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Sticky particles and the pressureless Euler equations in one spatial dimension

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Abstract
We consider the dynamics of finite systems of point masses which move along the real line. We suppose the particles interact pairwise and undergo perfectly inelastic collisions when they collide. In particular, once particles collide, they remain stuck together thereafter. Our main result is that if the interaction potential is semi convex, this sticky particle property can be quantified and is preserved upon letting the number of particles tend to infinity. This is used to show that solutions of the pressureless Euler equations exist for given initial conditions and satisfy an entropy inequality.

Keywords Conservation laws · Convergence of probability measures

1 Introduction

In this paper, we will study solutions of the *pressureless Euler equations* in one spatial dimension. This is a system of partial differential equations comprised of the *conservation of mass*

$$\partial_t \rho + \partial_x(\rho v) = 0 \tag{1.1}$$

and the *conservation of momentum*

$$\partial_t(\rho v) + \partial_x(\rho v^2) = -\rho(W' * \rho). \tag{1.2}$$

Both equations hold in $\mathbb{R} \times (0, \infty)$. This system governs the dynamics of collections of particles in one dimension whose pairwise interaction is determined by the potential W; these particles also may collide and they undergo perfectly inelastic collisions when they do. The unknowns are the density of particles ρ and an associated local velocity field v. Our main objective in this paper is to establish the existence of solutions for given initial conditions.

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We will suppose throughout this paper that $W : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and even

$$W(-x) = W(x), \quad x \in \mathbb{R}.$$

We note that W convex in (1.2) corresponds to particles interacting via an attractive pairwise force and W concave is associated with repulsive interaction. The principal assumption made in this work is that W is *semiconvex*. That is, there is $c > 0$ such that

$$x \mapsto W(x) + \frac{c}{2}x^2 \quad \text{is convex.} \tag{1.3}$$

In particular, we will study some types of interactions which are attractive and some which are repulsive.

In view of the conservation of mass (1.1), it will be natural for us to consider mass densities ρ as mappings with values in the space $\mathcal{P}(\mathbb{R})$ of Borel probability measures on $\mathbb{R}$. Recall that this space has a natural topology: $(\mu^k)_{k\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R})$ converges to $\mu \in \mathcal{P}(\mathbb{R})$ *narrowly* if

$$\lim_{k\to\infty} \int_{\mathbb{R}} g d\mu^k = \int_{\mathbb{R}} g d\mu$$

for any continuous and bounded function $g : \mathbb{R} \to \mathbb{R}$. Moreover, examples below will show that local velocities v will typically be discontinuous. However, we do expect local velocities to have reasonable integrability properties. These ideas motivate the following definition of a weak solution pair of the pressureless Euler equations.

Definition 1.1 Suppose $\rho_0 \in \mathcal{P}(\mathbb{R})$ and $v_0 : \mathbb{R} \to \mathbb{R}$ is continuous. A narrowly continuous $\rho : (0, \infty) \to \mathcal{P}(\mathbb{R}); t \mapsto \rho_t$ and Borel $v : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ is a *weak solution pair of the pressureless Euler equations* which satisfies the initial conditions

$$\rho|_{t=0} = \rho_0 \quad \text{and} \quad v|_{t=0} = v_0 \tag{1.4}$$

if

$$\int_0^\infty \int_{\mathbb{R}} (\partial_t \phi + v\partial_x \phi) d\rho_t dt + \int_{\mathbb{R}} \phi(\cdot, 0) d\rho_0 = 0 \tag{1.5}$$

and

$$\int_0^\infty \int_{\mathbb{R}} (v\partial_t \phi + v^2 \partial_x \phi) d\rho_t dt + \int_{\mathbb{R}} \phi(\cdot, 0) v_0 d\rho_0 = \int_0^\infty \int_{\mathbb{R}} \phi(W' * \rho_t) d\rho_t dt \tag{1.6}$$

for each $\phi \in C_c^\infty(\mathbb{R} \times [0, \infty))$.

Remark 1.2 Conditions (1.5) and (1.6) are weak formulations of (1.1) and (1.2), respectively.

We will construct weak solution pairs using finite particle systems. That is, we will study systems of particles with masses $m_1, \dots, m_N$ and respective trajectories $\gamma_1, \dots, \gamma_N : [0, \infty) \to \mathbb{R}$ that evolve in time according to Newton's second law

$$\ddot{\gamma}_i(t) = -\sum_{j=1}^{N} m_j W'(\gamma_i(t) - \gamma_j(t)). \tag{1.7}$$

This system of ODE will hold at each time where there is not a collision. When particles do collide, they experience perfectly inelastic collisions. For example, if the subcollection of particles with masses $m_1, \dots, m_k$ collide at time $s > 0$, then

$$m_1\dot{\gamma}_1(s-) + \dots + m_k\dot{\gamma}_k(s-) = (m_1 + \dots + m_k)\dot{\gamma}_i(s+)$$

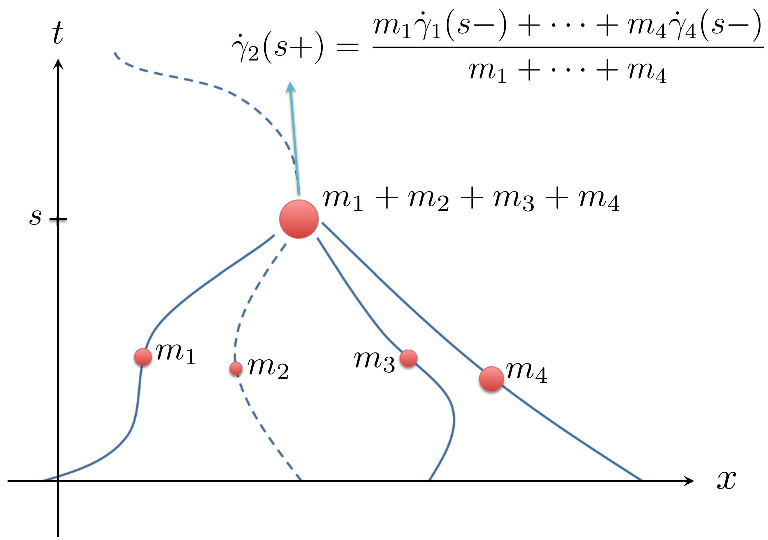

Fig. 1 Point masses m_1, m_2, m_3, and m_4 undergo a perfectly inelastic collision at time s. These masses are displayed larger than points to emphasize that they are allowed to be distinct. Note in particular that the trajectories coincide after time s and that the right limit of their slopes at time s also agree. Trajectory γ_2 is shown in dashed

for $i = 1, \ldots, k$. See Fig. 1 for a schematic diagram when $k = 4$.

When $\sum_{i=1}^N m_i = 1$, we can define a probability measure

$$\rho_t := \sum_{i=1}^N m_i \delta_{\gamma_i(t)} \in \mathcal{P}(\mathbb{R})$$

which represents the density of particles at time $t > 0$. We can also choose a Borel function $v : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ that satisfies

$$v(x, t) = \dot\gamma_i(t+)$$

whenever $x = \gamma_i(t)$. Here v is a local velocity field associated to particle trajectories. It turns out that ρ and v indeed comprise a weak solution pair of the pressureless Euler equations.

These particle trajectories satisfy

$$\gamma_i(s) = \gamma_j(s) \Longrightarrow \gamma_i(t) = \gamma_j(t).$$

for $i, j = 1, \ldots, N$ and $s \leq t$. We will actually establish the stronger *quantitative sticky particle property*: for $i, j = 1, \ldots, N$ and $0 < s \leq t$

$$\frac{|\gamma_i(t) - \gamma_j(t)|}{\sinh(\sqrt{c}t)} \leq \frac{|\gamma_i(s) - \gamma_j(s)|}{\sinh(\sqrt{c}s)}. \tag{1.8}$$

Here c is the constant in (1.3). Combining (1.8) with energy estimates derived using the semiconvexity of W, we will show that the collection of solutions obtained via finite particle systems are compact in a certain sense.

To this end, we shall assume for mathematical convenience that

$$\int_{\mathbb{R}} x^2 d\rho_0(x) < \infty \tag{1.9}$$

and

$$\sup_{x\in\mathbb{R}} \frac{|W'(x)|}{1+|x|} < \infty. \tag{1.10}$$

Our main theorem is as follows.

Theorem 1.3 *Assume $\rho_0 \in \mathcal{P}(\mathbb{R})$ satisfies (1.9), $v_0 : \mathbb{R} \to \mathbb{R}$ is absolutely continuous, and $W : \mathbb{R} \to \mathbb{R}$ satisfies (1.3) for some $c > 0$ and (1.10). There is a weak solution pair ρ and v of the pressureless Euler system which satisfies the initial conditions (1.4). Moreover,*

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}} v(x,t)^2 d\rho_t(x) + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} W(x-y)d\rho_t(x)d\rho_t(y) \\ &\quad \le \frac{1}{2}\int_{\mathbb{R}} v(x,s)^2 d\rho_s(x) + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} W(x-y)d\rho_s(x)d\rho_s(y) \end{aligned} \tag{1.11}$$

for almost every $0 \le s \le t$; and

$$(v(x,t) - v(y,t))(x-y) \le \frac{\sqrt{c}}{\tanh(\sqrt{c}t)}(x-y)^2 \tag{1.12}$$

for ρ_t almost every $x, y \in \mathbb{R}$ and almost every $t > 0$.

We note that the pressureless Euler equations arise in a one-dimensional version of a model used by Zel'dovich [9, 21] to study the formation of large scale structures in the universe. The existence of the corresponding weak solution pairs for $W(x) = |x|$ was first established by E, Rykov and Sinai [6] using a generalized variational principle. More recently, Brenier, Gangbo, Savaré and Westdickenberg [3] conducted a general study of pressureless Euler models with attractive and repulsive interactions; in particular, they recast the pressureless Euler equations in Lagrangian coordinates and derived differential inclusions for the associated flow map.

Other approaches to study weak solutions of the pressureless Euler system include the following works. Brenier and Grenier in [4] used a certain scalar conservation law to study the sticky particle system (which corresponds to $W \equiv 0$ in (1.2)); Natile and Savaré in [19] and Cavalletti, Sedjro, and Westdickenberg in [5] investigated a flow in the cone of monotone functions in $L^2([0, 1])$ associated with the sticky particle system; and the author in [12, 14] employed a trajectory map in Lagrangian coordinates to study weak solutions of the pressureless Euler equations in the presence of a semiconvex potential.

The main thrust of all the aforementioned works is that they find an auxiliary coordinate system to describe the pressureless Euler equations and then verify existence for the corresponding equations. This is often simpler than attacking the pressureless Euler equations directly. It should also be emphasized that most of this prior work makes use of discrete particle approximations. A notable exception is [5].

We also acknowledge that there have been important recent works on the pressureless Euler system in one spatial dimension involving the absence of shocks [8], entropy solutions in the presence of friction and dissipation [17, 20], and hydrodynamic limits [16]. The result of Jabin and Rey [16] is of particular note as it uses the Boltzmann equation in the limit of very frequent collisions to design a solution of the sticky particle system ($W \equiv 0$). It would be of interest to extend this result in analogy with Theorem 1.3 to general semiconvex potentials.

The outline of this paper is as follows. In Sect. 2, we use the solutions of (1.7) to design sticky particle trajectories $\gamma_1, \ldots, \gamma_N$ as mentioned above. In Sect. 3, we recast weak solution pairs of (1.1) and (1.2) as probability measures on the space of continuous paths; see also

[13] for how we treated the particular case $W \equiv 0$. Then in Sect. 4, we prove Theorem 1.3. Finally, we would like to express our gratitude to Sean Paul for inquiring for a good reason as to why the entropy inequality

$$(v(x,t)-v(y,t))(x-y) \leq \frac{1}{t}(x-y)^2 \tag{1.13}$$

holds when $W \equiv 0$. In trying to answer to his question, we were lead to the quantitative sticky particle property (1.8) and subsequently to the entropy inequality (1.12) and Theorem 1.3.

2 Sticky particle trajectories

In this section, we will study sticky particle trajectories $\gamma_1, \ldots, \gamma_N$ as described in the introduction. We will show they satisfy the quantitative sticky particle property (1.8) and also that they have an important averaging property. This averaging property is then used to show that $\gamma_1, \ldots, \gamma_N$ corresponds to a weak solution pair of the pressureless Euler equations. We begin by showing these paths exist.

Proposition 2.1 *Suppose $m_1, \ldots, m_N > 0$ with $\sum_{i=1}^N m_i = 1$, $x_1 \ldots, x_N \in \mathbb{R}$ and $v_1, \ldots, v_N \in \mathbb{R}$. There are piecewise C^2 paths*

$$\gamma_1, \ldots, \gamma_N : [0,\infty) \to \mathbb{R}$$

with the following properties.

(i) *For $i = 1, \ldots, N$ and all but finitely many $t \in (0,\infty)$,*

$$\ddot{\gamma}_i(t) = -\sum_{j=1}^N m_j W'(\gamma_i(t) - \gamma_j(t)). \tag{2.1}$$

(ii) *For $i = 1, \ldots, N$,*

$$\gamma_i(0) = x_i \quad and \quad \dot{\gamma}_i(0) = v_i. \tag{2.2}$$

(iii) *For $i, j = 1, \ldots, N$, $0 \leq s \leq t$ and $\gamma_i(s) = \gamma_j(s)$ imply*

$$\gamma_i(t) = \gamma_j(t).$$

(iv) *If $t > 0$, $\{i_1, \ldots, i_k\} \subset \{1, \ldots, N\}$, and*

$$\gamma_{i_1}(t) = \cdots = \gamma_{i_k}(t) \neq \gamma_i(t)$$

for $i \notin \{i_1, \ldots, i_k\}$, then

$$\dot{\gamma}_{i_j}(t+) = \frac{m_{i_1}\dot{\gamma}_{i_1}(t-) + \cdots + m_{i_k}\dot{\gamma}_{i_k}(t-)}{m_{i_1} + \cdots + m_{i_k}}$$

for $j = 1, \ldots, k$.

Proof We will prove the assertion by induction on N. Suppose $N = 2$ and let $\overline{\gamma}_1, \overline{\gamma}_2$ be a solution of (2.1) that satisfies (2.2); such solutions exist by Proposition A.1 in the appendix. If the trajectories $\overline{\gamma}_1$ and $\overline{\gamma}_1$ do not intersect, we take $\gamma_1 = \overline{\gamma}_1$ and $\gamma_2 = \overline{\gamma}_2$. Otherwise, let $s > 0$ be the first time such that $z := \overline{\gamma}_1(s) = \overline{\gamma}_2(s)$. We then set

$$\gamma_i(t) := \begin{cases} \overline{\gamma}_i(t), & t \in [0,s] \\ z + (t-s)(m_1\dot{\overline{\gamma}}_1(s-) + m_2\dot{\overline{\gamma}}_2(s-)), & t \in [s,\infty) \end{cases}$$

for $i = 1, 2$. It is easily verified that γ_1, γ_2 satisfy $(i) - (iv)$ above. We conclude that the assertion holds for $N = 2$.

Now suppose the claim has been established for some $N \geq 2$. Let $m_1, \dots, m_{N+1} > 0$ with $\sum_i m_i = 1$, $x_1 \dots, x_{N+1} \in \mathbb{R}$, $v_1, \dots, v_{N+1} \in \mathbb{R}$ and assume $\overline{\gamma}_1, \dots, \overline{\gamma}_{N+1}$ is a corresponding solution of (2.1) that satisfies the initial conditions (2.2) (Proposition A.1). If these trajectories never intersect, we take $\gamma_i = \overline{\gamma}_i$ for $i = 1, \dots, N+1$ and conclude. Otherwise, let $s > 0$ be the first time that trajectories intersect. First, we will assume that at time s a single subcollection of trajectories $\overline{\gamma}_{i_1}, \dots, \overline{\gamma}_{i_k}$ intersect. That is,

$$z := \overline{\gamma}_{i_1}(s) = \cdots = \overline{\gamma}_{i_k}(s) \neq \overline{\gamma}_i(s)$$

for $i \notin \{i_1, \dots, i_k\}$. We also define

$$v := \frac{m_{i_1}\dot{\overline{\gamma}}_{i_1}(s-) + \cdots + m_{i_k}\dot{\overline{\gamma}}_{i_k}(s-)}{m_{i_1} + \cdots + m_{i_k}}.$$

By induction, there are trajectories $\{\xi_i\}_{i \neq i_j}$ and ξ corresponding to the $N + 1 - (k - 1)$ masses $\{m_i\}_{i \neq i_j}$ and $m_{i_1} + \cdots + m_{i_k}$, initial positions $\{\overline{\gamma}_i(s)\}_{i \neq i_j}$ and z, and velocities $\{\dot{\overline{\gamma}}_i(s)\}_{i \neq i_j}$ and v which satisfy $(i) - (iv)$ above. We then set

$$\gamma_i(t) := \begin{cases} \overline{\gamma}_i(t), & t \in [0, s] \\ \xi_i(t - s), & t \in [s, \infty) \end{cases}$$

for $i \neq i_j$ and

$$\gamma_{i_j}(t) := \begin{cases} \overline{\gamma}_{i_j}(t), & t \in [0, s] \\ \xi(t - s), & t \in [s, \infty). \end{cases}$$

Using the induction hypothesis, it is now routine the check that $\gamma_1, \dots, \gamma_{N+1}$ satisfy $(i) - (iv)$ in the statement of this claim. It also not difficult to see how the argument given above can be extended to the case where there are more than one subcollection of paths that intersect at time s. We leave the details to the reader and conclude this assertion. □

Remark 2.2 The right $\dot{\gamma}_i(t+)$ and left $\dot{\gamma}_i(t-)$ limits of $\dot{\gamma}_i$ exist for each $t > 0$ and $i = 1, \dots, N$. Moreover, these limits can be computed as

$$\dot{\gamma}_i(t\pm) = \lim_{h \to 0^\pm} \frac{\gamma_i(t + h) - \gamma_i(t)}{h}.$$

These remarks follow from our proof of Proposition 2.1; they also can be established by appealing directly to property (i) of the proposition.

Definition 2.3 The paths $\gamma_1, \dots, \gamma_N$ shown to exist in Proposition 2.1 are called *sticky particle trajectories* corresponding to the masses $m_1, \dots, m_N$ (with $\sum_i m_i = 1$), initial positions $x_1, \dots, x_N$, and initial velocities $v_1, \dots, v_N$. We also call $t > 0$ a *first intersection time* whenever there are at least two paths γ_i and γ_j that agree for the first time at t. See Fig. 2.

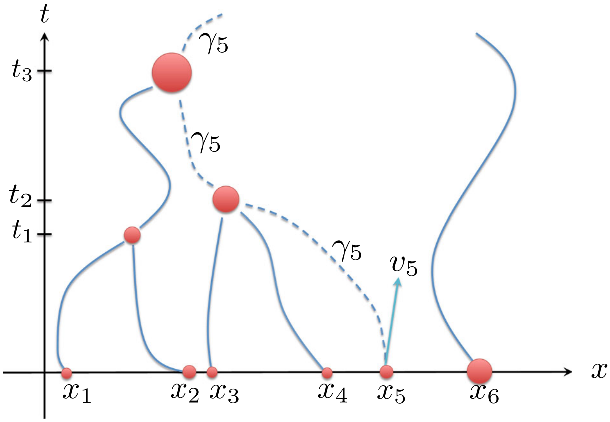

Fig. 2 Sticky particle trajectories when $N = 6$; here the initial positions $x_1, \dots, x_6$ are displayed along with the initial velocity v_5 of the trajectory starting out at x_5. The first intersection times $t_1 < t_2 < t_3$ are also shown along the time axis and trajectory γ_5 is exhibited in dashed

2.1 Quantitative sticky particle property and stability

For the remainder of this section, we will consider a single collection of sticky particle trajectories $\gamma_1, \dots, \gamma_N$ corresponding to a fixed but arbitrary collection of masses $m_1, \dots, m_N$ (with $\sum_i m_i = 1$), initial positions $x_1, \dots, x_N$, and initial velocities

$$v_i := v_0(x_i)$$

where $v_0 : \mathbb{R} \to \mathbb{R}$ is absolutely continuous. We will show they satisfy a quantitative version of the property (iii) in Proposition 2.1 and verify a stability property of these paths. First, we will need an elementary lemma.

Lemma 2.4 *Suppose $T > 0$ and $y : [0, T) \to [0, \infty)$ is continuous and piecewise C^2. Further assume*

$$\dot{y}(t+) \le \dot{y}(t-) \tag{2.3}$$

for each $t \in (0, T)$ and that for some $c > 0$

$$\ddot{y}(t) \le c y(t)$$

for all but finitely many $t \in (0, T)$. Then (i)

$$(0, T) \ni t \mapsto \frac{y(t)}{\sinh(\sqrt{c}t)} \quad \text{is nonincreasing}$$

and (ii)

$$y(t) \le \cosh(\sqrt{c}t) y(0) + \frac{1}{\sqrt{c}} \sinh(\sqrt{c}t) \dot{y}(0+), \quad t \in [0, T).$$

Proof (i) Without any loss of generality, we assume $c = 1$ in this proof. We will also suppose $0 < t_0 < \cdots < t_n < T$ are times such that y is C^2 on each of the intervals $(0, t_1), \dots (t_n, T)$. It then suffices to show the continuous function

$$u(t) := \frac{y(t)}{\sinh(t)}, \quad t \in (0, T)$$

is nonincreasing on each of these intervals.

Observe

$$\begin{aligned}\dot{u}(t+) &= \frac{\dot{y}(t+)}{\sinh(t)} - \frac{y(t)}{\sinh(t)^2}\cosh(t)\\ &\le \frac{\dot{y}(t-)}{\sinh(t)} - \frac{y(t)}{\sinh(t)^2}\cosh(t)\\ &= \dot{u}(t-)\end{aligned}$$

for each $t > 0$ by (2.3). Also note

$$\begin{aligned}\ddot{y}(t) &= \ddot{u}(t)\sinh(t) + 2\dot{u}(t)\cosh(t) + u(t)\sinh(t)\\ &= \ddot{u}(t)\sinh(t) + 2\dot{u}(t)\cosh(t) + y(t)\\ &\le y(t)\end{aligned}$$

for $t \in (0, T) \setminus \{t_1, \dots, t_n\}$. Consequently,

$$\frac{d}{dt}\left(\dot{u}(t)\sinh(t)^2\right) = \sinh(t)\left(\ddot{u}(t)\sinh(t) + 2\dot{u}(t)\cosh(t)\right) \le 0 \tag{2.4}$$

for $t \in (0, T) \setminus \{t_1, \dots, t_n\}$.

As y is nonnegative,

$$\dot{u}(t) \le \frac{\dot{y}(t)}{\sinh(t)}.$$

for $t \in (0, T) \setminus \{t_1, \dots, t_n\}$. As a result,

$$\limsup_{s\to 0^+}\left\{\dot{u}(s)\sinh(s)^2\right\} \le \limsup_{s\to 0^+}\{\dot{y}(s)\sinh(s)\} = \dot{y}(0+)\sinh(0) = 0.$$

In view of (2.4)

$$\dot{u}(t)\sinh(t)^2 \le \limsup_{s\to 0^+}\left\{\dot{u}(s)\sinh(s)^2\right\} = 0$$

for $t \in (0, t_1)$; we emphasize that this inequality is valid since $\dot{u}$ is C^1 on $(0, t_1)$ which allows us to integrate the left hand side of (2.4) on this interval. Therefore, $\dot{u}(t) \le 0$ for $t \in (0, t_1)$.

So far, we have that u is nonincreasing on $t \in (0, t_1]$ and

$$\dot{u}(t_1-) \le 0.$$

In addition, (2.4) gives

$$\begin{aligned}\dot{u}(t)\sinh(t)^2 &\le \dot{u}(t_1+)\sinh(t_1)^2\\ &\le \dot{u}(t_1-)\sinh(t_1)^2\\ &\le 0\end{aligned}$$

for $t \in (t_1, t_2)$. Thus, u is nonincreasing on $[t_1, t_2]$ and

$$\dot{u}(t_2-) \le 0.$$

It is now evident that we may repeat this argument to show that u is nonincreasing on $[t_2, t_3], [t_3, t_4], \dots, [t_n, T)$ and therefore on $(0, T)$.

Part (ii) of this lemma follows similarly. We will omit the analogous argument this claim has been established in Lemma 3.7 of [12]. □

We now verify the following quantitative sticky particle property and stability estimate of sticky particle trajectories. We recall that $W(x) + (c/2)x^2$ is convex for some $c > 0$.

Proposition 2.5 *Suppose* $i, j \in \{1, \ldots, N\}$.

(*i*) *For* $0 < s \leq t$,

$$\frac{|\gamma_i(t) - \gamma_j(t)|}{\sinh(\sqrt{c}t)} \leq \frac{|\gamma_i(s) - \gamma_j(s)|}{\sinh(\sqrt{c}s)}.$$

(*ii*) *If* $x_i \geq x_j$, *then*

$$0 \leq \gamma_i(t) - \gamma_j(t) \leq \cosh(\sqrt{c}t)(x_i - x_j) + \frac{1}{\sqrt{c}}\sinh(\sqrt{c}t)\int_{x_j}^{x_i} |v_0'(x)|dx$$

for all $t \geq 0$.

Proof Without any loss of generality, we may assume $\gamma_1 \leq \cdots \leq \gamma_N$. It then suffices to prove this proposition for $i = 1, \ldots, N-1$ and $j = i+1$. Indeed, if

$$\frac{\gamma_{i+1}(t) - \gamma_i(t)}{\sinh(\sqrt{c}t)} \text{ is nonincreasing} \tag{2.5}$$

for $i = 1, \ldots, N-1$, then

$$\frac{\gamma_k(t) - \gamma_j(t)}{\sinh(\sqrt{c}t)} = \sum_{i=j}^{k-1} \frac{\gamma_{i+1}(t) - \gamma_i(t)}{\sinh(\sqrt{c}t)}$$

is a sum of nonincreasing functions for $k > j$ which would prove assertion (i).

Likewise, for assertion (ii), it suffices to verify

$$\gamma_{i+1}(t) - \gamma_i(t) \leq \cosh(\sqrt{c}t)(x_{i+1} - x_i) + \frac{1}{\sqrt{c}}\sinh(\sqrt{c}t)|v_0(x_{i+1}) - v_0(x_i)| \tag{2.6}$$

for $t \geq 0$. In this case,

$$\begin{aligned}
\gamma_k(t) - \gamma_j(t) &= \sum_{i=j}^{k-1}(\gamma_{i+1}(t) - \gamma_i(t)) \\
&\leq \sum_{i=j}^{k-1}\left(\cosh(\sqrt{c}t)(x_{i+1} - x_i) + \frac{1}{\sqrt{c}}\sinh(\sqrt{c}t)|v_0(x_{i+1}) - v_0(x_i)|\right) \\
&\leq \cosh(\sqrt{c}t)(x_k - x_j) + \frac{1}{\sqrt{c}}\sinh(\sqrt{c}t)\sum_{i=j}^{k-1}\int_{x_i}^{x_{i+1}} |v_0'(x)|dx \\
&= \cosh(\sqrt{c}t)(x_k - x_j) + \frac{1}{\sqrt{c}}\sinh(\sqrt{c}t)\int_{x_j}^{x_k} |v_0'(x)|dx.
\end{aligned}$$

Consequently, we will fix $i \in \{1, \ldots, N-1\}$ and focus on establishing (2.5) and (2.6). Our plan is to verify the hypotheses of Lemma 2.4 with

$$y(t) := \gamma_{i+1}(t) - \gamma_i(t), \quad t \in [0, T),$$

where

$$T := \inf\{t > 0 : \gamma_{i+1}(t) = \gamma_i(t)\}.$$

Of course if $\gamma_{i+1}(t) > \gamma_i(t)$ for all $t > 0$, then $T = +\infty$. Otherwise, we have

$$\gamma_{i+1}(t) - \gamma_i(t) = 0$$

for $t \geq T$. In either case, it is enough to prove (2.5) and (2.6) on $[0, T)$. To this end, we will first show

$$\dot{\gamma}_{i+1}(s+) \leq \dot{\gamma}_{i+1}(s-) \tag{2.7}$$

and

$$\dot{\gamma}_i(s+) \geq \dot{\gamma}_i(s-) \tag{2.8}$$

for each $s \in (0, T)$.

We will only justify (2.7) as (2.8) can be proved similarly. Observe that if γ_{i+1} does not have a first intersection time at $s \in (0, T)$, then γ_{i+1} is C^1 in a neighborhood of s and thus

$$\dot{\gamma}_{i+1}(s) = \dot{\gamma}_{i+1}(s+) = \dot{\gamma}_{i+1}(s-).$$

If γ_{i+1} has a first intersection time at $s \in (0, T)$, then there are trajectories $\gamma_{i+2}, \dots, \gamma_{i+r}$ (some $r \geq 2$) such that

$$\gamma_{i+1}(s) = \gamma_{i+2}(s) = \cdots = \gamma_{i+r}(s)$$

and

$$\dot{\gamma}_{i+j}(s+) = \frac{m_{i+1}\dot{\gamma}_{i+1}(s-) + \cdots + m_{i+r}\dot{\gamma}_{i+r}(s-)}{m_{i+1} + \cdots + m_{i+r}} \tag{2.9}$$

$j = 1, \dots, r$.

Also note that as $\gamma_{i+1} \leq \gamma_{i+j}$ for $j = 2, \dots, r$,

$$\frac{\gamma_{i+1}(s+h) - \gamma_{i+1}(s)}{h} \geq \frac{\gamma_{i+j}(s+h) - \gamma_{i+j}(s)}{h}$$

for all $h < 0$ and sufficiently small. By Remark 2.2, we can send $h \to 0^-$ to find

$$\dot{\gamma}_{i+1}(s-) \geq \dot{\gamma}_{i+j}(s-)$$

for $j = 2, \dots, r$. It then follows from (2.9) that

$$\dot{\gamma}_{i+1}(s+) \leq \frac{m_{i+1}\dot{\gamma}_{i+1}(s-) + \cdots + m_{i+r}\dot{\gamma}_{i+1}(s-)}{m_{i+1} + \cdots + m_{i+r}} = \dot{\gamma}_{i+1}(s-),$$

which is (2.7). Putting (2.7) and (2.8) together gives

$$\dot{y}(s+) = \dot{\gamma}_{i+1}(s+) - \dot{\gamma}_i(s+) \leq \dot{\gamma}_{i+1}(s-) - \dot{\gamma}_i(s-) = \dot{y}(s-) \tag{2.10}$$

for all $s \in (0, T)$.

By part (i) of Proposition 2.1 and the semiconvexity of W,

$$\begin{aligned}\ddot{y}(t) &= \ddot{\gamma}_{i+1}(t) - \ddot{\gamma}_i(t)\\ &= -\sum_{j=1}^{N} m_j \left(W'(\gamma_{i+1}(t) - \gamma_j(t)) - W'(\gamma_i(t) - \gamma_j(t))\right)\\ &\leq \sum_{j=1}^{N} m_j c\,(\gamma_{i+1}(t) - \gamma_i(t))\end{aligned}$$

$$= c(\gamma_{i+1}(t) - \gamma_i(t))$$
$$= cy(t)$$

for all but finitely many $t > 0$. Combining this observation with (2.10) allows us to apply Lemma 2.4 and conclude. □

2.2 Averaging property

We will now discuss the averaging property of the paths $\gamma_1, \ldots, \gamma_N$. We shall see that it implies a statement about the conservation of momentum of collections of finitely many sticky particles.

Proposition 2.6 *Suppose $g : \mathbb{R} \to \mathbb{R}$ and $0 \le s < t$. Then*

$$\sum_{i=1}^{N} m_i g(\gamma_i(t))\dot{\gamma}_i(t+) = \sum_{i=1}^{N} m_i g(\gamma_i(t)) \left[\dot{\gamma}_i(s+) - \int_s^t \left(\sum_{j=1}^{N} m_j W'(\gamma_i(\tau) - \gamma_j(\tau)) \right) d\tau \right]. \tag{2.11}$$

Proof Let $t_0 = 0$ and $0 < t_1 < \cdots < t_\ell$ denote the first intersection times of the paths $\gamma_1, \ldots, \gamma_N$. As these paths satisfy the ODE (2.1) on $(0, \infty) \setminus \{t_1, \ldots, t_\ell\}$, it suffices to verify

$$\sum_{i=1}^{N} m_i g(\gamma_i(t))\dot{\gamma}_i(t_r+) = \sum_{i=1}^{N} m_i g(\gamma_i(t)) \left[\dot{\gamma}_i(t_k+) - \int_{t_k}^{t_r} \left(\sum_{j=1}^{N} m_j W'(\gamma_i(\tau) - \gamma_j(\tau)) \right) d\tau \right] \tag{2.12}$$

for each $t_k \le t_r$, where t_r is the largest element of $\{t_0, t_1, \ldots, t_\ell\}$ that is less than or equal to t. We will establish the identity (2.12) by induction on $k \le r$. Of course (2.12) is clear for $k = r$. So we assume it holds for some $k < r$ and then show it holds for $k - 1$.

At time t_k, let us initially suppose that a single subcollection $\gamma_{i_1}, \ldots, \gamma_{i_n}$ of paths intersect for the first time. This implies

$$\gamma_{i_1}(t_k) = \cdots = \gamma_{i_n}(t_k) \neq \gamma_i(t_k)$$

for each $i \notin \{i_1, \ldots, i_n\}$, $\overline{\gamma}(t) := \gamma_{i_1}(t) = \cdots = \gamma_{i_n}(t)$ since $t \ge t_k$, and also that

$$\dot{\gamma}_{i_p}(t_k+) = \frac{m_{i_1}\dot{\gamma}_{i_1}(t_k-) + \cdots + m_{i_n}\dot{\gamma}_{i_n}(t_k-)}{m_{i_1} + \cdots + m_{i_n}} \tag{2.13}$$

$p = 1, \ldots, n$. Furthermore, when $i \notin \{i_1, \ldots, i_n\}$, γ_i is continuously differentiable on (t_{k-1}, t_{k+1}) if $k < \ell$ or on (t_{k-1}, ∞) if $k = \ell$.

With these observations and the induction hypothesis, we have

$$\sum_{i=1}^{N} m_i g(\gamma_i(t))\dot{\gamma}_i(t_r+)$$
$$= \sum_{i=1}^{N} m_i g(\gamma_i(t)) \left[\dot{\gamma}_i(t_k+) - \int_{t_k}^{t_r} \left(\sum_{j=1}^{N} m_j W'(\gamma_i(\tau) - \gamma_j(\tau)) \right) d\tau \right]$$

$$
\begin{aligned}
&= \sum_{i\notin\{i_1,\dots,i_n\}} m_i g(\gamma_i(t))\left[\dot\gamma_i(t_k+) - \int_{t_k}^{t_r}\left(\sum_{j=1}^N m_j W'(\gamma_i(\tau)-\gamma_j(\tau))\right)d\tau\right]\\
&\quad + \sum_{p=1}^n m_{i_p} g(\gamma_{i_p}(t))\left[\dot\gamma_{i_p}(t_k+) - \int_{t_k}^{t_r}\left(\sum_{j=1}^N m_j W'(\gamma_{i_p}(\tau)-\gamma_j(\tau))\right)d\tau\right]\\
&= \sum_{i\notin\{i_1,\dots,i_n\}} m_i g(\gamma_i(t))\left[\dot\gamma_i(t_{k-1}+) - \int_{t_{k-1}}^{t_r}\left(\sum_{j=1}^N m_j W'(\gamma_i(\tau)-\gamma_j(\tau))\right)d\tau\right]\\
&\quad + \sum_{p=1}^n m_{i_p} g(\overline{\gamma}(t))\left[\dot\gamma_{i_p}(t_k+) - \int_{t_k}^{t_r}\left(\sum_{j=1}^N m_j W'(\overline{\gamma}(\tau)-\gamma_j(\tau))\right)d\tau\right]\\
&= \sum_{i\notin\{i_1,\dots,i_n\}} m_i g(\gamma_i(t))\left[\dot\gamma_i(t_{k-1}+) - \int_{t_{k-1}}^{t_r}\left(\sum_{j=1}^N m_j W'(\gamma_i(\tau)-\gamma_j(\tau))\right)d\tau\right]\\
&\quad + \sum_{p=1}^n m_{i_p} g(\overline{\gamma}(t))\left[\dot\gamma_{i_p}(t_k-) - \int_{t_k}^{t_r}\left(\sum_{j=1}^N m_j W'(\overline{\gamma}(\tau)-\gamma_j(\tau))\right)d\tau\right]\\
&= \sum_{i\notin\{i_1,\dots,i_n\}} m_i g(\gamma_i(t))\left[\dot\gamma_i(t_{k-1}+) - \int_{t_{k-1}}^{t_r}\left(\sum_{j=1}^N m_j W'(\gamma_i(\tau)-\gamma_j(\tau))\right)d\tau\right]\\
&\quad + \sum_{p=1}^n m_{i_p} g(\overline{\gamma}(t))\left[\dot\gamma_{i_p}(t_{k-1}+) - \int_{t_{k-1}}^{t_r}\left(\sum_{j=1}^N m_j W'(\overline{\gamma}(\tau)-\gamma_j(\tau))\right)d\tau\right]\\
&= \sum_{i=1}^N m_i g(\gamma_i(t))\left[\dot\gamma_i(t_{k-1}+) - \int_{t_{k-1}}^{t_r}\left(\sum_{j=1}^N m_j W'(\gamma_i(\tau)-\gamma_j(\tau))\right)d\tau\right].
\end{aligned}
$$

Note that we used (2.13) to derive the fourth equality above. Finally, we note that if more than one subcollection of trajectories intersect for the first time at t_k, we can argue as above on each subcollection to verify (2.12). Therefore, the conclusion follows by induction. □

Remark 2.7 Choosing

$$
g(x) = \begin{cases} 1, & x = \gamma_j(t) \\ 0, & x \neq \gamma_j(t) \end{cases}
$$

in (2.11) and sending $s \to t^-$ gives

$$
\left(\sum_{\gamma_i(t)=\gamma_j(t)} m_i\right)\dot\gamma_j(t+) = \sum_{\gamma_i(t)=\gamma_j(t)} m_i\dot\gamma_i(t-)
$$

for $j = 1, \dots, N$. Each summation above is taken over $i \in \{1, \dots, N\}$ such that $\gamma_i(t) = \gamma_j(t)$.

2.3 Energy estimates

We also can prove that the total energy of finite particle systems is non-increasing in time. In particular, the total energy will only be constant for systems where particles do not collide.

Proposition 2.8 *For each* $0 \leq s < t$

$$\frac{1}{2}\sum_{i=1}^{N} m_i\dot{\gamma}_i(t+)^2 + \frac{1}{2}\sum_{i,j=1}^{N} m_i m_j W(\gamma_i(t) - \gamma_j(t))$$
$$\leq \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\gamma}_i(s+)^2 + \frac{1}{2}\sum_{i,j=1}^{N} m_i m_j W(\gamma_i(s) - \gamma_j(s)). \tag{2.14}$$

Proof Let $t_0 = 0$ and $t_1 < \cdots < t_\ell$ denote the first intersection times of the paths $\gamma_1, \ldots, \gamma_N$. Recall that $\gamma_1, \ldots, \gamma_N$ satisfy the ODE (2.1) on (t_{k-1}, t_k) for $k = 1, \ldots, \ell$ and so the conservation of energy holds on these intervals. And in view of Remark 2.7, we can apply Jensen's inequality to derive

$$\frac{1}{2}\sum_{i=1}^{N} m_i\dot{\gamma}_i(t_k+)^2 \leq \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\gamma}_i(t_k-)^2$$

for $k = 1, \ldots, \ell$. Consequently,

$$\frac{1}{2}\sum_{i=1}^{N} m_i\dot{\gamma}_i(t_{k+1}+)^2 + \frac{1}{2}\sum_{i,j=1}^{N} m_i m_j W(\gamma_i(t_{k+1}) - \gamma_j(t_{k+1}))$$
$$\leq \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\gamma}_i(t_{k+1}-)^2 + \frac{1}{2}\sum_{i,j=1}^{N} m_i m_j W(\gamma_i(t_{k+1}) - \gamma_j(t_{k+1}))$$
$$= \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\gamma}_i(t_k+)^2 + \frac{1}{2}\sum_{i,j=1}^{N} m_i m_j W(\gamma_i(t_k) - \gamma_j(t_k)) \tag{2.15}$$

for $k = 0, \ldots, \ell - 1$.

Now suppose $0 \leq s < t$. If no $t_1, \ldots, t_\ell$ belongs to the interval (s, t), we conclude by the conservation of energy and the inequalities above. Otherwise, select t_k to be the smallest $\{t_1, \ldots, t_\ell\}$ belonging to (s, t) and select t_r to be the largest $\{t_1, \ldots, t_\ell\}$ belonging to (s, t). Then $\gamma_1, \ldots, \gamma_N$ satisfies (2.1) on (s, t_k) and on (t_r, t) so that the conservation holds on these intervals. Combining with (2.15) then gives

$$\frac{1}{2}\sum_{i=1}^{N} m_i\dot{\gamma}_i(t+)^2 + \frac{1}{2}\sum_{i,j=1}^{N} m_i m_j W(\gamma_i(t) - \gamma_j(t))$$
$$= \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\gamma}_i(t_r+)^2 + \frac{1}{2}\sum_{i,j=1}^{N} m_i m_j W(\gamma_i(t_r) - \gamma_j(t_r))$$
$$\leq \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\gamma}_i(t_k+)^2 + \frac{1}{2}\sum_{i,j=1}^{N} m_i m_j W(\gamma_i(t_k) - \gamma_j(t_k))$$

$$\leq \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\gamma}_i(t_k-)^2 + \frac{1}{2}\sum_{i,j=1}^{N} m_i m_j W(\gamma_i(t_k) - \gamma_j(t_k))$$
$$= \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\gamma}_i(s+)^2 + \frac{1}{2}\sum_{i,j=1}^{N} m_i m_j W(\gamma_i(s) - \gamma_j(s)).$$

□

Corollary 2.9 *Define*

$$\varphi(t) := e^{(c+1)t^2}\int_0^t e^{-(c+1)s^2} ds. \tag{2.16}$$

For each $0 \leq t_1 \leq t_2$*,*

$$\int_{t_1}^{t_2}\sum_{i=1}^{N} m_i \dot{\gamma}_i(s)^2 ds \leq (\varphi(t_2) - \varphi(t_1)) \left(\sum_{i=1}^{N} m_i v_0(x_i)^2 + \frac{1}{2}\sum_{i,j=1}^{N} m_i m_j W'(x_i - x_j)^2\right). \tag{2.17}$$

Proof As $x \mapsto W(x) + (c/2)x^2$ is convex,

$$\begin{aligned} W(\gamma_i(t) - \gamma_j(t)) &\geq W(x_i - x_j) + W'(x_i - x_j)(\gamma_i(t) - \gamma_j(t) - (x_i - x_j)) \\ &\quad - \frac{c}{2}(\gamma_i(t) - x_i - (\gamma_j(t) - x_j))^2 \\ &\geq W(x_i - x_j) - \frac{1}{2}W'(x_i - x_j)^2 - \frac{c+1}{2}(\gamma_i(t) - x_i - (\gamma_j(t) - x_j))^2 \\ &\geq W(x_i - x_j) - \frac{1}{2}W'(x_i - x_j)^2 - (c+1)((\gamma_i(t) - x_i)^2 + (\gamma_j(t) - x_j)^2) \\ &\geq W(x_i - x_j) - \frac{1}{2}W'(x_i - x_j)^2 - (c+1)t\left(\int_0^t \dot{\gamma}_i(s)^2 ds + \int_0^t \dot{\gamma}_j(s)^2 ds\right). \end{aligned}$$

Combining this lower bounds with (2.14) gives

$$\begin{aligned} \sum_{i=1}^{N} m_i \dot{\gamma}_i(t)^2 &\leq \sum_{i=1}^{N} m_i v_0(x_i)^2 + \frac{1}{2}\sum_{i,j=1}^{N} m_i m_j W'(x_i - x_j)^2 \\ &\quad + 2(c+1)t\int_0^t \sum_{i=1}^{N} m_i \dot{\gamma}_i(s)^2 ds \end{aligned} \tag{2.18}$$

almost everywhere. As a result,

$$\begin{aligned} &\frac{d}{dt}e^{-(c+1)t^2}\int_0^t \sum_{i=1}^{N} m_i \dot{\gamma}_i(s)^2 ds \\ &= e^{-(c+1)t^2}\left(\sum_{i=1}^{N} m_i \dot{\gamma}_i(t)^2 - 2(c+1)t\int_0^t \sum_{i=1}^{N} m_i \dot{\gamma}_i(s)^2 ds\right) \\ &\leq e^{-(c+1)t^2}\left(\sum_{i=1}^{N} m_i v_0(x_i)^2 + \frac{1}{2}\sum_{i,j=1}^{N} m_i m_j W'(x_i - x_j)^2\right). \end{aligned}$$

almost everywhere. Integrating from 0 to t gives,

$$\int_0^t \sum_{i=1}^N m_i \dot{\gamma}_i(s)^2 ds \le \varphi(t) \left(\sum_{i=1}^N m_i v_0(x_i)^2 + \frac{1}{2} \sum_{i,j=1}^N m_i m_j W'(x_i - x_j)^2 \right).$$

Upon substituting this inequality in (2.18), we find

$$\begin{aligned} \sum_{i=1}^N m_i \dot{\gamma}_i(t)^2 &\le (1 + 2(c+1)t\varphi(t)) \left(\sum_{i=1}^N m_i v_0(x_i)^2 + \frac{1}{2} \sum_{i,j=1}^N m_i m_j W'(x_i - x_j)^2 \right) \\ &= \varphi'(t) \left(\sum_{i=1}^N m_i v_0(x_i)^2 + \frac{1}{2} \sum_{i,j=1}^N m_i m_j W'(x_i - x_j)^2 \right) \end{aligned}$$

almost everywhere. Inequality (2.17) now follows from integrating from t_1 to t_2. □

3 Probability measures on the path space

We now consider $\Gamma := C([0,\infty))$, the space of continuous paths from $[0,\infty)$ into $\mathbb{R}$, equipped with the following distance

$$d(\gamma, \xi) := \sum_{n \in \mathbb{N}} \frac{1}{2^n} \left(\frac{\max_{0 \le t \le n} |\gamma(t) - \xi(t)|}{1 + \max_{0 \le t \le n} |\gamma(t) - \xi(t)|} \right) \qquad (\gamma, \xi \in \Gamma).$$

It is routine to check that $\lim_{k\to\infty} d(\xi_k, \xi) = 0$ if and only if $\xi_k \to \xi$ locally uniformly on $[0,\infty)$. It is also not difficult to verify that Γ is a complete and separable metric space (see for instance the appendix of [15]).

In this section, we will associate a Borel probability measure on Γ, which we will write as $\eta \in \mathcal{P}(\Gamma)$, to a given $\rho_0 \in \mathcal{P}(\mathbb{R})$ and absolutely continuous $v_0 : \mathbb{R} \to \mathbb{R}$. In particular, we will interpret the support of η as the set of trajectories of a collection of evolving point masses which interact pairwise with potential W and via perfectly inelastic collisions when collisions occur. To this end, we will employ the evaluation map

$$e_t : \Gamma \to \mathbb{R}; \gamma \mapsto \gamma(t)$$

and the push forward measure $e_{t\#}\eta \in \mathcal{P}(\mathbb{R})$

$$\int_{\mathbb{R}} g d(e_{t\#}\eta) := \int_\Gamma g(\gamma(t)) d\eta(\gamma),$$

for each $t \ge 0$.

We will also make use of the space

$$X := \left\{ \gamma : [0,\infty) \to \mathbb{R} : \gamma \text{ absolutely continuous}, \int_0^n \dot{\gamma}(t)^2 dt < \infty \text{ for all } n \in \mathbb{N} \right\}$$

and the function

$$\Psi(\gamma) := \begin{cases} \sum_{n=1}^\infty \frac{1}{2^n \varphi(n)} \left(\int_0^n \dot{\gamma}(t)^2 dt + \gamma(0)^2 \right), & \gamma \in X \\ +\infty, & \gamma \notin X \end{cases}$$

($\gamma \in \Gamma$). Recall that φ was defined in (2.16). It is a straightforward exercise to employ the Arzelà-Ascoli theorem and check that Ψ has compact sublevel sets within Γ. Our central existence assertion is as follows.

Theorem 3.1 *Assume $\rho_0 \in \mathcal{P}(\mathbb{R})$ satisfies (1.9) and $v_0 : \mathbb{R} \to \mathbb{R}$ is absolutely continuous. Further suppose $W : \mathbb{R} \to \mathbb{R}$ satisfies (1.3) for some $c > 0$ and (1.10). There is $\eta \in \mathcal{P}(\Gamma)$ which has the following properties.*

(i) *For η almost every $\gamma \in \Gamma$, $\Psi(\gamma) < \infty$.*
(ii) $\rho_0 = e_{0\#}\eta$.
(iii) *For each $0 < s \le t$ and $\gamma, \xi \in \operatorname{supp}(\eta)$,*

$$\frac{|\gamma(t) - \xi(t)|}{\sinh(\sqrt{c}t)} \le \frac{|\gamma(s) - \xi(s)|}{\sinh(\sqrt{c}s)}.$$

(iv) *For each $t \ge 0$ and $\gamma, \xi \in \operatorname{supp}(\eta)$ with $\gamma(0) \ge \xi(0)$,*

$$0 \le \gamma(t) - \xi(t) \le \cosh(\sqrt{c}t)(\gamma(0) - \xi(0)) + \frac{1}{\sqrt{c}} \sinh(\sqrt{c}t) \int_{\xi(0)}^{\gamma(0)} |v_0'(x)| dx$$

(v) *There is a Borel $v : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ such that*

$$\dot{\gamma}(t) = v(\gamma(t), t) \quad a.e. \ t > 0$$

for η almost every $\gamma \in \Gamma$.
(vi) *For almost every $t \ge 0$ and each Borel $h : \mathbb{R} \to \mathbb{R}$ with $\int_\Gamma h(\gamma(t))^2 d\eta(\gamma) < \infty$,*

$$\int_\Gamma \dot{\gamma}(t) h(\gamma(t)) d\eta(\gamma) = \int_\Gamma \left(v_0(\gamma(0)) - \int_0^t W' * (e_{s\#}\eta)(\gamma(s)) ds \right) h(\gamma(t)) d\eta(\gamma).$$

(vii) *For almost every $0 \le s \le t$,*

$$\begin{aligned} &\int_\Gamma \frac{1}{2}\dot{\gamma}(t)^2 d\eta(\gamma) + \frac{1}{2}\int_\Gamma \int_\Gamma W(\gamma(t) - \xi(t)) d\eta(\gamma) d\eta(\xi) \\ &\le \int_\Gamma \frac{1}{2}\dot{\gamma}(s)^2 d\eta(\gamma) + \frac{1}{2}\int_\Gamma \int_\Gamma W(\gamma(s) - \xi(s)) d\eta(\gamma) d\eta(\xi). \end{aligned}$$

Our first step in proving this theorem is showing that it holds when ρ_0 is a convex combination of Dirac measures. In this case, we also establish two key estimates. In proving estimate (3.2) below, we will recall that since v_0 is absolutely continuous,

$$\omega(r) := \sup\left\{ \int_a^b |v_0'(x)| dx : 0 \le b - a \le r \right\}, \quad r \ge 0$$

tends to 0 as $r \to 0^+$. In particular, ω is uniformly continuous and grows at most linearly; so we may choose $\alpha > 0$ for which

$$\omega(r) \le \alpha(r + 1), \quad r \ge 0.$$

Lemma 3.2 *Suppose*

$$\rho_0 = \sum_{i=1}^N m_i \delta_{x_i} \in \mathcal{P}(\mathbb{R})$$

and let $\gamma_1, \dots, \gamma_N$ *be a collection of sticky particle trajectories with masses* $m_1, \dots, m_N$*, initial positions* $x_1, \dots, x_N$*, and initial velocities* $v_0(x_1), \dots, v_0(x_N)$*. Then*

$$\eta = \sum_{i=1}^{N} m_i \delta_{\gamma_i} \in \mathcal{P}(\Gamma)$$

satisfies properties $(i) - (vii)$ *of Theorem (3.1). Moreover,*

$$\int_\Gamma \Psi(\gamma) d\eta(\gamma) \le \int_{\mathbb{R}} \left(x^2 + v_0(x)^2\right) d\rho_0(x) + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} W'(x-y)^2 d\rho_0(x) d\rho_0(y), \quad (3.1)$$

and

$$|\gamma(t)| \le \left(\cosh(\sqrt{c}t) + \frac{\alpha}{\sqrt{c}} \sinh(\sqrt{c}t)\right) \left(|\gamma(0)| + \int_{\mathbb{R}} |x| d\rho_0(x) + 1\right) + \sqrt{2\varphi(t)} \left(\int_{\mathbb{R}} \left(x^2 + v_0(x)^2\right) d\rho_0(x) + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} W'(x-y)^2 d\rho_0(x) d\rho_0(y)\right)^{1/2} \quad (3.2)$$

for each $t \ge 0$ *and* $\gamma \in \text{supp}(\eta)$.

Proof 1. By (2.17),

$$\int_\Gamma \left(\int_0^n \dot{\gamma}(t)^2 dt\right) d\eta(\gamma) \le \varphi(n) \left(\int_{\mathbb{R}} v_0(x)^2 d\rho_0(x) + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} W'(x-y)^2 d\rho_0(x) d\rho_0(y)\right)$$

for each $n \in \mathbb{N}$. As $\varphi(n) \ge 1$ for all $n \in \mathbb{N}$ and $\sum_{n \in \mathbb{N}} 1/2^n = 1$, it follows that

$$\begin{aligned}
\int_\Gamma \Psi(\gamma) d\eta(\gamma) &= \int_\Gamma \left[\sum_{n=1}^{\infty} \frac{1}{2^n \varphi(n)} \left(\int_0^n \dot{\gamma}(t)^2 dt + \gamma(0)^2\right)\right] d\eta(\gamma) \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n \varphi(n)} \int_\Gamma \left(\int_0^n \dot{\gamma}(t)^2 dt\right) d\eta(\gamma) + \sum_{n=1}^{\infty} \frac{1}{2^n \varphi(n)} \left(\int_\Gamma \gamma(0)^2 d\eta(\gamma)\right) \\
&\le \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\int_{\mathbb{R}} \left(x^2 + v_0(x)^2\right) d\rho_0(x) + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} W'(x-y)^2 d\rho_0(x) d\rho_0(y)\right) \\
&= \int_{\mathbb{R}} \left(x^2 + v_0(x)^2\right) d\rho_0(x) + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} W'(x-y)^2 d\rho_0(x) d\rho_0(y).
\end{aligned}$$

This proves (3.1) and that η satisfies property (i) of Theorem (3.1).

2. That η satisfies property (ii) follows from

$$\int_{\mathbb{R}} g d\rho_0 = \sum_{i=1}^{N} m_i g(x_i) = \sum_{i=1}^{N} m_i g(\gamma_i(0)) = \int_\Gamma g(\gamma(0)) d\eta(\gamma).$$

Properties (iii) and (iv) are a consequence of Proposition 2.5. As for property (v), set

$$v(x,t) = \begin{cases} \dot{\gamma}_i(t+), & x = \gamma_i(t) \\ 0, & \text{otherwise,} \end{cases}$$

and note that part (iii) of Proposition 2.1 implies that v is well defined. Property (vi) is a corollary of (2.11), and property (vii) follows from (2.14).

3. As η satisfies property (vi),

$$|\gamma(t)-\xi(t)| \le \cosh(\sqrt{c}t)|\gamma(0)-\xi(0)| + \frac{1}{\sqrt{c}}\sinh(\sqrt{c}t)\omega\left(|\gamma(0)-\xi(0)|\right)$$

for $\gamma, \xi \in \operatorname{supp}(\eta)$ and $t \ge 0$. It follows that

$$\begin{aligned}
|\gamma(t)| &\le \int_\Gamma |\gamma(t)-\xi(t)|d\eta(\xi) + \int_\Gamma |\xi(t)|d\eta(\xi)\\
&\le \cosh(\sqrt{c}t)\int_{\mathbb{R}} |\gamma(0)-x|d\rho(x) + \frac{1}{\sqrt{c}}\sinh(\sqrt{c}t)\int_{\mathbb{R}} \omega(|\gamma(0)-x|)d\rho(x)\\
&\quad + \left(\int_\Gamma \xi(t)^2 d\eta(\xi)\right)^{1/2}\\
&\le \cosh(\sqrt{c}t)\left(|\gamma(0)| + \int_{\mathbb{R}} |x|d\rho(x)\right) + \frac{\alpha}{\sqrt{c}}\sinh(\sqrt{c}t)\left(1+|\gamma(0)| + \int_{\mathbb{R}}|x|d\rho(x)\right)\\
&\quad + \sqrt{2}\left(\int_\Gamma\left(\xi(0)^2 + \int_0^t \dot{\xi}(s)^2 ds\right)d\eta(\xi)\right)^{1/2}\\
&\le \left(\cosh(\sqrt{c}t) + \frac{\alpha}{\sqrt{c}}\sinh(\sqrt{c}t)\right)\left(1+|\gamma(0)| + \int_{\mathbb{R}}|x|d\rho(x)\right)\\
&\quad + \sqrt{2\varphi(t)}\left(\int_{\mathbb{R}}\left(x^2+v_0(x)^2\right)d\rho_0(x) + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} W'(x-y)^2 d\rho_0(x)d\rho_0(y)\right)^{1/2}
\end{aligned}$$

which is (3.2). □

In the following proof of Theorem 3.1, we will say that $(\eta^k)_{k\in\mathbb{N}} \subset \mathcal{P}(\Gamma)$ *converges narrowly* to $\eta \in \mathcal{P}(\Gamma)$ provided

$$\lim_{k\to\infty}\int_\Gamma F(\gamma)d\eta^k(\gamma) = \int_\Gamma F(\gamma)d\eta(\gamma) \tag{3.3}$$

for each bounded, continuous $F : \Gamma \to \mathbb{R}$. If $(\eta^k)_{k\in\mathbb{N}} \subset \mathcal{P}(\Gamma)$ converges narrowly to η and the limit (3.3) exists and is finite for a continuous $F : \Gamma \to [0,\infty)$, we will say that F is *uniformly integrable* (with respect to $(\eta^k)_{k\in\mathbb{N}}$). In particular, we will make use of the fact that if F is uniformly integrable and $G : \Gamma \to \mathbb{R}$ is continuous with $|G| \le F$ on $\operatorname{supp}(\eta^k)$ for each $k \in \mathbb{N}$, then G is uniformly integrable, as well (this follows from Lemma 5.1.7 and the more general definition of uniformly integrability given in Sect. 5.1.1 of [1]).

We also note that if $(\eta^k)_{k\in\mathbb{N}} \subset \mathcal{P}(\Gamma)$ converges narrowly to η, then

$$\lim_{k\to\infty}\int_\Gamma H(\gamma,\xi)d\eta^k(\gamma)d\eta^k(\xi) = \int_\Gamma H(\gamma,\xi)d\eta(\gamma)d\eta(\xi)$$

bounded, continuous $H : \Gamma\times\Gamma \to \mathbb{R}$ (Theorem 2.8 [2]). In this case, we'll say $(\eta^k \times \eta^k)_{k\in\mathbb{N}} \subset \mathcal{P}(\Gamma\times\Gamma)$ converges narrowly to $\eta\times\eta$. The notion of uniform integrability analogously extends to narrow convergence on $\mathcal{P}(\Gamma\times\Gamma)$.

Proof of Theorem 3.1 Suppose $\rho_0 \in \mathcal{P}(\mathbb{R})$ satisfies (1.9). We may select a sequence $(\rho_0^k)_{k\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R})$ in which each ρ_0^k is a convex combination of Dirac measures, $\rho_0^k \to \rho_0$ narrowly, and

$$\lim_{k\to\infty}\int_{\mathbb{R}} x^2 d\rho_0^k(x) = \int_{\mathbb{R}} x^2 d\rho_0(x). \tag{3.4}$$

The existence of such an approximating sequence is well known and can be verified as in Appendix A of [13]. Note that (3.4) and assumption (1.10) allow us to choose B such that

$$\int_{\mathbb{R}}\left(x^2+v_0(x)^2\right)d\rho_0^k(x)+\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}W'(x-y)^2d\rho_0^k(x)d\rho_0^k(y)\le B$$

for $k\in\mathbb{N}$.

By Lemma 3.2, there is an $\eta^k\in\mathcal{P}(\Gamma)$ satisfying conditions $(i)-(vii)$ with ρ_0^k instead of ρ_0 for each $k\in\mathbb{N}$. In view of (3.1) and our selection of B,

$$\sup_{k\in\mathbb{N}}\int_\Gamma\Psi(\gamma)d\eta^k(\gamma)\le B.$$

As the sublevel sets of Ψ are compact, Prokhorov's theorem (Theorem 5.1.3 of [1]) implies there is a subsequence $(\eta^{k_j})_{j\in\mathbb{N}}$ which converges narrowly to some $\eta^\infty\in\mathcal{P}(\Gamma)$.

In view of (3.4), $\gamma\mapsto\gamma(0)^2$ is uniformly integrable with respect to $(\eta^{k_j})_{j\in\mathbb{N}}$. By (3.2),

$$\gamma(t)^2\le 4\left(\cosh(\sqrt{c}t)+\frac{\alpha}{\sqrt{c}}\sinh(\sqrt{c}t)\right)^2\left(\gamma(0)^2+B+1\right)+4\varphi(t)B \tag{3.5}$$

for $t\ge0$ and each $\gamma\in\operatorname{supp}(\eta^k)$. It follows that $\gamma\mapsto\gamma(t)^2$ is also uniformly integrable with respect to $(\eta^{k_j})_{j\in\mathbb{N}}$ for each $t\ge0$.

We now proceed to show that η^∞ has the claimed properties $(i)-(vii)$. Whenever necessary, we will use that η^{k_j} fulfills these conditions with $\rho_0^{k_j}$ instead of ρ_0 for each $j\in\mathbb{N}$.

Proof of (i). Recall that Ψ is lower semicontinuous and nonnegative. By the narrow convergence of $\eta^{k_j}\to\eta^\infty$,

$$\int_\Gamma\Psi(\gamma)d\eta^\infty(\gamma)\le\liminf_{j\to\infty}\int_\Gamma\Psi(\gamma)d\eta^{k_j}(\gamma)\le B$$

(Lemma 5.1.7 [1]). Therefore, $\Psi(\gamma)<\infty$ for η almost every γ.

Proof of (ii). Since $e_0:\Gamma\to\mathbb{R};\gamma\mapsto\gamma(0)$ is continuous,

$$\rho_0=\lim_{j\to\infty}\rho_0^{k_j}=\lim_{j\to\infty}e_{0\#}\eta^{k_j}=e_{0\#}\eta^\infty.$$

Proof of (iii) and (iv). For each $\gamma,\xi\in\operatorname{supp}(\eta^\infty)$, there are $\gamma^j,\xi^j\in\operatorname{supp}(\eta^{k_j})$ such that $\gamma^j\to\gamma$ and $\xi^j\to\xi$ as $j\to\infty$. For $0<s\le t$,

$$\begin{aligned}\frac{|\gamma(t)-\xi(t)|}{\sinh(\sqrt{c}t)}&=\lim_{j\to\infty}\frac{|\gamma^j(t)-\xi^j(t)|}{\sinh(\sqrt{c}t)}\\&\le\lim_{j\to\infty}\frac{|\gamma^j(s)-\xi^j(s)|}{\sinh(\sqrt{c}s)}\\&=\frac{|\gamma(s)-\xi(s)|}{\sinh(\sqrt{c}s)}.\end{aligned}$$

We can argue in the same way to establish (iv).

Proof of (v). Here we mimic the proof of part (iv) of Proposition 1.3 in [13]. For $x\in\mathbb{R}$ and $0<s\le t$, set

$$f(x,t,s):=\inf\left\{\xi(t)+\frac{\sinh(\sqrt{c}t)}{\sinh(\sqrt{c}s)}|x-\xi(s)|:\xi\in\operatorname{supp}(\eta^\infty)\right\}.$$

Note that if $x = \gamma(s)$ for some $\gamma \in \text{supp}(\eta^\infty)$, we can choose $\xi = \gamma$ in the above infimum to get $f(\gamma(s), t, s) \le \gamma(t)$. By part (iii), we also have that $\xi(t) + \frac{\sinh(\sqrt{c}t)}{\sinh(\sqrt{c}s)}|\gamma(s) - \xi(s)| \ge \gamma(t)$ for all $\xi \in \text{supp}(\eta^\infty)$. As a result,

$$f(\gamma(s), t, s) = \gamma(t)$$

for each $\gamma \in \text{supp}(\eta^\infty)$ and $s \le t$.

Next we set

$$v_n(x, t) = n(f(x, t + 1/n, t) - x)$$

for $(x, t) \in \mathbb{R} \times (0, \infty)$ and $n \in \mathbb{N}$. Notice that v_n is Borel measurable since f is upper semicontinuous. Furthermore,

$$v_n(\gamma(t), t) = n(\gamma(t + 1/n) - \gamma(t))$$

for each $\gamma \in \text{supp}(\eta^\infty)$.

The function

$$D(\gamma, t) = \begin{cases} \dot{\gamma}(t), & (\gamma, t) \in \mathcal{D} \\ 0, & \text{otherwise} \end{cases}$$

is Borel measurable since $\dot{\gamma}(t) = \lim_{n\to\infty} n\,(\gamma(t + 1/n) - \gamma(t))$ for $(\gamma, t) \in \mathcal{D}$. Here

$$\mathcal{D} = \{(\gamma, t) \in \Gamma \times (0, \infty) : \dot{\gamma}(t) \ \text{exists}\},$$

which is a Borel subset of $\Gamma \times (0, \infty)$. We also set

$$\Upsilon := \mathcal{D} \cap (\text{supp}(\eta^\infty) \times (0, \infty))$$

and note that the Borel subsets of Υ are of the form $\Upsilon \cap \mathcal{A}$ for Borel $\mathcal{A} \subset \Gamma \times (0, \infty)$.

Notice that for each $(\gamma, t) \in \Upsilon$,

$$\begin{aligned} D(\gamma, t) &= \lim_{n\to\infty} n\,(\gamma(t + 1/n) - \gamma(t)) \\ &= \lim_{n\to\infty} v_n(\gamma(t), t) \\ &= \lim_{n\to\infty} v_n \circ E(\gamma, t), \end{aligned}$$

where

$$E : \Gamma \times (0, \infty) \to \mathbb{R} \times (0, \infty); (\gamma, t) \mapsto (\gamma(t), t).$$

As E is continuous,

$$\mathcal{G} := \left\{\Upsilon \cap E^{-1}(B) : \ \text{Borel } B \subset \mathbb{R} \times (0, \infty)\right\}$$

is a sigma-algebra contained within the Borel subsets of Υ. We also note that any $\mathcal{G}$ measurable function on Υ is of the form $g \circ E$ for a Borel $g : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ (Lemma 1.13 of [18]).

Since the restriction of D to Υ is the pointwise limit of $\mathcal{G}$ measurable functions, this function is $\mathcal{G}$ measurable (Proposition 2.7 of [7] and Lemma 1.10 of [18]). That is,

$$D|_\Upsilon = v \circ E$$

for a Borel $v : \mathbb{R} \times (0, \infty) \to \mathbb{R}$. Moreover, for $\gamma \in \text{supp}(\eta^\infty) \cap \{\Psi < \infty\}$

$$\dot{\gamma}(t) = v(\gamma(t), t) \ \text{ a.e. } t > 0.$$

Proof of (vi). Suppose $h:\mathbb{R}\to\mathbb{R}$ is continuous with

$$|h(x)|\le C_0(1+|x|),\quad x\in\mathbb{R}. \tag{3.6}$$

We note that if η^∞ satisfies property (vi) for all h satisfying (3.6), then η^∞ satisfies property (vi) for all Borel $h:\mathbb{R}\to\mathbb{R}$ with $\int_\Gamma h(\gamma(t))^2 d\eta^\infty(\gamma)<\infty$ (Proposition 7.9 of [7]). Consequently, it suffices to send $j\to\infty$ in

$$\int_{t_1}^{t_2}\left(\int_\Gamma \dot\gamma(t)h(\gamma(t))d\eta^{k_j}(\gamma)\right)dt = \int_{t_1}^{t_2}\left(\int_\Gamma\left(v_0(\gamma(0))-\int_0^t W'*(e_{s\#}\eta^{k_j})(\gamma(s))ds\right)h(\gamma(t))d\eta^{k_j}(\gamma)\right)dt \tag{3.7}$$

for h which satisfies (3.6) and each $0\le t_1\le t_2$.

To this end, we first set $g(x):=\int_0^x h(y)dy$ and note

$$\int_{t_1}^{t_2}\left(\int_\Gamma \dot\gamma(t)h(\gamma(t))d\eta^{k_j}(\gamma)\right)dt=\int_\Gamma (g(\gamma(t_2))-g(\gamma(t_1)))d\eta^{k_j}(\gamma).$$

In view of (3.6), $|g(\gamma(t))|$ grows at most quadratically in $|\gamma(t)|$. As $\gamma\mapsto\gamma(t)^2$ is uniformly integrable, $g(\gamma(t))$ is also uniformly integrable and

$$\begin{aligned}\lim_{j\to\infty}\int_{t_1}^{t_2}\left(\int_\Gamma \dot\gamma(t)h(\gamma(t))d\eta^{k_j}(\gamma)\right)dt &= \lim_{j\to\infty}\int_\Gamma (g(\gamma(t_2))-g(\gamma(t_1)))d\eta^{k_j}(\gamma)\\ &=\int_\Gamma (g(\gamma(t_2))-g(\gamma(t_1)))d\eta^{\infty}(\gamma)\\ &=\int_{t_1}^{t_2}\left(\int_\Gamma \dot\gamma(t)h(\gamma(t))d\eta^{\infty}(\gamma)\right)dt.\end{aligned} \tag{3.8}$$

Next we note that since v_0 is absolutely continuous, we can choose a constant C_1 such that

$$|v_0(x)|\le C_1(1+|x|).$$

Thus

$$\begin{aligned}|v_0(\gamma(0))h(\gamma(t))|&\le C_0(1+|\gamma(t)|)C_1(1+|\gamma(0)|)\\ &\le \frac{1}{2}C_0^2(1+|\gamma(t)|)^2+\frac{1}{2}C_1(1+|\gamma(0)|)^2\\ &\le C_0^2(1+\gamma(t)^2)+C_1^2(1+\gamma(0)^2).\end{aligned} \tag{3.9}$$

Consequently, $v_0(\gamma(0))h(\gamma(t))$ uniformly integrable. Therefore,

$$\lim_{j\to\infty}\int_\Gamma v_0(\gamma(0))h(\gamma(t))d\eta^{k_j}(\gamma)=\int_\Gamma v_0(\gamma(0))h(\gamma(t))d\eta^{\infty}(\gamma) \tag{3.10}$$

for all $t\ge 0$. In view of (3.5) and (3.9), $\int_\Gamma v_0(\gamma(0))h(\gamma(t))d\eta^{k_j}(\gamma)$ is uniformly bounded for $t\in[t_1,t_2]$ and $j\in\mathbb{N}$; so we can apply dominated convergence to find

$$\lim_{j\to\infty}\int_{t_1}^{t_2}\left(\int_\Gamma v_0(\gamma(0))h(\gamma(t))d\eta^{k_j}(\gamma)\right)dt=\int_{t_1}^{t_2}\left(\int_\Gamma v_0(\gamma(0))h(\gamma(t))d\eta^{\infty}(\gamma)\right)dt. \tag{3.11}$$

By (3.6) and the at most linear growth of W', there is a constant C_2 such that

$$\begin{aligned}|h(\gamma(t))W'(\gamma(s)-\xi(s))| &\le C_0(1+|\gamma(t)|)|C_2(1+|\gamma(s)-\xi(s)|)\\ &\le C_0^2(1+\gamma(t)^2)+C_2^2(1+(\gamma(s)-\xi(s))^2)\\ &\le C_0^2(1+\gamma(t)^2)+2C_2^2(1+\gamma(s)^2+(\xi(s))^2). \end{aligned} \tag{3.12}$$

As a result $(\gamma,\xi)\mapsto h(\gamma(t))W'(\gamma(s)-\xi(s))$ is uniformly integrable with respect to $(\eta^{k_j}\times\eta^{k_j})_{j\in\mathbb{N}}$. It follows that

$$\begin{aligned}&\lim_{j\to\infty}\int_\Gamma\int_\Gamma h(\gamma(t))W'(\gamma(s)-\xi(s))d\eta^{k_j}(\gamma)d\eta^{k_j}(\xi)\\ &\quad=\int_\Gamma\int_\Gamma h(\gamma(t))W'(\gamma(s)-\xi(s))d\eta^{\infty}(\gamma)d\eta^{\infty}(\xi).\end{aligned}$$

Combining (3.5) with (3.12), we see $\int_\Gamma\int_\Gamma h(\gamma(t))W'(\gamma(s)-\xi(s))d\eta^{k_j}(\gamma)d\eta^{k_j}(\xi)$ is uniformly bounded for $j\in\mathbb{N}$ and $s\in[0,t]$. By dominated convergence,

$$\begin{aligned}&\lim_{j\to\infty}\int_\Gamma\left(\int_0^t W'*(e_{s\#}\eta^{k_j})(\gamma(s))ds\right)h(\gamma(t))d\eta^{k_j}(\gamma)\\ &\quad=\lim_{j\to\infty}\int_0^t\left(\int_\Gamma\int_\Gamma h(\gamma(t))W'(\gamma(s)-\xi(s))d\eta^{k_j}(\gamma)d\eta^{k_j}(\xi)\right)ds\\ &\quad=\int_0^t\left(\int_\Gamma\int_\Gamma h(\gamma(t))W'(\gamma(s)-\xi(s))d\eta^{\infty}(\gamma)d\eta^{\infty}(\xi)\right)ds\\ &\quad=\int_\Gamma\left(\int_0^t W'*(e_{s\#}\eta^{\infty})(\gamma(s))ds\right)h(\gamma(t))d\eta^{\infty}(\gamma)\end{aligned}\tag{3.13}$$

for each $t\ge 0$. In a very similar way, we conclude

$$\begin{aligned}&\lim_{j\to\infty}\int_{t_1}^{t_2}\int_\Gamma\left(\int_0^t W'*(e_{s\#}\eta^{k_j})(\gamma(s))ds\right)h(\gamma(t))d\eta^{k_j}(\gamma)dt\\ &\quad=\int_{t_1}^{t_2}\int_\Gamma\left(\int_0^t W'*(e_{s\#}\eta^{\infty})(\gamma(s))ds\right)h(\gamma(t))d\eta^{\infty}(\gamma)dt.\end{aligned}$$

Putting this limit together with (3.8) and (3.11) allows us to send $j\to\infty$ in (3.7) and find

$$\begin{aligned}&\int_{t_1}^{t_2}\left(\int_\Gamma\dot\gamma(t)h(\gamma(t))d\eta^{\infty}(\gamma)\right)dt\\ &\quad=\int_{t_1}^{t_2}\left(\int_\Gamma\left(v_0(\gamma(0))-\int_0^t W'*(e_{s\#}\eta^{\infty})(\gamma(s))ds\right)h(\gamma(t))d\eta^{\infty}(\gamma)\right)dt.\end{aligned}$$

Proof of (vii). Let us set

$$E^j(t):=\int_\Gamma\frac{1}{2}\dot\gamma(t+)^2d\eta^{k_j}(\gamma)+\frac{1}{2}\int_\Gamma\int_\Gamma W(\gamma(t)-\xi(t))d\eta^{k_j}(\gamma)d\eta^{k_j}(\xi)$$

for $j\in\mathbb{N}$ and every $t\ge 0$. We will show

$$\sup_{j\in\mathbb{N}}\max_{t\in[t_1,t_2]}|E^j(t)|<\infty\tag{3.14}$$

and

$$\lim_{j\to\infty}\int_{t_1}^{t_2}E^j(t)dt=\int_{t_1}^{t_2}E^{\infty}(t)dt\tag{3.15}$$

for each $t_1 \le t_2$, where

$$E^\infty(t) := \int_\Gamma \frac{1}{2}\dot{\gamma}(t)^2 d\eta^\infty(\gamma) + \frac{1}{2}\int_\Gamma\int_\Gamma W(\gamma(t)-\xi(t))d\eta^\infty(\gamma)d\eta^\infty(\xi)$$

for almost every $t \ge 0$. As $(E^j)_{j\in\mathbb{N}}$ is a sequence of nonincreasing functions (by (2.14)) which is uniformly bounded on each compact subinterval of $[0,\infty)$, Helly's selection theorem implies $\overline{E}(t) := \lim_{j\to\infty} E^j(t)$ exists for all $t \ge 0$. Clearly $\overline{E}$ is nonincreasing. By (3.15), $E^\infty(t) = \overline{E}(t)$ for almost every $t \ge 0$. We then would conclude that η^∞ satisfies (vii).

In order prove (3.14), we first note that since W is semiconvex and W' grows at most linearly, W grows at most quadratically. In particular, there is a constant $C_3 \ge 0$ for which

$$|W(x-y)| \le C_3(1+x^2+y^2). \tag{3.16}$$

Thus

$$\begin{aligned} E^j(t) &\le E^j(0) \\ &= \int_{\mathbb{R}} \frac{1}{2}v_0(x)^2 d\rho_0^k(x) + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} W(x-y)d\rho_0^k(x)d\rho_0^k(y) \\ &\le B + \frac{C_3}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}(1+x^2+y^2)d\rho_0^k(x)d\rho_0^k(y) \\ &\le B + C_3\left(\frac{1}{2}+\int_{\mathbb{R}} x^2 d\rho_0^k(x)\right) \\ &\le B + C_3(1/2+B) \end{aligned}$$

for each $t \ge 0$. Moreover,

$$\begin{aligned} E^j(t) &\ge \frac{1}{2}\int_\Gamma\int_\Gamma W(\gamma(t)-\xi(t))d\eta^{k_j}(\gamma)d\eta^{k_j}(\xi) \\ &\ge -\frac{C_3}{2}\int_\Gamma\int_\Gamma (1+\gamma(t)^2+\xi(t)^2)d\eta^{k_j}(\gamma)d\eta^{k_j}(\xi) \\ &\ge -C_3\left(\frac{1}{2}+\int_\Gamma \gamma(t)^2 d\eta^{k_j}(\gamma)\right) \end{aligned}$$

In view of (3.5), $\int_\Gamma \gamma(t)^2 d\eta^{k_j}(\gamma)$ is bounded above independently of $j \in \mathbb{N}$ and $t \in [t_1, t_2]$. These upper and lower bounds together prove (3.14).

In order to show (3.15), we note $\gamma \mapsto \gamma(t)^2$ is uniformly integrable and $|W(\gamma(t)-\xi(t))| \le C_3(1+\gamma(t)^2+\xi(t)^2)$. It follows that

$$\lim_{j\to\infty}\int_\Gamma\int_\Gamma W(\gamma(t)-\xi(t))d\eta^{k_j}(\gamma)d\eta^{k_j}(\xi) = \int_\Gamma\int_\Gamma W(\gamma(t)-\xi(t))d\eta(\gamma)d\eta(\xi).$$

It is also straightforward to combine (3.5) with (3.16) to show that the function $t \mapsto \int_\Gamma\int_\Gamma W(\gamma(t)-\xi(t))d\eta^{k_j}(\gamma)d\eta^{k_j}(\xi)$ is uniformly bounded for on the interval $[t_1, t_2]$. Dominated convergence then implies

$$\begin{aligned} &\lim_{j\to\infty}\textstyle\int_{t_1}^{t_2}\int_\Gamma\int_\Gamma W(\gamma(t)-\xi(t))d\eta^{k_j}(\gamma)d\eta^{k_j}(\xi)dt \\ &\quad= \textstyle\int_{t_1}^{t_2}\int_\Gamma\int_\Gamma W(\gamma(t)-\xi(t))d\eta(\gamma)d\eta(\xi)dt. \end{aligned} \tag{3.17}$$

Also observe that since η^{k_j} satisfy property (v) and (vi),

$$
\begin{aligned}
&\int_\Gamma\left(\int_{t_1}^{t_2}\dot\gamma(t)^2dt\right)d\eta^{k_j}(\gamma)\\
&=\int_{t_1}^{t_2}\left(\int_\Gamma\dot\gamma(t)^2d\eta^{k_j}(\gamma)\right)dt\\
&=\int_{t_1}^{t_2}\left(\int_\Gamma\left(v_0(\gamma(0))-\int_0^t W'*(e_{s\#}\eta^{k_j})(\gamma(s))ds\right)\dot\gamma(t)d\eta^{k_j}(\gamma)\right)dt\\
&=\int_\Gamma\left(\int_{t_1}^{t_2}\left(v_0(\gamma(0))-\int_0^t W'*(e_{s\#}\eta^{k_j})(\gamma(s))ds\right)\dot\gamma(t)dt\right)d\eta^{k_j}(\gamma)\\
&=\int_\Gamma\left\{\left(v_0(\gamma(0))-\int_0^t W'*(e_{s\#}\eta^{k_j})(\gamma(s))ds\right)\gamma(t)\Big|_{t_1}^{t_2}\right.\\
&\quad\left.+\int_{t_1}^{t_2}\gamma(t)W'*(e_{t\#}\eta^{k_j})(\gamma(t))dt\right\}d\eta^{k_j}(\gamma).
\end{aligned}
$$

The limit

$$
\lim_{j\to\infty}\int_\Gamma v_0(\gamma(0))\gamma(t)\Big|_{t_1}^{t_2}d\eta^{k_j}(\gamma)=\int_\Gamma v_0(\gamma(0))\gamma(t)\Big|_{t_1}^{t_2}d\eta^{\infty}(\gamma)
$$

follows from (3.10) while

$$
\begin{aligned}
&\lim_{j\to\infty}\int_\Gamma\left(\int_0^t W'*(e_{s\#}\eta^{k_j})(\gamma(s))ds\right)\gamma(t)\Big|_{t_1}^{t_2}d\eta^{k_j}(\gamma)\\
&\quad=\int_\Gamma\left(\int_0^t W'*(e_{s\#}\eta^{\infty})(\gamma(s))ds\right)\gamma(t)\Big|_{t_1}^{t_2}d\eta^{\infty}(\gamma)
\end{aligned}
$$

in a consequence of (3.13). Once we write,

$$
\begin{aligned}
&\textstyle\int_\Gamma\left(\int_{t_1}^{t_2}\gamma(t)W'*(e_{t\#}\eta^{k_j})(\gamma(t))dt\right)d\eta^{k_j}(\gamma)\\
&=\textstyle\int_{t_1}^{t_2}\int_\Gamma\int_\Gamma\gamma(t)W'(\gamma(t)-\xi(t))d\eta^{k_j}(\gamma)d\eta^{k_j}(\xi)dt,
\end{aligned}
$$

we see that the limit

$$
\begin{aligned}
&\lim_{j\to\infty}\int_\Gamma\left(\int_{t_1}^{t_2}\gamma(t)W'*(e_{t\#}\eta^{k_j})(\gamma(t))dt\right)d\eta^{k_j}(\gamma)\\
&\quad=\int_\Gamma\left(\int_{t_1}^{t_2}\gamma(t)W'*(e_{t\#}\eta^{\infty})(\gamma(t))dt\right)d\eta^{\infty}(\gamma)
\end{aligned}
$$

follows from a minor variation of our proof of (3.13).

As a result,

$$
\begin{aligned}
&\lim_{j\to\infty}\int_{t_1}^{t_2}\left(\int_\Gamma\dot\gamma(t)^2d\eta^{k_j}(\gamma)\right)dt\\
&\quad=\lim_{j\to\infty}\int_\Gamma\left\{\left(v_0(\gamma(0))-\int_0^t W'*(e_{s\#}\eta^{k_j})(\gamma(s))ds\right)\gamma(t)\Big|_{t_1}^{t_2}\right.\\
&\quad\quad\left.+\int_{t_1}^{t_2}\gamma(t)W'*(e_{t\#}\eta^{k_j})(\gamma(t))dt\right\}d\eta^{k_j}(\gamma)\\
&\quad=\int_\Gamma\left\{\left(v_0(\gamma(0))-\int_0^t W'*(e_{s\#}\eta^{\infty})(\gamma(s))ds\right)\gamma(t)\Big|_{t_1}^{t_2}\right.
\end{aligned}
$$

$$+\int_{t_1}^{t_2}\gamma(t)W'*(e_{t\#}\eta^\infty)(\gamma(t))dt\Bigg\}d\eta^\infty(\gamma)$$
$$=\int_{t_1}^{t_2}\left(\int_\Gamma\dot\gamma(t)^2d\eta^\infty(\gamma)\right)dt.$$

This limit combined with (3.17) implies (3.15), as desired. □

4 Solutions to the pressureless Euler system

This section is dedicated to the proof of Theorem 1.3. So we assume $\rho_0 \in \mathcal{P}(\mathbb{R})$ with $\int_{\mathbb{R}} x^2 d\rho_0(x) < \infty$, $v_0 : \mathbb{R} \to \mathbb{R}$ is absolutely continuous, $W(x) + (c/2)x^2$ is convex and $|W'(x)|$ grows at most linearly as $|x| \to \infty$. According to Theorem 3.1, there is $\eta \in \mathcal{P}(\Gamma)$ which satisfies parts $(i) - (vii)$ of that statement. We will simply refer to these parts by their respective numbers below.

Let us define

$$\rho : (0,\infty) \to \mathcal{P}(\mathbb{R});\, t \mapsto e_{t\#}\eta$$

and select a Borel $v : \mathbb{R}\times(0,\infty) \to \mathbb{R}$ from part (vi). By (i), (ii), and (v),

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}}(\partial_t\phi + v\partial_x\phi)d\rho_t dt &= \int_0^\infty\int_\Gamma(\partial_t\phi(\gamma(t),t) + v(\gamma(t),t)\partial_x\phi(\gamma(t),t))d\eta(\gamma)dt\\ &= \int_0^\infty\int_\Gamma(\partial_t\phi(\gamma(t),t) + \dot\gamma(t)\partial_x\phi(\gamma(t),t))d\eta(\gamma)dt\\ &= \int_\Gamma\int_0^\infty\frac{d}{dt}\phi(\gamma(t),t)dtd\eta(\gamma)\\ &= -\int_\Gamma\phi(\gamma(0),0)d\eta(\gamma)\\ &= -\int_{\mathbb{R}}\phi(\cdot,0)d\rho_0\end{aligned}$$

for any $\phi \in C_c^\infty(\mathbb{R}\times[0,\infty))$; and in view of (vi),

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}}(v\partial_t\phi + v^2\partial_x\phi)d\rho_t dt\\ &= \int_0^\infty\int_\Gamma\left[\partial_t\phi(\gamma(t),t) + v(\gamma(t),t)\partial_x\phi(\gamma(t),t)\right]v(\gamma(t),t)d\eta(\gamma)dt\\ &= \int_0^\infty\int_\Gamma\underbrace{\left[\partial_t\phi(\gamma(t),t) + v(\gamma(t),t)\partial_x\phi(\gamma(t),t)\right]}_{\frac{d}{dt}\phi(\gamma(t),t)}\dot\gamma(t)d\eta(\gamma)dt\\ &= \int_0^\infty\int_\Gamma\frac{d}{dt}\phi(\gamma(t),t)\left[v_0(\gamma(0)) - \int_0^t W'*(e_{s\#}\eta)(\gamma(s))ds\right]d\eta(\gamma)dt\\ &= \int_\Gamma\int_0^\infty\frac{d}{dt}\phi(\gamma(t),t)\left[v_0(\gamma(0)) - \int_0^t W'*(e_{s\#}\eta)(\gamma(s))ds\right]dtd\eta(\gamma)\\ &= \int_0^\infty\int_\Gamma\phi(\gamma(t),t)W'*(e_{t\#}\eta)(\gamma(t))d\eta(\gamma)dt - \int_\Gamma\phi(\gamma(0),0)v_0(\gamma(0))d\eta(\gamma)\\ &= \int_0^\infty\int_{\mathbb{R}}\phi\left(W'*\rho_t\right)d\rho_t dt - \int_{\mathbb{R}}\phi(\cdot,0)v_0 d\rho_0.\end{aligned}$$

As a result, ρ and v is a weak solution pair of the pressureless Euler equations which satisfies the initial conditions $\rho|_{t=0} = \rho_0$ and $v|_{t=0} = v_0$.

A consequence of (v) is that for almost every $t > 0$,

$$\dot{\gamma}(t) = v(\gamma(t), t) \tag{4.1}$$

for η almost every $\gamma \in \Gamma$. Combining this with part (vii) gives

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}} v(x,t)^2 d\rho_t(x) + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} W(x-y)d\rho_t(x)d\rho_t(y)\\
&\quad = \int_{\Gamma} \frac{1}{2}\dot{\gamma}(t)^2 d\eta(\gamma) + \frac{1}{2}\int_{\Gamma}\int_{\Gamma} W(\gamma(t)-\xi(t))d\eta(\gamma)d\eta(\xi)\\
&\quad \le \int_{\Gamma} \frac{1}{2}\dot{\gamma}(s)^2 d\eta(\gamma) + \frac{1}{2}\int_{\Gamma}\int_{\Gamma} W(\gamma(s)-\xi(s))d\eta(\gamma)d\eta(\xi)\\
&\quad = \frac{1}{2}\int_{\mathbb{R}} v(x,s)^2 d\rho_s(x) + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} W(x-y)d\rho_s(x)d\rho_s(y)
\end{aligned}$$

for almost every $0 \le s \le t$. This proves (1.11).

Choose a Borel subset $S \subset \Gamma$ with $\eta(S) = 1$ such that (4.1) holds for each $\gamma \in S$ and almost every $t > 0$. By (iii),

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\gamma(t)-\xi(t)}{\sinh(\sqrt{c}t)}\right)^2\\
&\quad = \frac{2}{\sinh(\sqrt{c}t)^2}\left((v(\gamma(t),t)-v(\xi(t),t))(\gamma(t)-\xi(t)) - \frac{\sqrt{c}}{\tanh(\sqrt{c}t)}(\gamma(t)-\xi(t))^2\right) \le 0
\end{aligned}$$

for $\gamma, \xi \in S$ and almost every $t > 0$. It follows that

$$(v(x,t)-v(y,t))(x-y) \le \frac{\sqrt{c}}{\tanh(\sqrt{c}t)}(x-y)^2, \quad x, y \in e_t(S) \tag{4.2}$$

for almost every $t > 0$.

Fix a time $t > 0$ such that (4.2) holds. For each $\epsilon > 0$, we may also select a closed $F \subset S$ such that $\eta(S \setminus F) \le \epsilon$ (Theorem 1.1 [2]). Observe that

$$e_t(F \cap \{\Psi < \infty\}) = \bigcup_{m=1}^{\infty} e_t(F \cap \{\Psi \le m\}).$$

And since the sublevel sets of Ψ are compact, $F \cap \{\Psi \le m\}$ is compact. It follows that $e_t(F \cap \{\Psi < \infty\})$ is a Borel subset of $\mathbb{R}$. In addition,

$$\rho_t(e_t(F \cap \{\Psi < \infty\})) = \eta\left(e_t^{-1}\left(e_t(F \cap \{\Psi < \infty\})\right)\right) \ge \eta\left(F \cap \{\Psi < \infty\}\right) \ge 1-\epsilon \tag{4.3}$$

as $\eta(\{\Psi < \infty\}) = 1$ and $\eta(F) \ge 1-\epsilon$.

Note that the inequality in (4.2) also holds for $x, y \in e_t(F \cap \{\Psi < \infty\})$. As a result, (4.3) gives

$$\begin{aligned}
1 &\ge (\rho_t \times \rho_t)\left(\left\{(x,y) \in \mathbb{R}\times\mathbb{R} : (v(x,t)-v(y,t))(x-y) \le \frac{\sqrt{c}}{\tanh(\sqrt{c}t)}(x-y)^2\right\}\right)\\
&\ge (\rho_t \times \rho_t)\left(e_t(F \cap \{\Psi < \infty\}) \times e_t(F \cap \{\Psi < \infty\})\right)\\
&\ge (\rho_t(e_t(F \cap \{\Psi < \infty\})))^2
\end{aligned}$$

$$\geq (1-\epsilon)^2.$$

Since $\epsilon > 0$ is arbitrary,

$$1 = (\rho_t \times \rho_t)\left(\left\{(x,y)\in\mathbb{R}\times\mathbb{R} : (v(x,t)-v(y,t))(x-y) \leq \frac{\sqrt{c}}{\tanh(\sqrt{c}t)}(x-y)^2\right\}\right)$$
$$= \int_{\mathbb{R}} \rho_t\left(\left\{x\in\mathbb{R} : (v(x,t)-v(y,t))(x-y) \leq \frac{\sqrt{c}}{\tanh(\sqrt{c}t)}(x-y)^2\right\}\right) d\rho_t(y).$$

Thus, for ρ_t almost every $y \in \mathbb{R}$

$$\rho_t\left(\left\{x\in\mathbb{R} : (v(x,t)-v(y,t))(x-y) \leq \frac{\sqrt{c}}{\tanh(\sqrt{c}t)}(x-y)^2\right\}\right) = 1.$$

That is, for ρ_t almost every every $x, y \in \mathbb{R}$,

$$(v(x,t)-v(y,t))(x-y) \leq \frac{\sqrt{c}}{\tanh(\sqrt{c}t)}(x-y)^2.$$

5 Concluding remarks

It remains to be addressed whether or not weak solution pairs of the pressureless-Euler equations discussed in this paper are uniquely determined by their initial conditions. When $W \equiv 0$, it has been deduced that weak solution pairs for a wide class of initial conditions are in fact unique [11]. It would be of great interest to extend this result to the semiconvex potentials considered in this work. It would even be of interest to know if weak solution pairs of the initial problem which are generated by probability measures on the path space (discussed in Theorem 3.1) are uniquely specified.

Another outstanding problem is to extend our existence theory to the pressureless-Euler equations in several space dimensions

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho v) = 0 \\ \partial_t(\rho v) + \nabla\cdot(\rho v\otimes v) = -\rho(\nabla W * \rho). \end{cases}$$

These equations hold in $\mathbb{R}^d \times (0,\infty)$ and model the dynamics of collections of particles that interact pairwise according a potential $W : \mathbb{R}^d \to \mathbb{R}$ and which experience inelastic collisions when they collide. Here ρ represents the time-dependent density of particles in $\mathbb{R}^d$ and $v : \mathbb{R}^d \times (0,\infty) \to \mathbb{R}^d$ is the local velocity field. The problem one encounters in trying to extend the approach developed in this paper is that analogs of the quantitative sticky particle property (1.8) or the entropy inequality (1.12) are not available for $d > 1$.

A Newton's equations

Here we show that the ODE system (1.7) has a solution on the interval $[0,\infty)$ for prescribed initial conditions. We recall the standing assumptions that $W : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, W is even, and that (1.3) holds.

Proposition A.1 *Suppose $m_1, \dots, m_N > 0$, $x_1, \dots, x_N \in \mathbb{R}$ and $v_1, \dots, v_N \in \mathbb{R}$. There are*

$$\gamma_1, \dots, \gamma_N \in C^2([0,\infty))$$

satisfying

$$\ddot{\gamma}_i(t) = -\sum_{j=1}^{N} m_j W'(\gamma_i(t) - \gamma_j(t)) \tag{A.1}$$

for $t > 0$ *and*

$$\gamma_i(0) = x_i \quad \textit{and} \quad \dot{\gamma}_i(0) = v_i \tag{A.2}$$

for $i = 1, \ldots, N$.

Proof By Peano's existence theorem, there is a solution $\gamma_1, \ldots, \gamma_N \in C^2([0, T))$ of the ODE (A.1) for some $T \in (0, \infty]$ which satisfies the initial conditions (A.2). We may assume that $[0, T)$ is the maximal interval of existence so that this solution cannot be continued to a larger interval if $T < \infty$. In this case, it must be that

$$\sup_{t\in[0,T)} |\dot{\gamma}_j(t)| = \infty \tag{A.3}$$

for some $j = 1, \ldots, N$. Otherwise, $\sup_{t\in[0,T)} |\dot{\gamma}_i(t)| < \infty$ and

$$\sup_{t\in[0,T)} |\gamma_i(t)| \leq |x_i| + T \sup_{t\in[0,T)} |\dot{\gamma}_i(t)| < \infty$$

for all $i = 1, \ldots, N$, and this solution $\gamma_1, \ldots, \gamma_N$ could then be continued to $[0, T + \epsilon)$ for some $\epsilon > 0$ (Chapter 1 of [10]).

Observe

$$\frac{1}{2}\sum_{i=1}^{N} m_i \dot{\gamma}_i(t)^2 + \frac{1}{2}\sum_{i,j=1}^{N} m_i m_j W(\gamma_i(t) - \gamma_j(t)) = \frac{1}{2}\sum_{i=1}^{N} m_i v_i^2 + \frac{1}{2}\sum_{i,j=1}^{N} m_i m_j W(x_i - x_j) \tag{A.4}$$

for $t \in [0, T)$. This can be verified by differentiating the left hand side of (A.4) and by using that $\gamma_1, \ldots, \gamma_N$ solves (A.1). Arguing as we did to prove Corollary 2.9, we find

$$\sum_{i=1}^{N} m_i \dot{\gamma}_i(t)^2 ds \leq \varphi'(t) \left(\sum_{i=1}^{N} m_i v_0(x_i)^2 + \frac{1}{2}\sum_{i,j=1}^{N} m_i m_j W'(x_i - x_j)^2 \right) \tag{A.5}$$

for $t \in [0, T)$. As $m_i > 0$ for each $i = 1, \ldots, N$, (A.3) could not hold for any $j = 1, \ldots, N$. We conclude that $T = \infty$. □

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