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EVOLUTION OF MIXED STRATEGIES IN MONOTONE GAMES*

RYAN HYND†

Abstract. We consider the basic problem of approximating Nash equilibria in noncooperative games. For monotone games, we design continuous time flows which converge in an averaged sense to Nash equilibria. We also study mean field equilibria, which arise in the large player limit of symmetric noncooperative games. In this setting, we will additionally show that the approximation of mean field equilibria is possible under a suitable monotonicity hypothesis.

Key words. monotone games, static mean field games, Gaussian measures

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1. Introduction.

We begin by recalling a noncooperative game in which players labeled $j=1,\ldots,N$ each have finitely many choices. For definiteness, let us suppose that each player selects an action among the first m natural numbers and that each player is unaware of the others' selections. If player i selects $s_i \in \{1,\ldots,m\}$ for $i=1,\ldots,N$, player j's cost is a number

$$f_j(s_1,\ldots,s_N).$$

Each player seeks to have as small a cost as possible. However, each player's cost depends on the other players' actions.

This type of game leads naturally to the notion of a Nash equilibrium. This is an N-tuple $s=(s_1,\ldots,s_N)$ for which

$$f_j(s)\le f_j(t_j,s_{-j})$$

for all $t_j=1,\ldots,m$ and $j=1,\ldots,N$. Here we have written

$$(t_j,s_{-j})=(s_1,\ldots,s_{j-1},t_j,s_{j+1},\ldots,s_N).$$

Note in particular that no player can pay a smaller cost by making a unilateral change. Simple examples can be found in which Nash equilibria do not exist. However, Nash showed such equilibria exist if mixed strategies are allowed [35, 36].

For any $m\in\mathbb{N}$, we will denote the standard m-simplex as

$$\Delta_m=\left\{z\in\mathbb{R}^m: z_j\ge 0,\ \sum_{j=1}^m z_j=1\right\}.$$

A *mixed strategy* for player i is an element $x_i=(x_{i,1},\ldots,x_{i,m})\in\Delta_m$, which corresponds to player i choosing action $j\in\{1,\ldots,m\}$ with probability $x_{i,j}$. If player i

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†Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104 USA (rhynd@math.upenn.edu).

selects the mixed strategy $x_i \in \Delta_m$ for each $i = 1, \ldots, N$, player j's expected cost is defined to be

$$F_j(x_1, \ldots, x_N) = \sum_{s_1=1}^{m} \cdots \sum_{s_N=1}^{m} f_j(s) x_{1,s_1} \cdots x_{N,s_N}. \tag{1.1}$$

We'll also say $x_i \in \Delta_m$ is a *pure strategy* if one of the entries of x_i is equal to 1.

We can extend the definition of Nash equilibria given above to incorporate mixed strategies as follows. A *Nash equilibrium* is an N-tuple $x = (x_1, \ldots, x_N)$ for which

$$F_j(x) \le F_j(y_j, x_{-j})$$

for each $y_j \in \Delta_m$ and $j = 1, \ldots, N$. Here (y_j, x_{-j}) is defined analogously to (t_j, s_{-j}) above. As with Nash equilibria for pure strategies, no player can improve her expected cost by deviating from her current choice. In this note, we will discuss the possibility of approximating Nash equilibrium for this type and more general types of games.

1.1. Previous work. The existence of a Nash equilibrium for the game discussed above follows from an application of Brouwer's fixed point theorem. Since proofs of Brouwer's fixed point theorem are nonconstructive, it seems unlikely that there would be an easy way to approximate Nash equilibria in general. This problem has been examined at length, and its complexity has been categorized as being equivalent to finding a fixed point in the conclusion of Brouwer's theorem [11, 14, 15]. In particular, there is no known efficient algorithm for approximating Nash equilibria.

Nevertheless, there is one class of games in which approximation is at least theoretically feasible. These games are called *monotone.* In the context described above, their expected cost functions $F_1, \ldots, F_N$ satisfy

$$\sum_{j=1}^{N} (F_j(x) + F_j(y)) \ge \sum_{j=1}^{N} (F_j(x_j, y_{-j}) + F_j(y_j, x_{-j})) \tag{1.2}$$

for $x, y \in \Delta_m^N$. For example, any two-player zero-sum game satisfies this monotonicity condition (see Corollary 3.4 below). This condition additionally extends more generally to cost functions $F_1, \ldots, F_N$ which are not necessarily of the form (1.1), and it is inspired by a uniqueness criterion discovered by Lasry and Lions when they initiated the study of mean field games [32]. We also note that there have been several recent studies on monotone games [3, 9, 19, 20, 26, 34, 38, 39, 40, 41, 42, 43].

In prior joint work [1], we argued that if the game is monotone, then for any $x^0 \in \Delta_m^N$, there is a unique absolutely continuous path $u : [0, \infty) \to \Delta_m^N$ such that

$$\begin{cases} \dot{u}_j(t) + \partial_{x_j} F_j(u(t)) \ni 0, \\ u_j(0) = x_j^0 \end{cases} \tag{1.3}$$

for almost every $t \ge 0$ and each $j = 1, \ldots, N$. Here

$$\partial_{x_j} F_j(x) = \Big\{ z \in \mathbb{R}^m : F_j(y_j, x_{-j}) \ge F_j(x) + z \cdot (y_j - x_j) \text{ for } y_j \in \Delta_m \Big\}, \tag{1.4}$$

and the dot "$\cdot$" denotes the standard dot product on $\mathbb{R}^m$. In particular, the evolution equation in (1.3) is equivalent to

$$F_j(y_j, u_{-j}(t)) \ge F_j(u(t)) - \dot{u}_j(t) \cdot (y_j - u_j(t)) \tag{1.5}$$

holding for all $y \in \Delta_m$ and almost every $t \ge 0$.

We also considered the *Cesàro mean* of u,

$$\frac{1}{t}\int_0^t u(s)ds.$$

Let us suppose for the moment that this mean converges to x as $t\to\infty$. In view of (1.2) and (1.3),

$$\begin{aligned}\sum_{i=1}^N (F_i(z)-F_i(u_i(s),z_{-i})) &\ge \sum_{i=1}^N (F_i(z_i,u_{-i}(s))-F_i(u(s)))\\ &\ge -\sum_{i=1}^N \dot u_i(s)\cdot(z_i-u_i(s))\\ &= \frac{d}{ds}\sum_{i=1}^N \frac{1}{2}|u_i(s)-z_i|^2\end{aligned}$$

for $z\in\Delta_m^N$. Integrating from $s=0$ to $s=t$ and dividing by t gives

$$\begin{aligned}\sum_{i=1}^N\left(F_i(z)-F_i\left(\frac{1}{t}\int_0^t u_i(s)ds,z_{-i}\right)\right) &= \sum_{i=1}^N\left(F_i(z)-\frac{1}{t}\int_0^t F_i\left(u_i(s),z_{-i}\right)ds\right)\\ &= \sum_{i=1}^N\frac{1}{t}\int_0^t \left(F_i(z)-F_i\left(u_i(s),z_{-i}\right)\right)ds\\ &\ge \sum_{i=1}^N \frac{1}{2t}\left(|u_i(t)-z_i|^2-|u_i^0-z_i|^2\right).\end{aligned}$$

Here we used that each F_i is multilinear; recall the definition (1.1).

As $u(t)\in\Delta_m^N$ is bounded, we can send $t\to\infty$ to find

$$\sum_{i=1}^N\left(F_i(z)-F_i\left(x_i,z_{-i}\right)\right)\ge 0.$$

Choosing $z=(y_j,x_{-j})$ for $y_j\in\Delta_m$ would then lead to

$$F_j(y_j,x_{-j})-F_j(x)\ge 0.$$

That is, x is a Nash equilibrium. Of course it remains to be shown that the Cesàro mean of u converges. This follows from a theorem due to Baillon and Brézis [2]. The goal of this study is to identify a general setting in game theory for which we can apply this result.

1.2. A general setting. In what follows, we will study a general version of the noncooperative game detailed above. To this end, we will consider a separable Banach space X with continuous dual space X^* and write

$$\mu(x)=\langle\mu,x\rangle$$

for $\mu\in X^*$ and $x\in X$. Let us suppose $K_1,\dots,K_N\subset X^*$ are each nonempty, convex, and weak* compact, and set

$$K=K_1\times\cdots\times K_N.$$

We will study collections of N functions $F_1,\dots,F_N : K \to \mathbb{R}$ which are weak* continuous and satisfy

$$K_j \ni \nu_j \mapsto F_j(\nu_j,\mu_{-j}) \text{ convex} \tag{1.6}$$

for each $\mu \in K$ and $j = 1,\dots,N$. We'll say $\mu \in K$ is a *Nash equilibrium* of $F_1,\dots,F_N$ provided that

$$F_j(\mu) \le F_j(\nu_j,\mu_{-j}) \text{ for all } \nu_j \in K_j \text{ and } j = 1,\dots,N.$$

Later in this note, we will briefly recall how to justify the existence of a Nash equilibrium.

The prototypical scenario of interest is when $X = C(S)$ for a compact metric space S and

$$K_1 = \cdots = K_N = \mathcal{P}(S).$$

Here $\mathcal{P}(S)$ is the collection of Borel probability measures on S. We recall that X^* is isometrically isomorphic to $M(S)$, the collection of Radon measures on S equipped with the total variation norm. Moreover, $\mathcal{P}(S) \subset M(S)$ is convex and weak* compact. Note that if $f_j : S^N \to \mathbb{R}$ is continuous, then

$$F_j(\mu) = \int_{S^N} f_j(s)d\mu_1(s_1)\cdots d\mu_N(s_N) \tag{1.7}$$

is weak* continuous on $\mathcal{P}(S)^N$ for $j = 1,\dots,N$. Moreover, F_j clearly satisfies (1.6).

These objects relate to game theory as follows. The set S represents an action space for players $1,\dots,N$ in a noncooperative game. An element $\mu_j \in \mathcal{P}(S)$ constitutes a mixed strategy for player j; that is, player j chooses from a given collection of actions $A \subset S$ with probability $\mu_j(A)$. Of course, $\mu_j = \delta_{s_j}$ is a pure strategy: player j always selects action $s_j \in S$. The value $f_j(s)$ represents player j's cost if the players collectively opt for action $s = (s_1,\dots,s_N) \in S^N$. And $F_j(\mu)$ indicates player j's expected cost if players $1,\dots,N$ respectively select the mixed strategies $\mu_1,\dots,\mu_N$. Note than when S is finite, this example corresponds to the N-player noncooperative game considered at the beginning of this note.

Let us return to the general setting involving the separable Banach space X. In analogy with (1.4), we define

$$\partial_{\mu_j} F_j(\mu) = \Big\{ x \in X : F_j(\nu_j,\mu_{-j}) \ge F_j(\mu) + \langle \nu_j - \mu_j, x\rangle \text{ for } \nu_j \in K_j \Big\}$$

for $\mu \in K$. Note that $\partial_{\mu_j} F_j(\mu)$ is not a subdifferential in the traditional sense as the inequality in the definition is only required to hold for $\nu_j \in K_j$ rather than for all $\nu_j \in X^*$. Nevertheless, these subdifferentials are suitable for our purposes.

Observe that if there happens to be $\delta F_j(\mu)/\delta\mu_j \in X$ such that

$$\lim_{t\to 0^+} \frac{F_j(\mu_j + t(\nu_j - \mu_j), \mu_{-j}) - F_j(\mu)}{t} = \left\langle \nu_j - \mu_j, \frac{\delta F_j(\mu)}{\delta \mu_j} \right\rangle \tag{1.8}$$

for each $\nu_j \in K_j$, then

$$\frac{\delta F_j(\mu)}{\delta\mu_j} \in \partial_{\mu_j} F_j(\mu).$$

This is due to our convexity hypothesis (1.6). In the case $X = C(\mathbb{T}^d)$ and $K_j = \mathcal{P}(\mathbb{T}^d)$, a notion of differentiability based on the limit (1.8) has been used with success to study master equations in mean field games (see Chapter 2 of [8] for a detailed discussion).

It is plain to see that $\mu \in K$ is a Nash equilibrium if and only if

$$0 \in \partial_{\mu_j} F_j(\mu) \quad \text{for all } j = 1, \ldots, N.$$

In addition, we'll say that $F_1, \ldots, F_N$ is *monotone* provided

$$\sum_{j=1}^{N} \langle \mu_j - \nu_j, x_j - y_j \rangle \geq 0 \tag{1.9}$$

whenever $x_j \in \partial_{\mu_j} F_j(\mu)$ and $y_j \in \partial_{\mu_j} F_j(\nu)$ for $j = 1, \ldots, N$. We will also see in Proposition 3.3 that the aforementioned type of monotonicity (1.2) is a special case of the notion just introduced.

1.3. Approximation result. We aim to use a flow along the lines of (1.3) to approximate Nash equilibria for $F_1, \ldots, F_N$ in the general setting outlined above. An important detail in (1.3) that we made use of is the natural embedding

$$\Delta_m \subset \mathbb{R}^m.$$

Here $\mathbb{R}^m$ is a Hilbert space with the usual dot product. With this goal in mind, we will employ

a centered, nondegenerate Gaussian measure η on X.

Recall that this means η is a Borel probability measure on X such that the push forward of η by any nonzero element of X^* is a centered, nondegenerate Gaussian measure on $\mathbb{R}$. It turns out that $X^* \subset L^2(X, \eta)$, and we will see that the Hilbert space

$$H = \text{the closure of } X^* \text{ in } L^2(X, \eta)$$

will play the role of $\mathbb{R}^m$ with the dot product for the flow we present below.

Building on our experience with (1.5), we will consider a path $\xi : [0, \infty) \to H^N$ that fulfills

$$F_j(\nu_j, \xi_{-j}(t)) \geq F_j(\xi(t)) - (\nu_j - \xi_j(t), \dot{\xi}_j(t)) \tag{1.10}$$

for each $\nu_j \in K_j$, almost every $t \geq 0$, and $j = 1, \ldots, N$. Here $(\cdot, \cdot)$ is the $L^2(X, \eta)$ inner product. In Proposition 2.2 below, we will recall a continuous linear mapping $\mathcal{J} : H \to X$ which satisfies

$$(\mu, \zeta) = \langle \mu, \mathcal{J}\zeta \rangle \ \text{ for } \mu \in X^* \text{ and } \zeta \in H.$$

As a result, (1.10) may be expressed as

$$\mathcal{J}\dot{\xi}_j(t) + \partial_{\mu_j} F_j(\xi(t)) \ni 0 \tag{1.11}$$

for almost every $t \geq 0$ and each $j = 1, \ldots, N$.

In the following theorem, we will show that for a given initial condition $\xi(0) = \mu^0$, the initial value problem associated with (1.11) is well-posed and the Cesàro mean of

ξ converges to a Nash equilibrium of $F_1,\dots,F_N$. The crucial hypothesis is that μ^0 belongs to the set

$$\mathcal{D}=\left\{\mu\in K: \text{there is } \sigma\in H^N \text{ such that } \mathcal{J}\sigma_j\in\partial_{\mu_j}F_j(\mu) \text{ for } j=1,\dots,N\right\}. \tag{1.12}$$

THEOREM 1.1. *Suppose* $F_1,\dots,F_N$ *satisfies* (1.6) *and* (1.9) *and that* $\mu^0\in\mathcal{D}$. *There is a unique Lipschitz continuous* $\xi:[0,\infty)\to H^N$ *with* $\xi(t)\in\mathcal{D}$ *for each* $t\geq 0$ *and*

$$\begin{cases}\mathcal{J}\dot{\xi}_j(t)+\partial_{\mu_j}F_j(\xi(t))\ni 0 & a.e.\ t\geq 0,\\ \xi_j(0)=\mu_j^0\end{cases}$$

for each $j=1,\dots,N$. *Moreover,*

$$\frac{1}{t}\int_0^t \xi(s)ds$$

converges weak to a Nash equilibrium of* $F_1,\dots,F_N$ *as* $t\to\infty$.

We will also present a related approximation theorem for symmetric games. A prototypical example occurs when $F_1,\dots,F_N$ is defined via (1.7) with

$$f_i(s)=f\left(s_i,\frac{1}{N-1}\sum_{j\neq i}\delta_{s_j}\right)$$

for $i=1,\dots,N$ and some continuous $f:S\times\mathcal{P}(S)\to\mathbb{R}$. It turns out that $F_1,\dots,F_N$ has a symmetric Nash equilibrium $(\mu^N,\dots,\mu^N)\in\mathcal{P}(S)^N$. Furthermore, when $N\to\infty$, $(\mu^N)_{N\in\mathbb{N}}$ has a subsequence which converges weak* to some μ that satisfies

$$\int_S f(s,\mu)d\mu(s)\leq\int_S f(s,\mu)d\nu(s)$$

for each $\nu\in\mathcal{P}(S)$ (as explained in Chapter 4 of [31]). Such a μ is called a *mean field equilibrium.* Finding mean field equilibria is a basic problem in the theory of mean field games [8, 10, 22, 31], and we will informally refer to this example as a *static mean field game.* In Theorem 4.4 below, we will employ a simpler version of the flow described in Theorem 1.1 to approximate symmetric and mean field equilibria.

Most approximation results for Nash equilibria which require some form of monotonicity, such as the ones verified in [5, 7, 12, 13, 24, 30, 37, 45], involve discrete time flows. The first study that used a continuous time flow to approximate Nash equilibria in monotone games set in finite dimensions was initiated by Flåm [18]. In our prior work [1], we extended Flåm's work to Hilbert spaces and highlighted the role of the Cesàro mean. The contribution of this paper is in verifying that theoretical approximation can be obtained with a continuous time flow for monotone games set in dual Banach spaces.

This paper is organized as follows. In section 2, we will recall some basic facts about Gaussian measures. Next, we will discuss general N-player games in section 3 and prove Theorem 1.1. Then in section 4, we will show how to approximate equilibria in symmetric and static mean field games provided that the appropriate monotonicity hypothesis is in place. In the appendix, we will show how our general theory reduces to the type of game discussed at the beginning of this introduction and work out an explicit example to illustrate why we can't expect to have better than convergence in the sense of the Cesàro mean.

2. Preliminaries. As in the introduction, we will suppose X is a separable Banach space over $\mathbb{R}$ with norm $\|\cdot\|$ and denote the space of continuous linear functionals $\mu : X \to \mathbb{R}$ as X^*. We will also express the dual norm as

$$\|\mu\|_* = \sup\{|\mu(x)| : \|x\| \le 1\}.$$

Note that since X is separable, the weak* topology on X^* is metrizable. In particular, $\mu^k \to \mu$ weak* whenever $\mu^k(x) \to \mu(x)$ for all $x \in X$. It will also be important for us to recall that the closed unit ball $\{\mu \in X^* : \|\mu\|_* \le 1\}$ is weak* compact by Alaoglu's theorem. That is, dual norm bounded sequences have weak* convergent subsequences.

2.1. Gaussian measures. We will assume throughout that η is a centered, nondegenerate Gaussian measure on X. Namely, for each $\mu \in X^* \setminus \{0\}$, there is $q > 0$ such that

$$\int_X g(\mu(x))d\eta(x) = \int_{\mathbb{R}} g(y)\frac{e^{-\frac{y^2}{2q}}}{\sqrt{2\pi q}}dy \tag{2.1}$$

for all bounded and continuous $g : \mathbb{R} \to \mathbb{R}$. Below we will recall some basic properties of Gaussian measures for the purposes of this paper, which can be found in [4, 16, 23].

We'll write (μ, ν) for the inner product between μ and ν in $L^2(X, \eta)$ and

$$\|\mu\|_{L^2} = (\mu, \mu)^{1/2}.$$

It is known that η has a finite second moment

$$\int_X \|x\|^2 d\eta(x) < \infty.$$

Note that if $\mu \in X^*$ and $x \in X$, $|\mu(x)| \le \|\mu\|_* \|x\|$. It follows that

$$\|\mu\|_{L^2} \le \|\mu\|_* \left(\int_X \|x\|^2 d\eta(x)\right)^{1/2}.$$

Therefore, $X^* \subset L^2(X, \eta)$.

As in the introduction, we denote H as the closure of X^* in the $L^2(X, \eta)$ norm. We can think of H as linear functionals on X which are merely square integrable with respect to η. With this choice of Hilbert space H,

$$X^* \subset H$$

is a dense subspace. Moreover, this embedding is compact.

It is evident from (2.1) that $(\mu, \mu) > 0$ for each $\mu \in X^* \setminus \{0\}$. Therefore, if $\mu_1, \mu_2 \in X^*$ are equal to η almost everywhere, they must agree everywhere on X. The following lemma is a consequence of this observation.

LEMMA 2.1. *Suppose $(\mu^k)_{k \in \mathbb{N}}$ is a bounded sequence in X^* which converges weakly in H to ξ. Then $(\mu^k)_{k \in \mathbb{N}}$ converges weak*, and its limit agrees η almost everywhere with ξ.*

Proof. Choose a subsequence $(\mu^{k_j})_{j \in \mathbb{N}}$ which converges weak* to some $\mu \in X^*$. Note that for a given $\zeta \in H$,

$$|\mu^k(x)\zeta(x)| \le c\|x\||\zeta(x)|$$

for some c independent of $k \in \mathbb{N}$ and $x \in X$. Observe that the right-hand side above is in $L^1(X, \eta)$. Dominated convergence implies

$$\lim_{j\to\infty} \int_X \mu^{k_j}(x)\zeta(x)d\eta(x) = \int_X \mu(x)\zeta(x)d\eta(x).$$

It follows that $(\mu^{k_j})_{j\in\mathbb{N}}$ converges weakly to μ in H. As a result, $\mu = \xi$ almost everywhere. If $(\mu^k)_{k\in\mathbb{N}}$ has another weak* subsequential limit $\tilde{\mu}$, then $\mu(x) = \tilde{\mu}(x)$ for η almost $x \in X$. Therefore, $\mu \equiv \tilde{\mu}$ and $(\mu^k)_{k\in\mathbb{N}}$ converges to μ since this limit is independent of the subsequence. ∎

2.2. The mapping $\mathcal{J}$. We now consider the linear mapping $\mathcal{J} : H \to X$ defined by the formula

$$\mathcal{J}\xi = \int_X x\xi(x)d\eta(x). \tag{2.2}$$

Observe that this Bochner integral is a well-defined element of X. Indeed, since ξ is the $L^2(X, \eta)$ limit of a sequence of continuous functions and since X is separable, the mapping $x \mapsto x\xi(x)$ from X into X is strongly measurable; this can be seen as a consequence of Pettis's theorem (Chapter V section 4 of [44]). Moreover, $x \mapsto \|x\xi(x)\|$ is clearly in $L^1(X, \mu)$.

A basic assertion regarding $\mathcal{J}$ is as follows.

PROPOSITION 2.2. (i) *For $\mu \in X^*$ and $\xi \in H$,*

$$\langle \mu, \mathcal{J}\xi\rangle = (\mu, \xi). \tag{2.3}$$

(ii) *$\mathcal{J} : H \to X$ is continuous and injective.*

Proof. (i) As $\mathcal{J}\xi$ is the Bochner integral (2.2),

$$\langle \mu, \mathcal{J}\xi\rangle = \left\langle \mu, \int_X x\xi(x)d\eta(x)\right\rangle = \int_X \mu(x)\xi(x)d\eta(x).$$

(ii) Since

$$\|\mathcal{J}\xi\| \le \left(\int_X \|x\|^2 d\eta(x)\right)^{1/2} \|\xi\|_{L^2}$$

for $\xi \in H$, $\mathcal{J}$ is bounded. And if $\mathcal{J}\xi = 0 \in X$, then $(\mu, \xi) = 0$ for each $\mu \in X^*$. Since X^* is dense in H, $(\mu, \xi) = 0$ for each $\mu \in H$. That is, $\xi = 0 \in H$. ∎

2.3. An initial value problem. We will now briefly recall a few technical assertions needed to proved Theorem 1.1. For simplicity, we will state these claims for the Hilbert space H we introduced above; however, they are valid for any Hilbert space. A mapping $B : H \to 2^H$ is *monotone* provided

$$(\xi - \zeta, \mu - \nu) \ge 0$$

for each $\xi \in B\mu$ and $\zeta \in B\nu$. Moreover, we will say that B is *maximally monotone* if the only monotone $C : H \to 2^H$ with $B\mu \subset C\mu$ for all μ is B itself. Minty's lemma

[33] asserts that B is maximally monotone if and only if for each $\sigma \in H$, there is μ for which

$$\sigma \in \mu + B\mu.$$

The following theorem is a consequence of the seminal works by Kato [27, 28] and Kōmura [29]. We also note that a more general statement is proved in Theorem 3.1 of the monograph by Brézis [6].

Theorem 2.3 (Kato–Kōmura theorem). *Assume that B is maximally monotone with $B\zeta^0 \neq \emptyset$. There exists a unique Lipschitz continuous $\zeta : [0,\infty) \to H$ which satisfies*

$$\begin{cases} \dot{\zeta}(t) + B\zeta(t) \ni 0 & \text{for a.e. } t \geq 0, \\ \zeta(0) = \zeta^0 \end{cases} \tag{2.4}$$

and $B\zeta(t) \neq \emptyset$ for all $t \geq 0$.

Any $\mu \in H$ for which $0 \in B\mu$ is an *equilibrium* for B. It turns out that a solution of the initial value problem can be used to approximate equilibria of B provided of course that B has equilibria. The subsequent theorem was proved by Baillon and Brézis [2].

Theorem 2.4 (Baillon–Brézis theorem). *Suppose that B is maximally monotone, B has an equilibrium, and ζ is a solution of the initial value problem* (2.4). *The limit*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \zeta(s)ds$$

exists weakly in H and is an equilibrium for B.

3. *N*-player games. The primary goal of this section is to prove Theorem 1.1. To this end, we will suppose $K_1, \ldots, K_N$ are each nonempty, convex, and compact subsets of X^* and set $K = K_1 \times \cdots \times K_N$. In addition, we will assume $F_j : K \to \mathbb{R}$ is weak* continuous and that $\nu_j \mapsto F_j(\nu_j, \mu_{-j})$ is convex for each $\mu \in K$ and $j = 1, \ldots, N$.

First let us recall that a Nash equilibrium exists.

Proposition 3.1. *$F_1, \ldots, F_N$ has a Nash equilibrium.*

Proof. By our assumptions, the mapping from K into 2^K

$$K \ni \mu \mapsto \operatorname{argmin}\left\{\sum_{j=1}^N F_j(\nu_j, \mu_{-j}) : \nu \in K\right\}$$

has nonempty and convex images. Moreover, the graph of this mapping is closed. It follows from the Kakutani–Fan–Glicksberg theorem that there is a fixed point

$$\mu \in \operatorname{argmin}\left\{\sum_{j=1}^N F_j(\mu_j, \nu_{-j}) : \nu \in K\right\},$$

which is also a Nash equilibrium of $F_1, \ldots, F_N$. ∎

Remark 3.2. Kakutani [25] extended Brouwer's fixed point theorem to set-valued mappings. Fan [17] and Glicksberg [21] independently generalized Kakutani's fixed point theorem [25] to locally convex spaces.

3.1. Monotonicity of $F_1,\dots,F_N$. Recall that $F_1,\dots,F_N$ is monotone provided (1.9) holds. There is also a simple sufficient condition for monotonicity as detailed in the proposition below.

PROPOSITION 3.3. *Suppose, for each $\mu,\nu\in K$,*

$$\sum_{j=1}^{N}(F_j(\mu)+F_j(\nu))\ge\sum_{j=1}^{N}(F_j(\nu_j,\mu_{-j})+F_j(\mu_j,\nu_{-j})). \tag{3.1}$$

Then $F_1,\dots,F_N$ is monotone.

Proof. Suppose $x_j\in\partial_{\mu_j}F_j(\mu)$ and $y_j\in\partial_{\mu_j}F_j(\nu)$. Then

$$F_j(\nu_j,\mu_{-j})\ge F_j(\mu)+\langle\nu_j-\mu_j,x_j\rangle$$

and

$$F_j(\mu_j,\nu_{-j})\ge F_j(\nu)+\langle\mu_j-\nu_j,y_j\rangle$$

for $j=1,\dots,N$. Adding these inequalities yields

$$\sum_{j=1}^{N}(F_j(\nu_j,\mu_{-j})+F_j(\mu_j,\nu_{-j}))\ge\sum_{j=1}^{N}(F_j(\mu)+F_j(\nu))-\sum_{j=1}^{N}\langle\mu_j-\nu_j,x_j-y_j\rangle.$$

Using (3.1) gives

$$\sum_{j=1}^{N}\langle\mu_j-\nu_j,x_j-y_j\rangle\ge 0.$$

∎

COROLLARY 3.4. *If $N=2$ and $F_1+F_2\equiv 0$, then F_1,F_2 is monotone. That is, two-person zero-sum games are monotone.*

Proof. For $\mu,\nu\in K$,

$$\sum_{j=1}^{2}(F_j(\mu)+F_j(\nu))=\sum_{j=1}^{2}F_j(\mu)+\sum_{j=1}^{2}F_j(\nu)=0+0=0,$$

and

$$\begin{aligned}&\sum_{j=1}^{2}(F_j(\nu_j,\mu_{-j})+F_j(\mu_j,\nu_{-j}))\\&\qquad=(F_1(\nu_1,\mu_2)+F_1(\mu_1,\nu_2))+(F_2(\mu_1,\nu_2)+F_2(\nu_1,\mu_2))\\&\qquad=(F_1(\nu_1,\mu_2)+F_2(\nu_1,\mu_2))+(F_1(\mu_1,\nu_2))+F_2(\mu_1,\nu_2))\\&\qquad=0+0\\&\qquad=0.\end{aligned}$$

∎

We also note that monotonicity can be verified somewhat more easily in the model case.

PROPOSITION 3.5. *Suppose S is a compact metric space and $f_j:S^N\to\mathbb{R}$ is continuous, and set*

$$F_j(\mu_1,\dots,\mu_N)=\int_{S^N}f_j(s)d\mu_1(s_1)\cdots d\mu_N(s_N)$$

for $\mu \in \mathcal{P}(S)^N$ and $j = 1, \ldots, N$. Then $F_1, \ldots, F_N$ satisfies (3.1) *if and only if*

$$\sum_{j=1}^{N} (f_j(s) + f_j(t)) \geq \sum_{j=1}^{N} (f_j(s_j, t_{-j}) + f_j(t_j, s_{-j})) \tag{3.2}$$

for all $s, t \in S^N$.

Proof. Suppose (3.1) holds and $s, t \in S^N$. If we select $\mu_j = \delta_{s_j}$ and $\nu_j = \delta_{t_j}$ for $j = 1, \ldots, N$, then (3.1) is the same inequality as (3.2). Alternatively, suppose (3.2) holds and $\mu, \nu \in \mathcal{P}(S)^N$. Integrating this inequality against $d\mu_j(s_j)d\nu_j(t_j)$ for $j = 1, \ldots, N$ leads to (3.1). ∎

3.2. Flow of mixed strategies. As previously mentioned, the closure of X^* in $L^2(X, \eta)$ is a Hilbert space H with inner product $(\cdot, \cdot)$. We also will employ the linear mapping $\mathcal{J} : H \to X$ defined in (2.2) and suppose for the rest of this section that $F_1, \ldots, F_N$ is monotone.

We will now consider the problem of finding a solution $\xi : [0, \infty) \to H^N$ of the initial value problem

$$\begin{cases} \mathcal{J}\dot{\xi}_j(t) + \partial_{\mu_j} F_j(\xi(t)) \ni 0 & \text{a.e. } t \geq 0, \\ \xi_j(0) = \mu_j^0 \end{cases} \tag{3.3}$$

for a given $\mu^0 \in K$. Here H^N is the N-fold product of H endowed with the inner product

$$(\mu, \nu) := \sum_{j=1}^{N} (\mu_j, \nu_j).$$

In order to verify the existence of a solution, we will define a mapping $A : H^N \to 2^{H^N}$ via

$$A\mu := \begin{cases} \mathcal{J}^{-1}\left(\partial_{\mu_1} F_1(\mu)\right) \times \cdots \times \mathcal{J}^{-1}\left(\partial_{\mu_N} F_N(\mu)\right), & \mu \in \mathcal{D}, \\ \emptyset, & \mu \notin \mathcal{D}, \end{cases}$$

for $\mu \in H^N$. Here we recall that $\mathcal{D}$ is defined in (1.12) and emphasize that $\sigma \in A\mu$ if and only if $\mu \in K$ and $\mathcal{J}\sigma_j \in \partial_{\mu_j} F_j(\mu)$ for $j = 1, \ldots, N$.

We will first show that A is maximally monotone.

LEMMA 3.6. *A is maximally monotone.*

Proof. Suppose $\sigma \in A\mu$ and $\tilde{\sigma} \in A\tilde{\mu}$. Then

$$(\mu - \tilde{\mu}, \sigma - \tilde{\sigma}) = \sum_{j=1}^{N} (\mu_j - \tilde{\mu}_j, \sigma_j - \tilde{\sigma}_j) = \sum_{j=1}^{N} \langle \mu_j - \tilde{\mu}_j, \mathcal{J}\sigma_j - \mathcal{J}\tilde{\sigma}_j \rangle \geq 0$$

as $F_1, \ldots, F_N$ is monotone. Thus, A is monotone.

In order to show that A is maximal, we will appeal to Minty's lemma. That is, it suffices to show for each $\sigma \in H^N$ that there is $\mu \in K$ with $\mu + A\mu \ni \sigma$. This is the case provided

$$\mathcal{J}(\mu_j - \sigma_j) + \partial_{\mu_j} F_j(\mu) \ni 0$$

for $j=1,\dots,N$. Furthermore, μ is the desired solution if and only if $\mu\in\Phi(\mu)$, where $\Phi:K\mapsto 2^K$ is defined as

$$\Phi(\mu):=\operatorname{argmin}\left\{\sum_{j=1}^{N}\langle\nu_j,\mathcal{J}(\mu_j-\sigma_j)\rangle+F_j(\nu_j,\mu_{-j}):\nu\in K\right\}$$

for each $\mu\in K$.

Note that

$$K\ni\nu\mapsto\sum_{j=1}^{N}\langle\nu_j,\mathcal{J}(\mu_j-\sigma_j)\rangle+F_j(\nu_j,\mu_{-j})$$

is weak* continuous for each $\mu\in K$. Since K is weak* compact, this function has a minimum. And as this function is convex, its set of minima is convex. Thus, $\Phi(\mu)$ is nonempty and convex. The continuity of $F_1,\dots,F_N$ and of $\mathcal{J}$ also implies that the graph of Φ is closed. Therefore, there is $\mu\in K$ such that $\mu\in\Phi(\mu)$ by the Kakutani–Fan–Glicksberg theorem [17, 21]. ◻

We can now verify that the initial value problem (3.3) has a solution whose Cesàro mean converges to a Nash equilibrium.

Proof of Theorem 1.1. We've established that A is maximally monotone, and our hypothesis on μ^0 is that $A\mu^0\neq\emptyset$. The Kato–Kōmura theorem then implies that there is a unique Lipschitz continuous solution $\xi:[0,\infty)\to H^N$ of the equation

$$\begin{cases}\dot\xi(t)+A\xi(t)\ni 0 & \text{a.e. } t\geq 0,\\ \xi(0)=\mu^0.\end{cases}$$

Moreover, $A\xi(t)\neq\emptyset$ for each $t\geq 0$. It follows that $\xi(t)\in\mathcal{D}$ for $t\geq 0$ and that

$$\mathcal{J}\dot\xi_j(t)+\partial_{\mu_j}F_j(\xi(t))\ni 0\quad\text{a.e. } t\geq 0,$$

for $j=1,\dots,N$. Consequently, ξ is a solution of the initial value problem (3.3) as claimed. The limit

$$\mu_j:=\lim_{t\to\infty}\frac{1}{t}\int_0^t\xi_j(s)ds$$

exists weakly in H by the Baillon–Brézis theorem for each $j=1,\dots,N$ and is an equilibrium of A. Therefore, μ is a Nash equilibrium of $F_1,\dots,F_N$. By Lemma 2.1, this limit also exists weak*. ◻

4. Symmetric games. Suppose now that $K=K_1=\cdots=K_N\subset X^*$ is weak* compact, $F_1,\dots,F_N:K^N\to\mathbb{R}$ is continuous, and $K\ni\nu_j\mapsto F_j(\nu_j,\mu_{-j})$ is convex for each $\mu\in K^N$ and $j=1,\dots,N$. We will say that $F_1,\dots,F_N$ is *symmetric* provided

$$\underbrace{F_i(\mu,\dots,\mu,\nu,\mu,\dots,\mu)}_{\nu\text{ is in the }i\text{th argument of }F_i}=\underbrace{F_j(\mu,\dots,\mu,\nu,\mu,\dots,\mu)}_{\nu\text{ is in the }j\text{th argument of }F_j}$$

for all $i,j=1,\dots,N$ and all $\mu,\nu\in K$. With these assumptions, it can be shown that $F_1,\dots,F_N$ has a symmetric Nash equilibrium $(\mu,\dots,\mu)\in K^N$. As we only need to find $\mu\in K$ such that

$$F_1(\mu,\dots,\mu)\leq F_1(\nu,\mu,\dots,\mu)\quad\text{for all }\mu\in K,$$

we can employ a simpler approximation method than discussed above.

The theorem we present below will also apply to mean field equilibria, which we recall are $\mu\in\mathcal{P}(S)$ that satisfy

$$\int_S f(s,\mu)d\mu(s)\le\int_S f(s,\mu)d\nu(s)$$

for each $\nu\in\mathcal{P}(S)$. For static mean field games, we'll always assume S is a compact metric space and $f:S\times\mathcal{P}(S)\to\mathbb{R}$ is continuous; here $S\times\mathcal{P}(S)$ is endowed with the product topology from the metric on S and the weak* topology on $\mathcal{P}(S)$. The key monotonicity condition that will be needed is

$$\int_S (f(s,\mu)-f(s,\nu))d(\mu-\nu)(s)\ge 0 \tag{4.1}$$

for $\mu,\nu\in\mathcal{P}(S)$. This condition was introduced by Lasry and Lions in their seminal work on mean field games as a way to establish uniqueness of mean field equilibria [32]. In particular, if (4.1) holds strictly for $\mu\ne\nu$, there can be at most one mean field equilibrium.

In order to address both scenarios, we will consider a weak* continuous $G:K\times K\to\mathbb{R}$ such that

$$K\ni\nu\mapsto G(\mu,\nu)\text{ is convex for each }\mu\in K.$$

We'll also say $\mu\in K$ is an *equilibrium* for G provided

$$G(\mu,\mu)\le G(\mu,\nu)$$

for all $\nu\in K$. Moreover, μ is an equilibrium if and only if $0\in\partial_\nu G(\mu,\mu)$. Here

$$\partial_\nu G(\mu,\mu)=\{x\in X: G(\mu,\nu)\ge G(\mu,\mu)+\langle\nu-\mu,x\rangle\text{ for }\nu\in K\}.$$

One checks that an equilibrium for G is a fixed point of the mapping from K into 2^K given by

$$K\ni\sigma\mapsto\operatorname{argmin}\{G(\sigma,\nu):\nu\in K\}.$$

Furthermore, the proof Proposition 3.1 can be adapted to conclude that an equilibrium exists. We leave the details to the reader.

4.1. Monotonicity of *G*. We will say that G is *monotone* provided

$$\langle\mu-\nu,x-y\rangle\ge 0$$

whenever $x\in\partial_\nu G(\mu,\mu)$ and $y\in\partial_\nu G(\nu,\nu)$. As in the case of N-player games, it sometimes is useful to identify a simple sufficient condition for monotonicity. The following lemma can be justified similarly to Proposition 3.3.

LEMMA 4.1. *Suppose*

$$G(\mu,\mu)+G(\nu,\nu)\ge G(\mu,\nu)+G(\nu,\mu)$$

for all $\mu,\nu\in K$. Then G is monotone.

Example 4.2. One of the model cases occurs when $F_1,\dots,F_N$ is symmetric. Here the relevant G function is

$$G(\mu,\nu)=F_1(\nu,\mu,\dots,\mu)$$

for $\mu,\nu\in K$. We note that if the collection $F_1,\dots,F_N$ is additionally monotone, then G is monotone. Using Lemma 4.1, it is also possible to show that a sufficient condition for the monotonicity of G is

$$F_1(\mu,\dots,\mu)+F_1(\nu,\dots,\nu)\ge F_1(\nu,\mu,\dots,\mu)+F_1(\mu,\nu,\dots,\nu)$$

for $\mu,\nu\in K$.

Example 4.3. Let us also briefly consider the case of a static mean field game $f:S\times\mathcal{P}(S)\to\mathbb{R}$. Here

$$G(\mu,\nu)=\int_S f(s,\mu)d\nu(s)\quad (\mu,\nu\in\mathcal{P}(S))$$

is monotone provided (4.1) holds. One example is $f(s,\mu)=\varphi(s)$ for a continuous $\varphi:S\to\mathbb{R}$. Another example is

$$f(s,\mu)=\int_S k(s,t)d\mu(t)$$

for a continuous, symmetric, and nonnegative definite kernel $k:S\times S\to\mathbb{R}$. That is, for any $s_1,\dots,s_N\in S$ and $c_1,\dots,c_N\in\mathbb{R}$,

$$\sum_{i,j=1}^N k(s_i,s_j)c_ic_j\ge 0.$$

It is easy to check that these assumptions imply that f satisfies (4.1).

4.2. Another flow of mixed strategies. For the remainder of this section, we will suppose G is monotone. We will show how to approximate an equilibrium for G. For a given $\mu^0\in K$, we consider the following initial value problem: find an absolutely continuous $\zeta:[0,\infty)\to H$ such that

$$\begin{cases}\mathcal{J}\dot\zeta(t)+\partial_\nu G(\zeta(t),\zeta(t))\ni 0 & \text{a.e. } t\ge 0,\\ \zeta(0)=\mu^0.\end{cases}\tag{4.2}$$

In order to establish the existence of a solution, we will introduce the operator $B:H\to 2^H$ defined by

$$B\mu:=\begin{cases}\mathcal{J}^{-1}\left(\partial_\nu G(\mu,\mu)\right), & \mu\in\mathcal{C},\\ \emptyset, & \mu\notin\mathcal{C}.\end{cases}$$

Here

$$\mathcal{C}=\Big\{\mu\in K:\text{there is }\sigma\in H\text{ with }\mathcal{J}\sigma\in\partial_\nu G(\mu,\mu)\Big\}.$$

We note that $\sigma\in B\mu$ if and only if $\mu\in K$ and $\mathcal{J}\sigma\in\partial_\nu G(\mu,\mu)$.

We can apply Minty's lemma and the Kakutani–Fan–Glicksberg theorem as we did in Lemma 3.6 to conclude that B is maximally monotone. Furthermore, we can apply The Kato–Kōmura and Baillon–Brézis theorems to establish the following theorem as we did in our proof of Theorem 1.1. Again, we leave the details to the reader.

THEOREM 4.4. *Suppose* $\mu^0 \in \mathcal{C}$. *There is a unique Lipschitz continuous* $\zeta : [0,\infty) \to H$ *with* $\zeta(t) \in \mathcal{C}$ *for all* $t \ge 0$ *that satisfies* (4.2). *Furthermore,*

$$\frac{1}{t}\int_0^t \zeta(s)ds$$

converges weak to an equilibrium of* G *as* $t \to \infty$.

Example 4.5. Suppose $F_1, \ldots, F_N$ is monotone and symmetric. In order to approximate a symmetric Nash equilibrium, we can use a solution $\zeta : [0,\infty) \to H$ of

$$\begin{cases} \mathcal{J}\dot{\zeta}(t) + \partial_{\mu_1} F_1(\zeta(t), \ldots, \zeta(t)) \ni 0 \quad \text{a.e. } t \ge 0, \\ \zeta(0) = \mu^0. \end{cases}$$

According to Theorem 4.4, there is a solution whose Cesàro mean converges weak* to a symmetric Nash equilibrium μ provided that

$$\mathcal{J}\sigma \in \partial_{\mu_1} F_1(\mu^0, \ldots, \mu^0)$$

for some $\sigma \in H$.

Example 4.6. Let us consider a static mean field game $f : S \times \mathcal{P}(S) \to \mathbb{R}$ such that (4.1) holds. By Theorem 4.4, there is a Lipschitz path $\zeta : [0,\infty) \to H$ which satisfies

$$\begin{cases} \displaystyle\int_S \Big(\mathcal{J}\dot{\zeta}(t) + f(\cdot, \zeta(t))\Big)\, d(\nu - \zeta(t)) \ge 0 \quad \text{for a.e. } t \ge 0 \text{ and all } \nu \in \mathcal{P}(S), \\ \zeta(0) = \mu^0, \end{cases}$$

and $\zeta(t) \in \mathcal{P}(S)$ for all $t \ge 0$ provided that there is $\sigma \in H$ with $\mathcal{J}\sigma \in \partial_\nu G(\mu^0, \mu^0)$. That is,

$$\int_S (-\mathcal{J}\sigma + f(\cdot, \mu^0))\, d(\nu - \mu^0) \ge 0 \quad \text{for all } \nu \in \mathcal{P}(S).$$

Moreover, the Cesàro mean of ζ converges weak* to a mean field equilibrium as $t \to \infty$.

5. Summary. Nash equilibria in N-player noncooperative games have proved to be quite difficult to approximate. In this note, we proposed a continuous time method to approximate these points under the hypothesis that the game is monotone. Our method involves a continuous flow in the space of mixed strategies and applies to games in which mixed strategies are selected from a dual Banach space. The prototypical space of mixed strategies is the collection of probability measures on a compact metric space.

Our method has two key theoretical components. First, we used a Gaussian measure to embed our space of mixed strategies into a Hilbert space. Then we employed the monotonicity of the game to apply existence and convergence results from theory of semigroups generated by a maximally monotone operator on a Hilbert space. Our main result is that for appropriately chosen initial conditions, the Cesàro mean of solutions to our flow will converge to a Nash equilibrium. In addition, we showed how our results extend to equilibria in symmetric N-player games in the limit as $N \to \infty$.

The Gaussian measure we utilized is a reference measure which is fixed throughout this work. It would be really interesting to deduce whether or not we can choose this measure to influence the approximation method. It would also be of interest to further develop and apply these ideas to approximate other types of equilibria in game theory such as those which arise in the theory of mean field games.

Appendix A. Finite action sets. We will consider a particular Gaussian measure on $X = C(S)$ with

$$S = \{s_1, \dots, s_m\}.$$

These considerations will be used to show how our general theory applies to games with finite action sets. In particular, we will informally argue below that the abstract flows considered in this paper reduce to much simpler flows on finite dimensional spaces.

To this end, it will be convenient to define $e_1, \dots, e_m : S \to \mathbb{R}$ via

$$e_j(s_i) = \delta_{ij} \quad \text{for } i, j = 1, \dots, m.$$

This allows us to represent each $f \in C(S)$ and $\mu \in M(S)$ as

$$f = \sum_{j=1}^{m} f(s_j) e_j \quad \text{and} \quad \mu = \sum_{j=1}^{m} \mu(e_j) \delta_{s_j}.$$

These representations can be used to verify that $C(S)$ is isometrically isomorphic to $\mathbb{R}^m$ endowed with the ∞-norm and that $M(S)$ is isometrically isomorphic to $\mathbb{R}^m$ endowed with the 1-norm. It is also plain to see that $\mu \in \mathcal{P}(S)$ if and only if

$$(\mu(e_1), \dots, \mu(e_m)) \in \Delta_m.$$

We will consider the Borel probability measure γ on $C(S)$ defined as

$$\int_{C(S)} h d\gamma = \int_{\mathbb{R}^m} h\left(\sum_{j=1}^{m} x_j e_j\right) \frac{1}{(2\pi)^{m/2}} e^{-\frac{1}{2}|x|^2} dx$$

for continuous and bounded $h : C(S) \to \mathbb{R}$.

Proposition A.1. *γ is a Gaussian measure. Moreover,*

$$(\mu, \nu) = \sum_{j=1}^{m} \mu(e_j) \nu(e_j) \tag{A.1}$$

for $\mu, \nu \in M(S)$, and

$$\mathcal{J}\nu = \sum_{j=1}^{m} \nu(e_j) e_j \tag{A.2}$$

for $\nu \in M(S)$.

Proof. Suppose $g:\mathbb{R}\to\mathbb{R}$ is bounded and continuous and $c=(c_1,\dots,c_m)\in\mathbb{R}^m\setminus\{0\}$. Observe that

$$\begin{aligned}\int_{C(S)} g\left(\sum_{j=1}^m c_j\delta_{s_j}\right)d\gamma &= \int_{\mathbb{R}^m} g\left(\sum_{j=1}^m c_jx_j\right)\frac{1}{(2\pi)^{m/2}}e^{-\frac{1}{2}|x|^2}dx\\ &= \int_{\mathbb{R}^m} g(c\cdot x)\frac{1}{(2\pi)^{m/2}}e^{-\frac{1}{2}|x|^2}dx\\ &= \int_{\mathbb{R}^m} g(|c|x_1)\frac{1}{(2\pi)^{m/2}}e^{-\frac{1}{2}|x|^2}dx\\ &= \int_{\mathbb{R}} g(|c|x_1)\frac{1}{(2\pi)^{1/2}}e^{-\frac{1}{2}x_1^2}dx_1\\ &= \int_{\mathbb{R}} g(y)\frac{1}{(2\pi)^{1/2}|c|}e^{-\frac{1}{2|c|^2}y^2}dy.\end{aligned}$$

Thus, γ is a Gaussian measure. Also note that

$$\begin{aligned}(\mu,\nu) &= \sum_{i,j=1}^m \mu(e_i)\nu(e_j)\int_{C(S)}\delta_{s_i}\delta_{s_j}d\gamma\\ &= \sum_{i,j=1}^m \mu(e_i)\nu(e_j)\int_{\mathbb{R}^m} x_ix_j\frac{e^{-\frac{1}{2}|x|^2}}{(2\pi)^{m/2}}dx\\ &= \sum_{i,j=1}^m \mu(e_i)\nu(e_j)\delta_{ij}\\ &= \sum_{j=1}^m \mu(e_j)\nu(e_j).\end{aligned}$$

This verifies (A.1). Formula (A.2) follows from (A.1) and the identity (2.3). ∎

Finite action spaces for *N*-player games. Suppose $F_j:\mathcal{P}(S)^N\to\mathbb{R}$ is continuous and that $\nu_j\mapsto F_j(\nu_j,\mu_{-j})$ is convex for each $\mu\in\mathcal{P}(S)^N$ and $j=1,\dots,N$. We wish to express the system

$$\mathcal{J}\dot{\xi}_j(t)+\partial_{\mu_j}F_j(\xi(t))\ni 0 \tag{A.3}$$

$(j=1,\dots,N)$ more concretely. With this goal in mind, we set

$$g_j(x_1,\dots,x_N):=F_j\left(\sum_{k=1}^m x_{1,k}\delta_{s_k},\dots,\sum_{k=1}^m x_{N,k}\delta_{s_k}\right)$$

for $x_i=(x_{i,1},\dots,x_{i,m})\in\Delta_m$ and $i=1,\dots,N$. We note that g_j is continuous and that $y_j\mapsto g_j(y_j,x_{-j})$ is convex for each $x\in\Delta_m^N$ and $j=1,\dots,N$.

It is not hard to see that if $\mu_i=\sum_{k=1}^m x_{i,k}\delta_{s_k}$ for $x_i\in\Delta_m$ and $i=1,\dots,N$, then

$$\sum_{k=1}^N z_{j,k}e_k\in\partial_{\mu_j}F_j(\mu)\text{ if and only if }z_j\in\partial_{x_j}g_j(x).$$

It follows that the system (A.3) is equivalent to

$$\dot{u}_j(t)+\partial_{x_j}g_j(u(t))\ni 0$$

($j=1,\dots,N$) for $u:[0,\infty)\to\Delta_m^N$. That is,

$$\xi_j(t)=\sum_{k=1}^m u_{j,k}(t)\delta_{s_k}$$

would solve (A.3) and vice versa. We finally note that the collection $F_1,\dots,F_N$ is monotone if and only if

$$\sum_{j=1}^N (x_j-y_j)\cdot(z_j-w_j)\ge 0$$

whenever $z_j\in\partial_{x_j}g_j(x)$ and $w_j\in\partial_{x_j}g_j(y)$ for $j=1,\dots,N$.

Finite action spaces in static mean field games. If $f:S\times\mathcal{P}(S)\to\mathbb{R}$ is continuous, then

$$g_j(x):=f\left(s_j,\sum_{i=1}^m x_i\delta_{s_i}\right)\quad (x\in\Delta_m)$$

is continuous for each $j=1,\dots,m$. We aim to reinterpret the condition

$$\int_S\left(\mathcal{J}\dot\zeta(t)+f(\cdot,\zeta(t))\right)\,d(\nu-\zeta(t))\ge 0\quad\text{ for }\nu\in\mathcal{P}(S) \tag{A.4}$$

in terms of $g_1,\dots,g_m$.

Observe that if

$$\nu=\sum_{j=1}^m y_j\delta_{s_j}\quad\text{and}\quad \zeta(t)=\sum_{j=1}^m u_j(t)\delta_{s_j},$$

then

$$\begin{aligned}\int_S\left(\mathcal{J}\dot\zeta(t)+f(\cdot,\zeta(t))\right)\,d(\nu-\zeta(t))&=\sum_{j=1}^m(\dot u_j(t)+g_j(u(t)))(y_j-u_j(t))\\&=(\dot u(t)+g(u(t)))\cdot(y-u(t)).\end{aligned}$$

Here we have written $g=(g_1,\dots,g_m)$. As a result,

$$(\dot u(t)+g(u(t)))\cdot(y-u(t))\ge 0\quad\text{ for } y\in\Delta_m.$$

In particular, this evolution is equivalent to (A.4). Finally, we note that f is monotone in the sense of (4.1) if and only if

$$(g(x)-g(y))\cdot(x-y)\ge 0$$

for $x,y\in\Delta_m$.

Appendix B. An explicit example. We will work out an example which suggests Cesàro mean convergence is the best one may expect from the type of flows considered in this article. Let us assume that $N=2$ and the cost functions are

$$\begin{cases}F_1(x_1,x_2)=3x_{1,1}x_{2,1}+x_{1,2}x_{2,1}+4x_{1,2}x_{2,2},\\ F_2(x_1,x_2)=-3x_{1,1}x_{2,1}-x_{1,2}x_{2,1}-4x_{1,2}x_{2,2}\end{cases}$$

for $x_i = (x_{i,1}, x_{i,2}) \in \Delta_2$, for $i = 1, 2$. Note this is a zero-sum game and $\Delta_2 \subset \mathbb{R}^2$, where $\mathbb{R}^2$ is equipped with the standard dot product. It is not hard to check that the unique Nash equilibrium for F_1, F_2 is the pair

$$((1/2, 1/2), (2/3, 1/3)) \in \Delta_2^2.$$

Evolution inequalities. The corresponding flow (1.3) takes the form

$$\begin{pmatrix} \dot{u}_{1,1}(t) + 3u_{2,1}(t) \\ \dot{u}_{1,2}(t) + u_{2,1}(t) + 4u_{2,2}(t) \end{pmatrix} \cdot \begin{pmatrix} z_{1,1} - u_{1,1}(t) \\ z_{1,2} - u_{1,2}(t) \end{pmatrix} \geq 0$$

and

$$\begin{pmatrix} \dot{u}_{2,1}(t) - 3u_{1,1}(t) - u_{1,2}(t) \\ \dot{u}_{2,2}(t) - 4u_{1,2}(t) \end{pmatrix} \cdot \begin{pmatrix} z_{2,1} - u_{2,1}(t) \\ z_{2,2} - u_{2,2}(t) \end{pmatrix} \geq 0$$

for almost every $t \geq 0$ and each $z_1, z_2 \in \Delta_2$. The unknown is an absolutely continuous path $u : [0, \infty) \to \Delta_2^2$, where $u(t) = (u_1(t), u_2(t))$.

If we put

$$v_1(t) = u_{1,1}(t),\ v_2(t) = u_{2,1}(t),\ w_1 = z_{1,1}, \text{ and } w_2 = z_{2,1},$$

we can re-express the above inequalities as

$$\begin{pmatrix} \dot{v}_1(t) + 3v_2(t) \\ -\dot{v}_1(t) + v_2(t) + 4(1 - v_2(t)) \end{pmatrix} \cdot \begin{pmatrix} w_1 - v_1(t) \\ -(w_1 - v_1(t)) \end{pmatrix} = (2\dot{v}_1(t) + 6v_2(t) - 4)(w_1 - v_1(t)) \geq 0$$

and

$$\begin{pmatrix} \dot{v}_2(t) - 3v_1(t) - (1 - v_1(t)) \\ -\dot{v}_2(t) - 4(1 - v_1(t)) \end{pmatrix} \cdot \begin{pmatrix} w_2 - v_2(t) \\ -(w_2 - v_2(t)) \end{pmatrix} = (2\dot{v}_2(t) + 3 - 6v_1(t))(w_2 - v_2(t)) \geq 0.$$

Therefore, our initial value problem is equivalent to finding an absolutely continuous pair $v_1, v_2 : [0, \infty) \to [0, 1]$ which satisfies

$$\begin{cases} (\dot{v}_1(t) + 3v_2(t) - 2)(w_1 - v_1(t)) \geq 0, \\ (\dot{v}_2(t) + 3/2 - 3v_1(t))(w_2 - v_2(t)) \geq 0 \end{cases} \tag{B.1}$$

for each $w_1, w_2 \in [0, 1]$ and given initial conditions

$$v_1(0) = v_1^0 \in [0, 1] \quad \text{and} \quad v_2(0) = v_2^0 \in [0, 1]. \tag{B.2}$$

Solution which parametrizes a circle. Observe that the solution of the system of ODEs

$$\dot{v}_1(t) + 3v_2(t) - 2 = 0 \text{ and } \dot{v}_2(t) + 3/2 - 3v_1(t) = 0$$

subject to the initial conditions (B.2) is

$$\begin{cases} v_1(t) = (v_1^0 - 1/2)\cos(3t) + (2/3 - v_2^0)\sin(3t) + 1/2, \\ v_2(t) = (v_2^0 - 2/3)\cos(3t) + (v_1^0 - 1/2)\sin(3t) + 2/3. \end{cases} \tag{B.3}$$

In particular, this solution parametrizes the circle

$$(v_1 - 1/2)^2 + (v_2 - 2/3)^2 = (v_1^0 - 1/2)^2 + (v_2^0 - 2/3)^2$$

counterclockwise in the v_1v_2 plane. It is easily checked that if

$$(v_1^0 - 1/2)^2 + (v_2^0 - 2/3)^2 \leq (1/3)^2, \tag{B.4}$$

then $v_1(t), v_2(t) \in [0,1]$ for all $t \geq 0$. In this case, the circular path (B.3) solves (B.1) and (B.2).

Convergence. Observe that since the path (B.3) lies on a circle centered at $(1/2, 2/3)$, it will not converge to the circle's center as $t \to \infty$. However, it's plain to see that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t v_1(s)ds = \frac{1}{2} \quad \text{and} \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t v_2(s)ds = \frac{2}{3}.$$

As a result, when (B.4) holds, the solution of (B.1) does not converge to the Nash equilibrium of F_1, F_2, but its Cesàro mean does.

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