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Continuous time approximation of Nash equilibria in monotone games

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We consider the problem of approximating Nash equilibria of N functions $f_1, \dots, f_N$ of N variables. In particular, we show systems of the form

$$\dot{u}_j(t) = -\nabla_{x_j} f_j(u(t))$$

($j = 1, \dots, N$) are well-posed and the large time limits of their solutions $u(t) = (u_1(t), \dots, u_N(t))$ are Nash equilibria for $f_1, \dots, f_N$ provided that these functions satisfy an appropriate monotonicity condition. To this end, we will invoke the theory of maximal monotone operators on a Hilbert space. We will also identify an application of these ideas in game theory and show how to approximate equilibria in some game theoretic problems in function spaces.

Keywords: Noncooperative games; maximal monotone operator.

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1. Introduction

Let us first recall the notion of a Nash equilibrium. Consider a collection of N sets $X_1, \dots, X_N$ and define

$$X := X_1 \times \cdots \times X_N.$$

For a given $x = (x_1, \dots, x_N) \in X$, $j \in \{1, \dots, N\}$, and $y_j \in X_j$, we will use the notation (y_j, x_{-j}) for the point in X in which y_j replaces x_j in the coordinates of x.

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That is,

$$(y_j, x_{-j}) := (x_1, \dots, x_{j-1}, y_j, x_{j+1}, \dots, x_N).$$

A collection of functions $f_1, \dots, f_N : X \to \mathbb{R}$ has a *Nash equilibrium* at $x \in X$ provided

$$f_j(x) \le f_j(y_j, x_{-j})$$

for all $y_j \in X_j$ and $j = 1, \dots, N$.

Nash recognized that an equilibrium is a fixed point of the set valued mapping

$$X \ni x \mapsto \operatorname*{arg\,min}_{y \in X} \left\{ \sum_{j=1}^{N} f_j(y_j, x_{-j}) \right\}.$$

In particular, he applied Kakutani's fixed point theorem [17] to show that if $X_1, \dots, X_N$ are nonempty, convex, compact subsets of Euclidean space and each $f_1, \dots, f_N$ is multilinear, then $f_1, \dots, f_N$ has a Nash equilibrium at some $x \in X$ [24, 25]. Nash was interested in multilinear $f_1, \dots, f_N$ as they correspond to the expected cost of players assuming mixed strategies in noncooperative games. More generally, his existence theorem holds if each $f_j : X \to \mathbb{R}$ is continuous and

$$X \ni y \mapsto \sum_{j=1}^{N} f_j(y_j, x_{-j}) \text{ is convex for each } x \in X. \tag{1.1}$$

As the existence of a Nash equilibrium is due to a nonconstructive fixed point theorem, it seems unlikely that there are good ways to approximate these points. Nevertheless, we contend that there is a nontrivial class of functions $f_1, \dots, f_N$ for which the approximation of Nash equilibria is at least theoretically feasible. We will explain below that a sufficient condition for the approximation of Nash equilibria is that $f_1, \dots, f_N$ satisfy

$$\sum_{j=1}^{N} (\nabla_{x_j} f_j(x) - \nabla_{x_j} f_j(y)) \cdot (x_j - y_j) \ge 0 \tag{1.2}$$

for each $x, y \in X$. Here, we are considering each X_j as a subset of a Euclidean space and use "$\cdot$" to denote the dot products on any of these spaces. We are also temporarily assuming that $f_1, \dots, f_N$ are each differentiable on some neighborhood of X.

Our approach starts with the observation that if (1.1) holds and $X_1, \dots, X_N$ are convex, then x is a Nash equilibrium if and only if

$$\sum_{j=1}^{N} \nabla_{x_j} f_j(x) \cdot (y_j - x_j) \ge 0 \tag{1.3}$$

for each $y \in X$. As a result, it is natural to consider the differential inequalities

$$\sum_{j=1}^{N}(\dot{u}_j(t) + \nabla_{x_j} f_j(u(t))) \cdot (y_j - u_j(t)) \geq 0 \tag{1.4}$$

for $t \geq 0$ and $y \in X$. Here,

$$u : [0, \infty) \to X; \quad t \mapsto (u_1(t), \dots, u_N(t))$$

is the unknown. We will below argue that for an appropriately chosen initial condition $u(0) \in X$, the *Cesàro mean* of u

$$t \mapsto \frac{1}{t}\int_0^t u(s)ds$$

will converge to a Nash equilibrium of $f_1, \dots, f_N$ as $t \to \infty$ provided that (1.2) holds.

1.1. *Two examples*

An elementary example which illustrates this approach may be observed when $X_1 = X_2 = [-1, 1]$, $X = [-1, 1]^2$,

$$f_1(x_1, x_2) = x_1 x_2, \quad \text{and} \quad f_2(x_1, x_2) = -x_1 x_2.$$

It is not hard to check that the origin $(0, 0)$ is a Nash equilibrium of f_1, f_2 and that f_1, f_2 satisfy (1.2) (with equality holding) for each $x, y \in [-1, 1]^2$. We now seek an absolutely continuous path

$$u : [0, \infty) \to [-1, 1]^2$$

such that

$$(\dot{u}_1(t) + u_2(t))(y_1 - u_1(t)) + (\dot{u}_2(t) - u_1(t))(y_2 - u_2(t)) \geq 0 \tag{1.5}$$

holds for almost every $t \geq 0$ and each $(y_1, y_2) \in [-1, 1]$.

It turns out that if

$$u_1(0)^2 + u_2(0)^2 \leq 1,$$

then the unique solution of these differential inequalities is

$$\begin{cases} u_1(t) = u_1(0)\cos(t) - u_2(0)\sin(t), \\ u_2(t) = u_2(0)\cos(t) + u_1(0)\sin(t). \end{cases}$$

This solution simply parametrizes the circle of radius $(u_1(0)^2 + u_2(0)^2)^{1/2}$. In particular,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t u(s)ds = (0, 0).$$

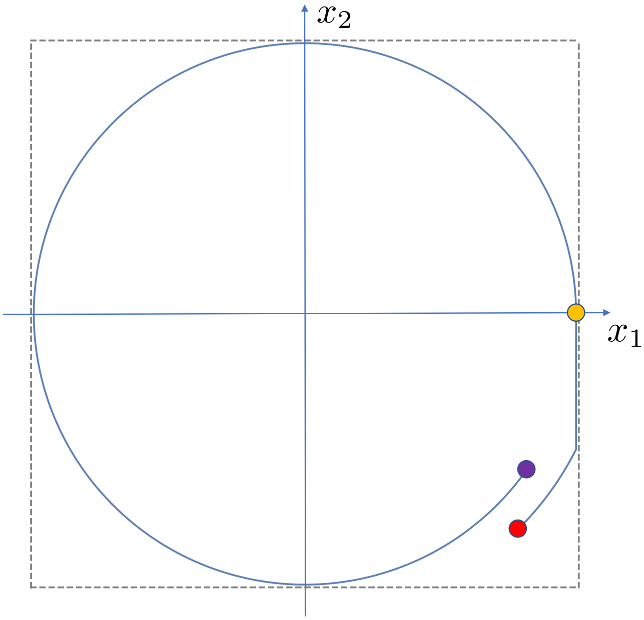

Fig. 1. (Color online) Plot of a solution u of (1.5) where the initial position of u is indicated in red. This solution starts out on a circle centered at the origin traversing it counterclockwise until it hits the boundary line segment ($x_1 = 1, -1 \leq x_2 \leq 0$). Then it proceeds upwards along this boundary line segment until it arrives at some time s at the orange marker which is located at $(1, 0)$. For times $t \geq s$, the position $u(t)$ as shown in purple remains on the unit circle traversing it counterclockwise.

For general initial conditions $(u_1(0), u_2(0)) \in [-1, 1]^2$, it can be shown that there is a solution u of (1.5) with $u_1(s)^2 + u_2(s)^2 \leq 1$ at some time $s \geq 0$. Refer to Fig. 1 for a schematic. It then follows that the same limit above holds for this solution.

The method we present can be extended in certain cases when $X_1, \ldots, X_N$ are not compact. For example, suppose $X_1 = X_2 = \mathbb{R}$,

$$f_1(x_1, x_2) = \frac{1}{2}x_1^2 - x_1x_2 - x_1, \quad \text{and} \quad f_2(x_1, x_2) = \frac{1}{2}x_2^2 + x_1x_2 - 2x_2.$$

It is easy to check that f_1, f_2 satisfy (1.2). Furthermore,

$$\begin{cases} \partial_{x_1} f_1(x_1, x_2) = x_1 - x_2 - 1 = 0, \\ \partial_{x_2} f_2(x_1, x_2) = x_2 + x_1 - 2 = 0 \end{cases}$$

provided

$$x_1 = \frac{3}{2} \quad \text{and} \quad x_2 = \frac{1}{2}.$$

As a result, this is the unique Nash equilibrium of f_1, f_2.

We note that the solution of

$$\begin{cases} \dot{u}_1(t) = -(u_1(t) - u_2(t) - 1), \\ \dot{u}_2(t) = -(u_2(t) + u_1(t) - 2) \end{cases}$$

is given by

$$\begin{cases} u_1(t) = \dfrac{3}{2} + \left(u_1(0) - \dfrac{3}{2}\right) e^{-t}\cos(t) + \left(u_2(0) - \dfrac{1}{2}\right) e^{-t}\sin(t), \\ u_2(t) = \dfrac{1}{2} + \left(u_2(0) - \dfrac{1}{2}\right) e^{-t}\cos(t) + \left(\dfrac{3}{2} - u_1(0)\right) e^{-t}\sin(t). \end{cases}$$

It is now clear that $\lim_{t\to\infty} u_1(t) = 3/2$ and $\lim_{t\to\infty} u_2(t) = 1/2$. Therefore,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t u(s)ds = \left(\frac{3}{2}, \frac{1}{2}\right).$$

1.2. *A general setting*

It just so happens that our method does not rely on the spaces $X_1, \ldots, X_N$ being finite dimensional. Consequently, we will consider the following version of our approximation problem. Let V be a real Hilbert space with continuous dual space V^*. Further assume

$$X_1, \ldots, X_N \subset V \text{ are closed and convex with nonempty interiors}$$

and suppose

$$f_1, \ldots, f_N : X \to \mathbb{R}$$

satisfy

$$\begin{cases} f_1, \ldots, f_N \text{ are weakly lowersemicontinuous,} \\ X \ni y \mapsto \displaystyle\sum_{j=1}^N f_j(y_j, x_{-j}) \text{ is convex for each } x \in X, \\ X \ni x \mapsto \displaystyle\sum_{j=1}^N f_j(y_j, x_{-j}) \text{ is weakly continuous for each } y \in X, \text{ and} \\ \displaystyle\sum_{j=1}^N (\partial_{x_j} f_j(x) - \partial_{x_j} f_j(y), x_j - y_j) \ge 0 \quad \text{for each } x, y \in X. \end{cases} \tag{1.6}$$

For the last condition listed in (1.6), we mean

$$\sum_{j=1}^N (p_j - q_j, x_j - y_j) \ge 0,$$

whenever

$$p_j \in \partial_{x_j} f_j(x) := \{\zeta \in V^* : f_j(z, x_{-j}) \ge f_j(x) + (\zeta, z - x_j),\ z \in X_j\}$$

and $q_j \in \partial_{x_j} f_j(y)$ for $j = 1, \ldots, N$. Here, $(\cdot, \cdot)$ denotes the natural pairing between V^* and V. Note in particular that $\partial_{x_j} f_j(x)$ is not a subdifferential in the usual sense as $f_j(z, x_{-j}) \geq f_j(x) + (\zeta, z - x_j)$ is only required to hold for $z \in X_j$. Also, note that $x \in X$ is a Nash equilibrium for $f_1, \ldots, f_N$ provided

$$0 \in \partial_{x_j} f_j(x)$$

for $j = 1, \ldots, N$.

When V is finite dimensional, we naturally identify V and V^* with Euclidean space of the same dimension. Alternatively, when V is infinite dimensional, we will suppose there is another real Hilbert space H for which

$$V \subset H \subset V^*$$

and $V \subset H$ is continuously embedded. It is with respect to this space H that we consider the following initial value problem: for a given $u^0 \in X$, find an absolutely continuous $u : [0, \infty) \to H^N$ such that

$$\begin{cases} u(0) = u^0, \quad \text{and} \\ \dot{u}_j(t) + \partial_{x_j} f_j(u(t)) \ni 0 \quad \text{for a.e. } t \geq 0 \text{ and each } j = 1, \ldots, N. \end{cases} \tag{1.7}$$

Note that the evolution equation above can be written explicitly as

$$(\dot{u}_j(t), y_j - u_j(t)) + f_j(y_j, u_{-j}(t)) - f_j(u(t)) \geq 0$$

for a.e. $t \geq 0$, each $y_j \in X_j$, and each $j = 1, \ldots, N$. It is routine to verify that this evolution is equivalent to (1.4) when $H = V = \mathbb{R}^d$ with the standard dot product and each $f_1, \ldots, f_N$ is differentiable on a neighborhood of X.

Our central result involving this initial value problem is as follows. In this statement, we will make use of the set

$$\mathcal{D} = \{x \in X : \partial_{x_j} f_j(x) \cap H \neq \emptyset \text{ for } j = 1, \ldots, N\}. \tag{1.8}$$

Theorem 1.1. *Assume $f_1, \ldots, f_N$ satisfy* (1.6).

(i) *For any $u^0 \in \mathcal{D}$ there is a unique Lipschitz continuous $u : [0, \infty) \to H^N$ satisfying $u(t) \in \mathcal{D}$ for each $t \geq 0$ and the initial value problem* (1.7).

(ii) *If $f_1, \ldots, f_N$ has a Nash equilibrium, then*

$$\frac{1}{t} \int_0^t u(s) ds$$

converges weakly in H^N to a Nash equilibrium of $f_1, \ldots, f_N$ as $t \to \infty$.

Remark 1.2. We will also explain that if $(\partial_{x_1} f_1, \ldots, \partial_{x_N} f_N)$ is single-valued, everywhere-defined, monotone, and hemicontinuous we can obtain the same result as above without assuming (1.6). This follows directly from a theorem of Browder and Minty.

We will phrase our initial value problem (1.7) as the evolution generated by a monotone operator on H^N with domain $\mathcal{D}$. According to pioneering work of

Kato [18] and Komura [20], the crucial task will be to verify the maximality of this operator. Regarding the large time behavior of solutions, the convergence to equilibria of maximal monotone operators by the Cesàro means of semigroups they generate originates in the work of Baillon and Brézis [3]. We also refer the reader to [2, Chap. 3] which gives a detailed discussion of this phenomenon. Moreover, the finite dimensional setting with smooth $f_1, \dots, f_N$ is an example of a differential variational inequality which is discussed in [2]. In addition, we note that Rockafellar verified Theorem 1.1 in the two-player, zero sum case ($N = 2$ and $f_1 + f_2 \equiv 0$) [28].

We also remark that in finite dimensions, similar results were known to Flåm [13]. In particular, Flåm seems to be the first author to identify the important role of the monotonicity condition (1.2) in approximating Nash equilibria. However, Rosen appears to be the first to consider using equations such as (1.4) to approximate equilibrium points [29]. Additionally, we would like to point out that the finite dimensional variational inequality (1.3) and its relation to Nash equilibrium is studied in depth in the monograph by Facchinei and Pang [11]. In infinite dimensions, there have also been several recent works which discuss discrete time approximations for related variational inequalities [7, 6, 1, 9, 16].

Ratliff *et al.* have analyzed games in which the strategy spaces for players are general topological spaces [26, 27]. When the spaces are smooth manifolds or Banach spaces, these authors focused on differential Nash equilibria which are points $x \in X$ such that $\nabla_{x_j} f_j(x) = 0$ for $j = 1, \dots, N$ and that the Hessian of $y_j \mapsto f_j(y_j, x_{-j})$ at x_j is positive definite. In particular, they used dynamical systems methods to study the limiting behavior of gradient-based algorithms to approximate differential Nash equilibria. We also recommend the work by Mazumdar *et al.* [21] for a recent discussion of how gradient-based algorithms perform in approximating equilibria and issues that may arise when employing them. The distinction with our work is that we focus almost entirely on exploiting monotonicity, we do not make the assumption that $f_1, \dots, f_N$ are smooth, and we restrict our attention to (global) Nash equilibrium in a Hilbert space setting.

As mentioned, the asymptotic statements we made above are predicated on the existence of a Nash equilibrium. This is guaranteed to be the case if X is weakly compact or if there is some $\theta > 0$ such that

$$\sum_{j=1}^{N} (\partial_{x_j} f_j(x) - \partial_{x_j} f_j(y), x_j - y_j) \geq \theta \|x - y\|^2$$

for each $x, y \in X$. Here, we are using $\|\cdot\|$ for the product norm on V^N. When such coercivity holds, $f_1, \dots, f_N$ has a unique Nash equilibrium and solutions of the initial value problem (1.7) converge exponentially fast in H^N to this equilibrium point.

We will prove Theorem 1.1 in the following section. Then we will apply this result to flows in Euclidean spaces of the form (1.4) in Sec. 3. Finally, we will consider examples in Lebesgue and Sobolev spaces in Sec. 4. A prototypical collection of functionals we will study is

$$f_j(v) = \int_\Omega \frac{1}{2}|\nabla v_j|^2 + F_j(v)dx$$

for $j = 1, \ldots, N$, where $v = (v_1, \ldots, v_N)$ and $\Omega \subset \mathbb{R}^d$ is a bounded domain. Here each $X_j = V = H_0^1(\Omega)$ and $H = L^2(\Omega)$. For this particular example, we will argue that if $F_1, \ldots, F_N : \mathbb{R}^N \to \mathbb{R}$ satisfy suitable growth and monotonicity conditions, then $f_1, \ldots, f_N$ has a unique Nash equilibrium which can be approximated by solutions $u_1, \ldots, u_N : \Omega \times [0, \infty) \to \mathbb{R}$ of the parabolic initial value problem

$$\begin{cases} \partial_t u_j = \Delta u_j - \partial_{z_j} F_j(u) & \text{in } \Omega \times (0, \infty), \\ u_j = 0 & \text{on } \partial\Omega \times [0, \infty), \\ u_j = u_j^0 & \text{on } \Omega \times \{0\} \end{cases}$$

for appropriately chosen initial conditions $u_1^0, \ldots, u_N^0 : \Omega \to \mathbb{R}$.

2. Approximation Theorem

In this section, we will briefly recall the notion of a maximal monotone operator on a Hilbert space and state a few key results for these operators. Then we will apply these results to prove Theorem 1.1 and a few related corollaries.

2.1. *Maximal monotone operators on a Hilbert space*

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. We will denote 2^H as the power set or collection of all subsets of H. We recall that $B : H \to 2^H$ is monotone if

$$\langle p - q, x - y \rangle \geq 0$$

for all $x, y \in H$, $p \in Bx$, and $q \in By$. Moreover, B is *maximally monotone* if the only monotone $C : H \to 2^H$ such that $Bx \subset Cx$ for all $x \in H$ is $C = B$. Minty's well-known maximality criterion is as follows [22].

Theorem (Minty's Lemma). *A monotone operator $B : H \to 2^H$ is maximal if and only if for each $y \in H$, there is a unique $x \in H$ such that*

$$x + Bx \ni y.$$

Let us now recall a fundamental theorem for the initial value problem

$$\begin{cases} \dot{u}(t) + Bu(t) \ni 0, & \text{a.e. } t \geq 0, \\ u(0) = u^0. \end{cases} \tag{2.1}$$

As discussed in [5, Chap. II] and [2, Chap. 3], the initial value problem is well-posed provided that B is maximally monotone and u^0 is an element of the domain of B

$$D(B) = \{x \in H : Bx \neq \emptyset\}.$$

We also have the following theorem, which is originally due to Kato [18] and Komura [20].

Theorem 2.1. *Suppose $B : H \to 2^H$ is maximally monotone and $u^0 \in D(B)$. Then there is a unique Lipschitz continuous solution*

$$u : [0, \infty) \to H$$

of (2.1) *such that $u(t) \in D(B)$ for all $t \geq 0$. Moreover,*

$$|\dot{u}(t)| \leq \inf\{|p| : p \in Bu^0\} \tag{2.2}$$

for almost every $t \geq 0$.

Remark 2.2. A more general version of 2.1 is described in [5, Theorem 3.1].

Remark 2.3. The infimum on the right-hand side of (2.2) is attained as the images of a maximal monotone operator are closed and convex subsets of H.

We also note that the operator B generates a contraction. Let u be a path as described in Theorem 2.1, and suppose $v : [0, \infty) \to H$ is any other Lipschitz continuous path with $v(t) \in D(B)$ for $t \geq 0$ and

$$\dot{v}(t) + Bv(t) \ni 0, \quad \text{a.e. } t \geq 0.$$

Then

$$|u(t) - v(t)| \leq |u(0) - v(0)|, \quad t \geq 0. \tag{2.3}$$

If, in addition, there is some $\lambda > 0$ such that

$$\langle p - q, x - y \rangle \geq \lambda |x - y|^2 \tag{2.4}$$

for all $x, y \in H$, $p \in Bx$, and $q \in By$, then (2.3) can be improved to

$$|u(t) - v(t)| \leq e^{-\lambda t} |u(0) - v(0)|, \quad t \geq 0. \tag{2.5}$$

Remarkably, it is also possible to use solutions of the initial value problem (as described in Theorem 2.1) to approximate equilibria of B. These are points $y \in H$ such that

$$By \ni 0.$$

The following theorem was proved in [3].

Theorem (The Baillon–Brézis Theorem). *Suppose B has an equilibrium point and u is a solution of the initial value problem* (2.1). *Then*

$$\frac{1}{t} \int_0^t u(s)ds$$

converges weakly in H to some equilibrium point of B.

Remark 2.4. If B satisfies (2.4), it is plain to see that $By \ni 0$ can have at most one solution. In this case,

$$|u(t) - y| \le e^{-\lambda t}|u(0) - y|, \quad t \ge 0$$

for any solution of the initial value problem (2.1). This follows from (2.5) as $v(t) = y$ is a solution of the initial value problem with $v(0) = y$.

2.2. *Proof of Theorem* 1.1

Let us now suppose the spaces V, $X_1, \dots, X_N$, and H are as described in Sec. 1.2 of the introduction. We will further assume $f_1, \dots, f_N : X \to \mathbb{R}$ are given functions which satisfy (1.6). An elementary but important observation is that a given $y \in H$ induces a linear form in V^* by the formula

$$(y, x) := \langle y, x\rangle, \quad (x \in V).$$

That this linear form is continuous is due to the continuity of the embedding $V \subset H$. For convenience, we will use $\|\cdot\|$ to denote both the product norm on V and the associated norm on V^N. Namely, we'll write

$$\|x\| = \left(\sum_{j=1}^N \|x_j\|^2\right)^{1/2}$$

for $x = (x_1, \dots, x_N) \in V^N$. We will also use this convention for the inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$ on H and on H^N and the induced norm $\|\cdot\|_*$ on V^* and on $(V^*)^N$.

Let us specify $A : H^N \to 2^{H^N}$ via

$$Ax = \begin{cases} \partial_{x_1} f_1(x) \cap H \times \cdots \times \partial_{x_N} f_N(x) \cap H, & x \in \mathcal{D}, \\ \emptyset, & x \notin \mathcal{D} \end{cases} \tag{2.6}$$

for $x \in H^N$. Here, $\mathcal{D}$ is defined in (1.8).

Lemma 2.5. *The domain of A is $\mathcal{D}$, and A is monotone. Moreover, an absolutely continuous $u : [0, \infty) \to H^N$ solves*

$$\begin{cases} \dot{u}(t) + Au(t) \ni 0, & \text{a.e. } t \ge 0, \\ u(0) = u^0 \end{cases} \tag{2.7}$$

if and only it solves (1.7).

Proof. By definition, the domain of A is $\mathcal{D}$. Suppose $x, y \in \mathcal{D}$,

$$\zeta_j \in \partial_{x_j} f_j(x) \cap H, \quad \text{and} \quad \xi_j \in \partial_{x_j} f_j(y) \cap H$$

for $j = 1, \dots, N$. In view of the last condition listed in (1.6),

$$\sum_{j=1}^N \langle \zeta_j - \xi_j, x_j - y_j\rangle = \sum_{j=1}^N (\zeta_j - \xi_j, x_j - y_j) \ge 0.$$

Thus, A is monotone.

Note that if u solves (2.7), then

$$-\dot{u}_j(t) \in \partial_{x_j} f_j(u(t)) \cap H \subset \partial_{x_j} f_j(u(t))$$

for almost every $t \geq 0$ and each $j = 1, \ldots, N$. Thus, u is a solution of (1.7). Conversely, if u solves (1.7) then

$$-\dot{u}_j(t) \in \partial_{x_j} f_j(u(t))$$

for almost every $t \geq 0$ and each $j = 1, \ldots, N$. As u is absolutely continuous, $-\dot{u}(t) \in H$ for almost every $t \geq 0$, and therefore, $-\dot{u}(t) \in Au(t)$ for almost every $t \geq 0$. □

In view of Theorem 2.1 and the Baillon–Brézis theorem, we can conclude Theorem 1.1 once we verify that A is maximal. To this end, we will verify the hypotheses of Minty's lemma.

Proof of Theorem 1.1. For a given $z \in H^N$, it suffices to show there is $x \in X$ such that

$$z_j \in x_j + \partial_{x_j} f_j(x) \tag{2.8}$$

for each $j = 1, \ldots, N$. In this case, $z_j - x_j \in \partial_{x_j} f_j(x)$ for $j = 1, \ldots, N$. Thus, $x \in \mathcal{D}$ and $z \in Ax + x$.

In order to solve (2.8), we will employ the auxiliary functions defined by

$$g_j(x) := \frac{1}{2}\|x_j\|^2 - (z_j, x_j) + f_j(x)$$

for $x \in X$ and $j = 1, \ldots, N$. In view of (1.6),

$$\begin{cases} g_1, \ldots, g_N \text{ are weakly lowersemicontinuous,} \\ X \ni y \mapsto \displaystyle\sum_{j=1}^{N} g_j(y_j, x_{-j}) \text{ is convex for each } x \in X, \\ X \ni x \mapsto \displaystyle\sum_{j=1}^{N} g_j(y_j, x_{-j}) \text{ is weakly continuous for each } y \in X, \text{ and} \\ \displaystyle\sum_{j=1}^{N} (\partial_{x_j} g_j(x) - \partial_{x_j} g_j(y), x_j - y_j) \geq \|x - y\|^2 \quad \text{for each } x, y \in X. \end{cases} \tag{2.9}$$

Direct computation also gives that

$$\partial_{x_j} g_j(x) = x_j - z_j + \partial_{x_j} f_j(x)$$

for each $x \in X$. Therefore, $x \in X$ is a Nash equilibrium of $g_1, \ldots, g_N$ if and only if x solves (2.8). Consequently, we now aim to show that $g_1, \ldots, g_N$ has a Nash equilibrium.

For each $r > 0$, set

$$X^r := \{x \in X : \|x\| \le r\}.$$

It is clear that $X^r \subset V$ is convex and weakly compact. And as each $X_1, \dots, X_N$ is nonempty, X^r is nonempty for all r greater or equal to some fixed $s > 0$. Let us consider the map $\Phi^r : X^r \to 2^{X^r}$ specified as

$$\Phi^r(x) := \operatorname*{arg\,min}_{y \in X^r} \left\{ \sum_{j=1}^N g_j(y_j, x_{-j}) \right\}$$

for $r > s$.

The first and second properties of $g_1, \dots, g_N$ listed in (2.9) imply that $\Phi^r(x) \neq \emptyset$ and convex for any $x \in X^r$ and $r > s$; and the first and third properties imply that the graph of Φ^r is weakly closed. Indeed let us suppose $(x^k)_{k\in\mathbb{N}}, (y^k)_{k\in\mathbb{N}} \subset X^r$, $x^k \rightharpoonup x$, $y^k \rightharpoonup y$, and

$$\sum_{j=1}^N g_j(y_j^k, x_{-j}^k) \le \sum_{j=1}^N g_j(z_j, x_{-j}^k)$$

for each $z \in X^r$ and all $k \in \mathbb{N}$. We can then send $k \to \infty$ to get

$$\sum_{j=1}^N g_j(y_j, x_{-j}) \le \sum_{j=1}^N g_j(z_j, x_{-j}).$$

The Kakutani–Glicksberg–Fan theorem [17, 14, 12] then implies there is $x^r \in \Phi^r(x^r)$. In particular,

$$\sum_{j=1}^N g_j(x^r) \le \sum_{j=1}^N g_j(y_j, x_{-j}^r) \tag{2.10}$$

for each $y \in X^r$. Define $\Psi : V^N \to (-\infty, \infty]$ by

$$\Psi(y) = \begin{cases} \displaystyle\sum_{j=1}^N g_j(y_j, x_{-j}^r), & y \in X, \\ +\infty, & y \notin X. \end{cases}$$

Then Ψ restricted to X^r has a minimum at x^r. We also note that Ψ is bounded from below on X since $X \ni y \mapsto \sum_{j=1}^N f_j(y_j, x_{-j}^r)$ is convex.

As each X_j has nonempty interior, X has nonempty interior. Therefore, X^r has nonempty interior for all r large enough. It follows from the Pshenichnii–Rockafellar conditions for the minimum of a proper, lowersemicontinuous, convex function on a closed subset of a Banach space [4, Theorem 4.3.6] that

$$0 \in \partial\Psi(x^r) + n_{X^r}(x^r)$$

for all r sufficiently large. Here,

$$n_{X^r}(x^r) := \left\{ \zeta \in (V^*)^N : \sum_{j=1}^{N} (\zeta_j, y_j - x_j^r) \le 0 \text{ for all } y \in X^r \right\}$$

is the normal cone of X^r at the point x^r. As a result, there is $p^r \in (V^*)^N$ for which

$$\sum_{j=1}^{N} g_j(y_j, x_{-j}^r) \ge \sum_{j=1}^{N} g_j(x^r) + (p_j^r, y_j - x_j^r)$$

for all $y \in X$ and

$$\sum_{j=1}^{N} (p_j^r, y_j - x_j^r) \ge 0 \tag{2.11}$$

for all $y \in X^r$.

By assumption, $u^0 \in \mathcal{D}$. It follows that there are

$$w_j \in \partial_{x_j} g_j(u^0)$$

for $j = 1, \ldots, N$. Note that the fourth listed property of $g_1, \ldots, g_N$ in (2.9) along with (2.11) gives

$$\begin{aligned} \|x^r - u^0\|^2 &\le \sum_{j=1}^{N} (p_j^r - w_j, x_j^r - u_j^0) \\ &\le \sum_{j=1}^{N} (-w_j, x_j^r - u_j^0) \\ &\le \|w\|_* \|x^r - u^0\| \end{aligned}$$

for all r sufficiently large. As a result, there is a sequence $(r_k)_{k\in\mathbb{N}}$ which increases to infinity such that $(x^{r_k})_{k\in\mathbb{N}} \subset X$ converges weakly to some $x \in X$. Upon sending $r = r_k \to \infty$ in (2.10), we find that x is the desired Nash equilibrium of $g_1, \ldots, g_N$. □

Remark 2.6. The conclusion that $g_1, \ldots, g_N$ has a Nash equilibrium could also have been deduced from the work of Williams [30].

Corollary 2.7. *If X is weakly compact, then $f_1, \ldots, f_N$ has a Nash equilibrium.*

Proof. If X is weakly compact, we can repeat the argument given in the proof above used to show $g_1, \ldots, g_N : X^r \to \mathbb{R}$ has a Nash equilibrium. All we would need to do is to replace X^r with X and $g_1, \ldots, g_N$ with $f_1, \ldots, f_N$. □

Corollary 2.8. *Suppose there is $\theta > 0$ such that*

$$\sum_{j=1}^{N}(\partial_{x_j} f_j(x) - \partial_{x_j} f_j(y), x_j - y_j) \geq \theta\|x - y\|^2$$

for each $x, y \in X$. Then $f_1, \dots, f_N$ has a unique Nash equilibrium $z \in X$. Moreover, if $u : [0, \infty) \to H^N$ is a solution of the initial value problem as described in Theorem 1.1, *there is $\lambda > 0$ such that*

$$|u(t) - z| \leq e^{-\lambda t}|u(0) - z| \tag{2.12}$$

for each $t \geq 0$.

Proof. We can repeat the argument given in the proof of Theorem 1.1 that shows $g_1, \dots, g_N$ has a Nash equilibrium to conclude that $f_1, \dots, f_N$ has a Nash equilibrium. Also, recall that if x and y are a Nash equilibria of $f_1, \dots, f_N$, then

$$0 \in \partial_{x_j} f_j(x) \quad \text{and} \quad 0 \in \partial_{x_j} f_j(y)$$

for $j = 1, \dots, N$. By hypothesis, this implies

$$\theta\|x - y\|^2 \leq \sum_{j=1}^{N}(0 - 0, x_j - y_j) = 0.$$

Therefore, $x = y$.

As $V^N \subset H^N$ is continuously embedded, there is a constant $C > 0$ such that

$$|x| \leq C\|x\|, \quad x \in V^N.$$

Setting

$$\lambda := \frac{\theta}{C^2},$$

gives

$$\langle Ax - Ay, x - y\rangle \geq \theta\|x - y\|^2 \geq \lambda|x - y|^2$$

for $x, y \in D(A)$. Here, $A : H^N \to 2^{H^N}$ is the maximal monotone operator defined in (2.6). The inequality (2.12) now follows from Remark 2.4. □

2.3. *A special case*

We now consider the special case mentioned in the introduction. In the statement below, we will suppose that V and H are as above. However, we will not assume

$f_1, \ldots, f_N$ satisfy (1.6). We also note that the proof essentially follows from an observation made by Brézis in [5, Remark 2.3.7].

Theorem 2.9. *Suppose* $f_1, \ldots, f_N : V^N \to \mathbb{R}$ *satisfy*

$$\begin{cases} \partial_{x_j} f_j(x) = \{\nabla_{x_j} f_j(x)\} \quad \textit{for each } x \in V^N, \\ \nabla_{x_j} f_j(x+ty) \rightharpoonup \nabla_{x_j} f_j(x) \quad \textit{as } t \to 0^+ \textit{ for each } x, y \in V^N, \textit{ and} \\ \displaystyle\sum_{j=1}^{N} (\nabla_{x_j} f_j(x) - \nabla_{x_j} f_j(y), x_j - y_j) \geq 0 \quad \textit{for each } x, y \in V^N, \end{cases}$$

and define

$$\mathcal{D} = \{x \in V^N : \nabla_{x_1} f_1(x), \ldots, \nabla_{x_N} f_N(x) \in H\}.$$

Then for each $u^0 \in \mathcal{D}$, *there is a unique Lipschitz continuous* $u : [0, \infty) \to H^N$ *such that* $u(t) \in \mathcal{D}$ *for each* $t \geq 0$ *and*

$$\begin{cases} u(0) = u^0, \quad \textit{and} \\ \dot{u}_j(t) + \nabla_{x_j} f_j(u(t)) = 0 \quad \textit{for a.e. } t \geq 0 \textit{ and each } j = 1, \ldots, N. \end{cases}$$

Proof. It suffices to show that $A : H^N \to 2^{H^N}$ defined via

$$Ax := \begin{cases} \{(\nabla_{x_1} f_1(x), \ldots, \nabla_{x_N} f_N(x))\}, & x \in \mathcal{D}, \\ \emptyset, & x \notin \mathcal{D} \end{cases}$$

$(x \in H^N)$ is maximal. To this end, we first consider the operator $B : V^N \to (V^*)^N$ defined via

$$Bx := (x_1 + \nabla_{x_1} f_1(x), \ldots, x_N + \nabla_{x_N} f_N(x))$$

for $x \in V^N$. Note that

$$B(x + ty) \rightharpoonup Bx$$

and

$$(Bx - By, x - y) \geq \|x - y\|^2$$

for all $x, y \in V^N$.

By a theorem due independently to Browder [8] and Minty [23], B is surjective; see also [19, Corollary 1.8]. In particular, if $y \in H^N$, there is $x \in V^N$ such that

$$Bx = y.$$

That is,

$$x_j + \nabla_{x_j} f_j(x) = y_j$$

for $j = 1, \ldots, N$. Then $\nabla_{x_j} f_j(x) = y_j - x_j \in H$ for $j = 1, \ldots, N$, which implies $x \in \mathcal{D}$ and

$$x + Ax \ni y. \qquad \square$$

Remark 2.10. If $f_1, \dots, f_N$ additionally satisfy

$$\sum_{j=1}^{N} (\nabla_{x_j} f_j(x) - \nabla_{x_j} f_j(y), x_j - y_j) \geq \theta \|x - y\|^2$$

($x, y \in V^N$), then $f_1, \dots, f_N$ has a unique Nash equilibrium and (2.12) holds for any $u(0) \in \mathcal{D}$.

3. Finite Dimensional Flows

In this section, we will present an implication of Theorem 1.1 in the particular case

$$V = H = \mathbb{R}^d$$

equipped with the standard Euclidean dot product and norm. Let $X = X_1 \times \cdots \times X_N$, where $X_1, \dots, X_N \subset \mathbb{R}^d$ are closed and convex subsets each with nonempty interior. Further assume

$$\begin{cases} f_1, \dots, f_N : X \to \mathbb{R} \quad \text{are continuous,} \\ X \ni y \mapsto \displaystyle\sum_{j=1}^{N} f_j(y_j, x_{-j}) \quad \text{is convex for each } x \in X, \text{ and} \\ \displaystyle\sum_{j=1}^{N} (\partial_{x_j} f_j(x) - \partial_{x_j} f_j(y)) \cdot (x_j - y_j) \geq 0 \quad \text{for each } x, y \in X, \end{cases}$$

where

$$\partial_{x_j} f_j(x) = \{z \in \mathbb{R}^d : f_j(y_j, x_{-j}) \geq f_j(x) + z_j \cdot (y_j - x_j) \text{ for all } y_j \in X_j\}$$

for $j = 1, \dots, N$. As before, we set

$$\mathcal{D} = \{x \in X : \partial_{x_j} f_j(x) \neq \emptyset,\ j = 1, \dots, N\},$$

Theorem 1.1 implies the following.

Proposition 3.1. *For each $u^0 \in \mathcal{D}$, there is a unique Lipschitz continuous $u : [0, \infty) \to \mathbb{R}^{Nd}$ with $u(t) \in \mathcal{D}$ for $t \geq 0$ and which satisfies*

$$\begin{cases} u(0) = u^0, \quad \textit{and} \\ \dot{u}_j(t) + \partial_{x_j} f_j(u(t)) \ni 0 \quad \textit{for a.e. } t \geq 0 \textit{ and } j = 1, \dots, N. \end{cases}$$

Furthermore, if $f_1, \dots, f_N$ has a Nash equilibrium, the limit

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t u(s) ds$$

exists and equals a Nash equilibrium of $f_1, \dots, f_N$.

Example 3.2 (Quadratic Cost Functions). A basic example we had in mind when considering Proposition 3.1 occurs when $X_1 = \cdots = X_N = \mathbb{R}^d$ and

$$f_j(x) = \sum_{k,\ell=1}^{N} \frac{1}{2} A^j_{k,\ell} x_k \cdot x_\ell$$

for $x = (x_1, \dots, x_N) \in \mathbb{R}^{Nd}$ and $j = 1, \dots, N$. Here, $A^j_{k,\ell}$ are $d \times d$ symmetric matrices which additionally satisfy $A^j_{k,\ell} = A^j_{\ell,k}$ for $j, k, \ell = 1, \dots, N$. The required monotonicity condition on $f_1, \dots, f_N$ is satisfied provided

$$\sum_{j,k=1}^{N} A^j_{k,j} y_j \cdot y_k \geq 0$$

for each $y \in \mathbb{R}^{Nd}$.

Example 3.3 (Application to Noncooperative Games). In noncooperative games, $X_1, \dots, X_N \subset \mathbb{R}^d$ are typically assumed to be convex, compact subsets with nonempty interior and $f_1, \dots, f_N : X \to \mathbb{R}$ are each multilinear. That is,

$$X_j \ni y_j \mapsto f_i(y_j, x_{-j}) \text{ is linear}$$

for each $i, j = 1, \dots, N$. This is the scenario considered by Nash in some of his foundational works in game theory [24, 25].

It is routine to verify that the above monotonicity assumption we made on $f_1, \dots, f_N$ is equivalent to

$$\sum_{j=1}^{N} (f_j(x) + f_j(y)) \geq \sum_{j=1}^{N} (f_j(y_j, x_{-j}) + f_j(x_j, y_{-j})) \tag{3.1}$$

for $x, y \in X$. And since each f_j is smooth on $\mathbb{R}^{Nd}$, it must be that $\mathcal{D} = X$. Thus, Proposition 3.1 holds for each $u^0 \in X$ provided that $f_1, \dots, f_N$ are multilinear functions which satisfy (3.1).

4. Examples in Function Spaces

In this final section, we will apply the ideas we have developed to consider functionals defined on Lebesgue and Sobolev spaces. As we have done above, we will consider both the existence of Nash equilibria and their approximation by continuous time flows. Throughout, we will assume $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary. We will also change notation by using v for the variables in which our functionals are defined. This allows us to reserve x for points in Ω. Finally, we will use $|\cdot|$ for any norm on a finite dimensional space.

4.1. *A few remarks on Sobolev spaces*

Before discussing examples, let us recall a few facts about Sobolev spaces and establish some notation. A good reference for this material is [10, Chaps. 5 and 6]. First, we note

$$H^1_0(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega),$$

where each inclusion is compact. Here, $L^2(\Omega)$ is the space of Lebesgue measurable functions on Ω which are square integrable equipped with the standard inner product. The space $H^1_0(\Omega)$ is the closure of all smooth functions having compact support in Ω in the norm

$$u \mapsto \left(\int_\Omega |\nabla u|^2 dx\right)^{1/2}.$$

The topological dual of $H^1_0(\Omega)$ is $H^{-1}(\Omega)$.

Recall that the Dirichlet Laplacian $-\Delta : H^1_0(\Omega) \to H^{-1}(\Omega)$ is defined via

$$(-\Delta u, v) = \int_\Omega \nabla u \cdot \nabla v dx$$

for $u, v \in H^1_0(\Omega)$. Here, we are using $(\cdot,\cdot)$ as the natural pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$. Moreover, $-\Delta$ is an isometry; in particular, this map is invertible. This allows us to define the following inner product on $H^{-1}(\Omega)$

$$(f, g) \mapsto (f, (-\Delta)^{-1} g).$$

We also note that if $f \in L^2(\Omega)$, the inner product between f and g simplifies to

$$(f, (-\Delta)^{-1} g) = \int_\Omega f (-\Delta)^{-1} g dx.$$

In addition, we will consider the space $H^2(\Omega) \cap H^1_0(\Omega)$ consisting of $H^1_0(\Omega)$ functions for which $-\Delta u \in L^2(\Omega)$. This is a Hilbert space endowed with the inner product

$$(u, v) \mapsto \int_\Omega \Delta u \, \Delta v dx.$$

4.2. $H = L^2(\Omega)$

For $v \in H^1_0(\Omega)^N$, define

$$f_j(v) = \int_\Omega H_j(\nabla v_j) + F_j(v) - h_j v_j \; dx$$

for $j = 1, \ldots, N$. Here each $h_j \in L^2(\Omega)$, and $H_j : \mathbb{R}^d \to \mathbb{R}$ is assumed to be smooth and satisfy

$$\theta |p - q|^2 \leq (\nabla H_j(p) - \nabla H_j(q)) \cdot (p - q) \leq \frac{1}{\theta} |p - q|^2 \tag{4.1}$$

for all $p, q \in \mathbb{R}^d$ and some $\theta \in (0,1]$. We'll also suppose $F_1, \dots, F_N : \mathbb{R}^N \to \mathbb{R}$ are smooth,

$$\sum_{j=1}^{N} (\partial_{z_j} F_j(z) - \partial_{z_j} F_j(w))(z_j - w_j) \geq 0$$

for $z, w \in \mathbb{R}^N$, and there is C such that

$$|F_j(z)| \leq C(1 + |z|^2) \quad \text{and} \quad |\partial_{z_j} F_j(z)| \leq C(1 + |z|) \tag{4.2}$$

for $z \in \mathbb{R}^N$ and $j = 1, \dots, N$.

Using these assumptions, it is straightforward to verify that $f_1, \dots, f_N$ fulfills (1.6) with $X_1 = \cdots = X_N = V = H_0^1(\Omega)$. The following lemma also follows by routine computations.

Lemma 4.1. *For each $v, w \in H_0^1(\Omega)^N$ and $j = 1, \dots, N$,*

$$\partial_{v_j} f_j(v) = \{\nabla_{v_j} f_j(v)\}$$

with

$$(\nabla_{v_j} f_j(v), w_j) = \int_\Omega \nabla H_j(\nabla v_j) \cdot \nabla w_j + \partial_{z_j} F_j(v) w_j - h_j w_j dx$$

and

$$\sum_{j=1}^{N} (\nabla_{v_j} f_j(v) - \nabla_{v_j} f_j(w), v_j - w_j) \geq \theta \int_\Omega |\nabla v - \nabla w|^2 dx.$$

Moreover, v is a Nash equilibrium of $f_1, \dots, f_N$ if and only if it is a weak solution of

$$\begin{cases} -\mathrm{div}(\nabla H_j(\nabla v_j)) + \partial_{z_j} F_j(v) = h_j & \text{in } \Omega, \\ v_j = 0 & \text{on } \partial\Omega \end{cases} \tag{4.3}$$

$(j = 1, \dots, N)$.

Remark 4.2. In this statement, $(\cdot, \cdot)$ denotes the natural pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. And a weak solution v of (4.3) satisfies $(\nabla_{v_j} f_j(v), w_j) = 0$ for each $w_j \in H_0^1(\Omega)$ and $j = 1, \dots, N$.

We now can state a result involving the approximation of Nash equilibrium for $f_1, \dots, f_N$.

Proposition 4.3. *Let $u^0 \in (H^2(\Omega) \cap H_0^1(\Omega))^N$. There is a unique Lipschitz continuous*

$$u : [0, \infty) \to L^2(\Omega)^N; t \mapsto u(\cdot, t)$$

such that $u(\cdot,t) \in (H^2(\Omega)\cap H^1_0(\Omega))^N$ *for each* $t \ge 0$ *and*

$$\begin{cases} \partial_t u_j = \operatorname{div}(\nabla H_j(\nabla u_j)) - \partial_{z_j} F_j(u) + h_j & \text{in } \Omega\times(0,\infty),\\ u_j = 0 & \text{on } \partial\Omega\times(0,\infty),\\ u_j = u_j^0 & \text{on } \Omega\times\{0\} \end{cases} \tag{4.4}$$

$(j = 1,\ldots,N)$. *Furthermore, there is* $\lambda > 0$ *such that*

$$\int_\Omega |u(x,t)-v(x)|^2 dx \le e^{-2\lambda t}\int_\Omega |u^0(x)-v(x)|^2 dx$$

for $t \ge 0$. *Here,* v *is the unique Nash equilibrium of* $f_1,\ldots,f_N$.

Remark 4.4. For almost every $t \ge 0$, the PDE in (4.4) holds almost everywhere in Ω; the boundary condition holds in the trace sense; and $u(\cdot,0) = u^0$ as $L^2(\Omega)^N$ functions.

Proof. Suppose that $\zeta \in L^2(\Omega)^N$ and $v \in H^1_0(\Omega)^N$ is a weak solution of

$$\begin{cases} -\operatorname{div}(\nabla H_j(\nabla v_j)) + \partial_{z_j} F_j(v) = h_j + \zeta_j & \text{in } \Omega,\\ v_j = 0 & \text{on } \partial\Omega. \end{cases}$$

As $\partial\Omega$ is assumed to be smooth and in view of the hypothesis (4.1), elliptic regularity ([10, Theorem 1, §8.3]) implies $v \in (H^2(\Omega)\cap H^1_0(\Omega))^N$. Therefore,

$$\{v \in H^1_0(\Omega)^N : \nabla_{v_1} f_1(v),\ldots,\nabla_{v_N} f_N(v) \in L^2(\Omega)\} = (H^2(\Omega)\cap H^1_0(\Omega))^N.$$

Theorem 1.1 and Corollary 2.8 now allow us to conclude. ☐

4.3. *Another example with* $H = L^2(\Omega)$

Define

$$f_j(v) = \int_\Omega L_j(\nabla v_1,\ldots,\nabla v_N) - h_j v_j dx$$

for $v \in H^1_0(\Omega)^N$. Here, $h_1,\ldots,h_N \in L^2(\Omega)$ and each $L_j : \mathbb{R}^{Nd} \to \mathbb{R}$ is smooth with

$$\theta|p-q|^2 \le \sum_{j=1}^N (\nabla_{p_j} L_j(p) - \nabla_{p_j} L_j(q))\cdot(p_j - q_j) \le \frac{1}{\theta}|p-q|^2 \tag{4.5}$$

for all $p,q \in \mathbb{R}^{Nd}$ and some $\theta \in (0,1]$. We'll also assume there is $C > 0$ such that

$$|\nabla_{p_j} L_j(p)| \le C(1+|p|)$$

for each $p \in \mathbb{R}^{Nd}$ and $j = 1,\ldots,N$.

Direct computation gives the following lemma.

Lemma 4.5. *For each $v, w \in H_0^1(\Omega)^N$ and $j = 1, \ldots, N$,*

$$\partial_{v_j} f_j(v) = \{\nabla_{v_j} f_j(v)\}$$

with

$$(\nabla_{v_j} f_j(v), w_j) = \int_\Omega \nabla_{p_j} L_j(\nabla v_1, \ldots, \nabla v_N) \cdot \nabla w_j - h_j w_j dx$$

and

$$\sum_{j=1}^N (\nabla_{v_j} f_j(v) - \nabla_{v_j} f_j(w), v_j - w_j) \geq \theta \int_\Omega |\nabla v - \nabla w|^2 dx.$$

Moreover,

$$\nabla_{v_j} f_j(v + tw) \rightharpoonup \nabla_{v_j} f_j(v)$$

as $t \to 0^+$, and v is a Nash equilibrium of $f_1, \ldots, f_N$ if and only if it is a weak solution of

$$\begin{cases} -\mathrm{div}(\nabla_{p_j} L_j(\nabla v_1, \ldots, \nabla v_N)) = h_j & \text{in } \Omega, \\ v_j = 0 & \text{on } \partial\Omega \end{cases} \tag{4.6}$$

$(j = 1, \ldots, N)$.

Remark 4.6. A weak solution v of (4.6) satisfies $(\nabla_{v_j} f_j(v), w_j) = 0$ for each $w_j \in H_0^1(\Omega)$ and $j = 1, \ldots, N$.

With this lemma, we can now establish the subsequent assertion about the Nash equilibrium of $f_1, \ldots, f_N$.

Proposition 4.7. *Let $u^0 \in (H^2(\Omega) \cap H_0^1(\Omega))^N$. There is a unique Lipschitz continuous*

$$u : [0, \infty) \to L^2(\Omega)^N; \quad t \mapsto u(\cdot, t)$$

such that $u(\cdot, t) \in (H^2(\Omega) \cap H_0^1(\Omega))^N$ for each $t \geq 0$ and

$$\begin{cases} \partial_t u_j = \mathrm{div}(\nabla_{p_j} L_j(\nabla u_1, \ldots, \nabla u_N)) + h_j & \text{in } \Omega \times (0, \infty), \\ u_j = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u_j = u_j^0 & \text{on } \Omega \times \{0\} \end{cases}$$

$(j = 1, \ldots, N)$. *Furthermore, there is $\lambda > 0$ such that*

$$\int_\Omega |u(x, t) - v(x)|^2 dx \leq e^{-2\lambda t} \int_\Omega |u^0(x) - v(x)|^2 dx$$

for $t \geq 0$. Here, v is the unique Nash equilibrium of $f_1, \ldots, f_N$.

Proof. Suppose that $\xi \in L^2(\Omega)^N$ and $v \in H_0^1(\Omega)^N$ is a weak solution of

$$\begin{cases} -\mathrm{div}(\nabla_{p_j} L_j(\nabla v_1, \dots, \nabla v_N)) = h_j + \xi_j & \text{in } \Omega, \\ v_j = 0 & \text{on } \partial\Omega. \end{cases}$$

By (4.5), elliptic regularity implies $v \in (H^2(\Omega) \cap H_0^1(\Omega))^N$ (see the remark at the end of [10, §9.1]). As a result

$$\{v \in H_0^1(\Omega)^N : \nabla_{v_1} f_1(v), \dots, \nabla_{v_N} f_N(v) \in L^2(\Omega)\} = (H^2(\Omega) \cap H_0^1(\Omega))^N.$$

We then conclude by Theorem 2.9 and Remark 2.10. □

4.4. $H = H^{-1}(\Omega)$

For a given $v \in L^2(\Omega)^N$, set

$$f_j(v) = \int_\Omega F_j(v)dx - \langle h_j, v_j \rangle$$

for $j = 1, \dots, N$. Here $h_1, \dots, h_N \in H^{-1}(\Omega)$, and $\langle \cdot, \cdot \rangle$ denotes the inner product in $H^{-1}(\Omega)$. We will assume $F_1, \dots, F_N : \mathbb{R}^N \to \mathbb{R}$ are smooth and that they satisfy (4.2) and

$$\theta|z - w|^2 \leq \sum_{i=1}^{N} (\partial_{z_i} F_i(z) - \partial_{z_i} F_i(w))(z_i - w_i) \tag{4.7}$$

for $z, w \in \mathbb{R}^N$ and some $\theta > 0$. For convenience, we will also assume

$$\partial_{z_j} F_j(0) = 0 \tag{4.8}$$

for $j = 1, \dots, N$.

The following observations can be verified by routine computations.

Lemma 4.8. *For each $v, w \in L^2(\Omega)^N$ and $j = 1, \dots, N$,*

$$\partial_{v_j} f_j(v) = \{\nabla_{v_j} f_j(v)\}$$

with

$$\nabla_{v_j} f_j(v) = \partial_{z_j} F_j(v) - (-\Delta)^{-1} h_j.$$

Moreover, $v \in H_0^1(\Omega)^N$ is a Nash equilibrium of $f_1, \dots, f_N$ if and only

$$\begin{cases} -\Delta(\partial_{z_j} F_j(v)) = h_j & \text{in } \Omega, \\ v_j = 0 & \text{on } \partial\Omega \end{cases} \tag{4.9}$$

$(j = 1, \dots, N)$.

By Proposition A.1 in Appendix A, there are smooth functions $G_1, \ldots, G_N$ such that

$$\begin{cases} \partial_{z_j} F_j(z) = y_j \text{ if and only if } \partial_{y_k} G_k(y) = z_k, \\ |G_k(y)| \le B(1 + |y|^2) \quad \text{and} \quad |\partial_{y_k} G_k(y)| \le B(1 + |y|), \\ 0 \le \displaystyle\sum_{i=1}^{N} (\partial_{y_i} G_i(y) - \partial_{y_i} G_i(z))(y_i - z_i), \\ |\partial_{y_j} \partial_{y_k} G_k(y)| \le \dfrac{1}{\theta} \end{cases} \tag{4.10}$$

for all $y, z \in \mathbb{R}^N$, $j, k = 1, \ldots, N$ and some constant B. Note that in view of (4.8)

$$\partial_{y_k} G_k(0) = 0$$

for $k = 1, \ldots, N$. Thus, it is possible to solve (4.9) by choosing

$$w_j = (-\Delta)^{-1} h_j \in H_0^1(\Omega)$$

for $j = 1, \ldots, N$ and then setting

$$v_k = \partial_{y_k} G_k(w)$$

for $k = 1, \ldots, N$. In particular, we have just shown that $f_1, \ldots, f_N$ has a Nash equilibrium, and it is not hard to employ (4.7) in order to conclude that it is unique.

Let us now see how to approximate this Nash equilibrium.

Proposition 4.9. *Let $u^0 \in H_0^1(\Omega)^N$. There is a unique Lipschitz continuous*

$$u : [0, \infty) \to H^{-1}(\Omega)^N; t \mapsto u(\cdot, t)$$

such that $u(\cdot, t) \in H_0^1(\Omega)^N$ for each $t \ge 0$ and

$$\begin{cases} \partial_t u_j = \Delta(\partial_{z_j} F_j(u)) - h_j & \text{in } \Omega \times (0, \infty), \\ u_j = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u_j = u_j^0 & \text{on } \Omega \times \{0\} \end{cases}$$

($j = 1, \ldots, N$). Furthermore, there is $\lambda > 0$ such that

$$\|u(\cdot, t) - v\| \le e^{-\lambda t} \|u^0 - v\|$$

for $t \ge 0$. Here, $\|\cdot\|$ denotes the norm on $H^{-1}(\Omega)^N$ and v is the Nash equilibrium of $f_1, \ldots, f_N$.

Proof. Define $A : H^{-1}(\Omega)^N \to 2^{H^{-1}(\Omega)^N}$ as

$$Av = \begin{cases} \{(-\Delta(\partial_{z_1} F_1(v)) - h_1, \ldots, -\Delta(\partial_{z_N} F_N(v)) - h_N)\}, & v \in H_0^1(\Omega)^N, \\ \emptyset & \text{otherwise} \end{cases}$$

for $v \in H^{-1}(\Omega)^N$. We note that $D(A) = H_0^1(\Omega)^N$, and $0 \in Av$ if and only if $v \in H_0^1(\Omega)^N$ is a Nash equilibrium for $f_1, \ldots, f_N$. By (4.7),

$$\begin{aligned} \langle Av - Aw, v - w\rangle &= \int_\Omega \sum_{j=1}^N (\partial_{z_j} F_j(v) - \partial_{z_j} F_j(w))(v_j - w_j)dx \\ &\geq \theta \int_\Omega |v - w|^2 dx \\ &\geq \lambda \|v - w\|^2 \end{aligned}$$

for $v, w \in H_0^1(\Omega)^N$ and some $\lambda > 0$.

As a result, it suffices to check that A is maximal. This amounts to finding a weak solution $v \in H_0^1(\Omega)^N$ of

$$\begin{cases} v_j - \Delta(\partial_{z_j} F_j(v)) = h_j + \zeta_j & \text{in } \Omega, \\ v_j = 0 & \text{on } \partial\Omega \end{cases} \tag{4.11}$$

for a given $\zeta \in H^{-1}(\Omega)^N$. To this end, we consider the auxiliary problem of finding $w \in H_0^1(\Omega)^N$ such that

$$\begin{cases} \partial_{y_k} G_k(w) - \Delta w_k = h_k + \zeta_k & \text{in } \Omega, \\ w_k = 0 & \text{on } \partial\Omega \end{cases} \tag{4.12}$$

for $k = 1, \ldots, N$. If we can find w, then $v \in H_0^1(\Omega)^N$ defined by

$$v_k = \partial_{y_k} G_k(w)$$

$(j = 1, \ldots, N)$ is a solution of (4.11).

Note that (4.12) has a solution if and only if

$$g_k(w) = \int_\Omega G_k(w) + \frac{1}{2}|\nabla w_k|^2 dx - (h_k + \zeta_k, w_k) \tag{4.13}$$

$(w \in H_0^1(\Omega)^N,\ k = 1, \ldots, N)$ has a Nash equilibrium. Here, $(\cdot, \cdot)$ is the natural pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Using (4.10), it is straightforward to check that $g_1, \ldots, g_N$ satisfies the properties listed in (1.6) with $X_1 = \cdots = X_N = V = H_0^1(\Omega)$ along with

$$\sum_{k=1}^N (\partial_{w_k} g_k(w) - \partial_{w_k} g_k(v), w_k - v_k) \geq \int_\Omega |\nabla w - \nabla v|^2 dx$$

for $w, v \in H_0^1(\Omega)^N$. As a result, Corollary 2.8 implies that $g_1, \ldots, g_N$ has a Nash equilibrium. □

4.5. $H = H_0^1(\Omega)$

Set

$$f_j(v) = \int_\Omega F_j(\Delta v) - \nabla h_j \cdot \nabla v_j dx$$

for $v \in (H^2(\Omega) \cap H_0^1(\Omega))^N$ and $j = 1, \ldots, N$. Here $h_1, \ldots, h_N \in H_0^1(\Omega)$ and

$$\Delta v = (\Delta v_1, \ldots, \Delta v_N).$$

We will suppose $F_1, \ldots, F_N : \mathbb{R}^N \to \mathbb{R}$ are smooth and satisfy (4.2), (4.7), and (4.8). A few key observations regarding $f_1, \ldots f_N$ are as follows.

Lemma 4.10. *For each* $v, w \in (H^2(\Omega) \cap H_0^1(\Omega))^N$ *and* $j = 1, \ldots, N$,

$$\partial_{v_j} f_j(v) = \{\nabla_{v_j} f_j(v)\}$$

with

$$(\nabla_{v_j} f_j(v), w_j) = \int_\Omega \partial_{z_j} F_j(\Delta v) \Delta w_j - \nabla h_j \cdot \nabla w_j dx.$$

Moreover, v *is a Nash equilibrium of* $f_1, \ldots, f_N$ *if and only if it is a solution of*

$$\begin{cases} -\partial_{z_j} F_j(\Delta v) = h_j & \text{in } \Omega, \\ v_j = 0 & \text{on } \partial\Omega \end{cases} \tag{4.14}$$

$(j = 1, \ldots, N)$.

Remark 4.11. Above, $(\cdot, \cdot)$ denotes the natural pairing between $(H^2(\Omega) \cap H_0^1(\Omega))^*$ and $H^2(\Omega) \cap H_0^1(\Omega)$. The PDE in (4.14) holds almost everywhere in Ω.

Using the functions $G_1, \ldots, G_N$ from the previous section, we can solve

$$\begin{cases} -\Delta v_k = -\partial_{y_k} G_k(-h) & \text{in } \Omega, \\ v_k = 0 & \text{on } \partial\Omega \end{cases}$$

for $k = 1, \ldots, N$ to obtain a solution of (4.14). It follows that $f_1, \ldots, f_N$ has a unique Nash equilibrium v, which belongs to the space

$$\mathcal{D}_1 = \{v \in (H^2(\Omega) \cap H_0^1(\Omega))^N : \Delta v \in H_0^1(\Omega)^N\}.$$

As with our previous examples, we can approximate this Nash equilibrium with a continuous time flow.

Proposition 4.12. *Let* $u^0 \in \mathcal{D}_1$. *There is a unique Lipschitz continuous*

$$u : [0, \infty) \to H_0^1(\Omega)^N; t \mapsto u(\cdot, t)$$

such that $u(\cdot, t) \in \mathcal{D}_1$ *for each* $t \geq 0$ *and*

$$\begin{cases} \partial_t u_j = \partial_{z_j} F_j(\Delta u) + h_j & \text{in } \Omega \times (0, \infty), \\ u_j = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u_j = u_j^0 & \text{on } \Omega \times \{0\} \end{cases}$$

$(j = 1, \ldots, N)$. Furthermore, there is $\lambda > 0$ such that

$$\int_\Omega |\nabla u(x,t) - \nabla v(x)|^2 dx \le e^{-2\lambda t} \int_\Omega |\nabla u^0(x) - \nabla v(x)|^2 dx$$

for $t \ge 0$. Here, v is the unique Nash equilibrium of $f_1, \ldots, f_N$.

Proof. Define $A : H_0^1(\Omega)^N \to 2^{H_0^1(\Omega)^N}$ by

$$Av = \begin{cases} \{(-\partial_{z_1} F_1(\Delta v) - h_1, \ldots, -\partial_{z_N} F_N(\Delta v) - h_N)\}, & v \in \mathcal{D}_1, \\ \emptyset & \text{otherwise} \end{cases}$$

for $v \in H_0^1(\Omega)^N$. Note that $D(A) = \mathcal{D}_1$, and $0 \in Av$ if and only if $v \in \mathcal{D}_1$ is a Nash equilibrium for $f_1, \ldots, f_N$. In view of (4.7),

$$\begin{aligned} \langle Av - Aw, v - w \rangle &= \int_\Omega \sum_{j=1}^N (\partial_{z_j} F_j(\Delta v) - \partial_{z_j} F_j(\Delta w))(\Delta v_j - \Delta w_j) dx \\ &\ge \theta \int_\Omega |\Delta v - \Delta w|^2 dx \\ &\ge \lambda \int_\Omega |\nabla v - \nabla w|^2 dx \end{aligned}$$

for $v, w \in \mathcal{D}_1$ and some $\lambda > 0$.

Therefore, it suffices to show

$$\begin{cases} v_j - \partial_{z_j} F_j(\Delta v) = h_j + \zeta_j & \text{in } \Omega, \\ v_j = 0 & \text{on } \partial\Omega \end{cases}$$

has a solution $v \in (H^2(\Omega) \cap H_0^1(\Omega))^N$ for a given $\zeta \in H_0^1(\Omega)^N$. We note that any such solution v will automatically belong to $\mathcal{D}_1$. Moreover, this is equivalent to finding a weak solution $v \in H_0^1(\Omega)^N$ of

$$\begin{cases} \partial_{y_k} G_k(v - h - \zeta) - \Delta v_k = 0 & \text{in } \Omega, \\ v_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.15}$$

We emphasize that elliptic regularity implies that any such weak solution actually belongs to $(H^2(\Omega) \cap H_0^1(\Omega))^N$.

Any $v \in H_0^1(\Omega)^N$ which solves (4.15) is a Nash equilibrium of $g_1, \ldots, g_N$ defined by

$$g_k(v) = \int_\Omega G_k(v - h - \zeta) + \frac{1}{2}|\nabla v_k|^2 dx.$$

As we did when considering the functions defined in (4.13), we can apply Corollary 2.8 and conclude that $g_1, \ldots, g_N$ has a unique Nash equilibrium. □

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Appendix A. Dual Functions

In this appendix, we will verify a technical assertion that was used to establish the existence of a Nash equilibrium in a few of the examples we considered above.

Proposition A.1. *Suppose $F_1, \dots, F_N : \mathbb{R}^N \to \mathbb{R}$ are smooth and satisfy*

$$|F_j(z)| \le C(1 + |z|^2) \tag{A.1}$$

for $z \in \mathbb{R}^N$ and $j = 1, \dots, N$ and

$$\theta|z - w|^2 \le \sum_{j=1}^{N} \left(\partial_{z_j} F_j(z) - \partial_{z_j} F_j(w)\right)(z_j - w_j) \tag{A.2}$$

for $z, w \in \mathbb{R}^N$ and some $\theta > 0$. There are smooth functions $G_1, \dots, G_N : \mathbb{R}^N \to \mathbb{R}$ with the following properties.

(i) *For $z, y \in \mathbb{R}^N$, $\partial_{z_j} F_j(z) = y_j$ for $j = 1, \dots, N$ if and only if $\partial_{y_k} G_k(y) = z_k$ for $k = 1, \dots, N$.*

(ii) *For all $j, k = 1, \dots, N$ and $y \in \mathbb{R}^N$,*

$$|\partial_{y_j} \partial_{y_k} G_k(y)| \le \frac{1}{\theta}.$$

(iii) *For each $y, w \in \mathbb{R}^N$,*

$$0 \le \sum_{k=1}^{N} (\partial_{y_k} G_k(y) - \partial_{y_k} G_k(w))(y_k - w_k).$$

(iv) *There is a constant B such that*

$$|G_k(y)| \le B(1 + |y|^2) \quad \text{and} \quad |\partial_{y_k} G_k(y)| \le B(1 + |y|)$$

for $y \in \mathbb{R}^N$ and $k = 1, \dots, N$.

Proof. First, we note that the mapping of $\mathbb{R}^N$

$$z \mapsto (\partial_{z_1} F_1(z), \dots, \partial_{z_N} F_N(z)) \tag{A.3}$$

is invertible. This follows from the theorems of Browder [8] and Minty [23] as (A.2) implies that this map is both monotone and coercive. Invertibility is a consequence of Hadamard's global invertibility theorem [15, Theorem A], as well. The condition (A.2) also gives

$$\theta|v|^2 \leq \sum_{j,k=1}^{N} \partial_{z_k}\partial_{z_j}F_j(z)v_k v_j \tag{A.4}$$

for each $z, v \in \mathbb{R}^N$.

Now fix $y \in \mathbb{R}^N$ and let $z \in \mathbb{R}^N$ be the unique Nash equilibrium of

$$\mathbb{R}^N \ni x \mapsto F_j(x) - y_j x_j$$

($j = 1, \ldots, N$). Such a Nash equilibrium exists for these N functions by Corollary 2.8. As a result

$$\partial_{z_j}F_j(z) = y_j, \quad j = 1, \ldots, N. \tag{A.5}$$

Since the mapping (A.3) is invertible and smooth, (A.5) determines z as a smooth function of y. Let us now define

$$G_k(y) := z_k y_k - F_k(z) \tag{A.6}$$

for $k = 1, \ldots, N$. Differentiating with respect to y_k and using (A.5), we find

$$\partial_{y_k}G_k(y) = z_k.$$

This proves (i) and that

$$y \mapsto (\partial_{y_1}G_1(y), \ldots, \partial_{y_N}G_N(y))$$

is the inverse mapping of (A.3).

It follows from part (*i*) and the inverse function theorem that whenever $\partial_{y_k}G_k(y) = z_k$ for $k = 1, \ldots, N$ and $\partial_{z_j}F_j(z) = y_j$ for $j = 1, \ldots, N$, the matrices

$$(\partial_{z_k}\partial_{z_j}F_j(z))_{j,k=1}^N \quad \text{and} \quad (\partial_{y_j}\partial_{y_k}G_k(y))_{j,k=1}^N$$

are inverses. Consequently, for a given $w \in \mathbb{R}^N$, we can choose $v \in \mathbb{R}^N$ defined as

$$v_j = \sum_{k=1}^{N} \partial_{y_j}\partial_{y_k}G_k(y)w_k$$

in (A.4) to get

$$0 \le \theta \sum_{j=1}^{N} \left(\sum_{k=1}^{N} \partial_{y_j} \partial_{y_k} G_k(y) w_k \right)^2$$
$$\le \sum_{j,k=1}^{N} \partial_{y_j} \partial_{y_k} G_k(y) w_k w_j$$
$$= \sum_{j=1}^{N} \left(\sum_{k=1}^{N} \partial_{y_j} \partial_{y_k} G_k(y) w_k \right) w_j$$
$$\le \sqrt{\sum_{j=1}^{N} \left(\sum_{k=1}^{N} \partial_{y_j} \partial_{y_k} G_k(y) w_k \right)^2} \cdot |w|. \tag{A.7}$$

As a result

$$\sum_{j=1}^{N} \left(\sum_{k=1}^{N} \partial_{y_j} \partial_{y_k} G_k(y) w_k \right)^2 \le \frac{1}{\theta^2} |w|^2.$$

Upon selecting $w = e_k$, the kth standard unit vector in $\mathbb{R}^N$, we find

$$(\partial_{y_j} \partial_{y_k} G_k(y))^2 = \left(\sum_{\ell=1}^{N} \partial_{y_j} \partial_{y_\ell} G_\ell(y) w_\ell \right)^2 \le \sum_{i=1}^{N} \left(\sum_{\ell=1}^{N} \partial_{y_i} \partial_{y_\ell} G_\ell(y) w_\ell \right)^2 \le \frac{1}{\theta^2}$$

for $j, k = 1, \ldots, N$. We conclude assertion (ii).

Part (iii) follows directly from (A.7). Indeed, for any $y, w \in \mathbb{R}^N$, we can use the fundamental theorem of calculus to derive

$$\sum_{k=1}^{N} (\partial_{y_k} G_k(y) - \partial_{y_k} G_k(w))(y_k - w_k)$$
$$= \int_0^1 \left(\sum_{j,k=1}^{N} \partial_{y_j} \partial_{y_k} G_k(w + t(y - w))(y_k - w_k)(y_j - w_j) \right) dt \ge 0.$$

We are left to verify assertion (iv). First note that by part (ii),

$$|\partial_{y_k} G_k(y)| \le |\partial_{y_k} G_k(y) - \partial_{y_k} G_k(0)| + |\partial_{y_k} G_k(0)|$$
$$= \left| \int_0^1 \sum_{j=1}^{N} \partial_{y_j} \partial_{y_k} G_k(ty) y_j \, dt \right| + |\partial_{y_k} G_k(0)|$$
$$\le \sum_{j=1}^{N} \left(\int_0^1 |\partial_{y_j} \partial_{y_k} G_k(ty)| dt \right) |y_j| + |\partial_{y_k} G_k(0)|$$

$$\leq \frac{1}{\theta}\sum_{j=1}^{N}|y_j| + |\partial_{y_k}G_k(0)|$$
$$\leq D(1+|y|)$$

for some $D > 0$ independent of $k = 1, \ldots, N$ and y.

For a given $y \in \mathbb{R}^N$ there is a unique z such that $\partial_{z_j}F_j(z) = y_j$ and $\partial_{y_k}G_k(y) = z_j$ for $j, k = 1, \ldots, N$. We just showed above that

$$|z_k| = |\partial_{y_k}G_k(y)| \leq D(1+|y|).$$

In view of hypothesis (A.1) and (A.6), we additionally have

$$\begin{aligned}|G_k(y)| &= |z_k y_k - F_k(z)|\\ &\leq |z_k||y_k| + |F_k(z)|\\ &\leq D(1+|y|)|y| + C(1+|z|^2)\\ &\leq D(1+|y|)|y| + C(1+ND^2(1+|y|)^2)\\ &\leq B(1+|y|^2)\end{aligned}$$

for an appropriately chosen constant B. □

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