Arch. Rational Mech. Anal.
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-021-01668-x

Extremal Functions for Morrey's Inequality

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Communicated by A. FIGALLI

Abstract

We give a qualitative description of extremals for Morrey's inequality. Our theory is based on exploiting the invariances of this inequality, studying the equation satisfied by extremals, and the observation that extremals are optimal for a related convex minimization problem.

1. Introduction

The Sobolev inequality is arguably the most important functional inequality in the theory of Sobolev spaces. It asserts that for each natural number n and

$$1 < p < n,$$

there is a constant C such that

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{1/p^*} \leqq C \left(\int_{\mathbb{R}^n} |Du|^p dx\right)^{1/p} \tag{1.1}$$

for each continuously differentiable $u : \mathbb{R}^n \to \mathbb{R}$ with compact support. Here

$$p^* = \frac{np}{n-p}.$$

This estimate is ubiquitous in PDE theory and also played a crucial role in the solution of Yamabe's problem in Riemannian geometry [22]. For Yamabe's problem, it was essential to know the *sharp* or smallest constant $C = C^*$ for which (1.1) holds and to know the *extremals* or functions for which equality is attained in (1.1).

Ryan Hynd: Partially supported by NSF Grant DMS-1554130.

Published online: 28 May 2021

Using symmetrization methods, Talenti derived the sharp constant

$$C^* = \frac{1}{\sqrt{\pi}\, n^{1/p}} \left(\frac{p-1}{n-p}\right)^{1-1/p} \left(\frac{\Gamma\left(1+\frac{n}{2}\right)\Gamma(n)}{\Gamma\left(\frac{n}{p}\right)\Gamma\left(1+n-\frac{n}{p}\right)}\right)^{1/n}$$

and found nonnegative extremals of the form

$$u(x) = \left(a + b|x-z|^{\frac{p}{p-1}}\right)^{1-\frac{n}{p}}$$

for $a, b > 0$ and $z \in \mathbb{R}^n$ [33]. Aubin made similar insights around the same time in his influential work on Yamabe's problem [2,3]. It was later verified, in [26] and [6] for $p = 2$ and in [11] for all $1 < p < n$, that extremals for the Sobolev inequality must be of the above form and therefore are radially symmetric.

The counterpart for Sobolev's inequality when

$$n < p < \infty \tag{1.2}$$

is known as Morrey's inequality. This inequality asserts that there is a constant C for which the inequality

$$\sup_{x\neq y}\left\{\frac{|u(x)-u(y)|}{|x-y|^{1-n/p}}\right\} \leqq C\left(\int_{\mathbb{R}^n}|Du|^p dz\right)^{1/p} \tag{1.3}$$

holds for every $u : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable. It is of great interest to deduce the sharp constant $C = C_*$ and to explicitly express the corresponding extremal functions. These problems are easy to solve when $n = 1$ but are unsolved for $n \geqq 2$. In particular, symmetrization methods do not seem to yield insightful information for Morrey's inequality the way they do for other well known functional inequalities as detailed in [7,9,10,28,34].

In this paper, we show extremals for Morrey's inequality exist and establish several qualitative properties of these functions. Specifically, we study their symmetry properties and show that they are unique up to the natural invariances associated with Morrey's inequality. We also verify Morrey extremals are bounded and smooth on $\mathbb{R}^n$ minus two points, and their level sets bound convex regions in $\mathbb{R}^n$. Our approach is based on the analysis of a certain PDE that extremals satisfy. We will informally summarize these properties below and then verify precise versions of these statements in the discussion to follow. Hereafter, the term "extremal" will only apply to Morrey's inequality and we will always assume (1.2).

Maximized Hölder ratio. For each extremal function u, there are distinct points $x_0, y_0 \in \mathbb{R}^n$ that maximize its $1 - n/p$ Hölder ratio. That is,

$$\sup_{x\neq y}\left\{\frac{|u(x)-u(y)|}{|x-y|^{1-n/p}}\right\} = \frac{|u(x_0)-u(y_0)|}{|x_0-y_0|^{1-n/p}}. \tag{1.4}$$

This property also holds more generally for functions $v : \mathbb{R}^n \to \mathbb{R}$ which satisfy

$$\int_{\mathbb{R}^n}|Dv|^p dx < \infty. \tag{1.5}$$

Curiously, a stability estimate for Morrey's inequality follows from this insight. In particular, for any v satisfying (1.5), there is an extremal u such that

$$\left(\frac{C_*}{2}\right)^p \int_{\mathbb{R}^n} |Du - Dv|^p dx + \sup_{x \neq y} \left\{ \frac{|v(x) - v(y)|}{|x - y|^{1-n/p}} \right\}^p \leqq C_*^p \int_{\mathbb{R}^n} |Dv|^p dx$$

when $2 < p < \infty$; here C_* is the sharp constant for Morrey's inequality. There is also an analogous inequality for $1 < p \leqq 2$.

PDE. Suppose u is an extremal which satisfies (1.4). Then the PDE

$$-\Delta_p u = \frac{|u(x_0) - u(y_0)|^{p-2}(u(x_0) - u(y_0))}{C_*^p |x_0 - y_0|^{p-n}} (\delta_{x_0} - \delta_{y_0}) \tag{1.6}$$

holds in $\mathbb{R}^n$. Here Δ_p is the p-Laplace operator

$$\Delta_p u = \operatorname{div}(|Du|^{p-2} Du).$$

Moreover, the converse is true. If v satisfies

$$-\Delta_p v = c(\delta_{x_0} - \delta_{y_0})$$

in $\mathbb{R}^n$ for some $c \in \mathbb{R}$, then v is necessarily an extremal and the Hölder ratio of v is maximized at x_0 and y_0.

Uniqueness. For each distinct pair of points $x_0, y_0 \in \mathbb{R}^n$ and distinct pair of values $\alpha, \beta \in \mathbb{R}$, there is a unique extremal u which satisfies (1.4) and

$$u(x_0) = \alpha \quad \text{and} \quad u(y_0) = \beta.$$

Consequently, the dimension of the space of extremals is $2n + 2$. Furthermore,

$$\int_{\mathbb{R}^n} |Du|^p dx \leqq \int_{\mathbb{R}^n} |Dv|^p dx$$

for every $v : \mathbb{R}^n \to \mathbb{R}$ which satisfies the pointwise constraints

$$v(x_0) = \alpha \quad \text{and} \quad v(y_0) = \beta.$$

This observation allows us to numerically approximate extremals with coordinate gradient descent (as discussed in Appendix B); see Fig. 1.

Cylindrical symmetry. Assume u is an extremal which satisfies (1.4). Then

$$u(x) = u(O(x - x_0) + x_0)$$

for any orthogonal transformation O of $\mathbb{R}^n$ which satisfies

$$O(y_0 - x_0) = y_0 - x_0.$$

That is, u is invariant under rigid transformations of $\mathbb{R}^n$ that fix the line passing through x_0 and y_0. In particular, extremals are cylindrical about this line and are not radially symmetric.

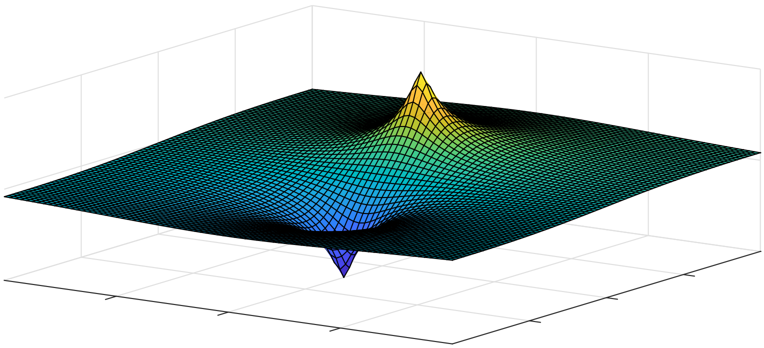

Fig. 1. The graph of an extremal for Morrey's inequality in two spatial dimensions

Reflectional antisymmetry. If u is an extremal whose Hölder ratio is maximized at distinct $x_0, y_0 \in \mathbb{R}^n$, then

$$u\left(x - 2\frac{\left((x_0 - y_0)\cdot(x - \frac{1}{2}(x_0 + y_0)\right)}{|x_0 - y_0|^2}(x_0 - y_0)\right) - \frac{u(x_0) + u(y_0)}{2}$$
$$= -\left(u(x) - \frac{u(x_0) + u(y_0)}{2}\right).$$

In other words, the function

$$u - \frac{1}{2}(u(x_0) + u(y_0))$$

is antisymmetric with respect to reflection about the hyperplane with normal $x_0 - y_0$ that contains the point $\frac{1}{2}(x_0 + y_0)$.

Boundedness. Suppose that u is an extremal which satisfies (1.4). Then u is bounded with

$$\inf_{x\in\mathbb{R}^n} u(x) = \min\{u(x_0), u(y_0)\}$$

and

$$\sup_{x\in\mathbb{R}^n} u(x) = \max\{u(x_0), u(y_0)\}.$$

In particular, u assumes its minimum and maximum values at points which maximize its Hölder seminorm. Furthermore, if $n \geqq 2$ and u is nonconstant, u attains its maximum and minimum values uniquely at x_0 and y_0 (Fig. 2).

Quasiconcavity. Suppose that u is an extremal whose Hölder ratio is maximized at distinct $x_0, y_0 \in \mathbb{R}^n$. Also consider the two half-spaces

$$\Pi_\pm = \left\{x \in \mathbb{R}^n : \pm\left(x - \frac{1}{2}(x_0 + y_0)\right)\cdot(x_0 - y_0) > 0\right\}$$

separated by the hyperplane with normal $x_0 - y_0$ that contains $\frac{1}{2}(x_0 + y_0)$. If $u(x_0) > u(y_0)$, then

$$u|_{\Pi_+} \text{ is quasiconcave and } u|_{\Pi_-} \text{ is quasiconvex;}$$

alternatively if $u(x_0) < u(y_0)$, then

$$u|_{\Pi_+} \text{ is quasiconvex and } u|_{\Pi_-} \text{ is quasiconcave.}$$

In either case, each level set of u is the boundary of a convex subset of $\mathbb{R}^n$. Refer to Fig. 3.

Nonvanishing gradient. Suppose $n \geqq 2$. If u is a nonconstant extremal whose Hölder ratio is maximized at distinct $x_0, y_0 \in \mathbb{R}^n$, then

$$|Du| > 0 \quad \text{in} \quad \mathbb{R}^n \backslash \{x_0, y_0\}.$$

This positivity combined with equation (1.6) will imply that u has continuous derivatives of all orders and is in fact real analytic on $\mathbb{R}^n \backslash \{x_0, y_0\}$. It also implies there are no points where u has a local minimum or maximum except for x_0 and y_0. In addition, this property will allow us to show that C_* cannot be found from the local version of Morrey's inequality which can be used to establish (1.3) for some C[13,14,18,31].

We will begin by indicating the natural invariances associated with Morrey's inequality and by exploiting these invariances to establish the existence of extremal functions. Then we will proceed to verify each of the above assertions roughly in the order that they are presented with the exception of property (1.4); we will save this technical assertion for the last section of this paper. Finally, we would like to thank Eric Carlen, Yat Tin Chow, Elliott Lieb, Bob Kohn, Erik Lindgren, Stan Osher, and Tak Kwong Wong for their advice and insightful discussions related to this work.

2. Preliminaries

In what follows, it will be convenient for us to adopt the notation

$$[u]_{C^{1-n/p}(\mathbb{R}^n)} := \sup_{x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^{1-n/p}} \right\}$$

and

$$\|Du\|_{L^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |Du|^p dx \right)^{1/p}.$$

This allows us to restate Morrey's inequality (1.3) more concisely as

$$[u]_{C^{1-n/p}(\mathbb{R}^n)} \leqq C \|Du\|_{L^p(\mathbb{R}^n)}.$$

For the remainder of this paper, we will also reserve C for any constant such that Morrey's inequality holds and C_* for the sharp constant.

A natural class of functions for us to consider is the homogeneous Sobolev space

$$\mathcal{D}^{1,p}(\mathbb{R}^n) := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n) : u_{x_i} \in L^p(\mathbb{R}^n) \text{ for } i = 1, \dots, n \right\}.$$

In particular, $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ means that u is weakly differentiable and all of its weak first order partial derivatives are $L^p(\mathbb{R}^n)$ functions. What's more is that we can interpret Morrey's inequality to hold on $\mathcal{D}^{1,p}(\mathbb{R}^n)$ once we recall that each $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ has a Hölder continuous representative $u^* : \mathbb{R}^n \to \mathbb{R}$. Namely, $u(x) = u^*(x)$ for Lebesgue almost every $x \in \mathbb{R}^n$ and

$$[u^*]_{C^{1-n/p}(\mathbb{R}^n)} \leqq C\|Du\|_{L^p(\mathbb{R}^n)}.$$

As such an argument is now routine within the theory of Sobolev spaces, we omit the details. Moreover, we will always identify u with u^* going forward.

2.1. Existence of a Nonconstant Extremal

Note that the seminorms $u \mapsto [u]_{C^{1-n/p}(\mathbb{R}^n)}$ and $u \mapsto \|Du\|_{L^p(\mathbb{R}^n)}$ are invariant under the following transformations:

- Multiplication by -1: $u(x) \mapsto -u(x)$
- Addition by a constant $c \in \mathbb{R}$: $u(x) \mapsto u(x) + c$
- Translation by a vector $a \in \mathbb{R}^n$: $u(x) \mapsto u(x + a)$
- Application of an orthogonal transformation O of $\mathbb{R}^n$: $u(x) \mapsto u(Ox)$
- Scaling by $\lambda > 0$: $u(x) \mapsto \lambda^{n/p-1}u(\lambda x)$

We will exploit these invariances to verify that extremals for Morrey's inequality exist.

Lemma 2.1. *There exists a nonconstant $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ such that*

$$[u]_{C^{1-n/p}(\mathbb{R}^n)} = C_*\|Du\|_{L^p(\mathbb{R}^n)}.$$

Proof. Let us define

$$\Lambda := \inf \left\{ \|Du\|_{L^p(\mathbb{R}^n)} \ : u \in \mathcal{D}^{1,p}(\mathbb{R}^n),\ [u]_{C^{1-n/p}(\mathbb{R}^n)} = 1 \right\}$$

and choose a minimizing sequence $(u_k)_{k \in \mathbb{N}}$ for which

$$\Lambda = \lim_{k \to \infty} \|Du_k\|_{L^p(\mathbb{R}^n)}.$$

For each $k \in \mathbb{N}$, we can select $x_k, y_k \in \mathbb{R}^n$ with $x_k \neq y_k$ such that

$$1 = [u_k]_{C^{1-n/p}(\mathbb{R}^n)} < \frac{u_k(y_k) - u_k(x_k)}{|x_k - y_k|^{1-n/p}} + \frac{1}{k}.$$

We may also find an orthogonal transformation O_k satisfying

$$O_k e_n = \frac{y_k - x_k}{|x_k - y_k|},$$

where $e_n = (0, \ldots, 0, 1)$.

In addition, we define

$$v_k(z) := |x_k - y_k|^{n/p-1}\Big\{u_k\left(|x_k - y_k|O_k z + x_k\right) - u_k(x_k)\Big\}$$

for $z \in \mathbb{R}^n$ and $k \in \mathbb{N}$. By the invariances of the seminorms $u \mapsto [u]_{C^{1-n/p}(\mathbb{R}^n)}$ and $u \mapsto \|Du\|_{L^p(\mathbb{R}^n)}$,

$$[v_k]_{C^{1-n/p}(\mathbb{R}^n)} = 1 \quad \text{and} \quad \Lambda = \lim_{k\in\mathbb{N}} \|Dv_k\|_{L^p(\mathbb{R}^n)}.$$

Moreover,

$$v_k(0) = 0 \quad \text{and} \quad 1 - \frac{1}{k} < v_k(e_n) \leqq 1.$$

We can now employ a standard variant of the Arzelà-Ascoli theorem to obtain a subsequence $(v_{k_j})_{j\in\mathbb{N}}$ converging locally uniformly to a continuous function $v : \mathbb{R}^n \to \mathbb{R}$.

Local uniform convergence immediately implies

$$v(0) = 0, \quad v(e_n) = 1, \quad \text{and} \quad [v]_{C^{1-n/p}(\mathbb{R}^n)} \leqq 1.$$

As

$$1 = \frac{v(e_n) - v(0)}{|e_n - 0|^{1-n/p}} \leqq [v]_{C^{1-n/p}(\mathbb{R}^n)},$$

we actually have

$$[v]_{C^{1-n/p}(\mathbb{R}^n)} = 1.$$

Since $\|Dv_k\|_{L^p(\mathbb{R}^n)}$ is bounded, we may as well suppose that $(Dv_{k_j})_{j\in\mathbb{N}}$ converges weakly in $L^p(\mathbb{R}^n; \mathbb{R}^n)$. Using the local uniform convergence of v_{k_j}, it is easy to verify that the weak limit of Dv_{k_j} in $L^p(\mathbb{R}^n; \mathbb{R}^n)$ is the weak derivative of v. It then follows that $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ and

$$\Lambda = \liminf_{j\in\mathbb{N}} \|Dv_{k_j}\|_{L^p(\mathbb{R}^n)} \geqq \|Dv\|_{L^p(\mathbb{R}^n)}.$$

Since v is nonconstant, Λ is positive. By the definition of Λ, $C = 1/\Lambda$ is a constant for which Morrey's inequality holds. As

$$1 = [v]_{C^{1-n/p}(\mathbb{R}^n)} \leqq \frac{1}{\Lambda}\|Dv\|_{L^p(\mathbb{R}^n)} \leqq 1,$$

$1/\Lambda = C_*$ and v is a nonconstant extremal of Morrey's inequality. □

Corollary 2.2. *Assume that $x_0, y_0 \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$ are distinct. There is an extremal u which satisfies $u(x_0) = \alpha$ and $u(y_0) = \beta$ and whose $1 - n/p$ Hölder ratio is maximized at x_0 and y_0.*

Proof. In the proof of the above lemma, we showed there is an extremal v which satisfies $v(0)=0$ and $v(e_n)=1$ and whose $1-n/p$ Hölder ratio is maximized at 0 and e_n. We now select an orthogonal transformation O of $\mathbb{R}^n$ for which

$$O\left(\frac{y_0-x_0}{|y_0-x_0|}\right)=e_n$$

and set

$$u(x)=(\beta-\alpha)v\left(\frac{O\,(x-x_0)}{|y_0-x_0|}\right)+\alpha$$

for $x\in\mathbb{R}^n$. Clearly $u(x_0)=\alpha$ and $u(y_0)=\beta$, and by the scaling invariance of the Hölder seminorm

$$[u]_{C^{1-n/p}(\mathbb{R}^n)}=\frac{|\beta-\alpha|}{|x_0-y_0|^{1-n/p}}=\frac{|u(x_0)-u(y_0)|}{|x_0-y_0|^{1-n/p}}.$$

□

2.2. *PDE for Extremals*

We have established the existence of an extremal whose $1-n/p$ Hölder ratio attains its maximum at a pair of distinct points. We will later show that every function in $\mathcal{D}^{1,p}(\mathbb{R}^n)$ has this property. For now, we will assume this property and use it to derive a PDE satisfied by extremal functions.

Proposition 2.3. *Suppose $u\in\mathcal{D}^{1,p}(\mathbb{R}^n)$ is an extremal whose $1-n/p$ Hölder ratio attains its maximum at two distinct points $x_0,y_0\in\mathbb{R}^n$. Then, for each $\phi\in\mathcal{D}^{1,p}(\mathbb{R}^n)$*

$$C_*^p\int_{\mathbb{R}^n}|Du|^{p-2}Du\cdot D\phi dx=\frac{|u(x_0)-u(y_0)|^{p-2}(u(x_0)-u(y_0))}{|x_0-y_0|^{p-n}}(\phi(x_0)-\phi(y_0)). \tag{2.1}$$

Proof. By assumption,

$$\frac{|u(x_0)-u(y_0)|^p}{|x_0-y_0|^{p-n}}=C_*^p\int_{\mathbb{R}^n}|Du|^p dx; \tag{2.2}$$

and in view of Morrey's inequality,

$$\frac{|u(x_0)-u(y_0)+t(\phi(x_0)-\phi(y_0))|^p}{|x_0-y_0|^{p-n}}\leqq C_*^p\int_{\mathbb{R}^n}|Du+tD\phi|^p dx \tag{2.3}$$

for each $\phi\in\mathcal{D}^{1,p}(\mathbb{R}^n)$ and $t>0$. Moreover, the convexity of the function $\mathbb{R}\ni w\mapsto|w|^p$ can be used to derive

$$\begin{aligned}&\frac{|u(x_0)-u(y_0)+t(\phi(x_0)-\phi(y_0))|^p}{|x_0-y_0|^{p-n}}\\ &\geqq\frac{|u(x_0)-u(y_0)|^p}{|x_0-y_0|^{p-n}}\end{aligned}$$

$$+ tp\frac{|u(x_0)-u(y_0)|^{p-2}(u(x_0)-u(y_0))}{|x_0-y_0|^{p-n}}(\phi(x_0)-\phi(y_0)). \tag{2.4}$$

As a result, we can subtract (2.2) from (2.3) and use (2.4) to obtain

$$C_*^p\int_{\mathbb{R}^n}\left(\frac{|Du+tD\phi|^p-|Du|^p}{pt}\right)dx \\ \geqq \frac{|u(x_0)-u(y_0)|^{p-2}(u(x_0)-u(y_0))}{|x_0-y_0|^{p-n}}(\phi(x_0)-\phi(y_0)). \tag{2.5}$$

It is also possible to find a constant c_p such that

$$0 \leqq \frac{|Du+tD\phi|^p-|Du|^p}{pt} - |Du|^{p-2}Du\cdot D\phi \\ \leqq c_p\begin{cases} t^{p-1}|D\phi|^p, & 1<p<2 \\ t|D\phi|^2(|Du|+|D\phi|)^{p-2}, & 2\leqq p<\infty \end{cases}$$

for $t \in (0,1]$; see for example Lemma 10.2.1 in [1]. With this estimate, we can pass to the limit as $t \to 0^+$ in (2.5) to conclude that

$$C_*^p\int_{\mathbb{R}^n}|Du|^{p-2}Du\cdot D\phi dx \geqq \frac{|u(x_0)-u(y_0)|^{p-2}(u(x_0)-u(y_0))}{|x_0-y_0|^{p-n}}(\phi(x_0)-\phi(y_0)). \tag{2.6}$$

We arrive at (2.1) once we replace ϕ with $-\phi$ in (2.6). □

If $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ satisfies (2.1) for each $\phi \in \mathcal{D}^{1,p}(\mathbb{R}^n)$, we can choose $\phi = u$ to find that u is a necessarily an extremal function for Morrey's inequality and the $1-n/p$ Hölder seminorm of u is attained at (x_0, y_0). Also note that (2.1) implies that u is a weak solution of the PDE (1.6)

$$-\Delta_p u = \frac{|u(x_0)-u(y_0)|^{p-2}(u(x_0)-u(y_0))}{C_*^p|x_0-y_0|^{p-n}}(\delta_{x_0}-\delta_{y_0})$$

in $\mathbb{R}^n$. In particular, u is p-harmonic

$$-\Delta_p u = 0$$

in $\mathbb{R}^n\backslash\{x_0, y_0\}$. As a result, Du is a locally Hölder continuous mapping from $\mathbb{R}^n\backslash\{x_0, y_0\}$ into $\mathbb{R}^n$ [12,25,35].

We will use the weak form of (1.6) to deduce various properties of extremals below. Our first observation is that there are at least two other ways to identify that a function $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is an extremal. The first is as a solution of a PDE of the type (1.6); the second is as a minimizer of a related convex optimization problem.

Theorem 2.4. *Suppose $x_0, y_0 \in \mathbb{R}^n$ are distinct and $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$. Then the following are equivalent:*

(i) u is an extremal with

$$[u]_{C^{1-n/p}(\mathbb{R}^n)} = \frac{|u(x_0)-u(y_0)|}{|x_0-y_0|^{1-n/p}}.$$

(ii) There is a constant $c \in \mathbb{R}$ for which u satisfies

$$-\Delta_p u = c(\delta_{x_0} - \delta_{y_0}) \tag{2.7}$$

in $\mathbb{R}^n$.

(iii) For each $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ with $v(x_0) = u(x_0)$ and $v(y_0) = u(y_0)$,

$$\int_{\mathbb{R}^n} |Du|^p dx \leqq \int_{\mathbb{R}^n} |Dv|^p dx. \tag{2.8}$$

Remark 2.5. In (ii), (2.7) means that

$$\int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot D\phi dx = c(\phi(x_0) - \phi(y_0)) \tag{2.9}$$

for each $\phi \in \mathcal{D}^{1,p}(\mathbb{R}^n)$.

Proof. We have already shown that (i) implies (ii) in Proposition 2.3. Let us assume (ii), and suppose $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ satisfies $v(x_0) = u(x_0)$ and $v(y_0) = u(y_0)$. Applying (2.9), we find

$$\begin{aligned}\int_{\mathbb{R}^n} |Dv|^p dx &\geqq \int_{\mathbb{R}^n} |Du|^p dx + p\int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot (Dv - Du) dx \\ &= \int_{\mathbb{R}^n} |Du|^p dx + pc((v-u)(x_0) - (v-u)(y_0)) \\ &= \int_{\mathbb{R}^n} |Du|^p dx.\end{aligned}$$

Therefore, (ii) implies (iii).

Now suppose that u satisfies (iii) and w is an extremal for which

$$[w]_{C^{1-n/p}(\mathbb{R}^n)} = \frac{|w(x_0) - w(y_0)|}{|x_0 - y_0|^{1-n/p}}$$

and

$$w(x_0) = u(x_0) \quad \text{and} \quad w(y_0) = u(y_0). \tag{2.10}$$

Choosing $v = u + t(w - u)$ in (2.8) gives

$$\begin{aligned}\int_{\mathbb{R}^n} |Du|^p dx &\leqq \int_{\mathbb{R}^n} |Du + tD(w-u)|^p dx \\ &= \int_{\mathbb{R}^n} |Du|^p dx + pt\int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot D(w-u) dx + o(t)\end{aligned}$$

as $t \to 0$. It follows that

$$\int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot D(w-u) dx = 0.$$

By Proposition 2.3, we also have

$$\int_{\mathbb{R}^n} |Dw|^{p-2} Dw \cdot D(w-u) dx = 0.$$

In particular,

$$\int_{\mathbb{R}^n} (|Du|^{p-2}Du - |Dw|^{p-2}Dw) \cdot D(u-w)dx = 0.$$

As the mapping $z \mapsto |z|^{p-2}z$ of $\mathbb{R}^n$ is strictly monotone, $Du \equiv Dw$. In view of (2.10), $u \equiv w$ and we conclude that (iii) implies (i). $\square$

2.3. Extremals in One Spatial Dimension

Suppose $n = 1$ and $u \in \mathcal{D}^{1,p}(\mathbb{R})$. As $u' \in L^p(\mathbb{R})$, u is absolutely continuous and

$$|u(x) - u(y)| = \left|\int_y^x u'dz\right| \leqq \left(\int_y^x |u'|^p dz\right)^{1/p} |x-y|^{1-1/p} \tag{2.11}$$

for $y \leqq x$. As a result

$$[u]_{C^{1-1/p}(\mathbb{R})} \leqq \|u'\|_{L^p(\mathbb{R})}, \tag{2.12}$$

and Morrey's inequality holds with $C_* \leqq 1$. Moreover, it is easy to check that equality holds for

$$u(x) = \begin{cases} -1, & x \in (-\infty, -1) \\ x, & x \in [-1, 1] \\ 1, & x \in (1, \infty). \end{cases} \tag{2.13}$$

Therefore, $C_* = 1$. We also note that the $1 - 1/p$ Hölder ratio of u is maximized at ± 1.

Conversely, if equality holds in (2.12) for some $u \in \mathcal{D}^{1,p}(\mathbb{R})$ and if

$$[u]_{C^{1-1/p}(\mathbb{R})} = \frac{|u(x_0) - u(y_0)|}{|x_0 - y_0|^{1-1/p}} \tag{2.14}$$

for some $y_0 < x_0$, then (2.11) gives

$$\left(\int_{y_0}^{x_0} |u'|^p dz\right)^{1/p} = \left(\int_{\mathbb{R}} |u'|^p dz\right)^{1/p}.$$

It would then follow that u' vanishes on $\mathbb{R}\setminus(y_0, x_0)$ and that u' is necessarily constant on $[y_0, x_0]$. As a result, u would be of the form (2.13). In the last section of this paper, we will verify (2.14) and use a higher dimensional analog of (2.11) to verify that the Hölder ratio of each extremal is maximized at distinct points for $n \geqq 2$.

3. Symmetry and Pointwise Bounds

We will now use PDE (1.6) to verify a variety of assertions for extremals. We start off by showing that extremals are unique up to the natural invariances associated with Morrey's inequality. This uniqueness is employed to derive some symmetry and antisymmetry properties. Then we will use the strong maximum principle and Hopf's boundary point lemma for p-harmonic functions to bound extremals above and below and to obtain a sign of a directional derivative along an antisymmetry plane for extremals.

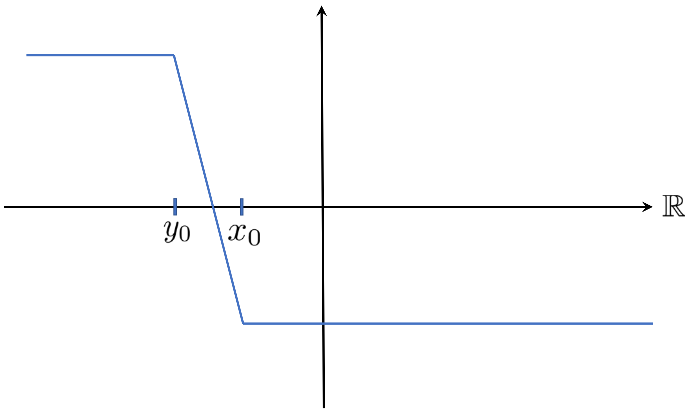

Fig. 2. The graph of an extremal in one spatial dimension whose $1 - 1/p$ Hölder ratio is maximized at x_0 and y_0

3.1. Uniqueness

In Corollary 2.2, we showed that for distinct $\alpha, \beta \in \mathbb{R}$ and $x_0, y_0 \in \mathbb{R}^n$, there is an extremal $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ which satisfies $u(x_0) = \alpha$ and $u(x_0) = \beta$ and whose Hölder ratio is maximized at x_0 and y_0. It turns out that this extremal is the only one that satisfies these constraints. The key observation needed to justify this statement is as follows:

Proposition 3.1. *Suppose $x_0, y_0 \in \mathbb{R}^n$ and $x_1, y_1 \in \mathbb{R}^n$ are distinct, and assume $u, v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ are nonconstant extremals with*

$$[u]_{C^{1-n/p}(\mathbb{R}^n)} = \frac{u(x_0) - u(y_0)}{|x_0 - y_0|^{1-n/p}} \quad \textit{and} \quad [v]_{C^{1-n/p}(\mathbb{R}^n)} = \frac{v(x_1) - v(y_1)}{|x_1 - y_1|^{1-n/p}}.$$

Then for each orthogonal transformation O of $\mathbb{R}^n$ which satisfies

$$O\left(\frac{y_0 - x_0}{|y_0 - x_0|}\right) = \frac{y_1 - x_1}{|y_1 - x_1|}$$

and each $x \in \mathbb{R}^n$,

$$u(x) = \frac{u(y_0) - u(x_0)}{v(y_1) - v(x_1)} \cdot \left\{ v\left(\frac{|x_1 - y_1|}{|x_0 - y_0|} O(x - x_0) + x_1\right) - v(x_1)\right\} + u(x_0).$$

Proof. Set

$$\tilde{u}(x) := \frac{u(y_0) - u(x_0)}{v(y_1) - v(x_1)} \cdot \left\{ v\left(\frac{|x_1 - y_1|}{|x_0 - y_0|} O(x - x_0) + x_1\right) - v(x_1)\right\} + u(x_0)$$

for $x \in \mathbb{R}^n$. By design,

$$\tilde{u}(x_0) = u(x_0) \quad \text{and} \quad \tilde{u}(y_0) = u(y_0), \tag{3.1}$$

and in view of the invariances of the seminorms associated with Morrey's inequality, $\tilde{u}$ is a nonconstant extremal with

$$[\tilde{u}]_{C^{1-n/p}(\mathbb{R}^n)} = \frac{\tilde{u}(x_0) - \tilde{u}(y_0)}{|x_0 - y_0|^{1-n/p}}.$$

Consequently, both u and $\tilde{u}$ are both weak solutions of the PDE (1.6).

Evaluating (2.1) at $\phi = u - \tilde{u}$ gives

$$\int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot (Du - D\tilde{u}) dx = 0,$$

and similarly,

$$\int_{\mathbb{R}^n} |D\tilde{u}|^{p-2} D\tilde{u} \cdot (Du - D\tilde{u}) dx = 0.$$

Subtracting these equalities gives

$$\int_{\mathbb{R}^n} (|Du|^{p-2} Du - |D\tilde{u}|^{p-2} D\tilde{u}) \cdot (Du - D\tilde{u}) dx = 0.$$

Therefore, $u - \tilde{u}$ is constant and by (3.1) must in fact vanish identically. □

Corollary 3.2. *Suppose $u_1, u_2 \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ are nonconstant extremals whose $1 - n/p$ Hölder seminorms are both maximized at a pair of distinct points $x_0, y_0 \in \mathbb{R}^n$. If, in addition,*

$$u_1(x_0) = u_2(x_0) \quad \textit{and} \quad u_1(y_0) = u_2(y_0),$$

then $u_1(x) = u_2(x)$ for all $x \in \mathbb{R}^n$.

Proof. We can assume that $u_1(x_0) > u_1(y_0)$, or else we can prove this corollary for $-u_1$ and $-u_2$. The assertion is then immediate once we realize we can select $O = I_n$ in the statement of Proposition 3.1 with $u = u_1$ and $v = u_2$. □

A closer inspection of the previous assertion implies that any extremal u whose $1 - n/p$ Hölder ratio achieves its maximum at a pair of distinct points possesses a strong symmetry property. Specifically, u will be cylindrically symmetric and its axis of symmetry is the line through the points at which its $1 - n/p$ Hölder seminorm is achieved.

Corollary 3.3. *Suppose $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is an extremal with*

$$[u]_{C^{1-n/p}(\mathbb{R}^n)} = \frac{|u(x_0) - u(y_0)|}{|x_0 - y_0|^{1-n/p}}.$$

Then

$$u(x) = u(O(x - x_0) + x_0), \quad x \in \mathbb{R}^n$$

for any orthogonal transformation O which satisfies

$$O\,(y_0 - x_0) = y_0 - x_0.$$

Proof. Set

$$v(x) := u(O(x - x_0) + x_0), \quad x \in \mathbb{R}^n.$$

The claim follows once we observe that v is an extremal with $[v]_{C^{1-n/p}(\mathbb{R}^n)} = [u]_{C^{1-n/p}(\mathbb{R}^n)}$, $v(x_0) = u(x_0)$, and $v(y_0) = u(y_0)$. □

Extremals also have a certain antisymmetry property. We recall that the orthogonal reflection about the hyperplane $\{x \in \mathbb{R}^n : x \cdot a = c\}$ is given by the transformation

$$x \mapsto x - 2\frac{(x \cdot a - c)}{|a|^2}a.$$

Note in particular that this mapping is a composition of an orthogonal transformation and a translation. We will show below that $u - \frac{1}{2}(u(x_0) + u(y_0))$ is antisymmetric about the hyperplane with normal pointing in the same direction as $x_0 - y_0$ and that passes through the midpoint of x_0 and y_0.

Proposition 3.4. *Suppose $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is an extremal with*

$$[u]_{C^{1-n/p}(\mathbb{R}^n)} = \frac{|u(x_0) - u(y_0)|}{|x_0 - y_0|^{1-n/p}}.$$

Then

$$u\left(x - 2\frac{\left((x_0 - y_0) \cdot (x - \frac{1}{2}(x_0 + y_0)\right)}{|x_0 - y_0|^2}(x_0 - y_0)\right) - \frac{u(x_0) + u(y_0)}{2}$$
$$= -\left(u(x) - \frac{u(x_0) + u(y_0)}{2}\right)$$

for each $x \in \mathbb{R}^n$.

Proof. Define

$$v(x) := u(x_0) + u(y_0) - u\left(x - 2\frac{\left((x_0 - y_0) \cdot (x - \frac{1}{2}(x_0 + y_0)\right)}{|x_0 - y_0|^2}(x_0 - y_0)\right).$$

We note that

$$x \mapsto x - 2\frac{\left((x_0 - y_0) \cdot (x - \frac{1}{2}(x_0 + y_0)\right)}{|x_0 - y_0|^2}(x_0 - y_0)$$

is orthogonal reflection about the hyperplane with normal $x_0 - y_0$ which passes through $(x_0 + y_0)/2$. Since this map is a composition of an orthogonal transformation and a translation, v is an extremal with $[u]_{C^{1-n/p}(\mathbb{R}^n)} = [v]_{C^{1-n/p}(\mathbb{R}^n)}$. As

$$v(x_0) = u(x_0) \quad \text{and} \quad v(y_0) = u(y_0),$$

$v(x) = u(x)$ for all $x \in \mathbb{R}^n$ by Corollary 3.2. □

3.2. Pointwise Bounds

As mentioned above, we will argue that each extremal is bounded. In particular, if $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is an extremal with

$$[u]_{C^{1-n/p}(\mathbb{R}^n)} = \frac{|u(x_0) - u(y_0)|}{|x_0 - y_0|^{1-n/p}},$$

we will show that

$$\min\{u(x_0), u(y_0)\} \leqq u(x) \leqq \max\{u(x_0), u(y_0)\} \tag{3.2}$$

for all $x \in \mathbb{R}^n$. When $n = 1$, these inequalities are immediate in view of the explicit extremal (2.13). Consequently, we will focus on the case $n \geqq 2$ and also prove a refinement of (3.2).

Proposition 3.5. *Suppose $n \geqq 2$, $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is an extremal with*

$$[u]_{C^{1-n/p}(\mathbb{R}^n)} = \frac{u(x_0) - u(y_0)}{|x_0 - y_0|^{1-n/p}} > 0,$$

and set

$$\Pi_\pm := \left\{x \in \mathbb{R}^n : \pm\left(x - \frac{1}{2}(x_0 + y_0)\right) \cdot (x_0 - y_0) > 0\right\}.$$

Then

$$\frac{u(x_0) + u(y_0)}{2} < u(x) < u(x_0) \tag{3.3}$$

for $x \in \Pi_+$ and

$$u(y_0) < u(x) < \frac{u(x_0) + u(y_0)}{2} \tag{3.4}$$

for $x \in \Pi_-$.

Proof. We will only prove (3.3) as a similar argument can be used to justify (3.4). Moreover, we will suppose $x_0 = e_n$, $y_0 = -e_n$, $u(e_n) = 1$, and $u(-e_n) = -1$. In this case, $\Pi_+ = \{x \in \mathbb{R}^n : x_n > 0\}$. The general case would then follow by an appropriate change of variables.

Observe that Proposition 3.4 implies that

$$u|_{\partial\Pi_+} = 0.$$

We claim that

$$\int_{\Pi_+} |Du|^p dx \leqq \int_{\Pi_+} |Dw|^p dx \tag{3.5}$$

for each $w \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ which satisfies

$$w(e_n) = 1 \quad \text{and} \quad w|_{\partial\Pi_+} = 0 \tag{3.6}$$

and that equality holds in (3.5) if and only if $w|_{\Pi_+} = u|_{\Pi_+}$.

For a given $w \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ satisfying (3.6), we can set

$$v(x) = \begin{cases} w(x), & x \in \Pi_+ \\ -w\left(x - 2x_n e_n\right), & x \notin \Pi_+ \end{cases}$$

and check that $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$. As

$$v(e_n) = w(e_n) = 1 \quad \text{and} \quad v(-e_n) = -w(e_n) = -1,$$

Theorem 2.4 gives

$$2\int_{\Pi_+} |Du|^p dx = \int_{\mathbb{R}^n} |Du|^p dx \leqq \int_{\mathbb{R}^n} |Dv|^p dx = 2\int_{\Pi_+} |Dw|^p dx.$$

This proves (3.5). If equality holds, then v is an extremal (by Theorem 2.4) which is necessarily equal to u by Corollary 3.2.

Let us now choose

$$w(x) = \min\{u(x), 1\}, \quad x \in \mathbb{R}^d.$$

As w fulfills the constraints (3.6), (3.5) gives

$$\int_{\Pi_+} |Du|^p dx \leqq \int_{\Pi_+} |Dw|^p dx = \int_{\Pi_+ \cap \{u \leqq 1\}} |Du|^p dx \leqq \int_{\Pi_+} |Du|^p dx.$$

Consequently,

$$u(x) = w(x) = \min\{u(x), 1\} \leqq 1$$

for each $x \in \Pi_+$.

Since u is p-harmonic in the domain $\Pi_+ \backslash \{e_n\}$, either $u < 1$ in or $u \equiv 1$ throughout $\Pi_+ \backslash \{e_n\}$ by the strong maximum principle (Chapter 2 of [27]). Since $u|_{\partial \Pi_+} = 0$, it must be that $u < 1$ in $\Pi_+ \backslash \{e_n\}$. Analogously, we can employ $w(x) = \max\{u(x), 0\}$ for $x \in \Pi_+$ to deduce $u > 0$ in Π_+. We leave the details to the reader. □

We finally assert that a certain directional derivative of extremals always has a sign. This observation will be useful when we discuss the analyticity of extremal functions.

Proposition 3.6. *Assume $n \geq 2$, $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is a nonconstant extremal with*

$$[u]_{C^{1-n/p}(\mathbb{R}^n)} = \frac{u(x_0) - u(y_0)}{|x_0 - y_0|^{1-n/p}},$$

and set

$$\Pi_+ := \left\{x \in \mathbb{R}^n : \left(x - \frac{1}{2}(x_0 + y_0)\right) \cdot (x_0 - y_0) > 0\right\}.$$

Then

$$Du(x) \cdot (x_0 - y_0) > 0 \tag{3.7}$$

for $x \in \partial \Pi_+$.

Proof. Without loss of generality, we can focus on the case where $x_0 = e_n$, $y_0 = -e_n$, $u(e_n) = 1$, and $u(-e_n) = -1$. In this case, $\Pi_+ = \{x \in \mathbb{R}^n : x_n > 0\}$. By the previous proposition, $u > 0$ in Π_+ and $u|_{\partial\Pi_+} = 0$. Observe that for each $x \in \partial\Pi_+$, u is p-harmonic in the ball

$$D_x := B_{\frac{1}{2}}\left(x + \frac{1}{2}e_n\right)$$

and

$$u(x) = 0 < u(y), \quad y \in \overline{D_x}\backslash\{x\}.$$

The latter observation follows from the fact that $D_x \subset \Pi_+$ and $\overline{D_x} \cap \partial\Pi_+ = \{x\}$. By Hopf's boundary point lemma for p-harmonic functions (Lemma A.3 of [32]),

$$Du(x) \cdot (-e_n) < 0.$$

□

4. Regularity

In this section, we will consider the smoothness properties of extremals. In particular, we will establish that nonconstant extremals are smooth except for at the two points which maximize their Hölder seminorms. We also verify that their level sets bound convex regions in $\mathbb{R}^n$.

4.1. Nondifferentiability Points

It will be useful for us to recall some properties of p-harmonic functions in punctured domains. In particular, the behavior of these functions near their singular points has been deduced by Kichenassamy and Véron [20,21]. We will adapt their results to our setting as follows:

Lemma 4.1. *Suppose $n \geqq 2$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain with $x_0 \in \Omega$. Further suppose that $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is p-harmonic in $\Omega\backslash\{x_0\}$ with $u(x) \leqq u(x_0)$ for $x \in \Omega$. Then*

$$\lim_{x \to x_0} \frac{u(x_0) - u(x)}{|x - x_0|^{\frac{p-n}{p-1}}} = \gamma$$

for some $\gamma \geqq 0$, and

$$-\Delta_p u = n\omega_n \left(\frac{p-n}{p-1}\gamma\right)^{p-1} \delta_{x_0}$$

in Ω.

Remark 4.2. Here ω_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

Proof. Choose $r > 0$ so small that $B_r(x_0) \subset \Omega$. Note that $u(x_0) - u$ is p-harmonic in the punctured ball $B_r(x_0)\backslash\{x_0\}$. Moreover,

$$\begin{aligned}
u(x_0) - u(x) &\leqq [u]_{C^{1-n/p}(\mathbb{R}^n)} r^{1-\frac{n}{p}} \\
&= [u]_{C^{1-n/p}(\mathbb{R}^n)} r^{1-\frac{n}{p}-\left(\frac{p-n}{p-1}\right)} r^{\frac{p-n}{p-1}} \\
&= [u]_{C^{1-n/p}(\mathbb{R}^n)} r^{1-\frac{n}{p}-\left(\frac{p-n}{p-1}\right)} |x - x_0|^{\frac{p-n}{p-1}} \\
&=: L_r |x - x_0|^{\frac{p-n}{p-1}}
\end{aligned}$$

for $x \in \partial B_r(x_0)$. Since both $u(x_0) - u(x)$ and $L_r|x - x_0|^{\frac{p-n}{p-1}}$ are p-harmonic in $B_r(x_0)\backslash\{x_0\}$ and vanish at $x = x_0$, the maximum principle implies

$$u(x_0) - u(x) \leqq L_r|x - x_0|^{\frac{p-n}{p-1}}$$

for $x \in B_r(x_0)$. With this estimate, the assertion follows from Theorem 1.1 and Remark 1.6 of [20]. □

We have already noted that if u is an extremal whose $1 - n/p$ Hölder ratio attains its maximum at two distinct points $x_0, y_0 \in \mathbb{R}^n$, then u is continuously differentiable in $\mathbb{R}^n\backslash\{x_0, y_0\}$. It follows from the above lemma that a nonconstant extremal is not differentiable at its maximum and minimum points.

Corollary 4.3. *Suppose $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is a nonconstant extremal whose $1 - n/p$ Hölder ratio attains its maximum at two distinct points $x_0, y_0 \in \mathbb{R}^n$. Then u is not differentiable at x_0 or y_0.*

Proof. When $n = 1$, we may conclude the nondifferentiability of u at the points where its $1 - 1/p$ Hölder ratio attains its maximum by simply inspecting the graph of (2.13).

Let's now consider the case $n \geqq 2$ and additionally suppose $u(x_0) > u(y_0)$. As u is an extremal, we have by Proposition 2.3 that for sufficiently small $r > 0$,

$$C_*^p \int_{B_r(x_0)} |Du|^{p-2} Du \cdot D\phi dx = \frac{|u(x_0) - u(y_0)|^{p-2}(u(x_0) - u(y_0))}{|x_0 - y_0|^{p-n}} \phi(x_0)$$

for each $\phi \in C_c^\infty(B_r(x_0))$. That is,

$$-\Delta_p u = \frac{|u(x_0) - u(y_0)|^{p-2}(u(x_0) - u(y_0))}{C_*^p |x_0 - y_0|^{p-n}} \delta_{x_0}$$

in $B_r(x_0)$. As $u(x) \leqq u(x_0)$ in $B_r(x_0)$, we can appeal to Lemma 4.1 to find

$$n\omega_n \left(\frac{p-n}{p-1}\gamma\right)^{p-1} = \frac{|u(x_0) - u(y_0)|^{p-2}(u(x_0) - u(y_0))}{C_*^p |x_0 - y_0|^{p-n}} > 0.$$

In particular, $\gamma > 0$.

If u is differentiable at x_0,

$$u(x) - u(x_0) = Du(x_0) \cdot (x - x_0) + o(|x - x_0|)$$

as $x \mapsto x_0$. It would then follow that

$$\gamma = \lim_{x \to x_0} \frac{u(x_0) - u(x)}{|x - x_0|^{\frac{p-n}{p-1}}} \leqq \lim_{x \to x_0} \left((|Du(x_0)| + o(1))|x - x_0|^{1-\frac{p-n}{p-1}}\right) = 0,$$

which is a contradiction. We can also argue similarly for y_0. □

4.2. Quasiconcavity

We will now investigate the quasiconcavity and quasiconvexity properties of extremal functions. Recall that for a given convex $\Omega \subset \mathbb{R}^n$, a function $f : \Omega \to \mathbb{R}$ is quasiconcave if $\{x \in \Omega : f(x) \geqq t\}$ is convex for each $t \in \mathbb{R}$. Likewise, f is quasiconvex if $-f$ is quasiconcave. We note that f is quasiconcave if and only if

$$f((1-\lambda)x + \lambda y) \geqq \min\{f(x), f(y)\} \tag{4.1}$$

for $x, y \in \Omega$ and $\lambda \in [0, 1]$. Our central assertion is

Proposition 4.4. *Suppose that $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is an extremal with*

$$[u]_{C^{1-n/p}(\mathbb{R}^n)} = \frac{|u(x_0) - u(y_0)|}{|x_0 - y_0|^{1-n/p}}$$

and set

$$\Pi_\pm = \left\{x \in \mathbb{R}^n : \pm\left(x - \frac{1}{2}(x_0 + y_0)\right) \cdot (x_0 - y_0) > 0\right\}.$$

If $u(x_0) > u(y_0)$, then

$$u|_{\Pi_+} \text{ is quasiconcave and } u|_{\Pi_-} \text{ is quasiconvex;}$$

alternatively if $u(x_0) < u(y_0)$, then

$$u|_{\Pi_+} \text{ is quasiconvex and } u|_{\Pi_-} \text{ is quasiconcave.}$$

In our proof of the above proposition, we will make use of the subsequent approximation lemma.

Lemma 4.5. *Let $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ be an extremal with $u(e_n) = 1$, $u(-e_n) = -1$, and*

$$[u]_{C^{1-n/p}(\mathbb{R}^n)} = \frac{|u(e_n) - u(-e_n)|}{|e_n - (-e_n)|^{1-n/p}}.$$

For each $\epsilon > 0$, there is $w \in C_c^\infty(\{x_n > 0\})$ such that

$$\left(\int_{\{x_n>0\}} |Du - Dw|^p dx\right)^{1/p} \leqq \epsilon.$$

Proof. 1. Choose $f \in C_c^\infty(B_2)$ radial with $0 \leqq f \leqq 1$ and $f \equiv 1$ on B_1. Next, set $f^\delta(x) := f(\delta x)$ and

$$u^\delta := f^\delta u$$

for $\delta > 0$. In addition, we may select $r > 0$ so large that

$$\left(\int_{|x|\geqq r} |Du|^p dx\right)^{1/p} \leqq \frac{1}{4}\epsilon$$

and $\delta > 0$ so small that $\delta r < 1$. Since $Du^\delta = Df^\delta u + f^\delta Du$ and $|u| \le 1$ by (3.2),

$$
\begin{aligned}
\|Du^\delta - Du\|_{L^p(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} |Du^\delta - Du|^p dx\right)^{1/p} \\
&= \left(\int_{\mathbb{R}^n} |(f^\delta - 1)Du + uDf^\delta|^p dx\right)^{1/p} \\
&\leqq \left(\int_{\mathbb{R}^n} |(f^\delta - 1)Du|^p dx\right)^{1/p} + \left(\int_{\mathbb{R}^n} |uDf^\delta|^p dx\right)^{1/p} \\
&\leqq \left(\int_{\mathbb{R}^n} (1 - f^\delta)^p |Du|^p dx\right)^{1/p} + \left(\int_{\mathbb{R}^n} |Df^\delta|^p dx\right)^{1/p} \\
&\leqq \left(\int_{|x| \geqq r} |Du|^p dx\right)^{1/p} + \left(\int_{\mathbb{R}^n} \delta^p |Df(\delta x)|^p dx\right)^{1/p} \\
&\leqq \frac{1}{4}\epsilon + \delta^{1-n/p} \|Df\|_{L^p(\mathbb{R}^n)}.
\end{aligned}
$$

Choosing δ smaller if necessary, we have

$$\|Du^\delta - Du\|_{L^p(\mathbb{R}^n)} \leqq \frac{1}{2}\epsilon.$$

2. Let η be a standard, radial mollifier on $\mathbb{R}^n$ and set

$$v := \eta^\tau * u^\delta,$$

where $\eta^\tau(x) := \tau^{-n}\eta(x/\tau)$ for $\tau > 0$. As the support of u^δ is bounded, $v \in C_c^\infty(\mathbb{R}^n)$. We also recall that

$$\|Dv - Du^\delta\|_{L^p(\mathbb{R}^n)} \leqq \frac{1}{2}\epsilon$$

for all $\tau > 0$ small enough (Theorem 4.22 in [5] and Theorem 1 of section 5.3 in [13]). For such τ,

$$\|Dv - Du\|_{L^p(\mathbb{R}^n)} \leqq \|Dv - Du^\delta\|_{L^p(\mathbb{R}^n)} + \|Du^\delta - Du\|_{L^p(\mathbb{R}^n)} \leqq \epsilon.$$

According to Proposition 3.4, u is antisymmetric with respect to reflection about the $x_n = 0$ hyperplane. Using that η and f are radial, it is routine to verify that v is also antisymmetric with respect to reflection about the $x_n = 0$ hyperplane. In particular,

$$v|_{x_n=0} = 0.$$

3. Select $s > 0$ so that v is supported in the ball B_s. It follows v vanishes on the boundary of the Lipschitz domain

$$U = B_s \cap \Pi_+ = \{x \in \mathbb{R}^n : x_n > 0,\ |x| < s\}.$$

As a result, $v \in W_0^{1,p}(U)$. Theorem 18.7 in [23] then implies that there is $w \in C_c^\infty(U)$ for which

$$\|Dv - Dw\|_{L^p(U)} \leqq \frac{1}{2}\epsilon.$$

Therefore,

$$\begin{aligned}\|Dw - Du\|_{L^p(\{x_n>0\})} &\leqq \|Dw - Dv\|_{L^p(\{x_n>0\})} + \|Dv - Du\|_{L^p(\{x_n>0\})}\\ &= \|Dw - Dv\|_{L^p(U)} + \frac{1}{2}\|Dv - Du\|_{L^p(\mathbb{R}^n)}\\ &\leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon\\ &= \epsilon.\end{aligned}$$

□

Proof of Proposition 4.4. Without loss of generality, we may verify this assertion for $x_0 = e_n$, $y_0 = -e_n$, $u(e_n) = 1$, and $u(-e_n) = -1$. In this case,

$$\Pi_\pm = \{x \in \mathbb{R}^n : \pm x_n > 0\}.$$

By Proposition 3.4 and the fact that the reflection of a convex set about the hyperplane $\partial\Pi_+ = \{x \in \mathbb{R}^n : x_n = 0\}$ is still convex, we only need to verify that $u|_{\Pi_+}$ is quasiconcave.

1. Recall that u is p-harmonic in $\Pi_+\backslash\{e_n\}$, $u(e_n) = 1$ and $u|_{\partial\Pi_+} = 0$. We will use this information to build an approximation scheme of quasiconcave functions. For each $r > 0$, set

$$D_r := B_r(\sqrt{1+r^2}e_n) = \left\{x \in \mathbb{R}^n : x_1^2 + \cdots + x_{n-1}^2 + (x_n - \sqrt{1+r^2})^2 < r^2\right\}. \tag{4.2}$$

We note that $x \in D_r$ if and only if

$$x_1^2 + \cdots + x_n^2 + 1 < 2\sqrt{1+r^2}\, x_n. \tag{4.3}$$

This implies $e_n \in D_r$, $D_r \subset \Pi_+$, and also that $D_r \subset D_s$ for $r < s$. Moreover, $\Pi_+ = \bigcup_{r>0} D_r$; indeed for a given $x \in \mathbb{R}^n$ with $x_n > 0$, (4.3) holds for all large enough $r > 0$.

Now let $u_r \in W_0^{1,p}(D_r)$ be the unique solution of the boundary value problem

$$\begin{cases} -\Delta_p v = 0 & \text{in } D_r\backslash\{e_n\}\\ v(e_n) = 1 & \\ v = 0 & \text{on } \partial D_r. \end{cases} \tag{4.4}$$

This function u_r can be found by minimizing the integral

$$\int_{D_r} |Dv|^p dx$$

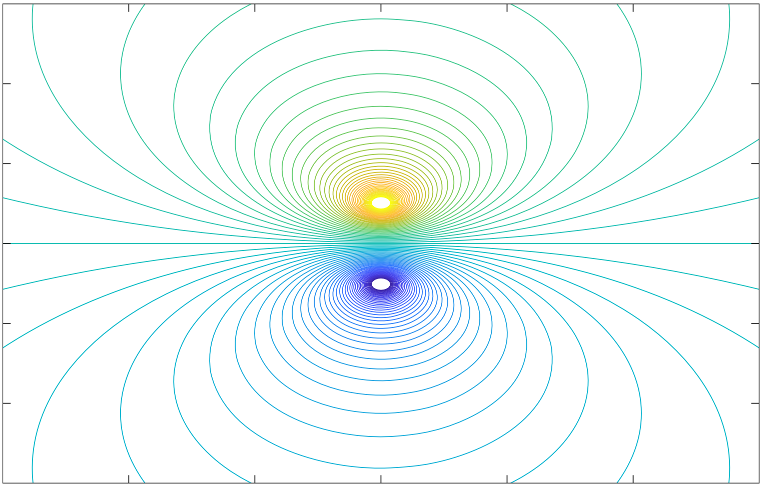

Fig. 3. Various contours of a numerical approximation for the extremal $u : \mathbb{R}^2 \to \mathbb{R}$ of Morrey's inequality whose Hölder seminorm is maximized at $(0, 1)$ and $(0, -1)$ and which satisfies $u(0, 1) = 1$ and $u(0, -1) = -1$. Here $p = 4$

among $v \in W_0^{1,p}(D_r)$ which satisfy $v(e_n) = 1$. In view of (4.4) and Lemma 4.1, we also have

$$-\Delta_p u_r = a_r \delta_{e_n}$$

in D_r for some $a_r \in \mathbb{R}$. Multiplying this equation by u_r and integrating by parts shows that in fact

$$a_r = \int_{D_r} |Du_r|^p dx. \tag{4.5}$$

2. Extending u_r by 0 to $\mathbb{R}^n \backslash D_r$, we see $u_r \in \mathcal{D}^{1,p}(\mathbb{R}^n)$. By the results of Lewis on capacitary functions in convex rings [24], u_r is quasiconcave. In order to complete this proof, we only need to prove that

$$u_r(x) \to u(x) \quad \text{for} \quad x \in \Pi_+ \tag{4.6}$$

as $r \to \infty$. If this is the case, we would have by (4.1) that

$$\begin{aligned} u((1-\lambda)x + \lambda y) &= \lim_{r\to\infty} u_r((1-\lambda)x + \lambda y) \\ &\geq \lim_{r\to\infty} \min\{u_r(x), u_r(y)\} = \min\{u(x), u(y)\} \end{aligned}$$

for each $x, y \in \Pi_+$ and $\lambda \in [0, 1]$. So now we focus on proving (4.6).

Recall that $D_1 \subset D_r$ when $r > 1$. It follows that $u_1 \in W_0^{1,p}(D_r)$ and

$$\int_{\mathbb{R}^n} |Du_r|^p dx = \int_{D_r} |Du_r|^p dx \leqq \int_{D_r} |Du_1|^p dx = \int_{\mathbb{R}^n} |Du_1|^p dx$$

for all $r > 1$. Since $u_r(e_n) = 1$, there is $u_\infty \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ and a sequence of positive numbers r_j increasing to ∞ such that

$$\begin{cases} u_{r_j} \to u_\infty \text{ locally uniformly in } \mathbb{R}^n \\ \\ Du_{r_j} \rightharpoonup Du_\infty \text{ in } L^p(\mathbb{R}^n; \mathbb{R}^n). \end{cases}$$

The local uniform convergence also gives that

$$u_\infty(e_n) = 1 \quad \text{and} \quad u_\infty|_{\partial\Pi_+} = 0. \tag{4.7}$$

Passing to a further subsequence if necessary, the limit $a := \lim_{j\to} a_{r_j}$ exists, where a_r is defined in (4.5).

3. In order to conclude (4.6), it suffices to show

$$u_\infty|_{\Pi_+} = u|_{\Pi_+}. \tag{4.8}$$

To this end, we note for $w \in C_c^\infty(\Pi_+)$ and $r > 0$ sufficiently large

$$\begin{aligned} \int_{\Pi_+} \frac{1}{p}|Dw|^p dx &= \int_{D_r} \frac{1}{p}|Dw|^p dx \\ &\geqq \int_{D_r} \frac{1}{p}|Du_r|^p dx + \int_{D_r} |Du_r|^{p-2} Du_r \cdot (Dw - Du_r) dx \\ &= \int_{D_r} \frac{1}{p}|Du_r|^p dx + a_r(w(e_n) - u_r(e_n)) \\ &= \int_{\Pi_+} \frac{1}{p}|Du_r|^p dx + a_r(w(e_n) - 1). \end{aligned}$$

Sending $r = r_j \to \infty$ gives that

$$\int_{\Pi_+} \frac{1}{p}|Dw|^p dx \geqq \int_{\Pi_+} \frac{1}{p}|Du_\infty|^p dx + a(w(e_n) - 1). \tag{4.9}$$

By Lemma 4.5, there is a sequence $(w^k)_{k\in\mathbb{N}} \subset C_c^\infty(\Pi_+)$ such that $Dw^k \to Du$ in $L^p(\Pi_+; \mathbb{R}^n)$. Also note that $Dw^k \to Du^+$ in $L^p(\mathbb{R}^n; \mathbb{R}^n)$ as $u^+ = 0$ on Π_-. We can then employ Morrey's inequality to find that

$$\begin{aligned} |w^k(e_n) - u(e_n)| &= |w^k(e_n) - u^+(e_n) - (w^k(0) - u^+(0))| \\ &\leqq C|e_n|^{1-n/p} \left(\int_{\mathbb{R}^n} |Dw^k - Du^+|^p dx \right)^{1/p} \\ &\leqq C \left(\int_{\Pi_+} |Dw^k - Du|^p dx \right)^{1/p}. \end{aligned}$$

Therefore, $w^k(e_n) \to u(e_n) = 1$ as $k \to \infty$. Substituting w^k for w in (4.9) and sending $k \to \infty$ gives that

$$\int_{\Pi_+} |Du|^p dx \geqq \int_{\Pi_+} |Du_\infty|^p dx.$$

As noted in the proof of Proposition 3.5, this inequality combined with (4.7) implies (4.8). □

Remark 4.6. The conclusion of the above proof also gives that $u_r \to u$ locally uniformly on Π_+ and $Du_r \to Du$ in $L^p(\Pi_+; \mathbb{R}^n)$.

4.3. Analyticity

It follows from a theorem of Hopf [19] that a p-harmonic function is locally analytic in a neighborhood of any point for which its gradient does not vanish. Therefore, the analyticity of extremal functions is an immediate corollary of the following assertion:

Proposition 4.7. *Suppose $n \geqq 2$ and $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is a nonconstant extremal with*

$$[u]_{C^{1-n/p}(\mathbb{R}^n)} = \frac{|u(x_0) - u(y_0)|}{|x_0 - y_0|^{1-n/p}}.$$

Then

$$|Du| > 0 \quad in \quad \mathbb{R}^n \backslash \{x_0, y_0\}.$$

Proof. We will assume without loss of generality that $x_0 = e_n$, $y_0 = -e_n$ and $u(e_n) = 1$, $u(-e_n) = -1$. By the antisymmetry of u across $\{x \in \mathbb{R}^n : x_n = 0\}$ and the fact that $u_{x_n} > 0$ on this hyperplane by (3.7), it suffices to show that $|Du| > 0$ on $\Pi_+ := \{x \in \mathbb{R}^n : x_n > 0\}$. To this end, we will employ the family of functions $(u_r)_{r>0}$ defined in the proof of Proposition 4.4.

In view of (4.4), u_r is a capacitary function on the convex ring $D_r \backslash \{e_n\}$ for each $r > 0$. Lewis showed that for each ball B with $\overline{B} \subset D_r \backslash \{e_n\}$, there is a constant $\rho = \rho(B) \geqq 1$ for which

$$\max_{\overline{B}} |Du_r| \leqq \rho \min_{\overline{B}} |Du_r|$$

(section 5 of [24]). In particular,

$$|Du_r(y)| \leqq \rho |Du_r(z)|$$

for every $y, z \in \overline{B}$.

Since $Du_r \to Du$ in $L^p(B)$, it must be that $Du_{r_j}(x) \to Du(x)$ for almost every $x \in B$ for an appropriate sequence $(r_j)_{j \in \mathbb{N}}$ tending to 0. Therefore,

$$|Du(y)| \leqq \rho |Du(z)|$$

for Lebesgue almost every $y, z \in B$. Since u is p-harmonic in B, u is continuously differentiable in B and this inequality holds for every $y, z \in B$. That is,

$$\max_{\overline{B}} |Du| \leqq \rho \min_{\overline{B}} |Du|. \tag{4.10}$$

Suppose there is some $y \in \Pi_+$ such that $Du(y) = 0$. Then for every $\delta > 0$ for which $B_\delta(y) \subset \Pi_+ \backslash \{e_n\}$, $Du|_{B_\delta(y)} = 0$. This follows from inequality (4.10). Choosing

$$\delta := \mathrm{dist}(y, \partial(\Pi_+ \backslash \{e_n\})),$$

we have that either $e_n \in \partial B_\delta(y)$ or $\partial B_\delta(y) \cap \partial \Pi_+ \neq \emptyset$. In the case $e_n \in \partial B_\delta(y)$, $u \equiv 1$ in $B_\delta(y)$; otherwise $u \equiv 0$ in $B_\delta(y)$. Either conclusion is a contradiction to Proposition 3.3. As a result, no such point y exists. □

5. Morrey's Estimate and the Sharp Constant C_*

We will now argue that the sharp constant C_* for Morrey's inequality (1.3) cannot be derived the way other constants for Morrey's inequality typically are. To this end, we will recall Morrey's estimate: there is a constant $C > 0$ depending only on p and n such that

$$|u(x) - u(y)| \leqq Cr^{1-n/p} \left(\int_{B_r(z_0)} |Du|^p dz \right)^{1/p} \tag{5.1}$$

for each $x, y \in B_r(z_0)$ and $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$. This inequality was verified by Evans and Gariepy (Theorem 4.10 of [14]) and is based on earlier work by Morrey (Lemma 1 of Section 2 of [29], Theorem 3.5.2 of [30]). We also note that other good presentations of versions of this inequality can be found in Nirenberg's Lecture II of [31], section 5.6.2 of Evans' textbook [13] and in section 7.8 of Gilbarg and Trudinger's monograph [18].

It is not hard to see that Morrey's inequality holds with the constant C in (5.1). Indeed, we can choose $z_0 = x$ and $r = |x - y|$ to find that

$$\frac{|u(x) - u(y)|}{|x - y|^{1-n/p}} \leqq C \left(\int_{B_r(x)} |Du|^p dz \right)^{1/p} \leqq C \left(\int_{\mathbb{R}^n} |Du|^p dz \right)^{1/p} \tag{5.2}$$

for each $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$. However, we claim that any such C must be larger than C_*. In particular, it is not possible to find the best constant for Morrey's inequality from Morrey's estimate.

Proposition 5.1. *Suppose $n \geqq 2$, C is a constant for which* (5.1) *holds, and C_* is the sharp constant for Morrey's inequality. Then*

$$C_* < C.$$

Proof. Let $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ be a nonconstant extremal whose Hölder ratio is maximized at 0 and e_n. In view of (5.2),

$$C_* \left(\int_{\mathbb{R}^n} |Du|^p dz \right)^{1/p} = \frac{|u(e_n) - u(0)|}{|e_n - 0|^{1-n/p}} \leqq C \left(\int_{B_1(0)} |Du|^p dz \right)^{1/p}.$$

By Proposition 4.7, $|Du| > 0$ in $\mathbb{R}^n \backslash B_1(0)$. Therefore,

$$C_* \left(\int_{\mathbb{R}^n} |Du|^p dz \right)^{1/p} < C \left(\int_{\mathbb{R}^n} |Du|^p dz \right)^{1/p}.$$

□

6. Maximizing the $1 - n/p$ Hölder Ratio

We will finally argue that the Hölder ratio of any function belonging to $\mathcal{D}^{1,p}(\mathbb{R}^n)$ always attains its maximum at a pair of distinct points. In particular, every extremal also has this property. We will also show how the following theorem implies a type of stability for Morrey's inequality.

Theorem 6.1. *Assume $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is nonconstant. Then*

$$[u]_{C^{1-n/p}(\mathbb{R}^n)} = \frac{|u(x_0) - u(y_0)|}{|x_0 - y_0|^{1-n/p}}$$

for some $x_0, y_0 \in \mathbb{R}^n$ with $x_0 \neq y_0$.

In order to verify this assertion, it will be convenient for us to employ the following estimate:

$$|u(x) - u(y)| \leqq C r^{1-n/p} \left(\int_{B_{r/2}\left(\frac{x+y}{2}\right)} |Du|^p dz \right)^{1/p}, \tag{6.1}$$

for each $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ and $x, y \in \mathbb{R}^n$. Here $r = |x - y|$ and C is a constant depending only on n and p. This estimate follows from Morrey's estimate (5.1) by choosing $z_0 = \frac{1}{2}(x + y)$ as $x, y \in \partial B_r(z_0)$.

We will prove Theorem 6.1 as follows. We first select any pair of sequences $(x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}}$ such that

$$[u]_{C^{1-n/p}(\mathbb{R}^n)} = \lim_{k\to\infty} \frac{|u(x_k) - u(y_k)|}{|x_k - y_k|^{1-n/p}},$$

and then show

$$\liminf_{k\to\infty} |x_k - y_k| > 0 \tag{6.2}$$

and

$$\sup_{k\in\mathbb{N}} |x_k|, \ \sup_{k\in\mathbb{N}} |y_k| < \infty. \tag{6.3}$$

It follows that $(x_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$ have convergent subsequences $(x_{k_j})_{j\in\mathbb{N}}$ and $(y_{k_j})_{j\in\mathbb{N}}$ that converge to distinct points x_0 and y_0. We could then conclude by noting

$$[u]_{C^{1-n/p}(\mathbb{R}^n)} = \lim_{k\to\infty} \left\{ \frac{|u(x_{k_j}) - u(y_{k_j})|}{|x_{k_j} - y_{k_j}|^{1-n/p}} \right\} = \frac{|u(x_0) - u(y_0)|}{|x_0 - y_0|^{1-n/p}}.$$

It turns out that (6.2) is easy to prove. For if it fails, we may as well assume $\lim_{k\to\infty} |x_k - y_k| = 0$. Applying (6.1) gives

$$\begin{aligned} [u]^p_{C^{1-n/p}(\mathbb{R}^n)} &= \limsup_{k\to\infty} \left\{ \frac{|u(x_k) - u(y_k)|^p}{|x_k - y_k|^{p-n}} \right\} \\ &\leqq C^p \limsup_{k\to\infty} \int_{B_{r_k/2}\left(\frac{x_k+y_k}{2}\right)} |Du|^p dz \qquad (r_k := |x_k - y_k|) \\ &= 0, \end{aligned}$$

so u would be constant. The last equality follows from the fact that $|Du|^p$ is an integrable function on $\mathbb{R}^n$ and the Lebesgue measure of $B_{r_k/2}\left(\frac{x_k+y_k}{2}\right)$ tends to 0 as $k \to \infty$. Therefore, we only are left to verify (6.3) in order to conclude Theorem 6.1. We will write separate proofs for $n = 1$ and for $n \geqq 2$.

6.1. The $1 - n/p$ Hölder Ratio is Maximized for $n = 1$

Suppose $n = 1$ and that (6.3) fails. Then there are three possibilities to consider:

(i) $x_k, y_k \to \infty$
(ii) $x_k \to \infty$, $y_k \to y$
(iii) $x_k \to \infty$, $y_k \to -\infty$.

We will argue that these three cases cannot occur.

Case (i): Fix $\epsilon > 0$. Since $|u'|^p$ is integrable, there is $R > 0$ such that

$$\left(\int_{\mathbb{R}\setminus[-R,R]} |u'|^p dx\right)^{1/p} \leqq \epsilon. \tag{6.4}$$

As the $1 - 1/p$ Hölder ratio of u is a symmetric function, we may assume

$$x_k > y_k \geqq R$$

for all $k \in \mathbb{N}$ sufficiently large. For all such $k \in \mathbb{N}$, we can combine (2.11) and (6.4) to find

$$\begin{aligned}\frac{|u(x_k) - u(y_k)|}{|x_k - y_k|^{1-1/p}} &\leqq \left(\int_{y_k}^{x_k} |u'|^p dx\right)^{1/p}\\ &\leqq \left(\int_{R}^{\infty} |u'|^p dx\right)^{1/p}\\ &\leqq \epsilon.\end{aligned}$$

As a result,

$$[u]_{C^{1-1/p}(\mathbb{R})} = \lim_{k\to\infty} \frac{|u(x_k) - u(y_k)|}{|x_k - y_k|^{1-1/p}} \leqq \epsilon$$

which forces u to be constant.

Case (ii): We assume $y = 0$ and $u(0) = 0$. With this assumption, it is routine to check

$$[u]_{C^{1-1/p}(\mathbb{R})} = \lim_{k\to\infty} \frac{|u(x_k) - u(0)|}{(x_k - 0)^{1-1/p}} = \lim_{k\to\infty} \frac{|u(x_k)|}{x_k^{1-1/p}}.$$

We may also suppose that

$$\frac{|u(x_k)|}{x_k^{1-1/p}} \leqq \frac{|u(x_{k+1})|}{x_{k+1}^{1-1/p}}$$

and

$$0 < 2x_k \leqq x_{k+1}$$

for each $k \in \mathbb{N}$. If not, we can pass to appropriate subsequences to achieve these inequalities.

Notice

$$\begin{aligned}\frac{|u(x_{k+1}) - u(x_k)|}{(x_{k+1} - x_k)^{1-1/p}} &\geqq \frac{|u(x_{k+1})| - |u(x_k)|}{(x_{k+1} - x_k)^{1-1/p}} \\ &= \frac{x_{k+1}^{1-1/p}(|u(x_{k+1})|/x_{k+1}^{1-1/p}) - x_k^{1-1/p}(|u(x_k)|/x_k^{1-1/p})}{(x_{k+1} - x_k)^{1-1/p}} \\ &\geqq \frac{|u(x_{k+1})|}{x_{k+1}^{1-1/p}} \frac{x_{k+1}^{1-1/p} - x_k^{1-1/p}}{(x_{k+1} - x_k)^{1-1/p}} \\ &= \frac{|u(x_{k+1})|}{x_{k+1}^{1-1/p}} \frac{1 - (x_k/x_{k+1})^{1-1/p}}{(1 - (x_k/x_{k+1}))^{1-1/p}} \\ &\geqq \frac{|u(x_{k+1})|}{x_{k+1}^{1-1/p}} \left(1 - (1/2)^{1-1/p}\right).\end{aligned}$$

Therefore,

$$\liminf_{k\to\infty} \frac{|u(x_{k+1}) - u(x_k)|}{|x_{k+1} - x_k|^{1-1/p}} \geqq [u]_{C^{1-1/p}(\mathbb{R})} \left(1 - (1/2)^{1-1/p}\right).$$

We can now argue as we did for case (i) to find that u is constant.

Case (iii): Without loss of generality we may assume $u(0) = 0$ and

$$y_k < 0 < x_k$$

for all $k \in \mathbb{N}$. Observe

$$|x_k - y_k| = x_k - y_k > x_k \quad \text{and} \quad |x_k - y_k| = x_k - y_k > -y_k.$$

Consequently,

$$\begin{aligned}\frac{|u(x_k) - u(y_k)|}{|x_k - y_k|^{1-1/p}} &\leqq \frac{|u(x_k)|}{|x_k - y_k|^{1-1/p}} + \frac{|u(y_k)|}{|x_k - y_k|^{1-1/p}} \\ &\leqq \frac{|u(x_k)|}{|x_k|^{1-1/p}} + \frac{|u(y_k)|}{|y_k|^{1-1/p}}.\end{aligned}$$

Therefore, it must be that either

$$\limsup_{k\to\infty} \frac{|u(x_k)|}{|x_k|^{1-1/p}} > 0 \quad \text{or} \quad \limsup_{k\to\infty} \frac{|u(y_k)|}{|y_k|^{1-1/p}} > 0.$$

In either scenario, we can argue as in our proof of case (ii) to conclude that u is constant.

6.2. An Estimate for the $1 - n/p$ Hölder Ratio

Now suppose $n \geqq 2$. Observe that if $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is nonconstant, there is an $R > 0$ such that

$$C\left(\int_{\mathbb{R}^n\setminus B_R(0)} |Du|^p dx\right)^{1/p} < \frac{1}{2}[u]_{C^{1-n/p}(\mathbb{R}^n)}.$$

If, in addition,

$$B_{r/2}\left(\frac{x+y}{2}\right) \subset \mathbb{R}^n\setminus B_R(0) \tag{6.5}$$

with $r = |x - y| > 0$, (6.1) would give

$$\frac{|u(x) - u(y)|}{|x - y|^{1-n/p}} \leqq C\left(\int_{\mathbb{R}^n\setminus B_R(0)} |Du|^p dx\right)^{1/p} < \frac{1}{2}[u]_{C^{1-n/p}(\mathbb{R}^n)}.$$

In particular, the supremum of the $1 - n/p$ Hölder ratio for u can't be achieved by such a pair $x, y \in \mathbb{R}^n$.

We will argue below that a similar estimate is available for x, y with large norm without requiring (6.5). Our strategy is based on the observation that there are $z_1, \ldots, z_m \in \mathbb{R}^n$ with large norm such that (6.5) holds for each of the consecutive pairs

$$(x, z_1), \ldots, (z_i, z_{i+1}), \ldots, (z_m, y).$$

Moreover, m can be estimated from above provided $|x|, |y|$ are sufficiently large. In summary, we have the following technical assertion which is proved in the appendix (Fig. 4).

Finite Chain Lemma 1. *Suppose $R > 0$ and $x, y \in \mathbb{R}^n\setminus B_{2R}(0)$. Then there are $m \in \{1, \ldots, 7\}$ and $z_1, \ldots, z_m \in \mathbb{R}^n\setminus B_{2R}(0)$ such that*

$$|x - z_1|, \ldots, |z_i - z_{i+1}|, \ldots, |z_m - y| \leqq |y - x| \tag{6.6}$$

and

$$\begin{cases} B_{r/2}\left(\frac{x+z_1}{2}\right) & \text{with} \quad r = |x - z_1| \\ \quad\vdots & \\ B_{r_i/2}\left(\frac{z_i+z_{i+1}}{2}\right) & \text{with} \quad r_i = |z_i - z_{i+1}| \\ \quad\vdots & \\ B_{s/2}\left(\frac{z_m+y}{2}\right) & \text{with} \quad s = |z_m - y| \end{cases} \tag{6.7}$$

are all subsets of $\mathbb{R}^n\setminus B_R(0)$.

The main application of the Finite Chain Lemma is the following assertion:

Lemma 6.2. *Suppose $R > 0$, $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$, and $x, y \in \mathbb{R}^n\setminus B_{2R}(0)$. Then*

$$|u(x) - u(y)| \leqq 8C\left(\int_{\mathbb{R}^n\setminus B_R(0)} |Du|^p dx\right)^{1/p} |x - y|^{1-n/p} \tag{6.8}$$

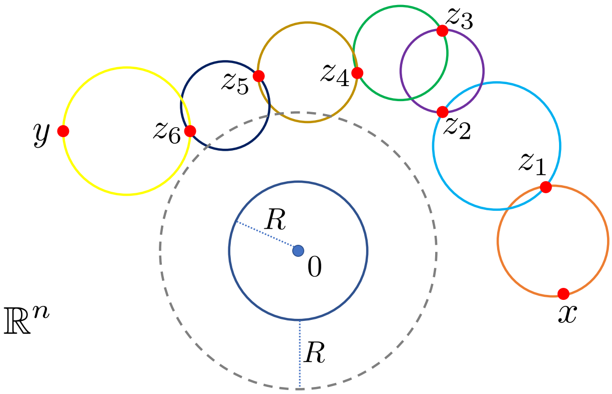

Fig. 4. This figure illustrates the conclusion of the Finite Chain Lemma. Note that $x, y \in \mathbb{R}^n \backslash B_{2R}(0)$ and $z_1, \ldots, z_6$ satisfy (6.6) and (6.7)

Proof. We apply the Finite Chain Lemma to obtain $z_1, \ldots, z_m \in \mathbb{R}^n \backslash B_{2R}(0)$ with $m \in \{1, \ldots, 7\}$ that satisfy (6.6) and (6.7). Next, we employ (6.1) to get

$$
\begin{aligned}
|u(x) - u(y)| &\leqq |u(x) - u(z_1)| + \sum_{j=1}^{m-1} |u(z_j) - u(z_{j+1})| + |u(z_m) - u(y)| \\
&= C \left(\int_{\mathbb{R}^n \backslash B_R(0)} |Du|^p dx \right)^{1/p} \\
&\quad \left(|x - z_1|^{1-n/p} + \sum_{j=1}^{m-1} |z_j - z_{j+1}|^{1-n/p} + |z_m - y|^{1-n/p} \right) \\
&\leqq (m+1) C \left(\int_{\mathbb{R}^n \backslash B_R(0)} |Du|^p dx \right)^{1/p} |x - y|^{1-n/p} \\
&\leqq 8C \left(\int_{\mathbb{R}^n \backslash B_R(0)} |Du|^p dx \right)^{1/p} |x - y|^{1-n/p}.
\end{aligned}
$$

□

6.3. The $1 - n/p$ *Hölder Ratio is Maximized for* $n \geqq 2$

We are now in position to verify (6.3) and in turn complete our proof of Theorem 6.1 for $n \geqq 2$. There are two cases to consider:

(*i*) $|x_k|, |y_k| \to \infty$;
(*ii*) $|x_k| \to \infty$, $y_k \to y$.

Case (*i*): Fix $\epsilon > 0$ and choose $R > 0$ so large that

$$
8C \left(\int_{\mathbb{R}^n \backslash B_R(0)} |Du|^p dx \right)^{1/p} < \epsilon. \tag{6.9}
$$

For all large enough $k \in \mathbb{N}$,

$$|y_k|, |x_k| \geqq 2R,$$

so we can combine (6.8) and (6.9) to get

$$\frac{|u(x_k) - u(y_k)|}{|x_k - y_k|^{1-n/p}} < \epsilon.$$

Therefore,

$$[u]_{C^{1-n/p}(\mathbb{R}^n)} = \lim_{k\to\infty} \frac{|u(x_k) - u(y_k)|}{|x_k - y_k|^{1-n/p}} \leq \epsilon.$$

It follows that u must be constant.

Case (ii): It is routine to verify

$$[u]_{C^{1-n/p}(\mathbb{R}^n)} = \lim_{k\to\infty} \frac{|u(x_k) - u(y)|}{|x_k - y|^{1-n/p}}.$$

We may also assume for each $k \in \mathbb{N}$

$$\frac{|u(x_k) - u(y)|}{|x_k - y|^{1-n/p}} \leq \frac{|u(x_{k+1}) - u(y)|}{|x_{k+1} - y|^{1-n/p}}$$

and

$$0 < 2|x_k - y| \leqq |x_{k+1} - y|,$$

as these inequalities hold along appropriately selected subsequences. Observe that

$$\begin{aligned}
\frac{|u(x_{k+1}) - u(x_k)|}{|x_{k+1} - x_k|^{1-n/p}} &\geq \frac{|u(x_{k+1}) - u(y)| - |u(x_k) - u(y)|}{|(x_{k+1} - y) - (x_k - y)|^{1-n/p}} \\
&\geq \frac{|u(x_{k+1}) - u(y)|}{|x_{k+1} - y|^{1-n/p}} \frac{|x_{k+1} - y|^{1-n/p} - |x_k - y|^{1-n/p}}{|(x_{k+1} - y) - (x_k - y)|^{1-n/p}} \\
&\geq \frac{|u(x_{k+1}) - u(y)|}{|x_{k+1} - y|^{1-n/p}} \frac{|x_{k+1} - y|^{1-n/p} - \frac{1}{2^{1-n/p}}|x_{k+1} - y|^{1-n/p}}{|(x_{k+1} - y) - (x_k - y)|^{1-n/p}} \\
&= \frac{|u(x_{k+1}) - u(y)|}{|x_{k+1} - y|^{1-n/p}} \left(1 - \left(\frac{1}{2}\right)^{1-n/p}\right) \frac{|x_{k+1} - y|^{1-n/p}}{|(x_{k+1} - y) - (x_k - y)|^{1-n/p}}.
\end{aligned}$$

As

$$|(x_{k+1} - y) - (x_k - y)| \leqq |x_{k+1} - y| + |x_k - y| \leqq \frac{3}{2}|x_{k+1} - y|,$$

it follows that

$$\frac{|x_{k+1} - y|^{1-n/p}}{|(x_{k+1} - y) - (x_k - y)|^{1-n/p}} \geq \left(\frac{2}{3}\right)^{1-n/p}.$$

As a result,

$$\liminf_{k\to\infty} \frac{|u(x_{k+1}) - u(x_k)|}{|x_{k+1} - x_k|^{1-n/p}} \geqq [u]_{C^{1-n/p}(\mathbb{R}^n)} \left[\left(\frac{2}{3}\right)^{1-n/p} - \left(\frac{1}{3}\right)^{1-n/p}\right].$$

We can now argue as in the case (i) to conclude that u must be constant.

6.4. Stability

It turns out that a type of stability for Morrey's inequality follows directly from Theorem 6.1. In order to establish this stability, we will make use of Clarkson's inequalities which state: for $f, g \in L^p(\mathbb{R}^n)$

$$\left\|\frac{f+g}{2}\right\|^p_{L^p(\mathbb{R}^n)} + \left\|\frac{f-g}{2}\right\|^p_{L^p(\mathbb{R}^n)} \leq \frac{1}{2}\|f\|^p_{L^p(\mathbb{R}^n)} + \frac{1}{2}\|g\|^p_{L^p(\mathbb{R}^n)} \tag{6.10}$$

for $2 < p < \infty$ and

$$\left\|\frac{f+g}{2}\right\|^{\frac{p}{p-1}}_{L^p(\mathbb{R}^n)} + \left\|\frac{f-g}{2}\right\|^{\frac{p}{p-1}}_{L^p(\mathbb{R}^n)} \leq \left(\frac{1}{2}\|f\|^p_{L^p(\mathbb{R}^n)} + \frac{1}{2}\|g\|^p_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{p-1}}$$

for $1 < p \leqq 2$. We would also like to emphasize that the method presented in our proof below is quite different from the way stability is typically pursued for Sobolev and related isoperimetric inequalities [4,8,9,15–17].

Corollary 6.3. *Suppose $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$. Then there is an extremal $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ such that*

$$\left(\frac{C_*}{2}\right)^p \|Du - Dv\|^p_{L^p(\mathbb{R}^n)} + [v]^p_{C^{1-n/p}(\mathbb{R}^n)} \leqq C_*^p \|Dv\|^p_{L^p(\mathbb{R}^n)}$$

when $2 < p < \infty$ and

$$\left(\frac{1}{2}\right)^{\frac{p}{p-1}} \|u' - v'\|^{\frac{p}{p-1}}_{L^p(\mathbb{R})} + [v]^{\frac{p}{p-1}}_{C^{1-1/p}(\mathbb{R})} \leqq \|v'\|^{\frac{p}{p-1}}_{L^p(\mathbb{R})}$$

when $1 < p \leqq 2$.

Proof. According to Theorem 6.1, there are distinct $x_0, y_0 \in \mathbb{R}^n$ such that

$$[v]_{C^{1-n/p}(\mathbb{R}^n)} = \frac{|v(x_0) - v(y_0)|}{|x_0 - y_0|^{1-n/p}}.$$

By Corollary 2.2, we can also select an extremal u with $u(x_0) = v(x_0)$, $u(y_0) = v(y_0)$ and

$$[u]_{C^{1-n/p}(\mathbb{R}^n)} = [v]_{C^{1-n/p}(\mathbb{R}^n)}.$$

Of course, we have

$$\|Du\|_{L^p(\mathbb{R}^n)} \leqq \|Dv\|_{L^p(\mathbb{R}^n)}.$$

It is also easy to check that

$$[v]_{C^{1-n/p}(\mathbb{R}^n)} = \left[\frac{u+v}{2}\right]_{C^{1-n/p}(\mathbb{R}^n)}.$$

First suppose $2 < p < \infty$. We note that inequality (6.10), while stated for functions, also holds for measurable mappings of $f, g : \mathbb{R}^n \to \mathbb{R}^n$. Therefore,

$$
\begin{aligned}
&\left(\frac{C_*}{2}\right)^p \|Du - Dv\|^p_{L^p(\mathbb{R}^n)} + [v]^p_{C^{1-n/p}(\mathbb{R}^n)} \\
&= C_*^p \left\|\frac{Du - Dv}{2}\right\|^p_{L^p(\mathbb{R}^n)} + [v]^p_{C^{1-n/p}(\mathbb{R}^n)} \\
&= C_*^p \left\|\frac{Du - Dv}{2}\right\|^p_{L^p(\mathbb{R}^n)} + \left[\frac{u+v}{2}\right]^p_{C^{1-n/p}(\mathbb{R}^n)} \\
&\leqq C_*^p \left\|\frac{Du - Dv}{2}\right\|^p_{L^p(\mathbb{R}^n)} + C_*^p \left\|\frac{Du + Dv}{2}\right\|^p_{L^p(\mathbb{R}^n)} \\
&= C_*^p \left(\left\|\frac{Du - Dv}{2}\right\|^p_{L^p(\mathbb{R}^n)} + \left\|\frac{Du + Dv}{2}\right\|^p_{L^p(\mathbb{R}^n)}\right) \\
&\leqq C_*^p \left(\frac{1}{2}\|Du\|^p_{L^p(\mathbb{R}^n)} + \frac{1}{2}\|Dv\|^p_{L^p(\mathbb{R}^n)}\right) \\
&\leqq C_*^p \|Dv\|^p_{L^p(\mathbb{R}^n)}.
\end{aligned}
$$

Recall that when $1 < p \leqq 2$, n is necessarily equal to 1. As a result, we can apply the other Clarkson's inequality to find that

$$
\begin{aligned}
\left(\frac{1}{2}\right)^{\frac{p}{p-1}} \|u' - v'\|^{\frac{p}{p-1}}_{L^p(\mathbb{R})} + [v]^{\frac{p}{p-1}}_{C^{1-1/p}(\mathbb{R})} &= \left\|\frac{u' - v'}{2}\right\|^{\frac{p}{p-1}}_{L^p(\mathbb{R})} + [v]^{\frac{p}{p-1}}_{C^{1-1/p}(\mathbb{R})} \\
&= \left\|\frac{u' - v'}{2}\right\|^{\frac{p}{p-1}}_{L^p(\mathbb{R})} + \left[\frac{u+v}{2}\right]^{\frac{p}{p-1}}_{C^{1-1/p}(\mathbb{R})} \\
&\leqq \left\|\frac{u' - v'}{2}\right\|^{\frac{p}{p-1}}_{L^p(\mathbb{R})} + \left\|\frac{u' + v'}{2}\right\|^{\frac{p}{p-1}}_{L^p(\mathbb{R})} \\
&\leqq \left(\frac{1}{2}\|u'\|^p_{L^p(\mathbb{R})} + \frac{1}{2}\|v'\|^p_{L^p(\mathbb{R})}\right)^{\frac{1}{p-1}} \\
&\leqq \|v'\|^{\frac{p}{p-1}}_{L^p(\mathbb{R})}.
\end{aligned}
$$

□

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A Finite Chain Lemma

We will first pursue the Finite Chain Lemma for $n = 2$. To this end, we will need to recall a few facts about isosceles triangles. Suppose $a > 0$ and consider an

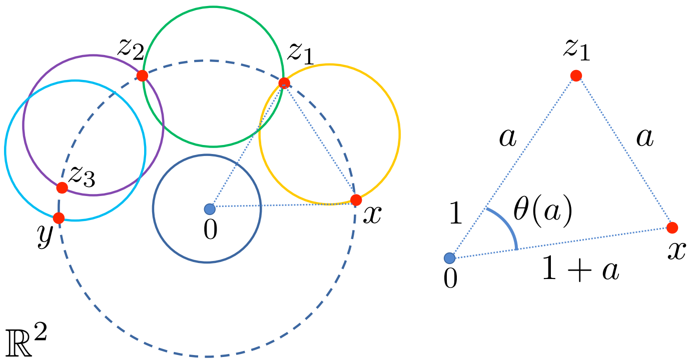

Fig. 5. This figure on the left is a schematic of the assertion made in Lemma A.1. This corresponds to the case where three points z_1, z_2, z_3 are needed to form a finite chain linking x to y on the circle centered at 0 of radius $1 + a$. The enlarged triangle on the right shows how $\theta(a)$ is defined and how it is related to the figure on the left

isosceles triangle with two sides equal to $1 + a$ and another equal to a. Using the law of cosines we find that the angle $\theta(a)$ between the two sides of length $1 + a$ satisfies (Fig. 5)

$$\cos(\theta(a)) = 1 - \frac{1}{2}\left(\frac{a}{1+a}\right)^2.$$

It is also easy to check that $0 < \theta(a) < \pi/3$ and that $\theta(a)$ is increasing in a. In particular, $\theta(1) \leqq \theta(a)$ for $a \geq 1$. It will also be useful to note

$$\theta(1) = \cos^{-1}\left(\frac{7}{8}\right) > \frac{\pi}{7}.$$

With these observations, we can derive the following assertion:

Lemma A.1. *Assume $a \geqq 1$ and $x, y \in \mathbb{R}^2$ with*

$$|y| = |x| = 1 + a.$$

If

$$|y - x| > a,$$

there are $z_1, \ldots, z_m \in \mathbb{R}^2$ with $m \in \{1, \ldots, 6\}$ such that

$$\begin{aligned} &|z_1| = \cdots = |z_m| = 1 + a, \\ &|x - z_1| = |z_1 - z_2| = \cdots = |z_{m-1} - z_m| = a, \end{aligned} \tag{A.1}$$

and

$$|y - z_m| \leqq a.$$

Proof. First assume $x = (1+a)e_1$. Note that

$$y = (1+a)(\cos\vartheta, \sin\vartheta)$$

for some $\vartheta \in (\theta(a), 2\pi - \theta(a))$. If $\vartheta \in (\theta(a), \pi]$, we set

$$z_j = (1+a)(\cos(j\theta(a)), \sin(j\theta(a)))$$

$j = 1, \dots, 6$. By the definition of $\theta(a)$, (A.1) holds.
As $7\theta(a) > \pi$, there is $m \in \{1, \dots, 6\}$ such that

$$m\theta(a) \leqq \vartheta \leqq (m+1)\theta(a).$$

It follows that

$$\begin{aligned} |y - z_m|^2 &= (1+a)^2 2\,[1 - \cos(m\theta(a) - \vartheta)] \\ &\leqq (1+a)^2 2\,[1 - \cos(\theta(a))] \\ &= a^2. \end{aligned}$$

If $\vartheta \in (\pi, 2\pi - \theta(a))$, we consider the reflection Ry of y about the x-axis. In particular, we can reason as in the case when $\vartheta \in (\theta(a), \pi]$ for Ry and obtain points $z_1, \dots, z_m \in \mathbb{R}^2$. In this case, $Rz_1, \dots, Rz_m$ satisfy the conclusion of this lemma. For a general x that is not necessarily equal to $(1+a)e_1$, we can find a rotation O of $\mathbb{R}^2$ so that $Ox = (1+a)e_1$. Then we can prove the assertion for $(1+a)e_1$ and Oy to get points $z_1, \dots, z_m \in \mathbb{R}^2$ satisfying the conclusion of the lemma as we argued above. Then $O^{-1}z_1, \dots, O^{-1}z_m$ satisfy the conclusion of this lemma for the given x and y. □

Corollary A.2. *Assume* $s \geqq t > 0$ *and* $x, y \in \mathbb{R}^2$ *with*

$$|y| = |x| = t + s.$$

If

$$|y - x| > s,$$

there are $z_1, \dots, z_m \in \mathbb{R}^2$ *with* $m \in \{1, \dots, 6\}$ *such that*

$$|z_1| = \cdots = |z_m| = t + s, \tag{A.2}$$

$$|x - z_1| = |z_1 - z_2| = \cdots = |z_{m-1} - z_m| = s, \tag{A.3}$$

and

$$|y - z_m| \leqq s. \tag{A.4}$$

Proof. We can apply the previous lemma with $a = s/t \geq 1$ and $x/t, y/t \in \mathbb{R}^2$. □

We are of course interested in the scenario where $|x|$ is not necessarily equal to $|y|$. Fortunately, we can just add an additional point to obtain an analogous statement. We also will need to use the following elementary fact: if $x, y \in \mathbb{R}^2$ with $|y| \geqq |x| > 0$, then

$$\left| |x| \frac{y}{|y|} - x \right| \leqq |x - y|. \tag{A.5}$$

Corollary A.3. *Assume $s \geqq t > 0$ and $x, y \in \mathbb{R}^2$ with*

$$|y| \geqq |x| = t + s.$$

Suppose

$$\left| |x| \frac{y}{|y|} - x \right| > s.$$

(i) Then there are $z_1, \dots, z_m \in \mathbb{R}^2$ with $m \in \{1, \dots, 7\}$ such that

$$|z_1| = \cdots = |z_m| = t + s,$$

and

$$|x - z_1|, \dots, |z_i - z_{i+1}|, \dots, |z_m - y| \leqq |y - x|. \tag{A.6}$$

(ii) Furthermore,

$$\begin{cases} B_{t_0/2}\left(\frac{x+z_1}{2}\right) & \textit{with} \quad t_0 = |x - z_1| \\ \quad \vdots & \\ B_{t_i/2}\left(\frac{z_i+z_{i+1}}{2}\right) & \textit{with} \quad t_i = |z_i - z_{i+1}| \\ \quad \vdots & \\ B_{t_m/2}\left(\frac{z_m+y}{2}\right) & \textit{with} \quad t_m = |z_m - y| \end{cases} \tag{A.7}$$

are all subsets of $\mathbb{R}^2 \backslash B_t(0)$.

Proof. (i) Corollary A.2 applied to x and $|x|\frac{y}{|y|}$ give at most six points $z_1, \dots, z_{m-1}$ such that (A.2) holds. We also observe that by (A.3) and inequality (A.5)

$$|x - z_1| = |z_1 - z_2| = \cdots = |z_{m-2} - z_{m-1}| = s < \left| |x| \frac{y}{|y|} - x \right| \leqq |y - x|.$$

Set

$$z_m := |x| \frac{y}{|y|},$$

and note by conclusion (A.4) of the previous corollary that $|z_{m-1} - z_m| \leqq s \leqq |y - x|$. Moreover, $|z_m| = |x| = t + s$. As

$$|z_m - y| = \left| |x| \frac{y}{|y|} - y \right| = |y| - |x| \leqq |y - x|,$$

we have verified each inequality in (A.6).

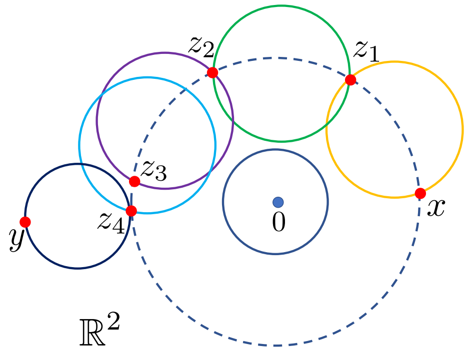

Fig. 6. This diagram shows how we can adapt our proof of Lemma A.1 to the case where $|y| > |x|$. We do so by simply adding another point $z_4 = (|x|/|y|)y$ to the chain we obtained by linking x and $(|x|/|y|)y$

(ii) Since $|x| = s + t$, $B_s(x) \subset \mathbb{R}^2 \backslash B_t(0)$. Moreover, if $z \in B_{s/2}\left(\frac{x+z_1}{2}\right)$ then

$$|z - x| \leqq \left|z - \frac{x+z_1}{2}\right| + \left|\frac{x+z_1}{2} - x\right| \leqq \frac{s}{2} + \frac{|x - z_1|}{2} = s.$$

Thus, $B_{s/2}\left(\frac{x+z_1}{2}\right) \subset B_s(x) \subset \mathbb{R}^2 \backslash B_t(0)$ with $s = |x - z_1|$. The inclusions for $i = 1, \ldots, m-1$ in (A.7) follow similarly.
As for the case $i = m$, set $t_m := |z_m - x|$. Note that the closest point in $B_{t_m/2}\left(\frac{z_m+y}{2}\right)$ to the origin is $z_m = |x|\frac{y}{|y|}$. Thus for any $z \in B_{t_m/2}\left(\frac{z_m+y}{2}\right)$, $|z| \geqq ||x|\frac{y}{|y|}| = |x| > t$. Hence, $B_{t_m/2}\left(\frac{z_m+y}{2}\right) \subset \mathbb{R}^2 \backslash B_t(0)$ and we conclude (A.7) (Fig. 6). $\square$

It turns out that we can easily generalize the ideas we developed for $n = 2$ to all $n \geqq 3$. The main insight is that for any two-dimensional subspace of $\mathbb{R}^n$ containing x and y, we can find a chain of points linking x to y that belong to this subspace. In particular, we can accomplish this task by applying Corollary A.3.

Corollary A.4. *Suppose $n \geqq 2$. Assume $s \geqq t > 0$ and $x, y \in \mathbb{R}^n$ with*

$$|y| \geqq |x| = t + s.$$

Suppose that

$$\left||x|\frac{y}{|y|} - x\right| > s.$$

(i) Then there are $z_1, \ldots, z_m \in \mathbb{R}^n$ with $m \in \{1, \ldots, 7\}$ such that

$$|z_1| = \cdots = |z_m| = t + s,$$

and

$$|x - z_1|, \ldots, |z_i - z_{i+1}|, \ldots, |z_m - y| \leqq |y - x|.$$

(ii) Furthermore,

$$\begin{cases} B_{t_0/2}\left(\frac{x+z_1}{2}\right) & \text{with} \quad t_0 = |x - z_1| \\ \vdots & \\ B_{t_i/2}\left(\frac{z_i+z_{i+1}}{2}\right) & \text{with} \quad t_i = |z_i - z_{i+1}| \\ \vdots & \\ B_{t_m/2}\left(\frac{z_m+y}{2}\right) & \text{with} \quad t_m = |z_m - y| \end{cases}$$

are all subsets of $\mathbb{R}^n \setminus B_t(0)$.

Proof. Choose a two-dimensional subspace in $\mathbb{R}^n$ that includes x and y. We can then apply Corollary A.3 to obtain $z_1, \dots, z_m \in \mathbb{R}^n$ that belong to this subspace and check that these points satisfy the desired conclusions. We leave the details to the reader. □

Proof of the Finite Chain Lemma. Without loss of generality, we may assume $|y| \geqq |x|$. Set $S := |x| - R$, and note $S \geqq R$ with

$$|y| \geqq |x| = R + S.$$

We will verify the claim by considering the following two cases:

(i) $\left| |x| \frac{y}{|y|} - x \right| \leqq S$

(ii) $\left| |x| \frac{y}{|y|} - x \right| > S$

Case (i): As $|x| = R + S$, $B_S(x) \subset \mathbb{R}^n \setminus B_R(0)$. And by adapting our the proof of (A.7), we find

$$B_{r/2}\left(\frac{x + (|x|/|y|)y}{2}\right) \subset B_S(x)$$

for $r = \left| |x| \frac{y}{|y|} - x \right|$ and

$$B_{s/2}\left(\frac{y + (|x|/|y|)y}{2}\right) \subset \mathbb{R}^n \setminus B_R(0)$$

for $s = \left| |x| \frac{y}{|y|} - y \right| = |y| - |x|$. We also have $r \leqq |x - y|$ by inequality A.5 and $s \leqq |y - x|$ by the triangle inequality. Therefore, the claim holds for $m = 1$ and

$$z_1 = |x| \frac{y}{|y|}.$$

Case (ii): We can apply Corollary A.4 with $s = S$ and $t = R$ to obtain an $m = \{1, \dots, 7\}$ and a finite sequence $z_1, \dots, z_m \in \mathbb{R}^n \setminus B_{2R}(0)$ which satisfy (6.6) and (6.7). □

B Coordinate Gradient Descent

In this section, we will change notation and use (x, y) to denote a point in $\mathbb{R}^2$. For a given $\ell > 1$, we seek to approximate minimizers of the two dimensional integral

$$\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} |Dv(x, y)|^p dxdy$$

among functions $v \in W^{1,p}([-\ell, \ell]^2)$ which satisfy

$$v(0, 1) = 1 \quad \text{and} \quad v(0, -1) = -1. \tag{B.1}$$

It can be shown that a unique minimizer $u_\ell \in W^{1,p}([-\ell, \ell]^2)$ exists and that u_ℓ is p-harmonic and thus continuously differentiable in $(-\ell, \ell)^2\backslash\{(0, \pm 1)\}$. Moreover, as $\ell \to \infty$, u_ℓ converges locally uniformly to an extremal of Morrey's inequality in $\mathbb{R}^2$ which satisfies (B.1). Therefore, our numerical approximation for u_ℓ will in turn serve as an approximation for the corresponding extremal.

To this end, we suppose that $\ell \in \mathbb{N}$ and divide the interval $[-\ell, \ell]$ into $N-1$ evenly spaced sub-intervals of length

$$h = \frac{2\ell}{N-1}.$$

Along the x-axis, we will label the endpoints of these intervals with

$$x_i = -\ell + (i-1)h$$

for $i = 1, \dots, N$ and along the y-axis we will use the labels

$$y_j = -\ell + (j-1)h$$

for $j = 1, \dots, N$.

Assuming that $v : [-\ell, \ell]^2 \to \mathbb{R}$ is continuously differentiable,

$$\begin{aligned}
&\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} |Dv(x, y)|^p dxdy \\
&\quad\approx \sum_{i,j=1}^{N-1} |Dv(x_i, y_j)|^p h^2 \\
&\quad= \sum_{i,j=1}^{N-1} \left(v_x(x_i, y_j)^2 + v_y(x_i, y_j)^2\right)^{p/2} h^2 \\
&\quad\approx \sum_{i,j=1}^{N-1} \left(\left(\frac{v(x_i + h, y_j) - v(x_i, y_j)}{h}\right)^2 + \left(\frac{v(x_i, y_j + h) - v(x_i, y_i)}{h}\right)^2\right)^{p/2} h^2 \\
&\quad= \sum_{i,j=1}^{N-1} \left(\left(\frac{v(x_{i+1}, y_j) - v(x_i, y_i)}{h}\right)^2 + \left(\frac{v(x_i, y_{j+1}) - v(x_i, y_i)}{h}\right)^2\right)^{p/2} h^2
\end{aligned}$$

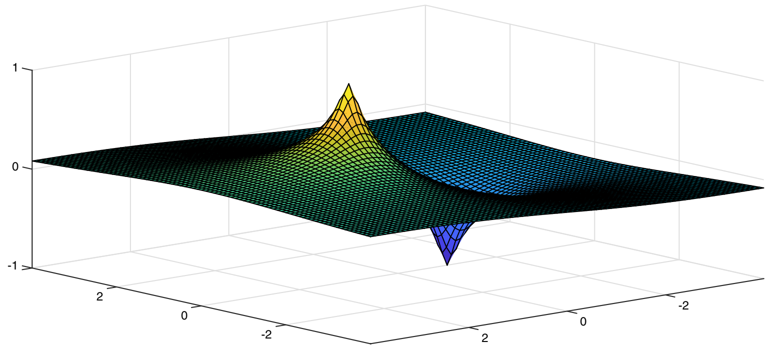

Fig. 7. A numerically computed approximation for an extremal of Morrey's inequality with $n = 2$ and $p = 4$. Here $\ell = 6$ and $k = 10$ (so that $N = 121$), $\tau = 10^{-10}$, and this approximation was obtained after 10^8 iterations. Our initial guess was $v^0_{i,j} = w(x_i, y_j)$, where $w(x, y) = c \ln[(x^2 + (y-1)^2 + 10^{-2})/(x^2 + (y+1)^2 + 10^{-2})]$ and c is chosen to ensure $w(0, 1) = 1$ and $w(0, -1) = -1$

$$= h^{2-p} \sum_{i,j=1}^{N-1} \left(\left(v(x_{i+1}, y_j) - v(x_i, y_i)\right)^2 + \left(v(x_i, y_{j+1}) - v(x_i, y_i)\right)^2 \right)^{p/2}$$

$$= h^{2-p} \sum_{i,j=1}^{N-1} \left(\left(v_{i+1,j} - v_{i,j}\right)^2 + \left(v_{i,j+1} - v_{i,j}\right)^2 \right)^{p/2}.$$

Here we have written

$$v_{i,j} = v(x_i, y_j).$$

We now suppose that N is of the form

$$N = 2\ell k + 1$$

for some $k \in \mathbb{N}$; this assumption is equivalent to $h = 1/k$. We can then attempt to minimize

$$E(v) := \sum_{i,j=1}^{N-1} \left(\left(v_{i+1,j} - v_{i,j}\right)^2 + \left(v_{i,j+1} - v_{i,j}\right)^2 \right)^{p/2}$$

among the $N^2 - 1$ variables

$$v = \begin{pmatrix} v_{1,1} & v_{1,2} & \dots & v_{1,N-1} & v_{1,N} \\ v_{2,1} & v_{2,2} & \dots & v_{2,N-1} & v_{2,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{N-1,1} & v_{N-1,2} & \dots & v_{N-1,N-1} & v_{N-1,N} \\ v_{N,1} & v_{N,2} & \dots & v_{N,N-1} & \end{pmatrix},$$

which satisfy

$$v_{\ell k+1,(\ell+1)k+1} = 1 \quad \text{and} \quad v_{\ell k+1,(\ell-1)k+1} = -1. \tag{B.2}$$

These constraints are natural as $x_{\ell k+1} = 0$, $y_{(\ell+1)k+1} = 1$, and $y_{(\ell-1)k+1} = -1$. We are now in position to use coordinate gradient descent to minimize E. That is, we choose an initial guess $v^0 = (v^0_{i,j})$ which satisfies (B.2), select a small parameter $\tau > 0$, and then run the iteration scheme

$$\begin{cases} v^m_{i,j} = v^{m-1}_{i,j} - \tau \dfrac{\partial E(v^{m-1})}{\partial v_{i,j}}, & (i,j) \neq (\ell k+1, (\ell \pm 1)k+1) \\ \\ v^m_{i,j} = v^{m-1}_{i,j}, & (i,j) = (\ell k+1, (\ell \pm 1)k+1) \end{cases}$$

for $m \in \mathbb{N}$. After we perform this scheme for large number of iterates $m = 1, \ldots, M$, for τ small and k sufficiently large, we can use $v^M_{i,j}$ as an approximation for $u_\ell(x_i, y_j)$. This is what we did to produce the graphs in Figs. 1 and 7 and the contour plot in Fig. 3.

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(*Received May 6, 2020 / Accepted May 13, 2021*)