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Two critical times for the SIR model

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ABSTRACT

We consider the SIR model and study the first time the number of infected individuals begins to decrease and the first time this population is below a given threshold. We interpret these times as functions of the initial susceptible and infected populations and characterize them as solutions of a certain partial differential equation. This allows us to obtain integral representations of these times and in turn to estimate them precisely for large populations.

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1. Introduction

The susceptible, infected, and recovered (SIR) model in epidemiology involves the system of ODE

$$\begin{cases} \dot{S} = -\beta S I \\ \dot{I} = \beta S I - \gamma I. \end{cases} \tag{1.1}$$

Here $S, I : [0,\infty) \to [0,\infty)$ denote the susceptible and infected compartments of a given population in the presence of an infectious disease. If N is the size of the population, then

$$R(t) = N - S(t) - I(t)$$

is the recovered compartment of the population at time t. The parameters $\beta > 0$ and $\gamma > 0$ are the infected and recovery rates per unit time, respectively. While this is a classical model in epidemiology [16], the SIR and its variants have played a crucial role in several recent studies of the COVID–19 epidemic [1–3,6,10,18,21,23].

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In what follows, we will study the first time the infected population I drops below a desired level and the first the I starts to decrease as a function of the initial conditions $S(0)$ and $I(0)$. We will use a partial differential equation (PDE) to assist our analysis, and we'll also identify the asymptotic behavior of the these functions when $S(0)+I(0)$ is large. We are not the first authors to study such times for compartmental models; recent work along these lines include [7,9,13,17,22,24]. Nevertheless, our work seems to be the first that considers how these times vary with initial conditions. Moreover, we have introduced a PDE that does not appear to have been considered before. In addition, we anticipate that this approach has a good chance of being successful when applied to more general compartmental models in epidemiology. For now, we will focus our efforts on the simple SIR model to get a better grasp on what type of analysis is needed.

Before stating our results, we will first need to recall some basic properties of solution pairs S, I to the SIR ODE (1.1). For more details on these facts, we refer the reader to standard references such as section 2.2 of [19], section 10.2 of [20], or section 9.2 of [8].

Positive and bounded solutions. For given initial conditions $S(0), I(0) > 0$, there is a unique solution of the SIR ODE $S, I : [0, \infty) \to \mathbb{R}$. Since

$$S(t) = S(0)e^{-\beta \int_0^t I(\tau)d\tau} \quad \text{and} \quad I(t) = I(0)e^{\int_0^t (\beta S(\tau) - \gamma)d\tau} \tag{1.2}$$

for $t \geq 0$, S and I are necessarily positive. Also note that as $\frac{d}{dt}(S + I)(t) = -\gamma I(t) \leq 0$,

$$S(t) + I(t) \leq S(0) + I(0)$$

for $t \geq 0$. It follows that S and I are additionally bounded.

Integrability. For each time $t \geq 0$,

$$S(t) + I(t) - \frac{\gamma}{\beta} \ln S(t) = S(0) + I(0) - \frac{\gamma}{\beta} \ln S(0).$$

That is, the path $t \mapsto (S(t), I(t))$ belongs to a level set of the function

$$\psi(x, y) := x + y - (\gamma/\beta) \ln x. \tag{1.3}$$

This observation also implies

$$t = \int_{S(t)}^{S(0)} \frac{dz}{\beta z \left((\gamma/\beta) \ln z - z + \psi(S(0), I(0)\right)}$$

for $t \geq 0$. In this sense, S and I are determined up to an integral. The formula above and related ways to represent solutions have been studied in detail in various works [4,5,9,11,13,17,22].

Decay of infected individuals. The number of infected individuals tends to 0 as $t \to \infty$

$$\lim_{t \to \infty} I(t) = 0.$$

In particular, for any given threshold $\mu > 0$, there is a finite time t such that the number of infected individuals falls below μ

$$I(t) \leq \mu.$$

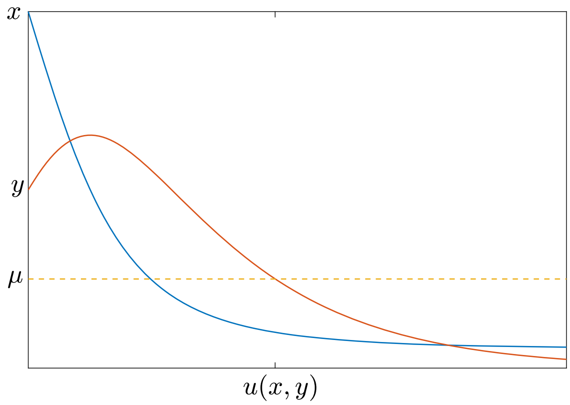

Fig. 1. Plot of the solution pair S, I of (1.1) with $S(0) = x$ and $I(0) = y$. $S(t)$ is shown in blue and $I(t)$ is shown in red. Note that $u(x, y)$ is the first time t such that $I(t) \leq \mu$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Below, we will write $u \geq 0$ for the first time in which I falls below μ.

Infected individuals eventually decrease. There is a time $v \geq 0$ for which

$$I \text{ is decreasing on } [v, \infty).$$

Since I is a positive function and $\dot{I}(t) = (\beta S(t) - \gamma)I(t)$, v can be taken to be the first time t that the number of susceptible individuals falls below the ratio of the recovery and infected rates

$$S(t) \leq \frac{\gamma}{\beta}.$$

As mentioned, we will study the times u and v mentioned above as functions of the initial conditions $S(0)$ and $I(0)$. First, let us fix a threshold

$$\mu > 0$$

and consider

$$u(x, y) := \inf\{t > 0 : I(t) \leq \mu\} \tag{1.4}$$

for $x, y \geq 0$. Here S, I are solutions of (1.1) with $S(0) = x$ and $I(0) = y$ (Fig. 1). As $u(x, y) = 0$ when $0 \leq y < \mu$, we will focus on the values of $u(x, y)$ for $x \geq 0$ and $y \geq \mu$.

We'll see that u is a smooth function on $(0, \infty) \times (\mu, \infty)$ which satisfies the PDE

$$\beta xy\partial_x u + (\gamma - \beta x)y\partial_y u = 1 \tag{1.5}$$

along with the boundary condition

$$u(x, \mu) = 0, \; x \in [0, \gamma/\beta]. \tag{1.6}$$

Moreover, we will also use ψ defined in (1.3) to write a representation formula for u. To this end, we note that for $x > 0, y \geq \mu$ and there is a unique $a(x, y) \in (0, \gamma/\beta]$ such that

$$\psi(x, y) = \psi(a(x, y), \mu).$$

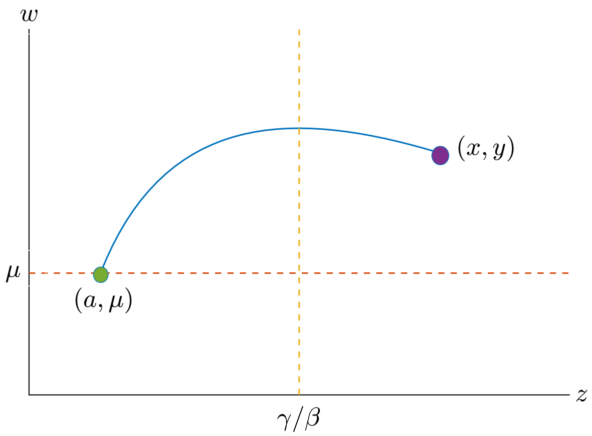

Fig. 2. Graph of $w = (\gamma/\beta)\ln z - z + \psi(x, y)$ for $a \le z \le x$. Here $x > 0$, $y > \mu$, and $a = a(x, y) \in (0, \gamma/\beta]$ is the unique solution of $\psi(x, y) = \psi(a, \mu)$.

Further, $a(x, y) < x$ when $x > \gamma/\beta$ or $y > \mu$. These facts follow easily from the definition of ψ; Fig. 2 also provides an illustration.

A fundamental result involving u is as follows.

Theorem 1.1. *The function u defined in (1.4) has the following properties.*

(i) u is continuous on $[0, \infty) \times [\mu, \infty)$ and is smooth in $(0, \infty) \times (\mu, \infty)$.
(ii) u is the unique solution of (1.5) in $(0, \infty) \times (\mu, \infty)$ which satisfies the boundary condition (1.6).
(iii) For each $x > 0$ and $y \ge \mu$,

$$u(x, y) = \int_{a(x,y)}^{x} \frac{dz}{\beta z\left((\gamma/\beta)\ln z - z + \psi(x, y)\right)}. \tag{1.7}$$

Next, we will study

$$v(x, y) := \inf\{t > 0 : S(t) \le \gamma/\beta\} \tag{1.8}$$

for $x \ge 0$ and $y \ge 0$. Again we are assuming that S, I is the solution pair of the SIR ODE (1.1) with $S(0) = x$ and $I(0) = y$ (Fig. 3). Since

$$\begin{cases} v(x, y) = 0 \ \text{ for } 0 \le x \le \gamma/\beta \text{ and} \\ \\ v(x, 0) = \infty \ \text{ for } x > \gamma/\beta, \end{cases}$$

we will focus on the values of $v(x, y)$ for $x > \gamma/\beta$ and $y > 0$. Also note that when $y > 0$, $v(x, y)$ records the first time t that $I(t)$ starts to decrease.

The methods we use to prove Theorem 1.1 extend analogously to v. In particular, v satisfies the same PDE as u with the boundary condition

$$v(\gamma/\beta, y) = 0, \ y \in (0, \infty). \tag{1.9}$$

Almost in parallel with Theorem 1.1, we have the subsequent assertion.

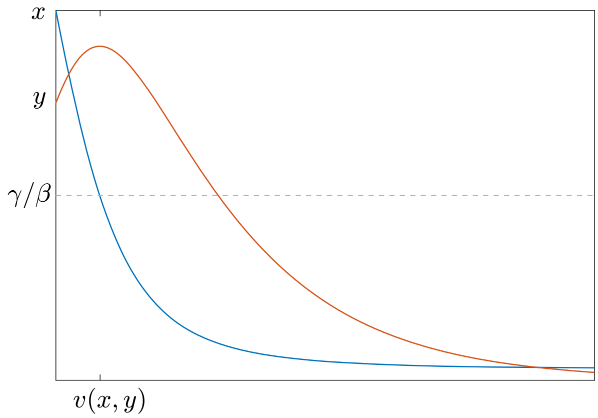

Fig. 3. The solution pair S, I of (1.1) with $S(0) = x$ and $I(0) = y$. $S(t)$ is shown in blue and $I(t)$ is shown in red. Note that $v(x, y)$ is the first time $S(t) \leq \gamma/\beta$ which is also the same time that I starts to decrease.

Theorem 1.2. *The function v defined in (1.8) has the following properties.*

(i) v is continuous on $[\gamma/\beta, \infty) \times (0, \infty)$ and is smooth in $(\gamma/\beta, \infty) \times (0, \infty)$.
(ii) v is the unique solution of (1.5) in $(\gamma/\beta, \infty) \times (0, \infty)$ which satisfies the boundary condition (1.9).
(iii) For each $x \geq \gamma/\beta$ and $y > 0$,

$$v(x, y) = \int_{\gamma/\beta}^{x} \frac{dz}{\beta z \left((\gamma/\beta) \ln z - z + \psi(x, y)\right)}. \tag{1.10}$$

We note that formula (1.10) has been previously established; see for example [9,22,24]. One novelty of our work is to consider the formula as a function of initial conditions x and y and to characterize this function as a solution of PDE (1.5) for given boundary conditions. Moreover, we will use this formula along with (1.7) to precisely estimate $u(x, y)$ and $v(x, y)$ when $x + y$ is large.

Theorem 1.3. *Define u by (1.4) and v by (1.8). Then*

$$\lim_{\substack{x+y\to\infty \\ x\geq 0,\ y\geq \mu}} \frac{u(x, y)}{\frac{1}{\gamma} \ln\left(\frac{x+y}{\mu}\right)} = 1 \tag{1.11}$$

and

$$\lim_{\substack{x+y\to\infty \\ x>\gamma/\beta,\ y>0}} \frac{\beta(x - \gamma/\beta + y)}{\ln\left[\left(\frac{x}{\gamma/\beta}\right)\left(\frac{x - \gamma/\beta}{y} + 1\right)\right]} \cdot v(x, y) = 1. \tag{1.12}$$

It follows from (1.11) that $u(x, y) \to \infty$ as $x + y \to \infty$. Consequently, we are implicitly considering solutions of the SIR ODE (1.1) on very long time intervals. However, the limits (1.11) and (1.12) are of a different nature than typical large time asymptotics results for solutions of (1.1) as these limits involve solutions with varying initial conditions. For example, in the seminal work of Hethcote [14], the classic text of Murray [25], and in recent papers such as [4,15,17,20], they all consider the large time behavior of a solution $t \mapsto (S(t), I(t))$ of (1.1) for a fixed set of initial conditions $S(0) = x$ and $I(0) = y$.

This paper is organized as follows. In section 2, we will study u and prove Theorem 1.1. Then in section 3, we will indicate what changes are necessary so that our proof of Theorem 1.1 adapts to Theorem 1.2. We will prove Theorem 1.3 in section 4 and offer some perspective on these results in section 5. Finally, we will explain how we obtained numerical approximations of u and v in the appendix.

2. The first time $I(t) \leq \mu$

This section is dedicated to proving Theorem 1.1. We will begin with an elementary upper bound on u.

Lemma 2.1. *For each $x \geq 0$ and $y \geq \mu$,*

$$u(x, y) \leq \frac{x+y}{\gamma\mu}.$$

Proof. Let S, I be the solution pair of the SIR ODE (1.1) with $S(0) = x$ and $I(0) = y$. Since $\frac{d}{dt}(S+I)(t) = -\gamma I(t)$,

$$\int_0^t \gamma I(\tau) d\tau + S(t) + I(t) = x + y.$$

Choosing $t = u(x, y)$ and noting that $I(\tau) \geq \mu$ for $0 \leq \tau \leq u(x, y)$ gives

$$u(x, y)\gamma\mu \leq \int_0^{u(x,y)} \gamma I(\tau) d\tau \leq x + y. \quad \square$$

Next we observe that I is always decreasing at the time it reaches the threshold μ.

Lemma 2.2. *Let $x > 0$, $y > \mu$, and suppose S, I is the solution of (1.1) with $S(0) = x$ and $I(0) = y$. Then*

$$\dot{I}(u(x, y)) < 0.$$

Proof. As $\dot{I}(t) = (\beta S(t) - \gamma)I(t)$, it suffices to show

$$\beta S(u(x, y)) - \gamma < 0. \tag{2.1}$$

If $\beta x \leq \gamma$, then $\beta S(t) < \gamma$ for all $t > 0$ since S is decreasing. Consequently, (2.1) holds for $t = u(x, y)$. Alternatively, if $\beta x > \gamma$, then I is increasing on the interval $[0, v(x, y)]$ and

$$\beta S(v(x, y)) - \gamma = 0.$$

In particular, $I(v(x, y)) \geq y > \mu$. Thus, $v(x, y) < u(x, y)$. As S is decreasing,

$$\beta S(u(x, y)) - \gamma < \beta S(v(x, y)) - \gamma = 0. \quad \square$$

Remark 2.3. Since $v(x, y) = 0$ for $x \in [0, \gamma/\beta]$ and $y > 0$, it also follows from this proof that

$$v(x, y) \leq u(x, y)$$

for each $x \geq 0$ and $y \geq \mu$.

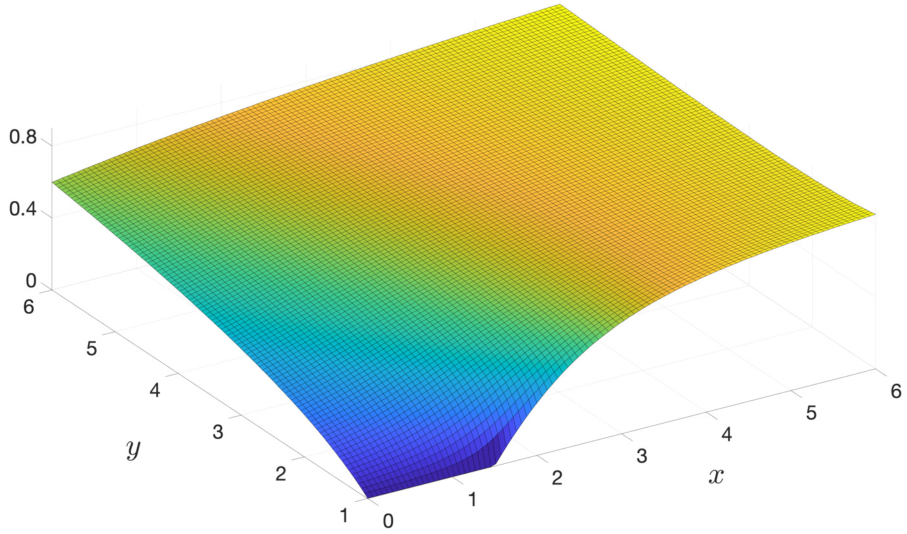

Fig. 4. Numerical approximation of the function $u(x, y)$ for $(x, y) \in [0, 6] \times [1, 5]$. Here $\mu = 1$, $\beta = 2$, and $\gamma = 3$. See the appendix for a description of how this graph was obtained.

In our proof of Theorem 1.1 below, we will employ the flow of the SIR ODE (1.1). This is the mapping

$$\Phi : [0, \infty)^3 \to [0, \infty)^2; (x, y, t) \mapsto (S(t), I(t))$$

where S, I is the solution pair of (1.1) with $S(0) = x$ and $I(0) = y$. We will also write $\Phi = (\Phi_1, \Phi_2)$ so that

$$\Phi_1(x, y, t) = S(t) \text{ and } \Phi_2(x, y, t) = I(t).$$

It is routine to verify that Φ is a continuous mapping. It can also be shown that $\Phi : (0, \infty)^3 \to [0, \infty)^2$ is in fact smooth. See for example Theorem 4.1 in Chapter V of [12].

Proof of Theorem 1.1. (i) Suppose $x^k \geq 0$ and $y^k \geq \mu$ with $x^k \to x$ and $y^k \to y$ as $k \to \infty$. By Lemma 2.1, $u(x^k, y^k)$ is bounded. We can then select a subsequence $u(x^{k_j}, y^{k_j})$ such that

$$t := \liminf_{k\to\infty} u(x^k, y^k) = \lim_{j\to\infty} u(x^{k_j}, y^{k_j}).$$

From the definition of u, we also have

$$\Phi_2(x^k, y^k, u(x^k, y^k)) = \mu \tag{2.2}$$

for each $k \in \mathbb{N}$. Since Φ_2 is continuous, we can send $k = k_j \to \infty$ in (2.2) to get

$$\Phi_2(x, y, t) = \mu.$$

And as $y \geq \mu$,

$$u(x, y) \leq t = \liminf_{k\to\infty} u(x^k, y^k).$$

We can also choose another subsequence $u(x^{k_\ell}, y^{k_\ell})$ for which

$$s := \limsup_{k\to\infty} u(x^k, y^k) = \lim_{\ell\to\infty} u(x^{k_\ell}, y^{k_\ell}).$$

The same argument above gives

$$\Phi_2(x, y, s) = \mu.$$

If $y > \mu$ or if $0 \le x \le \gamma/\beta$ and $y = \mu$, s is the unique time $\tau \ge 0$ such that $\Phi_2(x, y, \tau) = \mu$. In either case, $s = u(x, y)$. Alternatively, if $x > \gamma/\beta$ and $y = \mu$, then $\tau \mapsto \Phi_2(x, y, \tau)$ starts at $y = \mu$, increases for an interval of time, and then decreases to 0 as $\tau \to \infty$. It must be that either $s = 0 \le u(x, y)$ or $s = u(x, y)$. Consequently, in all scenarios

$$s = \limsup_{k\to\infty} u(x^k, y^k) \le u(x, y).$$

Therefore,

$$u(x, y) = \lim_{k\to\infty} u(x^k, y^k).$$

It follows that u is continuous on $[0, \infty) \times [\mu, \infty)$.

Let $x > 0$ and $y > \mu$ and recall that $\Phi_2(x, y, u(x, y)) = \mu$. By Lemma 2.2,

$$\partial_t \Phi_2(x, y, u(x, y)) < 0.$$

Since Φ_2 is smooth in a neighborhood of $(x, y, u(x, y))$, the implicit function theorem implies that u is smooth in a neighborhood of (x, y). We conclude that u is smooth in $(0, \infty) \times (\mu, \infty)$.

(ii) Fix $x > 0$ and $y > \mu$, and let S, I be the solution of (1.1) with $S(0) = x$ and $I(0) = y$. Observe that for each $0 \le t < u(x, y)$,

$$\begin{aligned}
u(S(t), I(t)) &= \inf\{\tau > 0 : I(t + \tau) \le \mu\} \\
&= \inf\{\tau : I(t + \tau) \le \mu \text{ and } \tau > 0\} \\
&= \inf\{s - t : I(s) \le \mu \text{ and } s > t\} \\
&= \inf\{s : I(s) \le \mu \text{ and } s > t\} - t \\
&= \inf\{s > t : I(s) \le \mu\} - t \\
&= u(x, y) - t.
\end{aligned}$$

The first equality above follows as the SIR ODE admits a unique solution; the last equality is due to our assumption that $t < u(x, y)$. Therefore,

$$\begin{aligned}
1 &= -\left.\frac{d}{dt} u(S(t), I(t))\right|_{t=0} \\
&= \left.\Big[\beta S(t)I(t)\partial_x u(S(t), I(t)) + (\gamma - \beta S(t))I(t)\partial_y u(S(t), I(t))\Big]\right|_{t=0} \\
&= \beta xy\partial_x u(x, y) + (\gamma - \beta x)y\partial_y u(x, y).
\end{aligned}$$

We conclude that u satisfies (1.5).

Now suppose w is a solution of (1.5) which satisfies the boundary condition (1.6). Note

$$\frac{d}{dt} w(S(t), I(t)) = -\Big[\beta S(t)I(t)\partial_x w(S(t), I(t)) + (\gamma - \beta S(t))I(t)\partial_y w(S(t), I(t))\Big] = -1$$

for $0 \le t < u(x, y)$. Integrating this equation from $t = 0$ to $t = u(x, y)$ gives

$$w(S(u(x,y)), I(u(x,y))) - w(x,y) = -u(x,y).$$

Since $I(u(x,y)) = \mu$, $\beta S(u(x,y)) \le \gamma$, and $w(S(u(x,y)), \mu) = 0$, it follows that $w(x,y) = u(x,y)$. Therefore, u is the unique solution of the PDE (1.5) which satisfies the boundary condition (1.6).

(iii) Suppose $x > 0$ and $y > \mu$ or $x > \gamma/\beta$ and $y = \mu$. We recall that $a = a(x,y) \in (0, \gamma/\beta]$ is the unique solution of

$$\psi(x,y) = \psi(a,\mu).$$

It is also easy to check that

$$\frac{\gamma}{\beta}\ln z - z + \psi(x,y) > \mu$$

for $a < z < x$. See Fig. 2 for an example.

Since u solves the PDE (1.5),

$$\begin{aligned}\frac{d}{dz}u\left(z, \frac{\gamma}{\beta}\ln z - z + \psi(x,y)\right) &= \frac{\beta zy\partial_x u(z,y) + (\gamma - \beta z)y\partial_y u(z,y)}{\beta zy}\Bigg|_{y=\frac{\gamma}{\beta}\ln z - z + \psi(x,y)} \\ &= \frac{1}{\beta z\left(\frac{\gamma}{\beta}\ln z - z + \psi(x,y)\right)}\end{aligned}$$

for $a < z < x$. Integrating from $z = a$ to $z = x$ and using the boundary condition (1.6) gives

$$u(x,y) = \int_a^x \frac{dz}{\beta z\left((\gamma/\beta)\ln z - z + \psi(x,y)\right)}. \quad \square$$

3. The first time $S(t) \le \gamma/\beta$

We will briefly point out what needs to be adapted from the previous section so that we can conclude Theorem 1.2 involving v. We first note that v is locally bounded in $[\gamma/\beta, \infty) \times (0, \infty)$.

Lemma 3.1. *For each $x \ge \gamma/\beta$ and $y > 0$,*

$$v(x,y) \le \frac{\ln x - \ln(\gamma/\beta)}{\beta y}.$$

Proof. Let S, I denote the solution of (1.1) with $S(0) = x$ and $I(0) = y$. Since $I(t)$ is increasing on $t \in [0, v(x,y)]$, $I(t) \ge y$ for $t \in [0, v(x,y)]$. In view of (1.2),

$$\frac{\gamma}{\beta} = S(v(x,y)) = xe^{-\beta \int_0^{v(x,y)} I(\tau)d\tau} \le xe^{-\beta y v(x,y)}.$$

Taking the natural logarithm and rearranging leads to $v(x,y) \le (\ln x - \ln(\gamma/\beta))/\beta y$, which is what we wanted to show. $\square$

The next assertion follows since S is decreasing whenever I is initially positive. The main point of stating this lemma is to make an analogy with Lemma 2.2.

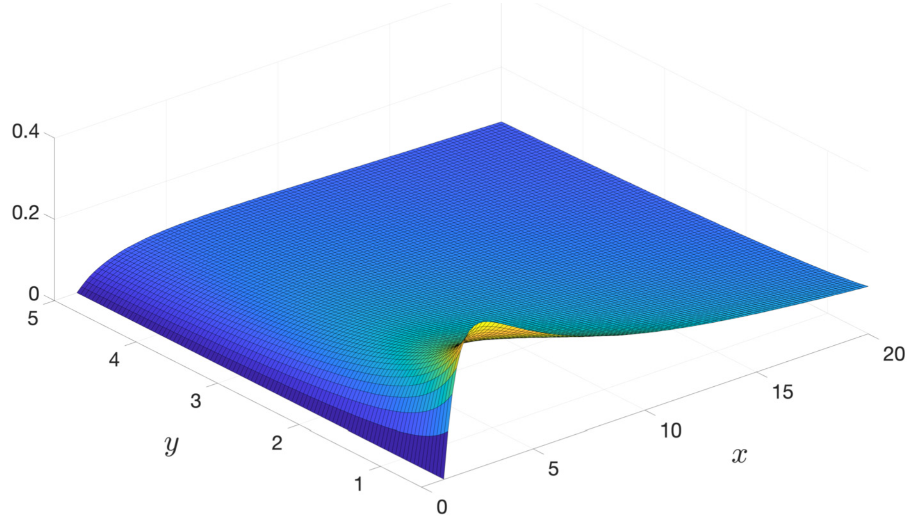

Fig. 5. Numerical approximation of the function $v(x, y)$ for $(x, y) \in [1, 20] \times [1/2, 5]$. Here $\beta = \gamma = 3$. Consult the appendix for an explanation of how we arrived at this graph.

Lemma 3.2. *Let $x \geq \gamma/\beta$ and $y > 0$, and suppose S, I is the solution of (1.1) with $S(0) = x$ and $I(0) = y$. Then*

$$\dot{S}(v(x, y)) < 0.$$

Having established Lemmas 3.1 and 3.2, we can now argue virtually the same way we did in the previous section to conclude Theorem 1.2. Consequently, we will omit a proof.

4. Asymptotics

In this section, we will derive a few estimates on $u(x, y)$ and $v(x, y)$ that we will need to prove Theorem 1.3. First, we record an upper and lower bound on u.

Lemma 4.1. *If $x \geq 0$ and $y \geq \mu$, then*

$$u(x, y) \geq \frac{1}{\gamma} \ln\left(\frac{x + y}{\gamma/\beta + \mu}\right). \tag{4.1}$$

If $x \in [0, \gamma/\beta)$ and $y \geq \mu$, then

$$u(x, y) \leq \frac{\ln(y/\mu)}{\gamma - \beta x}. \tag{4.2}$$

Proof. Set

$$w(x, y) = \frac{1}{\gamma} \ln\left(\frac{x + y}{\gamma/\beta + \mu}\right).$$

Observe that for each $x \geq 0$ and $y \geq \mu$,

$$\beta xy\partial_x w + (\gamma - \beta x)y\partial_y w = \frac{1}{\gamma}\frac{\beta xy + (\gamma - \beta x)y}{x + y} = \frac{y}{x + y} \leq 1.$$

Now fix $x \geq 0$ and $y \geq \mu$ and suppose S, I is the solution of (1.1) with $S(0) = x$ and $I(0) = y$. By our computation above,

$$\frac{d}{dt}w(S(t), I(t)) = -\Big[\beta S(t)I(t)\partial_x w(S(t), I(t)) + (\gamma - \beta S(t))I(t)\partial_y w(S(t), I(t))\Big] \geq -1$$

for $0 \leq t \leq u(x, y)$. And integrating this inequality from $t = 0$ to $t = u(x, y)$ gives

$$w(S(u(x, y)), \mu) - w(x, y) \geq -u(x, y). \tag{4.3}$$

Since $S(u(x, y)) \leq \gamma/\beta$,

$$w(S(u(x, y)), \mu) = \frac{1}{\gamma}\ln\left(\frac{S(u(x, y)) + \mu}{\gamma/\beta + \mu}\right) \leq 0.$$

Combined with (4.3) this implies $u(x, y) \geq w(x, y)$. We conclude (4.1).

Now suppose $\beta x < \gamma$ and $y > \mu$. By (1.2),

$$\mu = I(u(x, y)) = ye^{\int_0^{u(x,y)} (\beta S(\tau) - \gamma)d\tau} \leq ye^{(\beta x - \gamma)u(x,y)}.$$

Taking the natural logarithm and rearranging gives (4.2). □

Likewise, we can identify convenient upper and lower bounds for $v(x, y)$. To this end, we will exploit the fact that for each $x > \gamma/\beta$ and $y > 0$

$$g(z) = (\gamma/\beta)\ln z - z + \psi(x, y) \tag{4.4}$$

is concave on the interval $\gamma/\beta \leq z \leq x$. This implies

$$g(z) \geq \frac{g(\gamma/\beta) - y}{\gamma/\beta - x}(z - x) + g(x) = \left(\frac{\gamma}{\beta}\frac{\ln x - \ln(\gamma/\beta)}{x - \gamma/\beta} - 1\right)(z - x) + y \tag{4.5}$$

and

$$g(z) \leq g'(x)(z - x) + g(x) = \left(\frac{\gamma}{\beta x} - 1\right)(z - x) + y \tag{4.6}$$

for $\gamma/\beta \leq z \leq x$.

Lemma 4.2. *Suppose $x > \gamma/\beta$ and $y > 0$. Then*

$$v(x, y) \leq \frac{\ln x - \ln(\gamma/\beta) - \ln y + \ln\left(x - \gamma/\beta + y - \frac{\gamma}{\beta}(\ln x - \ln(\gamma/\beta))\right)}{\beta\left(y + x\left(1 - \frac{\gamma}{\beta}\frac{\ln x - \ln(\gamma/\beta)}{x - \gamma/\beta}\right)\right)} \tag{4.7}$$

and

$$v(x, y) \geq \frac{\ln x - \ln(\gamma/\beta) - \ln y + \ln\left(x - \gamma/\beta + y + \frac{\gamma}{\beta}(\frac{\gamma}{\beta x} - 1)\right)}{\beta(x - \gamma/\beta + y)}. \tag{4.8}$$

Proof. We will appeal to part (iii) of Theorem 1.2 which asserts

$$v(x,y)=\int\limits_{\gamma/\beta}^{x}\frac{dz}{\beta z g(z)}.$$

Here $g(z)$ is defined in (4.4). We will also employ the identity

$$\frac{d}{dz}\frac{\ln z-\ln(cz+d)}{d}=\frac{1}{z(cz+d)}. \tag{4.9}$$

By (4.5) and (4.9),

$$\begin{aligned}
v(x,y)&\le\int\limits_{\gamma/\beta}^{x}\frac{dz}{\beta z\left(\left(\frac{\gamma}{\beta}\frac{\ln x-\ln(\gamma/\beta)}{x-\gamma/\beta}-1\right)(z-x)+y\right)}\\
&=\left.\frac{\ln z-\ln\left(\left(\frac{\gamma}{\beta}\frac{\ln x-\ln(\gamma/\beta)}{x-\gamma/\beta}-1\right)(z-x)+y\right)}{\beta\left(y+x\left(1-\frac{\gamma}{\beta}\frac{\ln x-\ln(\gamma/\beta)}{x-\gamma/\beta}\right)\right)}\right|_{z=\gamma/\beta}^{z=x}\\
&=\frac{\ln x-\ln(\gamma/\beta)-\ln y+\ln\left(x-\gamma/\beta+y-\frac{\gamma}{\beta}(\ln x-\ln(\gamma/\beta))\right)}{\beta\left(y+x\left(1-\frac{\gamma}{\beta}\frac{\ln x-\ln(\gamma/\beta)}{x-\gamma/\beta}\right)\right)}.
\end{aligned}$$

Similarly, (4.6) and (4.9) give

$$\begin{aligned}
v(x,y)&\ge\int\limits_{\gamma/\beta}^{x}\frac{dz}{\beta z\left(\left(\frac{\gamma}{\beta x}-1\right)(z-x)+y\right)}\\
&=\left.\frac{\ln z-\ln\left(\left(\frac{\gamma}{\beta x}-1\right)(z-x)+y\right)}{\beta(x-\gamma/\beta+y)}\right|_{z=\gamma/\beta}^{z=x}\\
&=\frac{\ln x-\ln(\gamma/\beta)-\ln y+\ln\left(x-\gamma/\beta+y+\frac{\gamma}{\beta}(\frac{\gamma}{\beta x}-1)\right)}{\beta(x-\gamma/\beta+y)}. \qquad\square
\end{aligned}$$

Corollary 4.3. *For each $\delta>0$,*

$$\lim_{\substack{x+y\to\infty\\ x\ge0,y\ge\delta}}v(x,y)=0. \tag{4.10}$$

Proof. Choose sequences $x_k\ge0$ and $y_k\ge\delta$ with $x_k+y_k\to\infty$ such that

$$\limsup_{\substack{x+y\to\infty\\ x\ge0,y\ge\delta}}v(x,y)=\lim_{k\to\infty}v(x_k,y_k).$$

If $x_k\le\gamma/\beta$ for infinitely many $k\in\mathbb{N}$, then $v(x_k,y_k)=0$ for infinitely many k and

$$\lim_{k\to\infty} v(x_k, y_k) = 0. \tag{4.11}$$

Otherwise, we may as well suppose that $x_k > \gamma/\beta$ for all $k \in \mathbb{N}$. In this case, (4.7) implies

$$v(x_k, y_k) \le \frac{\ln x_k - \ln(\gamma/\beta) - \ln y_k + \ln\left(x_k - \gamma/\beta + y_k - \frac{\gamma}{\beta}(\ln x_k - \ln(\gamma/\beta))\right)}{\beta\left(y_k + x_k\left(1 - \frac{\gamma}{\beta}\frac{\ln x_k - \ln(\gamma/\beta)}{x_k - \gamma/\beta}\right)\right)}$$

for all $k \in \mathbb{N}$.

If $x_k \to \infty$, then

$$\frac{\gamma}{\beta}\frac{\ln x_k - \ln(\gamma/\beta)}{x_k - \gamma/\beta} \le \frac{1}{2}$$

for sufficiently large k. It follows that

$$\begin{aligned} v(x_k, y_k) &\le \frac{\ln x_k - \ln(\gamma/\beta) - \ln y_k + \ln\left(x_k - \gamma/\beta + y_k - \frac{\gamma}{\beta}(\ln x_k - \ln(\gamma/\beta))\right)}{\beta\left(y_k + \frac{1}{2}x_k\right)} \\ &\le \frac{\ln x_k - \ln(\gamma/\beta) + \ln\left(\frac{x_k - \gamma/\beta}{y_k} + 1\right)}{\beta\left(y_k + \frac{1}{2}x_k\right)} \\ &\le \frac{\ln x_k - \ln(\gamma/\beta) + \ln\left(\frac{x_k - \gamma/\beta}{\delta} + 1\right)}{\beta\left(\delta + \frac{1}{2}x_k\right)} \end{aligned}$$

for all large enough k. Therefore, (4.11) holds.

Alternatively, we can pass to a subsequence if necessary and suppose $x_k \le c$ for all $k \in \mathbb{N}$ and $y_k \to \infty$. Lemma 3.1 then gives

$$v(x_k, y_k) \le \frac{\ln c - \ln(\gamma/\beta)}{\beta y_k} \to 0.$$

As a result, (4.11) holds in all cases. It follows that

$$\limsup_{\substack{x+y\to\infty \\ x\ge 0, y\ge\delta}} v(x, y) = 0,$$

which in turn implies (4.10). $\square$

We are now ready to prove Theorem 1.3 which asserts

$$\lim_{\substack{x+y\to\infty \\ x\ge 0,\ y\ge\mu}} \frac{u(x, y)}{\frac{1}{\gamma}\ln\left(\frac{x+y}{\mu}\right)} = 1 \tag{4.12}$$

and

$$\lim_{\substack{x+y\to\infty \\ x>\gamma/\beta,\ y>0}} \frac{\beta(x - \gamma/\beta + y)}{\ln\left[\left(\frac{x}{\gamma/\beta}\right)\left(\frac{x-\gamma/\beta}{y} + 1\right)\right]} \cdot v(x, y) = 1. \tag{4.13}$$

Proof of (4.12). In view of (4.1),

$$\liminf_{\substack{x+y\to\infty\\ x\ge 0\ y\ge\mu}} \frac{u(x,y)}{\frac{1}{\gamma}\ln\left(\frac{x+y}{\mu}\right)} \ge 1.$$

It follows that $u(x,y)\to\infty$ as $x+y\to\infty$ with $x\ge 0$ and $y\ge\mu$. In view of Corollary (4.10), we may select $N\in\mathbb{N}$ so large that

$$v(x,y) < u(x,y)$$

for all $x+y\ge N$ with $x\ge 0$ and $y\ge\mu$.

Suppose $x+y\ge N$ with $x\ge 0$ and $y\ge\mu$ and choose a time t such that

$$v(x,y) < t < u(x,y).$$

Note that as $t > v(x,y)$,

$$S(t) < \frac{\gamma}{\beta}.$$

Here S, I is the solution of the SIR ODE (1.1) with $S(0)=x$ and $I(0)=y$. By (4.2), we also have

$$\begin{aligned} u(x,y) &= t + u(S(t), I(t))\\ &\le t + \frac{1}{\gamma-\beta S(t)}\ln\left(\frac{I(t)}{\mu}\right)\\ &\le t + \frac{1}{\gamma-\beta S(t)}\ln\left(\frac{x+y}{\mu}\right). \end{aligned} \tag{4.14}$$

In addition, we can use (1.2) to find

$$\begin{aligned} S(t) &= S(v(x,y))e^{-\beta\int_{v(x,y)}^{t} I(\tau)d\tau}\\ &= \frac{\gamma}{\beta}e^{-\beta\int_{v(x,y)}^{t} I(\tau)d\tau}\\ &\le \frac{\gamma}{\beta}e^{-\beta\mu(t-v(x,y))}. \end{aligned}$$

Here we used that $I(\tau)\ge\mu$ as $\tau\le t<u(x,y)$. Therefore,

$$\gamma - \beta S(t) \ge \gamma\left(1 - e^{-\beta\mu(t-v(x,y))}\right).$$

Combining this inequality with (4.14) gives

$$u(x,y) \le t + \frac{1}{1-e^{-\beta\mu(t-v(x,y))}}\frac{1}{\gamma}\ln\left(\frac{x+y}{\mu}\right).$$

As a result,

$$\limsup_{\substack{x+y\to\infty\\ x\geq 0\ y\geq\mu}} \frac{u(x,y)}{\dfrac{1}{\gamma}\ln\left(\dfrac{x+y}{\mu}\right)} \leq \frac{1}{1-e^{-\beta\mu t}}.$$

We conclude (4.12) upon sending $t\to\infty$. $\square$

In our argument below, we will employ the elementary inequalities

$$\frac{1}{x} \leq \frac{\ln x - \ln(\gamma/\beta)}{x-\gamma/\beta} \leq \frac{\beta}{\gamma}, \tag{4.15}$$

which hold for $x > \gamma/\beta$. They follow as the natural logarithm is concave.

Proof of (4.13). By the upper bound (4.7),

$$\begin{aligned}
&\frac{\beta(x-\gamma/\beta+y)}{\ln\left[\left(\dfrac{x}{\gamma/\beta}\right)\left(\dfrac{x-\gamma/\beta}{y}+1\right)\right]}\cdot v(x,y)\\
&= \frac{\beta(x-\gamma/\beta+y)}{\ln x - \ln(\gamma/\beta) - \ln y + \ln\left(x-\gamma/\beta+y\right)}\cdot v(x,y)\\
&\leq \frac{\beta(x+y)}{\ln x - \ln(\gamma/\beta) - \ln y + \ln\left(x-\gamma/\beta+y-\frac{\gamma}{\beta}(\ln x - \ln(\gamma/\beta))\right)}\cdot v(x,y)\\
&\leq \frac{x+y}{y + x\left(1-\dfrac{\gamma}{\beta}\dfrac{\ln x - \ln(\gamma/\beta)}{x-\gamma/\beta}\right)}\\
&= \frac{1}{1-\dfrac{x}{x+y}\dfrac{\gamma}{\beta}\left(\dfrac{\ln x - \ln(\gamma/\beta)}{x-\gamma/\beta}\right)}.
\end{aligned} \tag{4.16}$$

We may select sequences $x_k > \gamma/\beta$ and $y_k > 0$ with $x_k + y_k \to \infty$ and

$$\limsup_{\substack{x+y\to\infty\\ x>\gamma/\beta,\ y>0}} \frac{x}{x+y}\frac{\gamma}{\beta}\left(\frac{\ln x - \ln(\gamma/\beta)}{x-\gamma/\beta}\right) = \lim_{k\to\infty}\frac{x_k}{x_k+y_k}\frac{\gamma}{\beta}\left(\frac{\ln x_k - \ln(\gamma/\beta)}{x_k-\gamma/\beta}\right). \tag{4.17}$$

If $x_k\to\infty$, then

$$0 \leq \frac{x_k}{x_k+y_k}\frac{\gamma}{\beta}\left(\frac{\ln x_k - \ln(\gamma/\beta)}{x_k-\gamma/\beta}\right) \leq \frac{\gamma}{\beta}\frac{\ln x_k}{x_k-\gamma/\beta} \to 0.$$

Alternatively, x_k has a bounded subsequence. Passing to a subsequence if necessary, we may assume that $x_k \leq c$ for some constant c. In which case, $y_k\to\infty$. Employing (4.15), we find

$$0 \leq \frac{x_k}{x_k+y_k}\frac{\gamma}{\beta}\left(\frac{\ln x_k - \ln(\gamma/\beta)}{x_k-\gamma/\beta}\right) \leq \frac{x_k}{x_k+y_k} \leq \frac{c}{y_k} \to 0.$$

It follows that the limit in (4.17) is 0. And in view of (4.16),

$$\limsup_{\substack{x+y\to\infty\\ x>\gamma/\beta,\ y>0}} \frac{\beta(x-\gamma/\beta+y)}{\ln\left[\left(\frac{x}{\gamma/\beta}\right)\left(\frac{x-\gamma/\beta}{y}+1\right)\right]}\cdot v(x,y)\le 1.$$

By the lower bound (4.8),

$$\begin{aligned}\frac{\beta(x-\gamma/\beta+y)v(x,y)}{\ln\left[\left(\frac{x}{\gamma/\beta}\right)\left(\frac{x-\gamma/\beta}{y}+1\right)\right]} &= \frac{\beta(x-\gamma/\beta+y)v(x,y)}{\ln x-\ln(\gamma/\beta)-\ln y+\ln\left(x-\gamma/\beta+y\right)}\\ &\ge \frac{\ln x-\ln(\gamma/\beta)-\ln y+\ln\left(x-\gamma/\beta+y+\frac{\gamma}{\beta}(\frac{\gamma}{\beta x}-1)\right)}{\ln x-\ln(\gamma/\beta)-\ln y+\ln\left(x-\gamma/\beta+y\right)}\\ &= 1+\frac{\ln\left(\dfrac{x-\gamma/\beta+y+\frac{\gamma}{\beta}(\frac{\gamma}{\beta x}-1)}{x-\gamma/\beta+y}\right)}{\ln x-\ln(\gamma/\beta)-\ln y+\ln\left(x-\gamma/\beta+y\right)}\\ &= 1+\frac{\ln\left(1+\dfrac{\frac{\gamma}{\beta}(\frac{\gamma}{\beta x}-1)}{x-\gamma/\beta+y}\right)}{\ln\left(\dfrac{x}{\gamma/\beta}\right)+\ln\left(\dfrac{x-\gamma/\beta}{y}+1\right)}. \end{aligned} \tag{4.18}$$

Let us choose sequences $x_k>\gamma/\beta$ and $y_k>0$ such that $x_k+y_k\to\infty$ and

$$\liminf_{\substack{x+y\to\infty\\ x>\gamma/\beta,\ y>0}} \frac{\ln\left(1+\dfrac{\frac{\gamma}{\beta}(\frac{\gamma}{\beta x}-1)}{x-\gamma/\beta+y}\right)}{\ln\left(\dfrac{x}{\gamma/\beta}\right)+\ln\left(\dfrac{x-\gamma/\beta}{y}+1\right)} = \lim_{k\to\infty}\frac{\ln\left(1+\dfrac{\frac{\gamma}{\beta}(\frac{\gamma}{\beta x_k}-1)}{x_k-\gamma/\beta+y_k}\right)}{\ln\left(\dfrac{x_k}{\gamma/\beta}\right)+\ln\left(\dfrac{x_k-\gamma/\beta}{y_k}+1\right)}. \tag{4.19}$$

We recall $\ln(1+z)\ge\frac{3}{2}z$ for all nonpositive z sufficiently close to 0. Since

$$0\ge\frac{\dfrac{\gamma}{\beta}\left(\dfrac{\gamma}{\beta x_k}-1\right)}{x_k-\gamma/\beta+y_k}\to 0,$$

we then have

$$\ln\left(1+\frac{\frac{\gamma}{\beta}(\frac{\gamma}{\beta x_k}-1)}{x_k-\gamma/\beta+y_k}\right)\ge\frac{3}{2}\,\frac{\dfrac{\gamma}{\beta}\left(\dfrac{\gamma}{\beta x_k}-1\right)}{x_k-\gamma/\beta+y_k}$$

for all sufficiently large $k\in\mathbb{N}$. Furthermore,

$$\begin{aligned}0&\ge\frac{\dfrac{2}{3}\ln\left(1+\dfrac{\frac{\gamma}{\beta}(\frac{\gamma}{\beta x_k}-1)}{x_k-\gamma/\beta+y_k}\right)}{\ln\left(\dfrac{x_k}{\gamma/\beta}\right)+\ln\left(\dfrac{x_k-\gamma/\beta}{y_k}+1\right)}\\ &\ge\frac{\dfrac{\frac{\gamma}{\beta}\left(\frac{\gamma}{\beta x_k}-1\right)}{x_k-\gamma/\beta+y_k}}{\ln\left(\dfrac{x_k}{\gamma/\beta}\right)+\ln\left(\dfrac{x_k-\gamma/\beta}{y_k}+1\right)}\end{aligned}$$

$$\geq \frac{1}{\ln\left(\dfrac{x_k}{\gamma/\beta}\right)} \frac{\dfrac{\gamma}{\beta}\left(\dfrac{\gamma}{\beta x_k}-1\right)}{x_k-\gamma/\beta+y_k}$$
$$= -\frac{1}{x_k}\frac{x_k-\gamma/\beta}{\ln x_k-\ln(\gamma/\beta)}\frac{\gamma/\beta}{x_k-\gamma/\beta+y_k}$$
$$\geq -\frac{\gamma/\beta}{x_k-\gamma/\beta+y_k}.$$

In the last inequality, we used (4.15). We conclude the limit in (4.19) is 0. In view of inequality (4.18),

$$\liminf_{\substack{x+y\to\infty \\ x>\gamma/\beta,\ y>0}} \frac{\beta(x-\gamma/\beta+y)}{\ln\left[\left(\dfrac{x}{\gamma/\beta}\right)\left(\dfrac{x-\gamma/\beta}{y}+1\right)\right]}\cdot v(x,y)\geq 1. \quad \square$$

5. Discussion

In this paper, we studied two critical times related to the SIR model: the first time the infected population drops below a desired level and the first time the infected population starts to decrease. We considered these times as functions of the initial conditions to the SIR ODE and characterized them as solutions of a common PDE

$$\beta xy\partial_x w+(\gamma-\beta x)y\partial_y w=1$$

subject to appropriate boundary conditions. We anticipate that analogous questions (characterizing an important first instance or identifying when a qualitative change occurs) for more general compartmental models could be studied with a similar approach. This possibility is what is compelling about of this work. We sincerely hope that this work inspires other researchers to pursue related directions of inquiry.

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Appendix A. Approximating the graphs of u and v

Here we will describe the method used to approximate the graph of u in Fig. 4 and v in Fig. 5. Since the technique works for both functions, we will only focus on approximating the graph of u. As we explain our method, we will refer to the MALTAB functions `SIRfun`, `SIRsoln`, `PreImageTime`, and `SIRExitTime` and the MATLAB script we designed that are listed below.

We used MATLAB's `ode45` to integrate solutions to the SIR ODE (1.1) in `SIRsoln`; this function in turn relies on `SIRfun` which simply outputs the right hand side of the SIR ODE. We then approximated $u(x,y)$ with the function `SIRExitTime`. This was accomplished as follows. For a given input (x,y), we first used `ode45` to obtain an approximation for the corresponding solution of the SIR ODE $t\mapsto(S(t),I(t))$ on the interval

$$0\leq t\leq\frac{x+y}{\gamma\mu}. \tag{A.1}$$

Denoting this numerical solution as $(S(t_1), I(t_1)), \ldots, (S(t_N), I(t_N))$, we then used `PreImageTime` to find the first time t_k such that $I(t_k) \le \mu$; this time t_k also served as our approximation for $u(x, y)$. We note that since $u(x, y) \le (x + y)/\gamma\mu$, we only needed to consider times t in the interval above (A.1).

The last segment of code listed below is a script which approximates the graph of u for the parameters $\mu = 1, \beta = 2, \gamma = 3$ on the rectangle $0 \le x \le 6, \mu \le y \le 6$. In particular, it uses `SIRExitTime` to approximate $u(x_i, y_j)$ for

$$x_i = 6 \cdot \frac{i}{100} \quad \text{and} \quad y_j = \mu + (6 - \mu) \cdot \frac{j}{100}$$

where $i, j = 0, \ldots, 100$. The result of these computations is displayed in Fig. 4.

```
% Right hand side/ODE function in the SIR model

function Fun = SIRfun(X,beta,gamma)

Fun(1) =-beta*X(1)*X(2);
Fun(2) = beta*X(1)*X(2)-gamma*X(2);

Fun=Fun';

end
```

```
% Integrating SIR ODE on [0,T] with mesh size h and initial conditions X0

function [t,X] = SIRsoln(X0,beta,gamma,T,h)

tspan=linspace(0,T,1/h);

[t,X] = ode45(@(t,y) SIRfun(y,beta,gamma), tspan, X0);

end
```

```
% Finding the first time t(i) such that yvec(i)<mu

function tstar = PreImageTime(tvec,yvec,mu)

LogicVec=yvec<mu;

index=find(LogicVec,1);

tstar=tvec(index);

end
```

```
% Approximating u(X0)

function u = SIRExitTime(X0, mu,beta,gamma,h)

% An upper bound for u
T=sum(X0)/(mu*gamma);

[s,Y]=SIRsoln(X0,beta,gamma,T,h);

u=PreImageTime(s,Y(:,2),mu);

end
```

```
% This script plots an approximate graph of u for parameters specified below

% mu, beta, and gamma parameters
mmu=1;
```

```
bbeta=2;
ggamma=3;

% grid points
xmax=6;
ymax=6;
x=linspace(0,xmax,100);
y=linspace(mmu,ymax,100);

% initializing u
u=zeros(length(x),length(y));

for i=1:length(x)

  for j=1:length(y)

  % Computing the approximation for u(x(i),y(j))
  u(i,j)=SIRExitTime([x(i) y(j)], mmu,bbeta,ggamma,.001);

  end

end

% approximate graph
[X,Y]=meshgrid(x,y);
surf(X,Y,u')
```

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