$ such that

$$C_0(r)\le c\left\{\left(\frac{r}{\rho}\right)^3 A_0^{3/2}(\rho)+\left(\frac{\rho}{r}\right)^3 A_0^{3/4}(\rho)B_0^{3/4}(\rho)\right\},\quad 0<r\le\rho\le 1,$$

for all

$$v\in L^2((-1/2,1/2),H^1(B_1))\cap L^\infty((-1/2,1/2);L^2(B_1)),$$

where

$$\begin{cases} A_0(r):=\sup_{|t|\le r^2/2}\frac{1}{r}\int_{B_r}|v(y,s)|^2dy,\\ B_0(r):=\frac{1}{r}\iint_{Q_r}|\nabla v(y,s)|^2dyds,\\ C_0(r):=\frac{1}{r^2}\iint_{Q_r}|v(y,s)|^3dyds. \qquad\square\end{cases}$$

LEMMA 4.3. *There is a universal constant $c>0$ such that*

$$A(r/2)+B(r/2)\le c\left\{C^{2/3}(r)+C^{1/3}(r)D^{2/3}(r)+A^{1/2}(r)B^{1/2}(r)C^{1/3}(r)\right\}$$

for $0<r\le 1$.

Remark 4. Recall the energy inequality (2.11): $\phi\in C_c^\infty(Q_1),\phi\ge 0$,

$$\int_{B_1\times\{t\}}\phi\left(\frac{|u|^2}{2}+\frac{|F|^2}{2}\right)dx+\int_{-1/2}^t\int_{B_1}\phi\left(|\nabla u|^2+|\nabla F|^2\right)dxds$$
$$\le\int_{-1/2}^t\int_{B_1}\left\{(\phi_t+\Delta\phi)\left(\frac{|u|^2}{2}+\frac{|F|^2}{2}\right)-F^tu\cdot F^t\nabla\phi+\left(\frac{|u|^2}{2}+\frac{|F|^2}{2}+p\right)u\cdot\nabla\phi\right\}dxds,$$

$t\in[-1/2,1/2]$. As

$$\nabla\cdot u=0\quad\text{and}\quad\nabla\cdot F^t=0,$$

we can replace the right-hand side of the above inequality by

$$\int_{-1/2}^t\int_{B_1}\left\{(\phi_t+\Delta\phi)\left(\frac{|u|^2}{2}+\frac{|F|^2}{2}\right)+a(t)\cdot F^t\nabla\phi-F^tu\cdot F^t\nabla\phi\right.$$
$$\left.+\left(\frac{|u|^2}{2}+\frac{|F|^2}{2}-b(t)+p\right)u\cdot\nabla\phi\right\}dxds$$

for any $a \in L^1_{\mathrm{loc}}(-1/2, 1/2; \mathbb{R}^3)$ and $b \in L^1_{\mathrm{loc}}(-1/2, 1/2)$. In a crucial step below, we will use

$$\begin{cases} a(t) := [F^t u]_{B_r}(t) = \fint_{B_r} F^t(x,t)u(x,t)dx, \\ b(t) := \frac{1}{2}|u|^2_{B_r}(t) + \frac{1}{2}|F|^2_{B_r}(t) = \fint_{B_r}\Big(\frac{|u(x,t)|^2}{2} + \frac{|F(x,t)|^2}{2}\Big)\, dx. \end{cases} \tag{4.2}$$

Proof. Choose $\phi \in C_c^\infty(Q_1)$, nonnegative, such that $\phi \equiv 1$ on $Q_{1/2}$, and set

$$\phi_r(x,t) := \phi\left(\frac{x}{r}, \frac{t}{r^2}\right), \quad (x,t) \in Q_r.$$

Clearly $\phi_r \in C_c^\infty(Q_r)$ and

$$\begin{cases} |\nabla \phi_r| \le c/r, \\ |\partial_t \phi_r + \Delta \phi_r| \le c/r^2 \end{cases}$$

for a constant $c \geq 0$ independent of $r \in (0,1)$.

Using ϕ_r as the test function in the energy inequality (2.11) and choosing the mappings a and b as in (4.2) gives

$$\begin{aligned} A(r/2) + B(r/2) \le c \Bigg\{ &\frac{1}{r^3} \iint_{Q_r} (|u|^2 + |F|^2) + \frac{1}{r^2} \iint_{Q_r} ||u|^2 - |u|^2_{B_r}||u| \\ &+ \frac{1}{r^2} \iint_{Q_r} ||F|^2 - |F|^2_{B_r}||u| + \frac{1}{r^2} \iint_{Q_r} ||F^t u| - [F^t u]_{B_r}||F| \\ &+ \frac{1}{r^2}\left(\iint_{Q_r} |u|^3\right)^{1/3} \left(\iint_{Q_r} |p|^{3/2}\right)^{2/3} \Bigg\}. \end{aligned} \tag{4.3}$$

We now proceed to estimate each integral in the inequality above (4.3) in terms of A, B, C, D. First note that

$$\begin{aligned} \frac{1}{r^3} \iint_{Q_r} (|u|^2 + |F|^2) &\le c \frac{1}{r^3} \left(\iint_{Q_r} |u|^3 + |F|^3\right)^{2/3} |Q_r|^{1/3} \\ &\le c r^{5/3 - 3} \left(\iint_{Q_r} (|u|^3 + |F|^3)\right)^{2/3} \\ &= cC(r)^{2/3} \end{aligned}$$

for some universal constant $c > 0$. Next, observe

$$\begin{aligned} \frac{1}{r^2}\left(\iint_{Q_r} |u|^3\right)^{1/3} \left(\iint_{Q_r} |p|^{3/2}\right)^{2/3} &= \left(\frac{1}{r^2}\iint_{Q_r} |u|^3\right)^{1/3} \left(\frac{1}{r^2}\iint_{Q_r} |p|^{3/2}\right)^{2/3} \\ &\le C(r)^{1/3} D^{2/3}(r). \end{aligned}$$

We also have, by employing Hölder's and Poincaré's inequalities,

$$\begin{aligned}
\iint_{Q_r} ||u|^2 - |u|^2_{B_r}||u| &\le \int_{-r^2/2}^{r^2/2} \left(\int_{B_r} ||u|^2 - |u|^2_{B_r}|^{3/2} \right)^{2/3} \left(\int_{B_r} |u|^3 \right)^{1/3} dt \\
&\le \int_{-r^2/2}^{r^2/2} \left(\int_{B_r} |u||\nabla u| \right) \left(\int_{B_r} |u|^3 \right)^{1/3} dt \\
&\le \int_{-r^2/2}^{r^2/2} \left(\int_{B_r} |u|^2 \right)^{1/2} \left(\int_{B_r} |\nabla u|^2 \right)^{1/2} \left(\int_{B_r} |u|^3 \right)^{1/3} dt \\
&\le cr^{1/2} A(r)^{1/2} \int_{-r^2/2}^{r^2/2} \left(\int_{B_r} |\nabla u|^2 \right)^{1/2} \left(\int_{B_r} |u|^3 \right)^{1/3} dt \\
&\le cr^{1/2} A(r)^{1/2} \left(\iint_{Q_r} |u|^3 \right)^{1/3} \left(\int_{-r^2/2}^{r^2/2} \left(\int_{B_r} |\nabla u|^2 \right)^{3/4} dt \right)^{2/3} \\
&\le cr^{1/2+2/3} A(r)^{1/2} C^{1/3}(r) \left(\iint_{Q_r} |\nabla u|^2 \right)^{1/2} r^{1/2} \\
&\le cr^{1+2/3} A(r)^{1/2} C^{1/3}(r) \left(\iint_{Q_r} |\nabla u|^2 \right)^{1/2} \\
&\le cr^{1+2/3+1/2} A(r)^{1/2} C^{1/3}(r) B^{1/2}(r) \\
&\le cr^{2+1/6} A(r)^{1/2} C^{1/3}(r) B^{1/2}(r) \\
&\le cr^2 A(r)^{1/2} C^{1/3}(r) B^{1/2}(r)
\end{aligned}$$

for some universal constant c independent of $r \in (0, 1]$. Arguing in virtually the same manner shows

$$\frac{1}{r^2} \iint_{Q_r} ||F|^2 - |F|^2_{B_r}||u| + \frac{1}{r^2} \iint_{Q_r} ||F^t u| - [F^t u]_{B_r}||F| \le cA(r)^{1/2} C^{1/3}(r) B^{1/2}(r).$$

Substituting all of these bounds into inequality (4.3) gives the desired estimate:

$$A(r/2) + B(r/2) \le c \left\{ C^{2/3}(r) + C^{1/3} D^{2/3}(r) + A(r)^{1/2} C^{1/3}(r) B^{1/2}(r) \right\}. \qquad \square$$

LEMMA 4.4. *There is a universal constant $c > 0$ such that*

$$D(r) \le c \left\{ \frac{r}{\rho} D(\rho) + \left(\frac{\rho}{r} \right)^2 A^{3/4}(\rho) B^{3/4}(\rho) \right\},$$

$0 < r \le \rho$.

Proof. 1. Taking the divergence of the first equation in (2.1) gives

$$-\Delta p = \nabla \cdot \left[\nabla \cdot \left(uu^t - FF^t \right) \right] = \nabla \cdot \left[uu^t - [uu^t]_\rho - (FF^t - [FF^t]_\rho) \right] \tag{4.4}$$

as $(u \cdot \nabla)u = \nabla \cdot uu^t$. Here $[uu^t]_\rho$ denotes the average of uu^t on B_ρ, and likewise for $[FF^t]_\rho$.

Now let $Q_1 = Q_1(t) \in W_0^{2,3/2}(B_\rho; \mathbb{R}^{3\times 3})$ denote the unique weak solution of the PDE

$$\begin{cases} -\Delta Q_1 = uu^t - [uu^t]_\rho - (FF^t - [FF^t]_\rho), & x \in B_\rho, \\ \quad Q_1 = 0, & x \in \partial B_\rho, \end{cases}$$

for (almost every) $|t| < 1/2$. It is immediate that

$$p_1 = \nabla \cdot [\nabla \cdot Q_1] \in L^{3/2}(B_\rho)$$

satisfies (4.4), and by the Calderón–Zygmund inequality
(4.5)

$$|p_1|_{L^{3/2}(B_\rho)} \le |\nabla \cdot [\nabla \cdot Q_1]|_{L^{3/2}(B_\rho)} \le C(|uu^t - [uu^t]_\rho|_{L^{3/2}(B_\rho)} + |FF^t - [FF^t]_\rho|_{L^{3/2}(B_\rho)}).$$

It is also clear that

$$p_2 := p - p_1$$

is harmonic in B_ρ for almost every $|t| < 1/2$.

2. By the estimate (4.5) and Poincaré's inequality,

$$\begin{aligned}\left(\int_{B_\rho} |p_1|^{3/2} dx\right)^{2/3} &\le c \int_{B_\rho} \{|u||\nabla u| + |F||\nabla F|\} dx \\ &\le c \left(\int_{B_\rho} (|u|^2 + |F|^2) dx\right)^{1/2} \left(\int_{B_\rho} (|\nabla u|^2 + |\nabla F|^2) dx\right)^{1/2} \\ &\le c\rho^{1/2} A^{1/2}(\rho) \left(\int_{B_\rho} (|\nabla u|^2 + |\nabla F|^2) dx\right)^{1/2}.\end{aligned}$$

Integrating time from $-\rho^2/2$ to $+\rho^2/2$ gives

$$\begin{aligned}\iint_{Q_\rho} |p_1|^{3/2} &\le c\rho^{3/4} A^{3/4}(\rho) \int_{-\rho^2/2}^{\rho^2/2} \left(\int_{B_\rho} (|\nabla u|^2 + |\nabla F|^2) dx\right)^{3/4} \\ &\le c\rho^{3/4} A^{3/4} \left(\iint_{Q_\rho} (|\nabla u|^2 + |\nabla F|^2) dx dt\right)^{3/4} \rho^{1/2} \\ &\le c\rho^2 A^{3/4} B(\rho)^{3/4}.\end{aligned}$$

It also follows that

$$\begin{aligned}\iint_{Q_\rho} |p_2|^{3/2} &\le \iint_{Q_\rho} |p|^{3/2} + \iint_{Q_\rho} |p_1|^{3/2} \\ &\le c\rho^2 \left\{ D(\rho) + A^{3/4} B(\rho)^{3/4} \right\}.\end{aligned}$$

3. p_2 is harmonic and thus $|p_2|^{3/2}$ is subharmonic for almost every $|t| < 1/2$. In particular,

$$r \to \frac{1}{r^3} \int_{B_r} |p|^{3/2} dx$$

is nondecreasing. Consequently,

$$\frac{1}{r^2} \iint_{Q_r} |p_2|^{3/2} \le \frac{r}{\rho} \cdot \frac{1}{\rho^2} \iint_{Q_\rho} |p_2|^{3/2}$$

for $0 < r \le \rho$. Combining the preceding inequalities yields

$$\begin{aligned} D(r) &= \frac{1}{r^2}\iint_{Q_r} |p|^{3/2} dxdt \\ &\le \frac{1}{r^2}\iint_{Q_r} |p_1|^{3/2} dxdt + \frac{1}{r^2}\iint_{Q_r} |p_2|^{3/2} dxdt \\ &\le \frac{1}{r^2}\iint_{Q_\rho} |p_1|^{3/2} dxdt + \frac{r}{\rho}\cdot\frac{1}{\rho^2}\iint_{Q_\rho} |p_2|^{3/2} dxdt \\ &\le \left(\frac{\rho}{r}\right)^2 A^{3/4}(\rho)B^{3/4}(\rho) + c\frac{r}{\rho}\left[D(\rho) + A^{3/4}B(\rho)^{3/4}\right] \\ &\le c\left\{\frac{r}{\rho}D(\rho) + \left(\frac{\rho}{r}\right)^2 A^{3/4}B(\rho)^{3/4}\right\}, \end{aligned}$$

as desired. □

We are finally ready to prove Proposition 4.1. We remark that this argument follows very closely with that of Proposition 2.9 in [3]. Nevertheless, we provide it here for completion.

Proof of Proposition 4.1. 1. Set

$$\mathcal{E}(r) := A^{3/2}(r) + D^2(r), \quad 0 < r \le 1,$$

and observe

$$\begin{aligned} r\overline{E}(r) &= r\left\{\left(\iint\!\!\!\!\!\!-_{Q_r}|u|^3\right)^{1/3} + r\left(\iint\!\!\!\!\!\!-_{Q_r}|p|^{3/2}\right)^{2/3} + \left(\iint\!\!\!\!\!\!-_{Q_r}|F|^3\right)^{1/3}\right\} \\ &\le c\left\{C(r)^{1/3} + D(r)^{2/3}\right\} \\ &\le c\left\{C(r) + D^2(r)\right\}^{1/3} \\ &\le c\left\{C(r) + \mathcal{E}(r)\right\}^{1/3}. \end{aligned} \tag{4.6}$$

Recall that our aim is to show that there is $\epsilon > 0$ such that

$$\limsup_{r\to 0^+} B(r) < \epsilon \quad \Rightarrow \quad \liminf_{r\to 0^+} r\overline{E}(r) < \epsilon_5.$$

Therefore, we assume that $\limsup_{r\to 0^+} B(r) < \epsilon$ and choose ϵ below to establish the above implication.

2. Let $\theta, \rho \in (0, 1/2)$ be fixed and observe that easy corollaries of Lemmas 4.2, 4.3, and 4.4 are

$$\begin{cases} C(\theta\rho) \le c\left\{\theta^3 A^{3/2}(\rho) + \theta^{-3}A^{3/4}(\rho)B^{3/4}(\rho)\right\}, \\ A^{3/2}\left(\frac{1}{2}\theta\rho\right) \le c\left\{C(\theta\rho) + D^2(\theta\rho) + A^{3/2}(\theta\rho)B^{3/2}(\theta\rho)\right\}, \\ D(\theta\rho) \le c\left\{\theta D(\rho) + \theta^{-2}A^{3/4}(\rho)B^{3/4}(\rho)\right\}. \end{cases}$$

Using the inequalities (that are trivial to verify)

$$A(\theta\rho) \le \frac{1}{\theta}A(\rho) \quad \text{and} \quad B(\theta\rho) \le \frac{1}{\theta}B(\rho),$$

we also have

$$A^{3/2}\left(\frac{1}{2}\theta\rho\right) \le C\left\{\theta^2 D^2(\rho) + \theta^3 A^{3/2}(\rho) + \frac{1}{\theta^3}A^{3/4}(\rho)B^{3/4}(\rho) + \left(\frac{1}{\theta^3}+\frac{1}{\theta^4}\right)A^{3/2}(\rho)B^{3/2}(\rho)\right\}$$

and

$$D^2\left(\frac{1}{2}\theta\rho\right) \le c\left\{\theta^2 D^2(\rho) + \frac{1}{\theta^4}A^{3/2}(\rho)B^{3/2}(\rho)\right\}.$$

With the above estimates on $A^{3/2}$ and D^2,

$$\begin{aligned}\mathcal{E}\left(\frac{1}{2}\theta\rho\right) &= A^{3/2}\left(\frac{1}{2}\theta\rho\right) + D^2\left(\frac{1}{2}\theta\rho\right)\\ &\le c\left\{\theta^2 D^2(\rho) + \theta^3 A^{3/2}(\rho) + \frac{1}{\theta^3}A^{3/4}(\rho)B^{3/4}(\rho) + \frac{1}{\theta^4}A^{3/2}(\rho)B^{3/2}(\rho)\right\}\\ &\le c\left\{\theta^2\mathcal{E}(\rho) + \frac{1}{\theta^4}A^{3/2}(\rho)B^{3/2}(\rho) + \frac{B^{3/2}(\rho)}{\theta^9}\right\}\\ &\le c\left\{\theta^2\mathcal{E}(\rho) + \left(\frac{1}{\theta^9}+\frac{1}{\theta^4}A^{3/2}(\rho)\right)B^{3/2}(\rho)\right\}\\ &\le c\left\{\theta^2\mathcal{E}(\rho) + \frac{1}{\theta^9}B^{3/2}(\rho)\right\} \qquad (4.7)\end{aligned}$$

as $A(\rho) \le A(1/2) \le c$.

3. Now choose $\theta \in (0,1/2)$ so small that for each of the (universal) constants above,

$$c\theta^2 \le 1/2. \qquad (4.8)$$

We may also select $\delta_\epsilon \in (0,1/2)$ so small that

$$B(\rho) \le 2\epsilon \quad \text{for} \quad 0<\rho<\delta_\epsilon.$$

This can be done by hypothesis. With $\theta \in (0,1/2)$ chosen to satisfy (4.8), (4.7) gives

$$\mathcal{E}(\tau\rho) \le \frac{1}{2}\mathcal{E}(\rho) + c_\tau \epsilon^{3/2}, \quad 0<\rho<\delta_\epsilon,$$

where we have set

$$\begin{cases}\tau := \frac{1}{2}\theta,\\ c_\tau := c2^{3/2}/\theta^9.\end{cases}$$

We interpret (4.9) to be a decay estimate for $\mathcal{E}$, as a simple induction argument provides the sequence of inequalities

$$\mathcal{E}(\tau^k\rho) \le \frac{1}{2^{k+1}}\mathcal{E}(\rho) + c_\tau\epsilon^{3/2}\sum_{j=0}^{k}\frac{1}{2^j} \le \frac{1}{2^{k+1}}\mathcal{E}(\rho) + 2c_\tau\epsilon^{3/2} \qquad (4.9)$$

for any fixed $\rho \in (0,\delta_\epsilon)$ and $k \in \mathbb{N}$.

4. Employing Lemma 4.2 and inequality (4.9),

$$
\begin{aligned}
C(\tau^{k+1}\rho) &\le c\left\{\tau^3 A^{3/2}(\tau^k\rho) + \tau^{-3}A^{3/4}(\tau^k\rho)B^{3/4}(\tau^k\rho)\right\}\\
&\le c\left\{\tau^3\mathcal{E}(\tau^k\rho) + \tau^{-3}\mathcal{E}^{1/2}(\tau^k\rho)(2\epsilon)^{3/4}\right\}\\
&\le c\left\{\tau^3\left(\frac{1}{2^k}\mathcal{E}(\rho) + 2c_\tau\epsilon^{3/2}\right)\right.\\
&\qquad\left. + \tau^{-3}\left(\left(\frac{1}{2^k}\mathcal{E}(\rho) + 2c_\tau\epsilon^{3/2}\right)\right)^{1/2}\epsilon^{3/4}\right\}.
\end{aligned}
$$

In view of the above estimate and inequality (4.6),

$$
\begin{aligned}
\limsup_{k\to\infty}(\tau^k\rho)\overline{E}(\tau^k\rho) &\le c\limsup_{k\to\infty}\left[C(\tau^k\rho) + \mathcal{E}(\tau^k\rho)\right]^{1/3}\\
&\le c\left\{\tau^3 2c_\tau\epsilon^{3/2} + \tau^{-3}\left(2c_\tau\epsilon^{3/2}\right)^{1/2}\epsilon^{3/4} + 2c_\tau\epsilon^{3/2}\right\}^{1/3}\\
&\le c\left\{\tau^3 2c_\tau + \tau^{-3}\left(2c_\tau\right)^{1/2} + 2c_\tau\right\}^{1/3}\epsilon^{1/2}.
\end{aligned}
$$

We conclude by choosing $\epsilon > 0$ so small that

$$
\limsup_{k\to\infty}(\tau^k\rho)\overline{E}(\tau^k\rho) < \frac{1}{2}\epsilon_5. \qquad \square
$$

Remark 5. A close inspection of the work in this paper shows that the methods employed establish an analogous version of Theorem 1.1 for weak solutions of the system

$$
\begin{cases}
\partial_t u + (u\cdot\nabla)u = \Delta u - \nabla p + \nabla\cdot FF^t,\\
\partial_t F + (u\cdot\nabla)F = \Delta F + \nabla u F + (\nabla Q)^t,\\
\nabla\cdot u = 0,\\
\nabla\cdot F^t = 0,
\end{cases}
$$

where $Q = Q(x,t) \in \mathbb{R}^3$.

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