

Asymptotic flatness of Morrey extremals

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Abstract

We study the limiting behavior as $|x| \to \infty$ of extremal functions u for Morrey's inequality on \mathbb{R}^n . In particular, we compute the limit of u(x) as $|x| \to \infty$ and show |x||Du(x)| tends to 0. To this end, we exploit the fact that extremals are uniformly bounded and that they each satisfy a PDE of the form $-\Delta_p u = c(\delta_{x_0} - \delta_{y_0})$ for some $c \in \mathbb{R}$ and distinct $x_0, y_0 \in \mathbb{R}^n$. More generally, we explain how to quantitatively deduce the asymptotic flatness of bounded p-harmonic functions on exterior domains of \mathbb{R}^n for p > n.

1 Introduction

For each $n \in \mathbb{N}$ and p > n, Morrey's inequality asserts that there is a constant C > 0 such that

$$\sup_{x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^{1 - n/p}} \right\} \le C \left(\int_{\mathbb{R}^n} |Du|^p dx \right)^{1/p}$$
(1.1)

for all continuously differentiable functions $u : \mathbb{R}^n \to \mathbb{R}$. In particular, it provides control on the 1 - n/p Hölder seminorm of any function whose first partial derivatives belong to $L^p(\mathbb{R}^n)$. In recent work [6], we showed that there is a smallest constant $C_* > 0$ for which (1.1) holds and that there are nonconstant functions for which equality holds in (1.1) with $C = C_*$. We call any such function an *extremal*.

It turns out that for any nonconstant extremal function u, there is a unique pair of distinct points $x_0, y_0 \in \mathbb{R}^n$ such that

$$\sup_{x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^{1 - n/p}} \right\} = \frac{|u(x_0) - u(y_0)|}{|x_0 - y_0|^{1 - n/p}}.$$
(1.2)

Moreover, u satisfies the PDE

$$-\Delta_p u = c(\delta_{x_0} - \delta_{y_0}) \tag{1.3}$$

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Fig. 1 The graph of a numerically approximated extremal u with n = 2, p = 4, $x_0 = (0, 1)$, $y_0 = (0, -1)$, $u(x_0) = 1$ and $u(y_0) = -1$. Note that $u(x) \approx \frac{1}{2}(u(x_0) + u(y_0)) = 0$ for larger values of |x|

in \mathbb{R}^n for some nonzero constant *c*. Here

$$\Delta_p v := \operatorname{div}(|Dv|^{p-2}Dv)$$

is the *p*-Laplacian, and Eq. (1.3) is understood to mean

$$\int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot D\phi dx = c(\phi(x_0) - \phi(y_0))$$

for each $\phi \in C_c^{\infty}(\mathbb{R}^n)$.

Equation (1.3) can be used to show that each extremal is bounded and has various symmetry properties. In this note, we will make use of these facts to prove the following theorem. We interpret the existence of limit (1.4) below as asserting that extremals are *asymptotically flat*. This result was also confirmed by numerical computations as observed in Fig. 1.

Theorem 1.1 Suppose $n \ge 2$ and that p > n. If u is an extremal which satisfies (1.2), then

$$\lim_{|x| \to \infty} u(x) = \frac{1}{2}(u(x_0) + u(y_0))$$
(1.4)

and

$$\lim_{|x|\to\infty}|x||Du(x)|=0.$$

Furthermore,

$$r^{p-n} \int_{|x|>r} |Du|^p dx = p \int_{|x|>r} |x|^{p-n} |Du|^{p-2} \left(Du \cdot \frac{x}{|x|} \right)^2 dx$$

is nonincreasing in $r \in (s, \infty)$ for some s > 0 and tends to 0 as $r \to \infty$.

In proving Theorem 1.1, we will first verify that any bounded p-harmonic function u on the exterior domain

$$\mathbb{R}^n \setminus \overline{B_1} = \{x \in \mathbb{R}^n : |x| > 1\}$$

$$\beta = \lim_{|x| \to \infty} u(x).$$

By employing a Harnack inequality, we can quantify this assertion and show there are positive numbers A and α such that

$$|u(x) - \beta| \le \frac{A ||u||_{\infty}}{|x|^{\alpha}}, \quad |x| \ge 1.$$

In particular, we will be able to conclude that the limit (1.4) occurs with an (at least) algebraic rate of convergence.

The precise decay estimate we derive is described as follows.

Theorem 1.2 Suppose $n \ge 2$ and p > n. There are positive constants $\alpha > 0$ and A > 0 such that

$$\sup\left\{|u(x) - u(y)| : |x|, |y| \ge r\right\} \le \frac{A ||u||_{\infty}}{r^{\alpha}}, \quad r \ge 1$$

for each function u that is bounded and p-harmonic in $\mathbb{R}^n \setminus \overline{B_1}$.

Then we'll show how these results extend to solutions $u : \mathbb{R}^n \to \mathbb{R}$ of the multipole equation

$$-\Delta_p u = \sum_{i=1}^N c_i \delta_{x_i},$$

where $x_1, \ldots, x_N \in \mathbb{R}^n$ are distinct and $c_1, \ldots, c_N \in \mathbb{R}$ satisfy $\sum_{i=1}^N c_i = 0$. The main point is to establish that each solution u is bounded. Moreover, we will argue that each solution u is not differentiable at any x_i in which it has a strict local maximum or minimum. Finally, in the "Appendix", we will explain the numerical method we used to produce Fig. 1 as shown above.

2 Bounded *p*-harmonic functions on exterior domains

In what follows, we will suppose that

$$n \ge 2$$
 and $p > n$

are fixed. Even though we are primarily interested in functions defined on \mathbb{R}^n , we will also consider functions defined on bounded domains Ω or possibly on the complement of such subsets. Recall that each function in the Sobolev space $W^{1,p}(\Omega)$ has a 1 - n/p Hölder continuous representative (Theorem 5, Sect. 5.6 of [3]). Consequently, we will always identify a $W^{1,p}(\Omega)$ function with its continuous representative and consider $W^{1,p}(\Omega)$ as a subset of the continuous functions on Ω .

For a given domain $\Omega \subset \mathbb{R}^n$, we will say that *u* is *p*-harmonic in Ω and write

$$-\Delta_p u = 0$$
 in Ω

so long as $u \in W^{1,p}_{\text{loc}}(\Omega)$ and

$$\int_{\Omega} |Du|^{p-2} Du \cdot D\phi dx = 0$$

for each $\phi \in C_c^{\infty}(\Omega)$. Likewise, for a signed Borel measure ρ on Ω , we say that

$$-\Delta_p u = \rho$$
 in Ω

provided $u \in W^{1,p}_{\text{loc}}(\Omega)$ and

$$\int_{\Omega} |Du|^{p-2} Du \cdot D\phi dx = \int_{\Omega} \phi d\rho$$

for all $\phi \in C_c^{\infty}(\Omega)$.

In this section, we will establish three facts about bounded *p*-harmonic functions on $\mathbb{R}^n \setminus \overline{B_1}$. We first show that these functions are all asymptotically flat and their gradients tend to zero as $|x| \to \infty$ at a certain rate. Then we show that if one of these functions lies strictly between two values, its limit as $|x| \to \infty$ lies strictly between these two values, as well. Finally, we establish decay and monotonicity properties of two integral quantities involving these functions.

2.1 Asymptotic flatness

As mentioned above, our first order of business is to verify the asymptotic flatness of bounded p-harmonic functions on $\mathbb{R}^n \setminus \overline{B_1}$. This is the central goal of this subsection. We also note that the first part of following statement has essentially been verified by Serrin [15], who showed that a positive p-harmonic function on an exterior domain has a positive limit as $|x| \to \infty$ or tends to ∞ at a specific rate; this result was also extended recently by Fraas and Pinchover [4,5]. Our result is not as general, however our proof is simple and direct.

Proposition 2.1 Suppose u is a bounded p-harmonic function on $\mathbb{R}^n \setminus \overline{B_1}$. Then the limit

$$\lim_{|x|\to\infty}u(x)$$

exists and

$$\lim_{|x|\to\infty}|x||Du(x)|=0.$$

To this end, we will need to make use of a version of Caccioppoli's inequality and a Liouville-type assertion for *p*-harmonic functions on punctured domains.

Lemma 2.2 Suppose $\Omega \subset \mathbb{R}^n$ is a domain and $x_0 \in \Omega$. Further assume u satisfies

$$-\Delta_p u = c \delta_{x_0}$$

in Ω for some constant c. Then for each nonnegative $\zeta \in C^{\infty}_{c}(\Omega)$,

$$\int_{\Omega} \zeta^p |Du|^p dx \le p^p \int_{\Omega} |u - u(x_0)|^p |D\zeta|^p dx.$$
(2.1)

Proof Observe

$$\int_{\Omega} |Du|^{p-2} Du \cdot D\phi dx = c\phi(x_0)$$

for $\phi \in W_0^{1,p}(\Omega)$. Let $\phi = \zeta^p(u - u(x_0))$ and note $\phi(x_0) = 0$ and

$$D\phi = p\zeta^{p-1}D\zeta \ (u - u(x_0)) + \zeta^p Du.$$

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Substituting this test function above gives

$$\begin{split} \int_{\Omega} \zeta^{p} |Du|^{p} dx &= -p \int_{\Omega} \zeta^{p-1} |Du|^{p-2} Du \cdot (u - u(x_{0})) D\zeta dx \\ &\leq p \int_{\Omega} (\zeta |Du|)^{p-1} (|u - u(x_{0})| |D\zeta|) dx \\ &\leq p \left(\int_{\Omega} \zeta^{p} |Du|^{p} dx \right)^{1-1/p} \left(\int_{\Omega} |u - u(x_{0})|^{p} |D\zeta|^{p} dx \right)^{1/p} \\ &\text{is (2.1).} \end{split}$$

which is (2.1).

Corollary 2.3 Suppose Ω is a domain and $B_{2r}(x_0) \subset \Omega$. Further assume u satisfies

 $-\Delta_p u = c \delta_{x_0}$

in Ω for some constant c. Then

$$\int_{B_r(x_0)} |Du|^p dx \le \left(\frac{2p}{r}\right)^p \int_{B_{2r}(x_0)} |u - u(x_0)|^p dx.$$
(2.2)

Proof Choose $\varphi \in C_c^{\infty}(B_2(0))$ with $0 \le \varphi \le 1, \varphi \equiv 1$ in $B_1(0)$ and

$$\|D\varphi\|_{\infty} \le 2$$

Then set

$$\zeta(x) = \varphi\left(\frac{x-x_0}{r}\right), \quad x \in B_{2r}(x_0).$$

Clearly, $\zeta \in C_c^{\infty}(B_{2r}(x_0))$ is nonnegative, $\zeta \equiv 1$ in $B_r(x_0)$ and

$$\|D\zeta\|_{\infty} \leq \frac{2}{r}.$$

The conclusion follows from substituting this ζ in (2.1).

Corollary 2.4 *Suppose u is bounded and satisfies*

$$-\Delta_p u = c\delta_{x_0}$$

in \mathbb{R}^n for some constant c. Then u is necessarily constant and c = 0.

Proof In view of (2.2),

$$\begin{split} \int_{B_r(x_0)} |Du|^p dx &\leq \left(\frac{2p}{r}\right)^p \int_{B_{2r}(x_0)} |u - u(x_0)|^p dx \\ &\leq \left(\frac{2p}{r}\right)^p (2\|u\|_{\infty})^p \omega_n (2r)^n \\ &\leq \frac{(4p\|u\|_{\infty})^p \omega_n 2^n}{r^{p-n}} \end{split}$$

for each r > 0; here ω_n is the Lebesgue measure of B_1 . Sending $r \to \infty$ forces |Du| to vanish on \mathbb{R}^n .

We are now ready to employ these observations to fashion a proof of Proposition 2.1.

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Proof of Proposition 2.1 1. For t > 0, set

$$v_t(x) := u(tx), \quad x \in \mathbb{R}^n.$$

Note that v_t is *p*-harmonic on $\mathbb{R}^n \setminus \overline{B_{1/t}}$. Without loss of generality, suppose $|u(y)| \le 1$ for all |y| > 1, so that

 $|v_t(x)| \leq 1$

for |x| > 1/t. We will now proceed to send $t \to \infty$.

By a result of Ural'ceva [17] (see also Lewis [10] and Evans [2]), there is $\gamma \in (0, 1)$ depending on *p* and *n* such that

$$\|v_t\|_{C^{1,\gamma}(K)} \le A$$

for each compact $K \subset \mathbb{R}^n \setminus \{0\}$ and *t* sufficiently large. Here *A* depends on *p* and *n* and *K*. Consequently, there is a sequence $(v_{t_k})_{k \in \mathbb{N}}$ with $t_k \to \infty$ and $v_{\infty} \in C^1_{loc}(\mathbb{R}^n \setminus \{0\})$ such that

$$v_{t_k} \to v_{\infty}$$
 in $C^1(K)$

for each compact $K \subset \mathbb{R}^n \setminus \{0\}$. It follows easily that v_{∞} is *p*-harmonic on $\mathbb{R}^n \setminus \{0\}$.

By Theorem 1.1 and Remark 1.6 of [8] (see also [9]), there is a constant $\mu \in \mathbb{R}$ such that

$$-\Delta_p v_{\infty} = |\mu|^{p-2} \mu n \omega_n \delta_0$$

in \mathbb{R}^n . Moreover,

$$\lim_{|x| \to 0} \frac{|Dv_{\infty}(x)|}{|x|^{\left(\frac{p-n}{p-1}\right)-1}} = |\mu|.$$

This limit gives that $|Dv_{\infty}|^p$ is locally integrable in a neighborhood of 0. Since

$$|v_{\infty}(x)| \le 1$$

for all $x \in \mathbb{R}^n$, we have $v_{\infty} \in W^{1,p}_{loc}(\mathbb{R}^n)$. Corollary 2.4 then implies that v_{∞} is identically equal to a constant β and so

$$\lim_{k \to \infty} v_{t_k}(x) = \beta$$

locally uniformly on $\mathbb{R}^n \setminus \{0\}$.

2. Consider

$$m(t) := \min_{|y|=t} u(y)$$

for t > 1. By the comparison principle for *p*-harmonic functions,

$$u(z) \ge \min\{m(t), m(s)\}$$

for 1 < s < |z| < t. It follows that

$$m(\lambda t + (1 - \lambda)s) \ge \min\{m(t), m(s)\}$$

for $\lambda \in [0, 1]$. In particular, $m : (1, \infty) \rightarrow [-1, 1]$ is quasiconcave. So there is $r_1 \ge 1$ for which $m|_{(r_1,\infty)}$ is monotone (Theorem 17 in Chapter 3 of [12]) and thus

$$\lim_{t \to \infty} m(t) = \lim_{t \to \infty} \min_{|y|=t} u(y) = \lim_{t \to \infty} \min_{|x|=1} v_t(x)$$

exists.

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We can choose an $x_t \in \mathbb{R}^n$ with $|x_t| = 1$ so that

$$\min_{|x|=1} v_t(x) = v_t(x_t).$$

We may as well also suppose that $(x_{t_k})_{k \in \mathbb{N}}$ is convergent. In this case,

$$\lim_{t\to\infty}\min_{|x|=1}v_t(x)=\lim_{k\to\infty}\min_{|x|=1}v_{t_k}(x)=\lim_{k\to\infty}v_{t_k}(x_{t_k})=\beta.$$

With virtually the same argument, we find

$$\lim_{t \to \infty} \max_{|x|=1} v_t(x) = \beta$$

Consequently,

$$\lim_{t\to\infty}v_t(x)=\beta$$

uniformly for |x| = 1.

3. Now let $(y_k)_{k \in \mathbb{N}} \subset \mathbb{R}^n$ be a sequence such that $|y_k| \to \infty$. Without loss of generality, we will suppose $|y_k| > 0$ and that $(y_k/|y_k|)_{k \in \mathbb{N}}$ is convergent as these properties are true for a subsequence of $(y_k)_{k \in \mathbb{N}}$. Then

$$\lim_{k \to \infty} u(y_k) = \lim_{k \to \infty} u\left(|y_k| \frac{y_k}{|y_k|}\right) = \lim_{k \to \infty} v_{|y_k|}\left(\frac{y_k}{|y_k|}\right) = \beta,$$

and we conclude that

$$\lim_{|y|\to\infty} u(y) = \beta$$

We also have that

$$Dv_t(x) = Du(tx)t$$

tends to $0 \in \mathbb{R}^n$ uniformly for |x| = 1. Choosing $(y_k)_{k \in \mathbb{N}}$ as above, we find

$$\lim_{k \to \infty} |y_k| |Du(y_k)| = \lim_{k \to \infty} |y_k| \left| Du\left(|y_k| \frac{y_k}{|y_k|} \right) \right|$$
$$= \lim_{k \to \infty} \left| Dv_{|y_k|} \left(\frac{y_k}{|y_k|} \right) \right|$$
$$= 0$$

.

That is,

$$\lim_{|y| \to \infty} |y| |Du(y)| = 0$$

Remark 2.5 This theorem can be proved without appealing to the $C_{loc}^{1,\gamma}$ estimates for *p*-harmonic functions. Local uniform convergence of a subsequence of $(v_t)_{t>0}$ in $\mathbb{R}^n \setminus \{0\}$ would follow from Morrey's inequality, and convergence in $W_{loc}^{1,p}(\mathbb{R}^n \setminus \{0\})$ can be verified using the Browder and Minty method (as described in Sect. 9.1 of [3]).

Remark 2.6 In Corollary 4.2 below, we will show that $\min_{|x|=r} u(x)$ is nondecreasing and $\max_{|x|=r} u(x)$ is nonincreasing for all $r \in (1, \infty)$.

2.2 Strict bounds on limiting values

The next assertion states that the limit of a bounded *p*-harmonic function on $\mathbb{R}^n \setminus \overline{B_1}$ always lies strictly within the bounds observed by the function. In particular, any bounded and positive *p*-harmonic function on an exterior domain has a positive limit. Pinchover and Tintarev [13] established this conclusion using a different argument and for more general operators.

Proposition 2.7 Suppose *u* is *p*-harmonic in $\mathbb{R}^n \setminus \overline{B_1}$ and

$$a < u(x) < b, x \in \mathbb{R}^n \setminus B_1$$

for some $a, b \in \mathbb{R}$. Then

$$a < \lim_{|x| \to \infty} u(x) < b.$$

Proof Fix r > 1, and for R > r define

$$w_R(x) = \frac{R^{\frac{p-n}{p-1}} - |x|^{\frac{p-n}{p-1}}}{R^{\frac{p-n}{p-1}} - r^{\frac{p-n}{p-1}}}, \quad r \le |x| \le R.$$

Note that w_R is *p*-harmonic in the annulus $B_R \setminus \overline{B_r}$,

 $w_R|_{\partial B_r} = 1$ and $w_R|_{\partial B_R} = 0$.

Now choose $\delta > 0$ such that

$$\min_{x\in\partial B_r}u(x)-a\geq\delta.$$

By comparison,

$$u(x) - a \ge \delta w_R(x), \quad r \le |x| \le R.$$

Let $e_1 = (1, 0, ..., 0)$ and suppose R > 2r. Then $r < \frac{1}{2}R < R$ and so

$$u\left(\frac{R}{2}e_{1}\right) \geq a + \delta w_{R}\left(\frac{R}{2}e_{1}\right)$$

= $a + \delta \frac{R^{\frac{p-n}{p-1}} - (R/2)^{\frac{p-n}{p-1}}}{R^{\frac{p-n}{p-1}} - r^{\frac{p-n}{p-1}}}$
= $a + \delta \frac{1 - (1/2)^{\frac{p-n}{p-1}}}{1 - (r/R)^{\frac{p-n}{p-1}}}.$

As a result,

$$\lim_{|x|\to\infty}u(x)=\lim_{R\to\infty}u\left(\frac{R}{2}e_1\right)\geq a+\delta\left(1-(1/2)^{\frac{p-n}{p-1}}\right)>a.$$

Likewise, we find $\lim_{|x| \to \infty} u(x) < b$.

Remark 2.8 We will see in Corollary 4.2, that the same conclusion holds only assuming

$$a < u(x) < b, \quad |x| = r$$

for some r > 1. This improvement relies on a global comparison property of bounded *p*-harmonic functions on the exterior domain $\mathbb{R}^n \setminus \overline{B_1}$.

2.3 Integral decay and monotonicity

In Proposition 2.1, we showed that if *u* is a bounded *p*-harmonic function in $\mathbb{R}^n \setminus \overline{B_1}$, then

$$\lim_{|x| \to \infty} |x| |Du(x)| = 0.$$
(2.3)

This limit immediately implies the following decay property.

Corollary 2.9 Suppose u is bounded and p-harmonic in $\mathbb{R}^n \setminus \overline{B_1}$. Then

$$\int_{|x|>s} |Du|^p dx < \infty$$

for any s > 1. Moreover,

$$\lim_{r \to \infty} r^{p-n} \int_{|x| > r} |Du|^p dx = 0.$$

Proof Fix $\epsilon > 0$. By (2.3), there is r > s so large that

$$|Du(x)| \le \frac{\epsilon}{|x|}$$

for $|x| \ge r$. Then

$$\int_{|x|>r} |Du|^p dx \le \epsilon^p n \omega_n \int_r^\infty \tau^{-p} \tau^{n-1} d\tau = \epsilon^p \frac{n \omega_n}{(p-n)r^{p-n}}.$$

Since

$$\int_{s<|x|< r} |Du|^p dx < \infty,$$

the first assertion follows. As for the second claim,

$$\lim_{r \to \infty} r^{p-n} \int_{|x|>r} |Du|^p dx \le \epsilon^p \frac{n\omega_n}{(p-n)}.$$

The conclusion follows as $\epsilon > 0$ is arbitrary.

Using a certain identity for smooth *p*-harmonic functions, we can strengthen the conclusion of the previous corollary.

Proposition 2.10 Suppose *u* is smooth, bounded and *p*-harmonic in $\mathbb{R}^n \setminus \overline{B_1}$. Then

$$(1,\infty) \ni r \mapsto r^{p-n} \int_{|x|>r} |Du|^p dx$$

is nonincreasing. In particular,

$$\lim_{r \to \infty} r^{p-n} \int_{|x|>r} |Du|^p dx = \inf_{r>1} r^{p-n} \int_{|x|>r} |Du|^p dx = 0.$$
(2.4)

Moreover,

$$r^{p-n} \int_{|x|>r} |Du|^p dx = p \int_{|x|>r} |x|^{p-n} |Du|^{p-2} \left(Du \cdot \frac{x}{|x|} \right)^2 dx$$
(2.5)

for each r > 1.

Proof As u is smooth, direct computation gives

$$\operatorname{div}\left(\left(Du \cdot x + \left(\frac{n}{p} - 1\right)u\right)p|Du|^{p-2}Du - |Du|^{p}x\right) = 0$$
(2.6)

in $\mathbb{R}^n \setminus \overline{B_1}$ (Chapter 8, Sect. 6 of [3]). Integrating both sides of (2.6) over r < |x| < R gives

$$0 = \int_{r < |x| < R} \operatorname{div} \left(\left(Du \cdot x + \left(\frac{n}{p} - 1 \right) u \right) p |Du|^{p-2} Du - |Du|^{p} x \right) dx$$

$$= \int_{|x| = R} \left(\left(Du \cdot x + \left(\frac{n}{p} - 1 \right) u \right) p |Du|^{p-2} Du - |Du|^{p} x \right) \cdot \frac{x}{R} d\sigma$$

$$- \int_{|x| = r} \left(\left(Du \cdot x + \left(\frac{n}{p} - 1 \right) u \right) p |Du|^{p-2} Du - |Du|^{p} x \right) \cdot \frac{x}{r} d\sigma$$

$$= -R \int_{|x| = R} \left(|Du|^{p} - p |Du|^{p-2} (\partial_{r} u)^{2} \right) d\sigma + (n-p) \int_{|x| = R} u |Du|^{p-2} Du \cdot \frac{x}{R} d\sigma$$

$$+ r \int_{|x| = r} \left(|Du|^{p} - p |Du|^{p-2} (\partial_{r} u)^{2} \right) d\sigma + (n-p) \int_{|x| = r} u |Du|^{p-2} Du \cdot \frac{-x}{r} d\sigma.$$

(2.7)

Here

$$\partial_r u(x) := Du(x) \cdot \frac{x}{|x|}$$

is the radial derivative of u and σ is n - 1 dimensional Hausdorff measure. In view of (2.3),

$$-R \int_{|x|=R} \left(|Du|^p - p|Du|^{p-2} (\partial_r u)^2 \right) d\sigma$$
$$+ (n-p) \int_{|x|=R} u|Du|^{p-2} Du \cdot \frac{x}{R} d\sigma = o(R^{n-p})$$

as $R \to \infty$. So we can send $R \to \infty$ in (2.7) to conclude

$$0 = r \int_{|x|=r} \left(|Du|^p - p|Du|^{p-2} (\partial_r u)^2 \right) d\sigma + (n-p) \int_{|x|=r} u|Du|^{p-2} Du \cdot \frac{-x}{r} d\sigma$$

= $r \int_{|x|=r} \left(|Du|^p - p|Du|^{p-2} (\partial_r u)^2 \right) d\sigma + (n-p) \int_{|x|>r} |Du|^p dx.$

Now observe

$$\frac{d}{dr}\left\{r^{p-n}\int_{|x|>r}|Du|^{p}dx\right\} = (p-n)r^{p-n-1}\int_{|x|>r}|Du|^{p}dx - r^{p-n}\int_{|x|=r}|Du|^{p}d\sigma$$
$$= r^{p-n-1}\left\{(p-n)\int_{|x|>r}|Du|^{p}dx - r\int_{|x|=r}|Du|^{p}d\sigma\right\}$$
$$= r^{p-n-1}\left\{-rp\int_{|x|=r}|Du|^{p-2}(\partial_{r}u)^{2}d\sigma\right\}$$
$$= -pr^{p-n}\int_{|x|=r}|Du|^{p-2}(\partial_{r}u)^{2}d\sigma.$$
(2.8)

As a result,

$$(1,\infty) \ni r \mapsto r^{p-n} \int_{|x|>r} |Du|^p dx$$

is nonincreasing. This quantity tends to 0 as $r \to \infty$ by the previous corollary, so we conclude (2.4) by monotone convergence. Integrating the monotonicity formula (2.8) from r = s to $r = \infty$ gives

$$s^{p-n} \int_{|x|>s} |Du|^p dx = p \int_s^\infty r^{p-n} \int_{|x|=r} |Du|^{p-2} (\partial_r u)^2 d\sigma dr$$
$$= p \int_s^\infty \int_{|x|=r} |x|^{p-n} |Du|^{p-2} (\partial_r u)^2 d\sigma dr$$
$$= p \int_{|x|>s} |x|^{p-n} |Du|^{p-2} (\partial_r u)^2 dx$$

which is (2.5).

3 Asymptotics of extremals

This section is dedicated to the proof of Theorem 1.1. Let u be an extremal satisfying (1.2). In Proposition 3.5 of [6], we established that

$$\min\{u(x_0), u(y_0)\} \le u(x) \le \max\{u(x_0), u(y_0)\}$$
(3.1)

for each $x \in \mathbb{R}^n$; this inequality is also established in Lemma 5.4 below. As a result, *u* is uniformly bounded and is *p*-harmonic in $\mathbb{R}^n \setminus \overline{B_s}$ for

$$s := \max\{|x_0|, |y_0|\}.$$

It follows from Proposition 2.1 that the limit

$$\lim_{|x|\to\infty}u(x)$$

exists and

$$\lim_{|x|\to\infty}|x||Du(x)|=0.$$

As *u* is smooth in $\mathbb{R}^n \setminus \overline{B_s}$ (Sect. 4.3 of [6]), we can apply Proposition 2.10 to conclude

$$r^{p-n} \int_{|x|>r} |Du|^p dx = \int_{|x|>r} |x|^{p-n} |Du|^{p-2} \left(Du \cdot \frac{x}{|x|} \right)^2 dx$$

for r > s. Moreover, this quantity is nonincreasing on (s, ∞) and tends to 0 as $r \to \infty$.

In Proposition 3.4 of [6], we showed

$$u\left(x - 2\frac{\left((x_0 - y_0) \cdot \left(x - \frac{1}{2}(x_0 + y_0)\right)\right)}{|x_0 - y_0|^2}(x_0 - y_0)\right)$$
$$-\frac{u(x_0) + u(y_0)}{2} = -\left(u(x) - \frac{u(x_0) + u(y_0)}{2}\right)$$

for each $x \in \mathbb{R}^n$. This equality implies that $u - \frac{1}{2}(u(x_0) + u(y_0))$ is antisymmetric with respect to reflection about the hyperplane

$$\Pi := \left\{ x \in \mathbb{R}^n : (x_0 - y_0) \cdot \left(x - \frac{1}{2} (x_0 + y_0) \right) = 0 \right\}.$$

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Fig. 2 Level sets of the approximate extremal computed for Fig. 1. Each level set except the line $x_2 = 0$ bounds a convex, compact subset of \mathbb{R}^2

In particular,

$$u(x) = \frac{1}{2}(u(x_0) + u(y_0))$$

for each $x \in \Pi$. As Π is unbounded, it must be that

$$\lim_{|x| \to \infty} u(x) = \frac{1}{2}(u(x_0) + u(y_0)).$$

Remark 3.1 If *u* is an extremal which satisfies

$$\sup_{x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^{1 - n/p}} \right\} = \frac{u(x_0) - u(y_0)}{|x_0 - y_0|^{1 - n/p}} > 0$$

for distinct $x_0, y_0 \in \mathbb{R}^n$,

$$\{x \in \mathbb{R}^n : u(x) \ge t\}$$
 and $\{x \in \mathbb{R}^n : u(x) \le s\}$

are convex for

$$\frac{u(x_0) + u(y_0)}{2} < t \le u(x_0) \text{ and } \frac{u(x_0) + u(y_0)}{2} > s \ge u(y_0),$$

respectively. This was proved in Proposition 4.4 of [6]. An immediate corollary of Theorem 1.1 is that these subsets are *compact*, as displayed in Fig. 2.

4 Quantitative flatness

We will now establish a Harnack inequality for bounded, nonnegative *p*-harmonic functions on $\mathbb{R}^n \setminus \overline{B_1}$. We will then prove Theorem 1.2 similar to how Hölder continuity of *p*-harmonic functions can be established with a Harnack inequality (as explained in Sect. 2 of [11]). To this end, we will start with the following comparison principle.

Lemma 4.1 Suppose r > 1 and that u, v are bounded and p-harmonic in $\mathbb{R}^n \setminus \overline{B_1}$ with

 $u \leq v$

on ∂B_r . Then

 $u \leq v$

in $\mathbb{R}^n \setminus \overline{B_r}$.

Proof In view of the monotonicity of the mapping $z \mapsto |z|^{p-2}z$,

$$0 \leq \int_{\{u>v\} \cap \{|x|>r\}} (|Du|^{p-2}Du - |Dv|^{p-2}Dv) \cdot (Du - Dv)dx$$

=
$$\int_{|x|>r} (|Du|^{p-2}Du - |Dv|^{p-2}Dv) \cdot D(u - v)^{+}dx.$$
(4.1)

As $(u - v)^+$ is bounded and vanishes on ∂B_r , we can integrate by parts and appeal to (2.3) in order to deduce

$$\begin{split} &\int_{|x|>r} (|Du|^{p-2}Du - |Dv|^{p-2}Dv) \cdot D(u-v)^+ dx \\ &= \lim_{R \to \infty} \int_{r < |x| < R} (|Du|^{p-2}Du - |Dv|^{p-2}Dv) \cdot D(u-v)^+ dx \\ &= \lim_{R \to \infty} \int_{|x| = R} (u-v)^+ (|Du|^{p-2}Du - |Dv|^{p-2}Dv) \cdot \frac{x}{R} dx \\ &= \lim_{R \to \infty} o(R^{n-p}) \\ &= 0. \end{split}$$

Combining with (4.1) gives

$$\int_{\{u>v\}\cap\{|x|>r\}} (|Du|^{p-2}Du - |Dv|^{p-2}Dv) \cdot (Du - Dv)dx = 0.$$

As $z \mapsto |z|^{p-2}z$ is strictly monotone, it must either be that the Lebesgue measure of $\{u > v\} \cap \{|x| > r\}$ is zero or that Du = Dv almost everywhere in $\{u > v\} \cap \{|x| > r\}$. If the Lebesgue measure of $\{u > v\} \cap \{|x| > r\}$ is zero, $u(x) \le v(x)$ for almost every |x| > r; as u, v are continuous, this would imply that $u(x) \le v(x)$ for every |x| > r. Otherwise, $D(u - v)^+ = 0$ in $\mathbb{R}^n \setminus \overline{B_r}$ which would mean $(u - v)^+$ is constant throughout $\mathbb{R}^n \setminus \overline{B_r}$. Since $(u - v)^+$ vanishes on ∂B_r , we would have $(u - v)^+ \equiv 0$ in $\mathbb{R}^n \setminus \overline{B_r}$. That is, $u \le v$ in $\mathbb{R}^n \setminus \overline{B_r}$.

Corollary 4.2 Suppose *u* is *p*-harmonic and bounded in $\mathbb{R}^n \setminus \overline{B_1}$.

(i) For each r > 1,

$$\sup_{|x|>r} u(x) = \sup_{|x|=r} u(x) \text{ and } \inf_{|x|\ge r} u(x) = \inf_{|x|=r} u(x).$$

(ii) *For* $1 < r_1 \le r_2$,

 $\sup_{|x|=r_1} u(x) \ge \sup_{|x|=r_2} u(x) \text{ and } \inf_{|x|=r_1} u(x) \le \inf_{|x|=r_2} u(x).$

(iii) For $1 < r_1 \le r_2$,

$$\sup_{|x|=r_1} u(x) = \sup_{r_1 \le |x| \le r_2} u(x) \text{ and } \inf_{|x|=r_1} u(x) = \inf_{r_1 \le |x| \le r_2} u(x).$$

(iv) If r > 1 and a < u(x) < b for |x| = r, then

$$a < \lim_{|x| \to \infty} u(x) < b.$$

Proof We will only prove the statements involving suprema. (i) Let v denote the constant function on \mathbb{R}^n which is equal to $\sup_{|x|=r} u(x)$. As v is bounded, p-harmonic, and $u(y) \le v(y)$ for |y| = r, it follows that $u(y) \le v(y)$ for each $|y| \ge r$. That is,

$$u(y) \le \sup_{|x|=r} u(x), \quad |y| \ge r.$$

(ii) By part (i),

$$\sup_{|x|=r_1} u(x) = \sup_{|x|\ge r_1} u(x) \ge \sup_{|x|\ge r_2} u(x) = \sup_{|x|=r_2} u(x).$$

(iii) Part (i) also implies

$$\sup_{|x| \ge r_1} u(x) = \sup_{|x| = r_1} u(x) \le \sup_{r_1 \le |x| \le r_2} u(x) \le \sup_{|x| \ge r_1} u(x).$$

(iv) Choose $\delta > 0$ so small that $\sup_{|x|=r} u(x) < b - \delta$. By part (i), $u(x) < b - \delta$ for each $|x| \ge r$. Consequently, $\lim_{|x|\to\infty} u(x) \le b - \delta < b$.

The following harnack inequality is now an easy consequence of these observations.

Proposition 4.3 *There is a constant* C > 1 *such that*

$$\sup_{|x| \ge 2r} u(x) \le C \inf_{|x| \ge 2r} u(x) \tag{4.2}$$

for each r > 0 and bounded, nonnegative *p*-harmonic *u* in $\mathbb{R}^n \setminus \overline{B_r}$.

Remark 4.4 The example $u(x) = |x|^{\frac{p-n}{p-1}}$ shows that the boundedness assumption cannot be removed.

Proof First suppose r = 1 and choose C > 1 such that

$$\sup_{2 < |x| < 3} v \le C \inf_{2 < |x| < 3} v.$$
(4.3)

for each for each nonnegative *p*-harmonic function *v* on $\{1 < |x| < 4\}$. Such a constant *C* exists by the Harnack inequality proved by Serrin in Sect. 5 of [14]. In view of Corollary 4.2 and (4.3),

$$\sup_{|x|\ge 2} u(x) = \sup_{2\le |x|\le 3} u(x)$$

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$$\leq C \inf_{\substack{2 \leq |x| \leq 3}} u(x)$$
$$= C \inf_{|x| \geq 2} u(x).$$

For general r > 0, we set w(y) = u(ry) for |y| > 1. Then w is bounded, nonnegative, and p-harmonic in $\mathbb{R}^n \setminus \overline{B_1}$. By our computation above,

$$\sup_{|y|\ge 2} w(y) \le C \inf_{|y|\ge 2} w(y)$$

with the constant C from (4.3). It follows that (4.2) holds with this constant C.

Along with this Harnack inequality, we will need one more fact to prove Theorem 1.2.

Lemma 4.5 Suppose $f : [1, \infty) \to [0, \infty)$ is nonincreasing and satisfies

$$f(2r) \le \mu f(r), \quad r \ge 1$$

for some $\mu \in (0, 1)$. Then

$$f(r) \le \frac{1}{\mu} r^{\left(\frac{\ln \mu}{\ln 2}\right)} f(1)$$

for $r \geq 1$.

Remark 4.6 $\ln \mu < 0$, so f(r) decays like a power of r as $r \to \infty$.

Proof By induction,

$$f(2^k) \le \mu^k f(1)$$

for each nonnegative integer k. Choose $k \in \mathbb{N}$ so that

$$2^{k-1} \le r < 2^k$$

and

$$k-1 \le \frac{\ln r}{\ln 2} < k.$$

Then

$$\begin{split} f(r) &\leq f(2^{k-1}) \\ &\leq \frac{1}{\mu} \mu^k f(1) \\ &\leq \frac{1}{\mu} \mu^{\frac{\ln r}{\ln 2}} f(1) \\ &= \frac{1}{\mu} r^{\frac{\ln \mu}{\ln 2}} f(1). \end{split} \square$$

Proof of Theorem 1.2 Set

$$M(r) := \sup_{|x| \ge r} u(x), \quad m(r) := \inf_{|x| \ge r} u(x), \text{ and } \omega(r) := M(r) - m(r)$$

for $r \ge 1$. Observe that M(r), -m(r) and $\omega(r)$ are nonincreasing. Also note u(x) - m(r) is a bounded, nonnegative *p*-harmonic function for $|x| \ge r$. By Proposition 4.2,

$$M(2r) - m(r) = \sup_{|x| \ge 2r} (u(x) - m(r)) \le C \inf_{|x| \ge 2r} (u(x) - m(r)) = C(m(2r) - m(r))$$

for some C > 1 independent of r. Likewise M(r) - u(x) is a bounded, nonnegative p-harmonic function for $|x| \ge r$, so

$$M(r) - m(2r) = \sup_{|x| \ge 2r} (M(r) - u(x)) \le C \inf_{|x| \ge 2r} (M(r) - u(x)) = C(M(r) - M(2r)).$$

Adding these inequalities gives

$$\omega(2r) + \omega(r) \le C(-\omega(2r) + \omega(r)).$$

That is,

$$\omega(2r) \le \frac{C-1}{C+1}\omega(r), \quad r \ge 1$$

By the Lemma 4.5,

$$\omega(r) \le \frac{A \|u\|_{\infty}}{r^{\alpha}}, \quad r \ge 1$$

for some α , A > 0; here we used $\omega(1) \le 2 \|u\|_{\infty}$. In particular,

$$\sup\left\{|u(x) - u(y)| : |x|, |y| \ge r\right\} \le \frac{A ||u||_{\infty}}{r^{\alpha}}$$

for $r \ge 1$.

A minor variation of our proof of Theorem 1.2 combined with (3.1) gives the following conclusion.

Corollary 4.7 Assume u is an extremal which satisfies (1.2). There are positive A, α , and s such that

$$\left| u(x) - \frac{1}{2} (u(x_0) + u(y_0)) \right| \le \frac{A \max\{|u(x_0)|, |u(y_0)|\}}{|x|^{\alpha}}$$

for each |x| > s.

5 Multipole equation

We define

$$\mathcal{D}^{1,p}(\mathbb{R}^n) := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n) : u_{x_i} \in L^p(\mathbb{R}^n) \text{ for } i = 1, \dots, n \right\}$$

and suppose $x_1, \ldots, x_N \in \mathbb{R}^n$ are distinct and $a_1, \ldots, a_N \in \mathbb{R}$ are given. Let us consider the minimization problem: find $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ which minimizes the integral

$$\int_{\mathbb{R}^n} |Dv|^p dx \tag{5.1}$$

subject to the constraints

$$v(x_i) = a_i, \quad i = 1, \dots, N.$$
 (5.2)

Direct methods from the calculus of variations can be used to show that there is a minimizer $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$. Moreover, as the Dirichlet integral (5.1) is strictly convex, u is unique.

These observations were first noted by Kichenassamy in Sect. 2.3 of [7]. Discrete analogs of this minimization problem also arise in semi-supervised learning with labels as studied

recently by Calder [1] and by Slepčev and Thorpe [16]. We became interested in this problem when we noticed that the minimizer *u* above satisfies a generalized version of the PDE solved by Morrey extremals.

Proposition 5.1 (i) Suppose $u \in D^{1,p}(\mathbb{R}^n)$ minimizes (5.1) subject to the constraints (5.2). Then there are constants $c_1, \ldots, c_N \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot D\phi dx = \sum_{i=1}^N c_i \phi(x_i)$$
(5.3)

for all $\phi \in \mathcal{D}^{1,p}(\mathbb{R}^n)$.

(ii) Conversely, assume $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ satisfies (5.3) and the constraints (5.2). Then u minimizes (5.1) among all $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ which satisfy (5.2).

Remark 5.2 Choosing $\phi \equiv 1$ in (5.3), we see that $\sum_{i=1}^{N} c_i = 0$.

Remark 5.3 If $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ satisfies (5.3), then *u* is a solution of the multipole equation

$$-\Delta_p u = \sum_{i=1}^N c_i \delta_{x_i} \tag{5.4}$$

Proof (i) Let $\phi \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ and choose r > 0 so small that all of the balls $B_r(x_1), \ldots, B_r(x_N)$ are disjoint. It is straightforward to check that u is p-harmonic in $\mathbb{R}^n \setminus \bigcup_{i=1}^N B_r(x_i)$. Consequently, we can integrate by parts to find

$$\int_{\mathbb{R}^n \setminus \bigcup_{i=1}^N B_r(x_i)} |Du|^{p-2} Du \cdot D\phi dx = -\sum_{i=1}^N \int_{\partial B_r(x_i)} \phi |Du|^{p-2} Du \cdot \frac{x - x_i}{r} d\sigma.$$
(5.5)

By Theorem 1.1 and Remark 1.6 of [8],

$$\lim_{r \to 0} \left[-\int_{\partial B_r(x_i)} \phi |Du|^{p-2} Du \cdot \frac{x - x_i}{r} d\sigma \right] = c_i \phi(x_i)$$

for some $c_i \in \mathbb{R}$ independent of ϕ for each i = 1, ..., N. As a result, we can send $r \to 0^+$ in (5.5) and conclude (5.3).

(ii) Suppose $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ fulfills (5.3) and that $u, v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ satisfy (5.2). As

$$|z|^{p} \ge |w|^{p} + p|w|^{p-2}w \cdot (z-w)$$

for all $z, w \in \mathbb{R}^n$,

$$|Dv|^{p} \ge |Du|^{p} + p|Du|^{p-2}Du \cdot (Dv - Du)$$

holds almost everywhere in \mathbb{R}^n . Integrating this inequality gives

$$\begin{split} \int_{\mathbb{R}^n} |Dv|^p dx &\geq \int_{\mathbb{R}^n} |Du|^p dx + p \int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot D(v-u) dx \\ &= \int_{\mathbb{R}^n} |Du|^p dx + p \sum_{i=1}^N c_i (u-v)(x_i) \\ &= \int_{\mathbb{R}^n} |Du|^p dx. \end{split}$$

It also turns out that minimizers are uniformly bounded.

Lemma 5.4 Suppose $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ minimizes (5.1) subject to the constraints (5.2). Then

$$\min_{1 \le i \le N} a_i \le u(x) \le \max_{1 \le i \le N} a_i$$

for each $x \in \mathbb{R}^n$. Moreover, if not all of the a_i are identical,

$$\min_{1 \le i \le N} a_i < u(x) < \max_{1 \le i \le N} a_i$$
(5.6)

for each $x \in \mathbb{R}^n \setminus \{x_1, \ldots, x_N\}$.

Proof We will only establish the claimed upper bounds. Set

$$M := \max_{1 \le i \le N} a_i$$

and define

$$v(x) = \min\{u(x), M\}, \quad x \in \mathbb{R}^n.$$

It is plain to see that $v \leq M$ and that v satisfies (5.2). Moreover,

$$\int_{\mathbb{R}^n} |Dv|^p dx = \int_{u \le M} |Du|^p dx \le \int_{\mathbb{R}^n} |Du|^p dx.$$

So $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ minimizes (5.1) subject to (5.2). It follows that $u \equiv v \leq M$.

Observe that u - M is nonpositive and *p*-harmonic in the domain $\mathbb{R}^n \setminus \{x_1, \ldots, x_N\}$. By the strong maximum principle (Corollary 2.21 of [11]), it is either that $u \equiv M$ or u < M in $\mathbb{R}^n \setminus \{x_1, \ldots, x_N\}$. Since *u* is not constant in \mathbb{R}^n is must be that u < M in $\mathbb{R}^n \setminus \{x_1, \ldots, x_N\}$.

The following corollary is now easily seen as a consquence of Propositions 2.1 and 2.7.

Corollary 5.5 Suppose $u \in D^{1,p}(\mathbb{R}^n)$ minimizes (5.1) subject to the constraints (5.2). Then the limit

$$\lim_{|x| \to \infty} u(x) \tag{5.7}$$

exists and

$$\lim_{|x|\to\infty} |x||Du(x)| = 0.$$

Moreover, if not all of the a_i are identical,

 $\min_{1 \le i \le N} a_i < \lim_{|x| \to \infty} u(x) < \max_{1 \le i \le N} a_i.$

Remark 5.6 Using the estimate from Theorem 1.2, we can also conclude that the limit (5.7) occurs with at least an algebraic rate of convergence.

We can also make a few basic observations about a particular level set of solutions of Eq. (5.4).

Corollary 5.7 Suppose $u \in D^{1,p}(\mathbb{R}^n)$ minimizes (5.1) subject to the constraints (5.2) and

$$\lim_{|x|\to\infty}u(x)=\beta.$$

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Then

$$\{x \in \mathbb{R}^n : u(x) = \beta\}$$

is nonempty and noncompact. Furthermore, $c = \beta$ is the only value for which the level set

$$\{x \in \mathbb{R}^n : u(x) = c\}$$

has this property.

Proof We have established

$$\beta \in \left[\min_{1 \le i \le N} a_i, \max_{1 \le i \le N} a_i\right] = u(\mathbb{R}^n).$$

Since *u* is continuous, there is some $z \in \mathbb{R}^n$ for which $u(z) = \beta$. Consequently, $\{x \in \mathbb{R}^n : u(x) = \beta\} \neq \emptyset$.

If $\{x \in \mathbb{R}^n : u(x) = \beta\} \subset B_R$ for some R > 0, then either $u > \beta$ in $\mathbb{R}^n \setminus \overline{B_R}$ or $u < \beta$ in $\mathbb{R}^n \setminus \overline{B_R}$. If $u > \beta$ in $\mathbb{R}^n \setminus \overline{B_R}$, then $u - \beta$ is a bounded and positive *p*-harmonic function on an exterior domain. By Proposition 2.7, there is a $\eta > 0$ such that $u(x) - \beta \to \eta$ as $|x| \to \infty$. However, this contradicts $u(x) \to \beta$ as $|x| \to \infty$. Thus, no such *R* exists and $\{x \in \mathbb{R}^n : u(x) = \beta\}$ is noncompact.

Finally, we note that if there is a sequence $(x_k)_{k \in \mathbb{N}}$ with $|x_k| \to \infty$ and $u(x_k) = c$ then

$$\beta = \lim_{k \to \infty} u(x_k) = c.$$

That is, $\{x \in \mathbb{R}^n : u(x) = c\}$ is compact when $c \neq \beta$.

Remark 5.8 It would be really interesting to explicitly compute

$$\lim_{|x|\to\infty} u(x)$$

for solutions of the multipole PDE (5.4). Perhaps it is possible to do so in terms of the given data a_1, \ldots, a_N and x_1, \ldots, x_N . Recall that when N = 1,

$$\lim_{|x|\to\infty}u(x)=a_1$$

by Corollary 2.4; and when N = 2,

$$\lim_{|x| \to \infty} u(x) = \frac{a_1 + a_2}{2}$$

by Theorem 1.1. We wonder if there are analogous formulae for $N \ge 3$.

We conclude by studying the (non)differentiability of minimizers at the points x_1, \ldots, x_N . This and the other properties we have already discussed about solutions of the multipole PDE may be seen in Figs. 3 and 4.

Proposition 5.9 Suppose $u \in D^{1,p}(\mathbb{R}^n)$ minimizes (5.1) subject to the constraints (5.2) and $i \in \{1, ..., N\}$. If u has a strict local maximum or minimum at x_i , then u is not differentiable at x_i .

Proof We will prove that *u* is not differentiable at x_1 provided that it has a strict local max at x_1 . With this assumption, there is some r > 0 such that $u(x) < u(x_1)$ for $x \in \overline{B_r(x_1)} \setminus \{x_1\}$. In particular,

$$u(x_1) > \max_{\partial B_r(x_1)} u.$$
(5.8)

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Fig.3 The graph of the solution of the multipole Eq. (5.4) with n = 2, p = 3, $x_1 = (0, 1)$, $x_2 = (0, -1)$, $x_3 = (2, 0)$ and $c_1 = 1$, $c_2 = -3/2$ and $c_3 = 1/2$



Fig. 4 The graph of a solution of the multipole Eq. (5.4) with n = 2, p = 5, $x_1 = (0, 1)$, $x_2 = (0, -1)$, $x_3 = (2, 0)$, $x_4 = (-2, 0)$ and $c_1 = 2$, $c_2 = -2$, $c_3 = 1$ and $c_4 = -1$

Choosing *r* smaller if necessary, we may also suppose that *u* is *p*-harmonic in $B_r(x_1) \setminus \{x_1\}$. Set

$$v(x) := \left(u(x_1) - \max_{\partial B_r(x_1)} u\right) \left(1 - \frac{|x - x_1|^{\frac{p - n}{p - 1}}}{r^{\frac{p - n}{p - 1}}}\right) + \max_{\partial B_r(x_1)} u$$

for $x \in \overline{B_r(x_1)}$. Note that $v(x_1) = u(x_1)$ and

$$v|_{\partial B_r(x_1)} = \max_{\partial B_r(x_1)} u \ge u|_{\partial B_r(x_1)}$$

As v is p-harmonic in $B_r(x_1) \setminus \{x_1\}$, comparison gives $v \ge u$ in $B_r(x_1)$.

If *u* is differentiable at x_1 , then

$$v(x) = \left(u(x_1) - \max_{\partial B_r(x_1)} u\right) \left(1 - \frac{|x - x_1|^{\frac{p - n}{p - 1}}}{r^{\frac{p - n}{p - 1}}}\right) + \max_{\partial B_r(x_1)} u$$

$$\geq u(x)$$

$$= u(x_1) + Du(x_1) \cdot (x - x_1) + o(|x - x_1|)$$

$$\geq u(x_1) - (|Du(x_1)| + o(1)) |x - x_1|$$

as $x \to x_1$. Rearranging this inequality gives

$$(|Du(x_1)| + o(1))|x - x_1|^{1 - \left(\frac{p-n}{p-1}\right)} \ge \frac{1}{r^{\frac{p-n}{p-1}}} \left(u(x_1) - \max_{\partial B_r(x_1)} u \right).$$

And sending $x \to x_1$ leads us to

$$0 \ge u(x_1) - \max_{\partial B_r(x_1)} u,$$

which contradicts (5.8). Consequently, u is not differentiable at x_1 .

Corollary 5.10 Suppose $u \in D^{1,p}(\mathbb{R}^n)$ minimizes (5.1) subject to the constraints (5.2) and that a_1, \ldots, a_N are not all the same. Then u is not differentiable at any point in which it attains its global maximum or its global minimum.

Proof Suppose

$$a_1 = \max_{1 \le i \le N} a_i.$$

We noted that $u(x) < u(x_1) = a_1$ in $\mathbb{R}^n \setminus \{x_1, \ldots, x_N\}$ in (5.6). It follows that *u* has a strict local max at x_1 . By Proposition 5.9, *u* isn't differentiable at x_1 . As a result, *u* is not differentiable at any point in which it attains its global maximum. We can argue similarly for points at which *u* attains its global minimum.

Appendix A: Numerical method

We will now discuss the method used to approximate the solution of PDE (1.3) displayed in Fig. 1. It turns out that this method also can be adapted to obtain approximations of solutions of the multipole Eq. (5.4), as exhibited in Figs. 3 and 4. For simplicity, we will focus on the particular case of approximating a solution *u* of the PDE

$$-\Delta_p u = \delta_{(0,1)} - \delta_{(0,-1)} \tag{A.1}$$

in \mathbb{R}^2 . We will also change notation and use ordered pairs (x, y) to denote points in \mathbb{R}^2 so that u = u(x, y).

Observe that any solution $u \in \mathcal{D}^{1, p}(\mathbb{R}^2)$ of (A.1) minimizes

$$\iint_{\mathbb{R}^2} \frac{1}{p} |Dv|^p dx dy - (v(0,1) - v(0,-1)) \tag{A.2}$$

among all $v \in \mathcal{D}^{1,p}(\mathbb{R}^2)$. For a given $\ell \in \mathbb{N}$, we may also consider the problem of minimizing

$$\int_{-\ell}^{\ell} \int_{-\ell}^{\ell} \frac{1}{p} |Dv|^p dx dy - (v(0,1) - v(0,-1))$$

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amongst $v \in W^{1,p}([-\ell, \ell]^2)$. It is not hard to show this problem has a minimizer $u_{\ell} \in W^{1,p}([-\ell, \ell]^2)$. Further, it is routine to check that $u_{\ell}(x, y) - u_{\ell}(0, 0)$ converges to u(x, y) for each $(x, y) \in \mathbb{R}^2$ as $\ell \to \infty$, where *u* is the unique minimizer of (A.2) with u(0, 0) = 0. Consequently, we will focus on approximating u_{ℓ} .

Below we will show how to derive a discrete version of our minimization problem for u_{ℓ} . Then we will explain how to use an iteration scheme based on a quasi-Newton method. Again we emphasize that all of the figures in this article were based on this method or minor variants to account for differences in the particular examples we considered.

Appendix A.1: Discrete energy

Let us fix $\ell \in \mathbb{N}$ ($\ell \ge 2$) and discretize the interval $[-\ell, \ell]$ along the x-axis with

$$x_i = -\ell + (i-1)h$$

for i = 1, ..., M. Here

$$h = \frac{2\ell}{M-1},$$

and we note that each of the subintervals $[x_1, x_2], \ldots, [x_{M-1}, x_M]$ has length *h*. We can do the same for the interval $[-\ell, \ell]$ along the *y*-axis and obtain

$$y_j = -\ell + (j-1)h$$

for j = 1, ..., M. Our goal is to derive an appropriate energy specified for functions defined on the grid points (x_i, y_j) .

To this end, we observe that if $v : [-\ell, \ell]^2 \to \mathbb{R}$ is sufficiently smooth

$$\begin{split} &\int_{-\ell}^{\ell} \int_{-\ell}^{\ell} |Dv|^{p} dx dy \\ &\approx \sum_{i,j=1}^{M-1} |Dv(x_{i}, y_{j})|^{p} h^{2} \\ &= \sum_{i,j=1}^{M-1} \left(v_{x}(x_{i}, y_{j})^{2} + v_{y}(x_{i}, y_{j}) \right)^{p/2} h^{2} \\ &\approx \sum_{i,j=1}^{M-1} \left(\left(\frac{v(x_{i}+h, y_{j}) - v(x_{i}, y_{i})}{h} \right)^{2} + \left(\frac{v(x_{i}, y_{j}+h) - v(x_{i}, y_{i})}{h} \right)^{2} \right)^{p/2} h^{2} \\ &= \sum_{i,j=1}^{M-1} \left(\left(\frac{v(x_{i+1}, y_{j}) - v(x_{i}, y_{i})}{h} \right)^{2} + \left(\frac{v(x_{i}, y_{j+1}) - v(x_{i}, y_{i})}{h} \right)^{2} \right)^{p/2} h^{2} \\ &= h^{2-p} \sum_{i,j=1}^{M-1} \left(\left(v(x_{i+1}, y_{j}) - v(x_{i}, y_{i}) \right)^{2} + \left(v(x_{i}, y_{j+1}) - v(x_{i}, y_{i})^{2} \right)^{p/2} . \end{split}$$

We also suppose h = 1/k for some $k \in \mathbb{N}$ which gives

$$M = 2\ell k + 1$$

$$(x_{\ell k+1}, y_{(\ell+1)k+1}) = (0, 1)$$
 and $(x_{\ell k+1}, y_{(\ell-1)k+1}) = (0, -1).$

As a result, we will attempt to minimize

$$E(v) = \frac{1}{p} k^{p-2} \sum_{i,j=1}^{M-1} \left(\left(v_{i+1,j} - v_{i,j} \right)^2 + \left(v_{i,j+1} - v_{i,j} \right)^2 \right)^{p/2} - \left(v_{\ell k+1,(\ell+1)k+1} - v_{\ell k+1,(\ell-1)k+1} \right)$$
(A.3)

over the $M^2 - 1$ variables

$$v = \begin{pmatrix} v_{1,1} & v_{1,2} & \dots & v_{1,M-1} & v_{1,M} \\ v_{2,1} & v_{2,2} & \dots & v_{2,M-1} & v_{2,M} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{M-1,1} & v_{M-1,2} & \dots & v_{M-1,M-1} & v_{M-1,M} \\ v_{M,1} & v_{M,2} & \dots & v_{M,M-1} \end{pmatrix}$$

A minimizer $v = (v_{ij})$ for *E* would then form an approximation for u_ℓ on the grid points (x_i, y_j)

$$u_{\ell}(x_i, y_i) \approx v_{ii}$$

Appendix A.2: Quasi-Newton method

We used a multidimensional version of the secant method to approximate minimizers of the discrete energy *E* defined above in (A.3). In particular, since *E* is convex we only need to approximate a $v = (v_{ij})$ such that

 $\partial_{v_{ii}} E(v) = 0$

for each $i, j = 1, \dots, M$ with $(i, j) \neq (M, M)$.

First we chose the initial guesses

$$v_{ii}^0 = 0$$

and

$$v_{ij}^1 = g(x_i, y_j)$$

Here

$$g(x, y) = -\frac{1}{4\pi} \log \left[\frac{x^2 + (y-1)^2 + 10^{-2}}{x^2 + (y+1)^2 + 10^{-2}} \right]$$

is approximately equal to

$$g_0(x, y) = -\frac{1}{4\pi} \log \left[\frac{x^2 + (y-1)^2}{x^2 + (y+1)^2} \right],$$

which is a solution of the Dipole equation $-\Delta g_0 = \delta_{(0,1)} - \delta_{(0,-1)}$ in \mathbb{R}^2 .

Then we performed the iteration

$$\begin{cases} v_{ij}^{m+1} = v_{ij}^m - \tau_m \partial_{v_{ij}} E(v^m) \\ \tau_m := \frac{\sum_{ij} (v_{ij}^m - v_{ij}^{m-1}) (\partial_{v_{ij}} E(v^m) - \partial_{v_{ij}} E(v^{m-1}))}{\sum_{ij} (\partial_{v_{ij}} E(v^m) - \partial_{v_{ij}} E(v^{m-1}))^2} \end{cases}$$

for $m = 1, 2, 3, \ldots$ until the stopping criterion

$$\max_{ij} \left| \partial_{v_{ij}} E(v^m) \right| < 10^{-6}$$

was achieved. The iteration was computed for all i, j = 1, ..., M except for $(i, j) \neq (M, M)$.

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