

Optimal Regulation in the Presence of Reputation Concerns*

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Abstract

In all markets, firms go through a process of creative destruction: entry, random growth and exit. In many of these markets there are also regulations that restrict entry, possibly distorting this process. We study the public interest rationale for entry taxes in a general equilibrium model with free entry and exit of firms in which firm dynamics are driven by reputation concerns. In our model firms can produce high-quality output by making a costly but efficient initial unobservable investment. If buyers never learn about this investment, an extreme “lemons problem” develops, no firm invests, and the market shuts down. Learning introduces reputation incentives such that a fraction of entrants do invest. We show that, if the market operates with spot prices, entry taxes always enhance the role of reputation to induce investment, strengthening reputation in mitigating “lemons problems” and improving welfare despite the impact of these taxes on equilibrium prices and market size.

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1 Introduction

Creative destruction in capitalist economies is ubiquitous: even within narrowly defined industries, there is a continual process of churning through which heterogeneous firms enter, undergo random growth, and exit. At the same time, as documented by Djankov et al. (2002), attempts by governments to regulate this process of creative destruction with entry barriers are also ubiquitous. Is there a public interest rationale for this regulation of entry? And if so, what are the foundations of this rationale?

One leading view is that firm dynamics are driven by firms' heterogeneous productivities and that, rather than improving welfare, attempts to regulate the process of firm entry, growth, and exit are important impediments to aggregate productivity and welfare.¹ An alternative view is that firm dynamics are driven by buyers learning about the quality of the product or service being provided by young firms. In this view, information frictions play a key role: new firms enter small because demand for their product or service is held back by buyers' uncertainty about their quality and young firms' growth is driven by the dynamics of their reputation in the market.²

Learning about sellers' quality is indeed critical in almost all markets, where buyers interact with firms in spot markets and it is not usually feasible to provide appropriate or timely compensation for damages that poor quality products generate in terms of health, safety hazards, irreversibilities in consumption, opportunity losses, etc. Examples widely range from food, drugs, durable goods, to professional services, such as doctors, lawyers, financial advisors, etc. Even in markets where firms guarantee their products against defects, it is still costly for buyers to use warranties (it is not always clear which defects they cover, it takes time and resources to replace damaged goods, etc) and they prefer to pay higher prices for products from reputable firms rather than buy from firms with worse reputations and risk having to use warranties.

¹Hopenhayn (1992), Melitz (2003), and Luttmer (2007) present models of the process of creative destruction driven by the dynamics of heterogeneous firm productivities. Herrendorf and Teixeira (2011) examine the welfare costs of entry regulation in such a model. Hopenhayn and Rogerson (1993), Parente and Prescott (1999), Hsieh and Klenow (2009), Restuccia and Rogerson (2008), and Fattal Jaef (2013) study the welfare costs of alternative regulatory interventions into the process of creative destruction.

²Foster, Haltiwanger, and Syverson (2012) present evidence that demand rather than supply factors drive firm and plant dynamics. Klein and Leffler (1981), Bar-Isaac (2003), Board and Meyer-ter-Vehn (2010 and 2012), and Vial and Zurita (2012) present related models of firm dynamics driven by reputational considerations.

In this paper, we present a simple benchmark general equilibrium model of firm dynamics driven by reputational considerations and use this model to analyze the welfare implications of regulatory intervention into the process of firm entry, growth, and exit. We are motivated to do so to understand the Pigouvian or “public interest” foundations of the pervasive use by governments of regulation of firms entry into new markets with the stated aim of ensuring the “quality” of the goods offered in the market. We find that while reputational considerations mitigate the problem of adverse selection in equilibrium in an unregulated market, as a general matter, the use of entry fees or taxes to regulate firm entry is welfare enhancing.

In the model, a firm can produce high quality output only if it makes a costly initial investment in quality upon entering the market. Firms can also enter the market with low quality without making this initial investment, but the shoddy output of these low quality firms detracts from, rather than adds to, social welfare. In equilibrium, firms invest in quality only if they expect to recoup the investment costs through subsequent quasi-rents from selling their goods to buyers in spot markets at a price above marginal cost.

If buyers can observe firms’ investment and quality, then the equilibrium is fully efficient, as in Hopenhayn (1992). However, we assume that buyers cannot observe firms’ investments in quality upon entry and information about young firms’ quality diffuses only gradually over time. The entry of high quality firms in the market then necessarily also attracts some low-quality competitors who seek to extract information rents by pooling with their high-quality peers, at least initially, until their true quality becomes known. Thus, in our model, entry by low quality firms induces a lemons problem that impedes trade and reduces the quasi-rents high quality firms can extract, thus distorting investment incentives and firm entry. The laissez-faire reputational equilibrium outcome thus leads to lower welfare than the unconstrained first-best with perfect information because (i) there is lower average quality and (ii) there is a smaller overall market.

How do regulatory entry costs impact welfare in equilibrium? The Pigouvian argument in favor of entry regulation in the literature is that entry costs enhance the firms incentives to invest in quality, and discourage them from entering with low-quality products.³ But in the scenario described above, firms are already given some repu-

³See, for example, Klein and Leffler (1981) and Shapiro (1983 and 1986).

tational incentives to invest in quality, because they can anticipate that information about their quality will gradually diffuse over time, and hence their investment in quality will eventually be recognized and rewarded by the market. Thus, regulatory entry fees may not be needed to enhance quality. Moreover, in general equilibrium entry fees also have a social cost in that they raise equilibrium prices further above the marginal cost of production so as to compensate entering firms for the additional regulatory costs. Thus entry fees can reduce entry, market size and the consumer surplus generated from investment in quality. Our model allows us to explore precisely this interaction between reputational incentives and entry regulations at the micro level, and the resulting welfare tradeoff between enhancing quality and distorting market size in the aggregate.

We develop two principal results on the welfare enhancing role of regulation in this context. First, we show that very simple schemes of taxes and subsidies can implement an allocation arbitrarily close to the unconstrained first-best without adverse selection, even without altering the buyers information about product quality. The planner can use two instruments, entry fees and sale subsidies (which are subsidies to reputation), to target the two sources of inefficiencies. Through the appropriate design of these two instruments, the planner can simultaneously eliminate entry of low quality firms almost completely, while encouraging entry of high quality firms at an efficient scale.

Second, and more important, we show that even when taxes on firms' entry is the only available regulatory tool to target the two sources of inefficiency, under fairly general conditions, the optimal level of these taxes is strictly positive. The positive impact that regulatory taxes on entry have on improving the average quality of new entrants outweighs the negative impact of these taxes on the equilibrium gap between price and marginal cost, consumer surplus, and market size. Indeed for many signal structures governing the diffusion of information about sellers' qualities to buyers, in general equilibrium, positive but small entry costs both relax the lemons problem, which improves quality, and reduce the gap between price and marginal cost, which increases market size, thus leading to a first-order increase in social welfare along both dimensions.

One can gain intuition for these results and the main forces at play as follows. In equilibrium, the quasi-rents earned by a firm depend on its current reputation (which governs the price it can charge buyers for its product relative to the prices charged by

other firms in the market) and the overall number of firms in the market (which governs the overall level of prices that firms can charge holding fixed their reputation). In a steady state equilibrium with free-entry, the initial reputation of entering firms and the volume of entry adjust so that the discounted present value of quasi-rents anticipated by both high and low quality entrants cover their respective entry costs; both the cost of an investment in quality and regulatory entry fees for high quality firms and just regulatory entry fees for low quality firms.

When a regulatory entry fee is introduced, this increases the revenues a low quality firm must earn in order to find entry profitable, and allows the planner to recoup the information rents extracted by low quality firms. But higher rents for low quality entrants are possible only if there are relatively fewer of them, and they are entering with a higher initial reputation. This confirms the Pigouvian argument that entry fees indeed enhance the quality of entering firms.

But they also discourage entry by good quality firms, who must be compensated through an increase in quasi-rents to recoup the additional costs of entry fees. Through the price subsidy, the regulator can align the private and social returns of entering firms, and implement the optimal market scale, while using an entry fee to almost completely eliminate entry by low quality firms.⁴

On the other hand, when entry fees are the only regulatory instruments, their use results in a potential tradeoff between enhancing quality and restricting the number of firms in the market. It thus comes perhaps as a surprise that even without subsidies, the introduction of sufficiently small entry fees is always welfare-enhancing, and in many cases also increases further entry rather than discouraging it. This last effect however depends on the specific nature of how information about an entrant diffuses over time.

Specifically, we consider three different scenarios: a “good news” scenario, a “bad news” scenario, and a “Brownian diffusion” scenario. For each scenario, we solve

⁴We interpret this result as indicating that the lemons problem due to buyers’ lack of information about quality investments is, in fact, a problem of a lack of commitment on the part of buyers rather than a problem of incomplete information. With long term matching between sellers and buyers, the first best allocation can be achieved with bilateral contracts in which buyers commit to backload payments amounting to quasi-rents for the matched firm if and when it is revealed to be high quality. In our model, we assume that each pair of buyers and sellers interact only once in spot markets. A combination of entry fees and production subsidies serves to implement the same socially optimal allocation that firms and buyers could achieve if they were able to commit to long-term contracts. MacLeod (2007) also presents results along these lines.

for entry conditions for high- and low-quality firms analytically, making the welfare comparisons across different regulation policies tractable.

In the bad news scenario, a low quality product is eventually revealed to be flawed with a constant Poisson arrival rate. In this scenario, firms enter small, grow gradually as their reputations rise as they accumulate a performance record with no failures, and low quality firms exit suddenly once the first failure occurs. There is no tradeoff between enhancing quality and distorting market size: increasing entry fees always unambiguously raise steady-state quality at entry, volume of entry, and steady-state welfare.

In the Brownian diffusion scenario, positive or negative public signals about a firm's quality gradually accumulate over time. In this case, firms enter small and experience random growth, with high quality firms having a higher average growth rate and low quality firms exiting as they hit some lower size threshold. In this Brownian diffusion scenario, as in the bad news scenario, for low entry fees there is no welfare tradeoff, both steady-state quality and volume of entry increase with fees. But now, once the entry fee is sufficiently large, it starts to reduce the volume of entry and market size by high quality firms and hence there is an optimal finite level of entry fees. In both the bad news and Brownian diffusion scenarios, at least for low levels, an entry fee unambiguously raises both average quality and market size.

Finally, the good news scenario considers a case where a good product is revealed to be good at a constant Poisson arrival rate. In this scenario, firms enter small and either fail to grow or, in fact, shrink until a success occurs revealing that a firm is high quality, leading to a big increase in that firm's size. In the good news scenario, with regulatory entry fees, there is an immediate tradeoff between quality and market size: the imposition of a fee raises the quality of entrants but reduces the volume of entry. But we show that at least initially, the benefits of enhanced quality outweigh the costs of restricted market size, and hence steady-state welfare improves. Thus in all three cases, positive entry fees are welfare-enhancing, and in the first two cases, they lead to gains in both average quality and market size.

What accounts for the difference between these cases? An increase in the reputation at entry has two effects, (i) it speeds up learning about high quality firms, encouraging their entry and (ii) it delays exit of low-quality firms, also encouraging their entry. If an increase in the reputation at entry benefits high quality entrants disproportion-

ately more than low quality entrants, then overall prices may not need to increase to maintain incentives, and there may even be room for a decline in overall prices without adverse incentive effects. In contrast, if an increase in the reputation at entry benefits low quality entrants disproportionately more than high quality entrants, then the overall level of prices must increase to restore entry incentives, which leads to a tradeoff between the direct effect on average quality and the indirect effect on market size.

The stochastic process by which information diffuses governs the relative strength of these two effects. In the bad news scenario, an increase in the reputation at entry does not slow down exit of low-quality firms but speeds up learning about high-quality firms, benefiting disproportionately the latter and unequivocally improving welfare. In contrast, in the good news scenario, an increase in the reputation at entry slows down exit of low-quality firms and does not speed up learning about high-quality firms, benefiting disproportionately the former and generating a trade-off between average quality and market size. Finally, the Brownian diffusion scenario is a combination of both, mirroring results for the bad news case at low levels of regulations, and the good news case at high levels. An increase in the reputation at entry initially slow down exit of low-quality firms less than what it speeds up learning about high-quality firms, benefiting disproportionately the later, but for high levels of entry fees these effects reverse and an increase in the reputation at entry benefits disproportionately low-quality firms, inducing a trade-off between average quality and market size.

Our analysis connects two mechanisms that mitigate firms' incentives to engage in opportunistic behavior: reputation and regulation. There is a rich literature studying each of these mechanisms in isolation, but to our knowledge they have so far not been systematically connected. In combining these mechanisms, our paper shows how regulatory interventions can be used to leverage reputational incentives.

The literature on reputation concerns, surveyed recently in MacLeod (2007), interprets reputation as a valuable asset that the firm may lose if it is found to act opportunistically (Mailath and Samuelson (2001); Tadelis, (1999 and 2002)). In these models firms differ by an unobservable exogenous characteristic. The firm's reputation is defined as outsider's beliefs about this characteristic, and is updated based on signals about the firm's performance.

We contribute to this literature in two important dimensions. First, we embed the analysis of reputation dynamics in a general equilibrium context with endogenous entry and exit of firms. In our model, firms are free to enter or exit the market, and their initial investment determines the expected quality of their product.⁵ The firm's incentive and participation conditions then determine the number of high and low quality entrants, as well as their respective rates of exit, and the resulting firm dynamics determine the level of high- and low-quality firms (and their respective reputations) in general equilibrium. The endogenous entry margin allows us to fully endogenize the severity of the adverse selection problem. The general equilibrium context in turn allows us to study the resulting trade-off between market size and the quality of firms operating in the market.

Second, we offer two technical contributions to the existing characterizations of reputation dynamics. First, we show that under natural restrictions on buyer beliefs the equilibrium is unique as the model approaches continuous time.⁶ We then fully characterize value functions in the continuous time limit, when firms know their type and have the option to exit.⁷ With this analytical characterization, we are able to calculate how the ratio and difference of the expected discounted present value of quasi rents for low and high quality firms changes with the initial reputation of entrants for different stochastic signal processes, which characterizes the impact of regulatory entry costs on welfare.

The literature on regulation emphasizes how entry costs can be welfare improving in settings with adverse selection. In Leland (1979), extended later by Shaked and Sutton (1981) and Shapiro (1983 and 1986) entry costs can increase welfare as measured by a utilitarian social welfare function in settings with investment decisions and asymmetric information if there is enough heterogeneity in how consumers value quality. But these are not Pareto improvements since those consumers who prefer low quality are harmed. Similarly, Garcia-Fontes and Hopenhayn (2000) show entry restric-

⁵While the initial models considered exit to be exogenous, Hörner (2002), Bar-Isaac (2003), and Daley and Green (2012) introduce endogenous exit of firms, when these firms know their own type. None of these papers consider endogenous entry, or other investments that would fully endogenize firm reputations.

⁶This differs from a prior uniqueness result in Bar-Isaac (2003), which relied on a perturbation of exit strategies in discrete time.

⁷Prat and Alos-Ferrer (2010) solve similar value functions but without endogenous exit. Papers of reputation in continuous time are Board and Meyer-ter Vehn (2010) and Faingold and Sannikov (2011); however, they do not consider entry and exit decisions.

tions can improve welfare by their effects on the market size, but because the pool of consumers, heterogenous in their preferences, change. In our case, there is a representative consumer and the effects of information frictions and regulation on welfare come from their impact on aggregate production. Hence, in our case, we analyze the potential for Pareto improvements through regulation of entry.

Prescott and Townsend (1984) and Arnott, Greenwald, and Stiglitz (1993) discuss whether government interventions can be Pareto improving in a world of adverse selection and moral hazard, even if they cannot directly correct these information imperfections. We contribute to this discussion by showing that the market outcome with spot trade between producers and buyers is not constrained Pareto optimal. This result, however, does not arise from information asymmetry per se, but from the fact that buyers and sellers are unable to commit to dynamic contracts that generate prices that are different from spot prices. Our paper shows that government interventions are Pareto improving if the private sector cannot reproduce the commitment that a government can replicate with very simple taxes and subsidies. This, of course, does not preclude the possibility of other market-based solutions to the commitment problem through, e.g., longer-term contracts, back-loading of payments, posting of bonds, or market-provided intermediation and certification services.

Finally, Klein and Leffler (1981, p. 168) propose money burning as a way to align incentives. In contrast to their work, we focus on regulatory entry fees, not only because money burning is a social waste but also because it constitutes a focal point in beliefs, so there is no way to ensure that its amount in equilibrium is optimal. Furthermore, even though Klein and Leffler (1981) briefly mention in their conclusion that regulation may replace implicit contracts, we show regulation indeed complements implicit contracts.

In the following section, we describe the economy and characterize the spot market equilibrium for two extreme benchmarks: full information and no learning. In Section 3 we characterize the spot market equilibrium in steady-state with imperfectly informative signals. In Section 4 we study the role of regulation in improving welfare relative to a spot market economy under two settings: one where the regulator can observe entry and transactions, and another where the regulator can only observe entry. In Section 5 we make some final remarks. The Appendix contains the proofs.

2 The Model

In this section, we describe the economic environment, define a spot market equilibrium with regulation, and characterize the socially optimal allocation under full information. We next show that under full information, the spot market equilibrium without regulation implements the socially optimal allocation. Finally, we define the three signal structures driving the dynamics of reputation that we consider: good news, bad news, and Brownian diffusion.

2.1 The Economy

Time is discrete with time periods numbered $t = 0, 1, 2, \dots$. We denote the length of a time period in calendar time by Δ . For some calculations, we will consider the limit as Δ goes to zero.

At each time t , consumers in this economy derive utility from the consumption of two final goods: one that we term the *experience good* and one that we term the *numeraire good*. Let Y_t denote consumption of the experience good and N_t consumption of the numeraire good at t . Consumers' utility is given by

$$\sum_{t=0}^{\infty} \exp(-r\Delta t) [U(Y_t) + N_t] \Delta, \quad (1)$$

where $U' > 0$, $U'' < 0$, and r is the discount factor.

At each time t , there is an endowment of Δ units of the numeraire good (or one unit of this good per unit of calendar time). This numeraire good is not storable. The experience good is produced with a constant returns to scale technology that uses produced intermediate goods as the only inputs.

At each point in time t , there is a stock of incumbent firms in the economy, each with a capacity to produce Δ units of the intermediate good (or one unit of the good per unit of calendar time) at zero marginal cost for as long as that firm remains active. Each period, an intermediate good firm can become inactive for exogenous reasons or endogenously by the owner's decision. Firms that become inactive at t cannot return to production at later dates. We refer to the removal of active firms from production as *exit*.

Intermediate good firms can be one of two types, high-quality (H) or low-quality (L), depending on an initial investment made when the firm enters production. To create a high-quality firm at $t + 1$, an investment of C units of the numeraire good is required at t . Low-quality firms can be created at zero cost in any period. We refer to the creation of new intermediate good firms as *entry*.

The quality of the firm producing Δ units of the intermediate good per period determines the expected productivity of those units of the intermediate good in use as an input to produce the experience good. One unit of the intermediate good from a high-quality firm contributes $y(1) > 0$ units of output of the experience good at the margin, whereas one unit of output from a low-quality firm yields $y(0) < 0$ units of output of the experience good at the margin.⁸

Aggregate production of the experience good is then given by

$$Y_t = y(1)m_{Ht} + y(0)m_{Lt}, \quad (2)$$

where m_{Ht} is the measure of active high-quality firms at t and m_{Lt} is the corresponding measure of active low-quality firms.⁹

We denote the measure of new firms entering at t by $m_t^e \Delta \geq 0$. The fraction of those entrants who invest to become high-quality is denoted $\phi_t^e \in [0, 1]$. The corresponding resource constraint for the numeraire good is

$$N_t = 1 - C\phi_t^e m_t^e, \quad (3)$$

⁸The assumptions of zero marginal cost of production for the intermediate good and a negative marginal product of low-quality intermediate goods are normalizations that simplify the exposition, but not affect the main results.

⁹Our model of firms of uncertain quality providing inputs for the production of the experience good simplifies the computation of equilibrium and provides a straightforward measure of social welfare based on the consumer surplus of a representative household. Models of competition among firms producing differentiated products with this structure are common in the literature in international trade. This formulation allows us to consider an economy in which the marginal valuation of an additional unit of the experience good is decreasing in the aggregate quantity of the experience good produced while still proving tractable. In particular, we are able to abstract from the matching issues that would arise if individual producers of the experience good (with different reputations) sold a constrained quantity to individual consumers with diverse valuations of that good. By modeling the reputational good as an intermediate good, we are able to model purchasers of that good act who act as if they have a common valuation and are risk neutral (have constant marginal valuations of an additional unit of the reputational good) while still having the final consumer have diminishing marginal utility for aggregate output of that good. We also believe that this formulation is empirically relevant for many industries with supplier - final producer relationships.

where $C\phi_t^e m_t^e$ are the resources invested in creating high-quality firms at t .

Let ϕ denote the public belief regarding the probability that a given firm is high-quality. We refer to ϕ as the firm's *reputation*. The expected output of the experience good obtained from a unit of the intermediate good from a firm with reputation ϕ is denoted $y(\phi)$ and is given by the affine function

$$y(\phi) = \phi y(1) + (1 - \phi)y(0). \quad (4)$$

We denote the probability of continuation of a firm of quality $i = \{L, H\}$ and reputation ϕ_t at t by $\omega_{it}(\phi) \in [0, \exp(-\delta\Delta)]$, where $\delta > 0$ is the exogenous exit rate. (This probability is the complement of the probability that this firm will exit.)

The timing of events within a period is as follows. At the beginning of period t , firms that are incumbent from the previous period start with initial reputation designated by ϕ_t . These firms choose rates at which to continue $\omega_{it}(\phi)$ or exit and new firms enter, either investing in quality or not. A firm that enters at t starts production with reputation ϕ_t^e but does not produce until period $t+1$ and does so only if it survives the exogenous exit shock (with probability $\exp(-\delta\Delta)$). Buyers form *interim beliefs* $\phi_t^c(\phi)$ ($\phi^c : [0, 1] \rightarrow [0, 1]$) about the quality of those incumbent firms that started the period with reputation ϕ and that continue. Trade occurs at these interim beliefs, with the owners of active firms selling the output of those firms to producers of the experience good. After this trade, for each active firm, a *good* or *bad* public signal is realized. We let $\alpha_i(\Delta)$ denote the probability that an active firm of type $i = \{H, L\}$ generates a good signal. These signals lead to updating of reputations to ϕ_{t+1} to start period $t+1$ for all firms that were active or new entrants in period t .

The evolution of reputation for a firm is governed by Bayes' rule where applicable. A buyer who contemplates purchasing the output from a firm that began period t with reputation ϕ and that continues to operate in period t has interim beliefs about the quality of the firm consistent with firms' strategies for continuation

$$\phi_t^c(\phi) = \frac{\phi\omega_{Ht}(\phi)}{\phi\omega_{Ht}(\phi) + (1 - \phi)\omega_{Lt}(\phi)}, \quad (5)$$

where this expression is well defined.

Likewise, the updating of reputations given signals after trade takes place is also

governed by Bayes' rule. After a firm that started in period t with reputation ϕ_t operates and generates a signal, it starts period $t + 1$ with reputation ϕ_{t+1} given by either $\phi^g(\phi^c(\phi_t))$ or $\phi^b(\phi^c(\phi_t))$ depending on whether a good or a bad signal is realized. The functions ϕ^g and ϕ^b are defined by

$$\phi^g(\phi^c) = \frac{\phi^c \alpha_H(\Delta)}{\phi^c \alpha_H(\Delta) + (1 - \phi^c) \alpha_L(\Delta)} \quad (6)$$

and

$$\phi^b(\phi^c) = \frac{\phi^c(1 - \alpha_H(\Delta))}{\phi^c(1 - \alpha_H(\Delta)) + (1 - \phi^c)(1 - \alpha_L(\Delta))}. \quad (7)$$

At the beginning of each period t , there is a measure of reputations across high-quality firms $\nu_{Ht}(\phi)$ and across low-quality firms $\nu_{Lt}(\phi)$. Production is determined by the extent of trade. Hence, for an allocation to be feasible, we must have

$$m_{Ht} = \exp(-\delta\Delta) \phi_{t-1}^e m_{t-1}^e \Delta + \int_{\phi} \omega_{Ht}(\phi) d\nu_{Ht}(\phi), \quad (8)$$

$$m_{Lt} = \exp(-\delta\Delta) (1 - \phi_{t-1}^e) m_{t-1}^e \Delta + \int_{\phi} \omega_{Lt}(\phi) d\nu_{Lt}(\phi). \quad (9)$$

The evolution of the measures of reputations across high- and low-quality firms $\nu_{it}(\phi)$ from one period to the next is determined in the standard way. First, firms continuation strategies $\omega_{it}(\phi)$ reduce the measure of incumbent firms by reputation through exit. Second, the reputations of those firms that continue are updated according to buyers' interim beliefs $\phi_t^c(\phi)$, and a measure $\exp(-\delta\Delta) m_{t-1}^e \Delta$ of firms enters production at t with fraction ϕ_{t-1}^e of those firms being high-quality and $(1 - \phi_{t-1}^e)$ low-quality. Finally, trade occurs, good or bad public signals are observed, and both incumbent and entrant firms' reputations are updated according to Bayes' rule in (6) and (7).¹⁰

An *allocation* in this environment is a sequence of consumption of the experience and numeraire goods for the representative household $\{Y_t, N_t\}$, rates of entry of firms and initial reputations for entrants $\{m_t^e, \phi_t^e\}$, buyers' interim beliefs $\{\phi_t^c(\phi)\}$, and for

¹⁰Note that given our assumptions that time is discrete, that the number of signals is finite, and that there is a strictly positive exogenous death rate of firms, then the measures $\nu_{Ht}(\phi)$ and $\nu_{Lt}(\phi)$ can actually be represented as vectors in R^∞ since the support of reputations ϕ for any cohort of entering firms is countable. Moreover, as long as the sequence $\{m_{t-k}^e\}_{k=0}^\infty$ is uniformly bounded, then these measures will also have a finite sum. Hence, under these assumptions, the integrals in the expressions (8) and (9) are simple sums and hence are well defined.

$i = \{L, H\}$, continuation strategies $\{\omega_{it}(\phi)\}$, reputational distributions $\{\nu_{it}(\phi)\}$, and corresponding measures of active high- and low-quality firms $\{m_{it}\}$.

An allocation is *feasible* if it satisfies the final good resource constraints (2) and (3), the constraints on the evolution of reputation measures implied by continuation strategies, buyers' interim beliefs, the rates of entry and initial reputations for entrants, and the signals stated above, and the stocks of high- and low-quality firms satisfy (8)-(9).

2.2 Spot Market Equilibrium with Regulation

We now consider the equilibrium allocation in a market in which the owners of firms sell the intermediate goods obtained as output from their firms to producers of the experience good. Within this market we also allow for the use of two simple regulatory tools: (i) a tax $F \geq 0$ (in terms of the numeraire good) that is imposed on entrants into intermediate goods production, and (ii) a subsidy s per unit of the experience good purchased. We assume the net proceedings of these taxes and subsidies are rebated lump sum to consumers or obtained lump sum from consumers.

We assume that the experience and numeraire final goods and the intermediate goods are all transacted at spot prices in each period t . Hence, at each time t , experience good producers buy intermediate goods at spot market prices $p_t(\phi)$ that depend on the reputation of the intermediate firm selling. This spot market price is equal to the expected value marginal product of the intermediate good sold when used in production of the experience good, with expectations based on the reputation of the selling firm.¹¹ This expected value marginal product has two components: the relative price of the experience good with respect to the numeraire good, denoted by P_t , and the expected marginal product of the intermediate good from a firm with a given reputation, $y(\phi)$ from equation (4). Thus, the *spot market price* at t in units of the numeraire good, for a unit of the intermediate good from a firm that is believed to be of high-quality with probability ϕ , is given by

$$p_t(\phi) = y(\phi)P_t. \quad (10)$$

In what follows, we focus on *steady-state spot market equilibrium*, a spot market equi-

¹¹We assume that competition between experience good producers as buyers of the intermediate good drives the spot market price to their reservation value.

librium in which all prices and quantities are constant over time. To keep the notation simple, we suppress the time subscript. If $p(\phi)$ is the steady-state spot market price for intermediate goods based on reputation and P is the steady-state price of the experience good, we find it useful to directly use prices normalized by the price of the experience good. From the previous two equations, these normalized prices are just the expected marginal products of the intermediate good, $y(\phi)$. Given that firms produce Δ units of the intermediate good at zero marginal cost, $y(\phi)\Delta$ corresponds to the flow of normalized profits from an active firm with reputation ϕ at t .

In the next lemma, we show that in a steady-state spot market equilibrium, given buyers' interim beliefs $\phi^c(\phi)$, the discounted expected value of the profits earned by a firm of quality $i \in \{H, L\}$ and reputation ϕ can be characterized by a Bellman equation in which the normalized profits $y(\phi)\Delta$ are the current reward. We denote the fixed point of this Bellman equation by $V_i(\phi)$ and refer to it as the *normalized value function* of a firm of quality i and reputation ϕ . Because firms find it optimal to continue when they expect positive profits and exit if they expect negative profits from continuation, the actual value functions, denoted $W^i(\phi)$, are simply given by the normalized value functions scaled by the price of the experience good P in steady state, i.e., $W_i(\phi) = V_i(\phi)P$.

Lemma 1 *Normalized value functions of intermediate good producers.*

Take buyers' interim beliefs based on continuation $\phi^c(\phi)$ as given. The value of a firm with quality $i \in \{L, H\}$ and reputation ϕ is given as the unique solution $V_i(\phi)$ to the Bellman equation

$$V_i(\phi) = \max_{\omega \in [0, \exp(-\delta\Delta)]} \exp(-(r + \delta)\Delta)\omega V_i^c(\phi^c(\phi)), \quad (11)$$

where

$$V_i^c(\phi^c) = y(\phi^c)\Delta + \exp(-r\Delta) (\alpha_i(\Delta)V_i(\phi^g(\phi^c)) + (1 - \alpha_i(\Delta))V_i(\phi^b(\phi^c))) \quad (12)$$

with $\phi^g(\phi^c)$ and $\phi^b(\phi^c)$ defined by (6) and (7).

The proof of this lemma is given in the Appendix.

This Bellman equation (11) also defines the set of optimal continuation strategies for a firm of quality i given buyers' interim beliefs. Specifically, a continuation strategy

$\omega_i(\phi)$ is a *best response* to buyers' interim beliefs $\phi^c(\phi)$ only if $\omega_i(\phi) = \exp(-\delta\Delta)$ when $V_i(\phi) > 0$ and $V_i(\phi) \geq 0$ for all ϕ . Note that this second requirement implies that $\omega_i(\phi) = 0$ whenever $V_i^e(\phi^c(\phi)) < 0$.

In a steady-state spot market equilibrium, it must further be the case that the subsidy-adjusted price of the experience good is given by the marginal utility of the experience good:

$$P = sU'(Y), \quad (13)$$

where Y denotes the steady-state output level of the experience good and s denotes the steady-state subsidy to sales of the experience good to the final consumer.

In equilibrium, there must be non-positive profits associated with entry for both high- and low-quality firms. With subsidy to sales of the experience good of size s and a tax on entry of size F , this requirement is

$$sU'(Y)V_H(\phi^e) - C - F \leq 0, \quad (14)$$

with equality if $\phi^e m^e \Delta > 0$, and

$$sU'(Y)V_L(\phi^e) - F \leq 0, \quad (15)$$

with equality if $(1 - \phi^e)m^e \Delta > 0$.

We summarize this discussion with the following definition of a *steady-state spot market equilibrium*.

Definition 1 *Steady-state spot market equilibrium*

A *steady-state spot market equilibrium* consists of a feasible allocation in which all variables are constant over time $\{Y, N, m^e, \phi^e, \phi^c(\phi), \omega_i(\phi), \nu_i(\phi), m_i\}$ and normalized value functions $\{V_i(\phi)\}$ defined as in (11) such that

- (i) The continuation strategies $\omega_i(\phi)$ are a best response to buyers' interim beliefs ϕ^c .
- (ii) Buyers' interim beliefs $\phi^c(\phi)$ are consistent with the continuation strategies $\omega_i(\phi)$ as in (5) where Bayes' rule is defined, and
- (iii) The zero profits on entry conditions (14) and (15) are satisfied.

The case where $F = 0$ and $s = 1$ then corresponds to the *laissez-faire* benchmark without any regulatory interventions.

2.3 Full Information Benchmark: Social Optimum and Spot Market Equilibrium

Here we characterize the socially optimal allocation in the full information case, and show that it is attained by a steady-state spot market equilibrium without regulatory interventions.

In this case, the type of each firm is observable and hence the measure of reputation across firms has mass on $\phi = 0$ and on $\phi = 1$, with no firms with intermediate reputations. The evolution of the stocks of firms (8) and (9) in steady-state is given by

$$m_H = \exp(-\delta\Delta)\phi^e m^e \Delta + \omega_H(1)m_H \quad (16)$$

and

$$m_L = \exp(-\delta\Delta)(1 - \phi^e)m^e \Delta + \omega_L(0)m_L. \quad (17)$$

Clearly, since the output of a firm known to be low-quality is expected to subtract from production of the experience good ($y(0) < 0$), it is optimal to set $\omega_L(0) = 0$ and $\phi^e = 1$. Likewise, since an existing firm known to be of high-quality can contribute $y(1)$ to production of the experience good at zero cost as long as it continues in production, it is optimal to set $\omega_H(1) = \exp(-\delta\Delta)$, its maximum value. These results then characterize the optimal continuation decisions.

Now consider the optimal level of entry of high-quality firms. The marginal social cost, in terms of utility, of creating a new firm at t with probability $\phi^e = 1$ of being high-quality is given by C , whereas the marginal benefit is given by

$$\exp(-(r + \delta)\Delta) \sum_{\tau=t+1}^{\infty} \exp(-(r + \delta)\Delta(\tau - t))y(1)U'(\bar{Y})\Delta,$$

where \bar{Y} denotes the full information production of the experience good in steady-state. Then

$$\frac{\Delta \exp(-(r - \delta)\Delta)}{1 - \exp(-(r + \delta)\Delta)}y(1)U'(\bar{Y}) = C. \quad (18)$$

As $\Delta \rightarrow 0$, this equation converges to

$$y(1)U'(\bar{Y}) = C(r + \delta).$$

There is an optimal stock of high-quality firms in steady state determined by this equation and the resource constraint $\bar{m}_H = \bar{Y}/y(1)$.

The optimal dynamic choice of entry $m_t^e \Delta$ is the following. If $y(1)m_{H0}$ is less than the optimal steady-state level of production of the experience good \bar{Y} , the social planner creates an atom of new high-quality firms at $t = 0$ to attain the optimal stock \bar{m}_H of high-quality firms immediately (since utility is quasi-linear). If $y(1)m_{H0}$ exceeds this optimal level, the social planner creates no new firms until the stock of existing high-quality firms has depreciated down to this level at rate $\exp(-\delta\Delta)$. Once this optimal stock of high-quality firms is attained, the social planner chooses a flow of entry of new firms $m^e = \frac{1-\exp(-\delta\Delta)}{\Delta}\bar{m}_H$ to maintain the stock at a constant level. If $\Delta \rightarrow 0$ the flow of new firms necessary to maintain the stock at a constant level is then $m^e = \delta\bar{m}_H$.

Spot market equilibrium: Under full information, the normalized value function associated with a high-quality firm in the socially optimal allocation in steady-state is

$$V_H(1) = \exp(-(r + \delta)\Delta) \sum_{\tau=t+1}^{\infty} \exp(-(r + \delta)\Delta(\tau - t))y(1)U'(Y)\Delta > 0,$$

whereas that associated with operating a low-quality firm is

$$U'(Y)V_L(0) = 0.$$

This last result follows from the assumption that $y(0) < 0$, so it is always optimal to remove a low-quality firm from production as rapidly as possible. Note also that in the transition to steady state, $U'(Y)V_H(1) = C$ whenever there is positive entry and $U'(Y)V_H(1) < C$ when there is no entry. Clearly, the laissez-faire spot market prices ($p(1) = y(1)U'(Y)$ and $p(0) = y(0)U'(Y)$) implement efficient entry by high-quality firms, $\phi^e = 1$. Moreover, when there is no subsidy on the experience good, so $P = U'(\bar{Y})$, the spot market equilibrium leads to the efficient provision of the experience good, \bar{Y} .

Furthermore, in the full information benchmark, any regulatory intervention in the form of entry fees or subsidies on experience goods is distortionary, inducing an inefficient provision of those goods. Thus, under full information, entry regulation can only have detrimental welfare consequences, and typically does by raising the price

of the experience good, and generating scarcity rents to incumbent producers.¹²

Remark on no information case: In contrast to the full information case, in the extreme case of non-observable investment and no signals from which to learn, the adverse selection problem associated with free entry of low-quality firms is so severe that there is no production of the experience good in steady state.

This result follows from the observation that it is impossible to offer high-quality producers of the intermediate good a positive price for their good without attracting unbounded entry of low-quality firms. With no dependence of the public signal on the quality of the firm, reputation for high- and low-quality firms will not change over time if both types of firms have the same continuation rates $\omega_i(\phi) > 0$. Likewise, both types of firms will have the same exit rates if reputation does not evolve because, if reputation does not evolve, then they both expect the same profits, that is, $V_H(\phi) = V_L(\phi)$. Of course, this equality of value functions means that it is impossible to satisfy the entry condition for high-quality firms (14) as an equality (with positive entry of high-quality firms) without violating the entry condition (15) for low-quality firms. As a result, the steady-state equilibrium allocation with no information has no entry of firms and there can be no positive production of the experience good once the initial stock of high-quality firms dies out.

2.4 Signal Structures and Reputation

In what follows, for certain propositions, we consider three specific imperfectly informative signal structures on which reputation can be based, which we term *good news*, *bad news*, and *Brownian diffusion*. We define these signal structures here.

In the good news scenario, if the firm is of high-quality, a signal that reveals that quality arrives at rate $\lambda > 0$ per unit of time. No such signal can arrive if the firm is low-quality. This corresponds to $\alpha_L(\Delta) = 0$ and $\alpha_H(\Delta) = \lambda\Delta$. Note that in this scenario, $\phi^g(0)$ is not defined by Bayes' rule. We impose that $\phi^g(0) = \lim_{\phi \rightarrow 0} \phi^g(\phi) = 1$. This scenario captures the idea that advances in the design of a new product reveal themselves through sudden breakthroughs and successful jumps in performance.

In the bad news scenario, the assumption is reversed: if the firm is of low-quality, a signal that reveals that quality arrives at rate $\lambda > 0$ per unit of time. No such

¹²The only exception is setting both F and s exactly to neutralize each other, this is $F = (s - 1)C$.

signal can arrive if the firm is high-quality. This corresponds to $(1 - \alpha_H(\Delta)) = 0$ and $(1 - \alpha_L(\Delta)) = \lambda\Delta$. Note that in this scenario, $\phi^b(1)$ is not defined by Bayes' rule. We impose that $\phi^b(1) = \lim_{\phi \rightarrow 1} \phi^b(\phi) = 0$. This scenario captures the idea that flaws in the design of a new product reveal themselves through sudden safety hazards or accidents which may lead to a product recall, or otherwise force the seller to withdraw the product from the market.

Finally, for the Brownian diffusion scenario, to approximate a Brownian diffusion in discrete time, we choose $\alpha_H(\Delta)$ and $\alpha_L(\Delta)$ so that for all ϕ

$$\log\left(\frac{\phi^g(\phi)}{1 - \phi^g(\phi)}\right) - \log\left(\frac{\phi}{1 - \phi}\right) = \zeta\sqrt{\Delta}$$

and

$$\log\left(\frac{\phi^b(\phi)}{1 - \phi^b(\phi)}\right) - \log\left(\frac{\phi}{1 - \phi}\right) = -\zeta\sqrt{\Delta}.$$

This is achieved if

$$\alpha_H(\Delta) = \left(\frac{\exp(\zeta\sqrt{\Delta})}{1 + \exp(\zeta\sqrt{\Delta})}\right) > \frac{1}{2} \quad (19)$$

and $\alpha_L(\Delta) = 1 - \alpha_H(\Delta)$, where ζ is the signal-to-noise ratio. This scenario captures the idea that qualities of design reveal themselves through both positive and negative random signals of performance.

The signals in the good news, bad news, and Brownian diffusion scenarios are public signals of the quality of each firm that are observed by all potential buyers. These signals might be interpreted as ratings in some widely published guide derived from either specialized testing or noisy surveys of past customers' experiences with the intermediate good obtained from each firm. Under this interpretation, individual past buyers of the intermediate good from a particular firm have more precise information about that firm's quality from their past consumption experience, but this experience is not fully revealed to other buyers by a survey. We assume that this private information does not affect demand for a given firm's output because buyers do not buy repeatedly from the same firm.

Alternatively, one might interpret the signals as reflecting a noisy outcome of production of the experience good with the intermediate output supplied by a particular firm. For example, if the good signal is a positive contribution of 1 to the production of the experience good and the bad signal is a negative contribution of -1 , the ex-

pected contribution from a high-quality firm is $y(1) = 2\alpha_H(\Delta) - 1$ and the expected contribution from a low-quality firm is $y(0) = 2\alpha_L(\Delta) - 1$. Under this interpretation, the outcome of each individual buyer's experience is public information.

3 Reputation with Imperfectly Informative Signals

In this section, we solve for the steady-state spot market equilibrium without regulation when public signals about each intermediate goods producing firm are imperfectly informative. The steady-state spot market equilibrium can be solved for in a two-step recursive manner. In the first step, we solve for the buyers' interim beliefs $\phi^c(\phi)$, firms' normalized value functions $V_i(\phi)$, and continuation strategies $\omega_i(\phi)$ consistent with conditions (i) and (ii) in the definition of a steady-state spot market equilibrium. In the second step, we solve for the reputation of entering firms ϕ^e and the level of output of the experience good Y consistent with the entry conditions for high- and low-quality firms (14) and (15) expressed as equalities, with $s = 1$ and $F = 0$. The resource constraints that define a feasible allocation then give the remainder of the equilibrium quantities.

We impose two "reasonable" restrictions on buyers' interim beliefs to rule out multiplicity of equilibrium generated by off-equilibrium beliefs.

Assumption 1 *Monotonic Updating*

We say that buyers' beliefs show monotonic updating if $\phi^c(\phi)$ is non-decreasing in ϕ for all $\phi \in [0, 1]$.

Using the Bellman equations developed in Lemma 1, this restriction implies that value functions $V_i(\phi)$ are weakly increasing and that $V_H(\phi) \geq V_L(\phi)$.

Assumption 2 *High quality firms are more likely to continue than low-quality firms.*

We say that buyers' beliefs regarding firms' continuation strategies are consistent with the hypothesis that high-quality firms are more likely to continue than low-quality firms if $\phi^c(\phi) \geq \phi$ for all ϕ .

Note that this assumption is stronger than a restriction on firms' continuation strategies that $\omega_H(\phi) \geq \omega_L(\phi)$ for all ϕ . It further imposes this restriction on interim beliefs when both continuation strategies are equal to zero and hence Bayes Rule is not defined.

Without these two restrictions on buyers' interim beliefs, there is a generic multiplicity of equilibrium that arises from buyers' interpretation of off-equilibrium seller behavior. If buyers expect both types of firms to exit, then Bayes' rule does not discipline interim beliefs about firms that continue, and thus these beliefs can be specified in a manner that encourages firms to indeed exit. For example, one can generate any number of equilibria by setting beliefs $\phi^c(\phi) = 0$ for arbitrary subsets of $[0, 1]$. The multiplicity here arises from the standard problem that if buyers interpret continuation as a sure signal that a firm is low-quality, then all firms prefer to exit, and Bayes' rule cannot be used to restrict buyers' interpretation of out of equilibrium continuation of a firm.

Now we show that the steady-state spot market equilibrium that we focus on here is in fact the unique steady-state spot market equilibrium allocation as the model converges to continuous time (this is, as $\Delta \rightarrow 0$) given the three information structures and these two restrictions on buyers' interim beliefs. That result, and the structure of the *steady-state spot market equilibrium with imperfectly informative signals*, is described in the following proposition and proved in the Appendix.

Proposition 1 *Steady-state spot market equilibrium.*

Under assumptions 1 and 2 on buyers' beliefs, the steady-state spot market equilibrium implemented by normalized spot prices $y(\phi)$ is uniquely characterized, as $\Delta \rightarrow 0$, by the following four results:

1. *There is entry of some low-quality firms, ($\phi^e < 1$),*
2. *Reputations of all active firms in steady-state remain in an interval $[\bar{\phi}, 1]$ with $\bar{\phi} > 0$. High-quality firms always strive to remain active, i.e., $\omega_H(\phi) = \exp(-\delta\Delta)$ for $\phi > 0$. Low-quality firms also strive to remain active if $\phi > \bar{\phi}$, and otherwise they randomize continuation with a probability $\omega_L(\phi) \in (0, \exp(-\delta\Delta)]$ such that buyers' interim beliefs satisfy $\phi^c(\phi) = \bar{\phi}$.*
3. *The steady-state equilibrium entry reputation equals the exit threshold: $\phi^e = \bar{\phi}$.*

4. The steady-state equilibrium level of production of the experience good Y satisfies

$$V_H(\phi^e)U'(Y) = C. \quad (20)$$

The structure of this equilibrium allows for a two-step equilibrium construction procedure:

Equilibrium construction step 1: The equilibrium buyers' interim beliefs $\phi^c(\phi)$, firms' continuation strategies $\omega_i(\phi)$, and firms' normalized value functions $V_i(\phi)$ are all indexed by a reputation level $\bar{\phi} \in (0, 1)$ that we refer to as the *exit threshold* such that

(a) buyers' interim beliefs are given by $\phi^c(\phi) = \phi$ for $\phi \geq \bar{\phi}$, $\phi^c(\phi) = \bar{\phi}$ for $\phi \in (0, \bar{\phi})$, and $\phi^c(0) = 0$,

(b) firms' continuation strategies are given by $\omega_H(\phi) = \exp(-\delta\Delta)$ for all $\phi > 0$ and $\omega_H(0) = 0$ for high-quality firms, whereas, for low-quality firms $\omega_L(\phi) = \exp(-\delta\Delta)$ for all $\phi \geq \bar{\phi}$, $\omega_L(\phi)$ solves

$$\bar{\phi} = \frac{\phi \exp(-\delta\Delta)}{\phi \exp(-\delta\Delta) + (1 - \phi)\omega_L(\phi)}$$

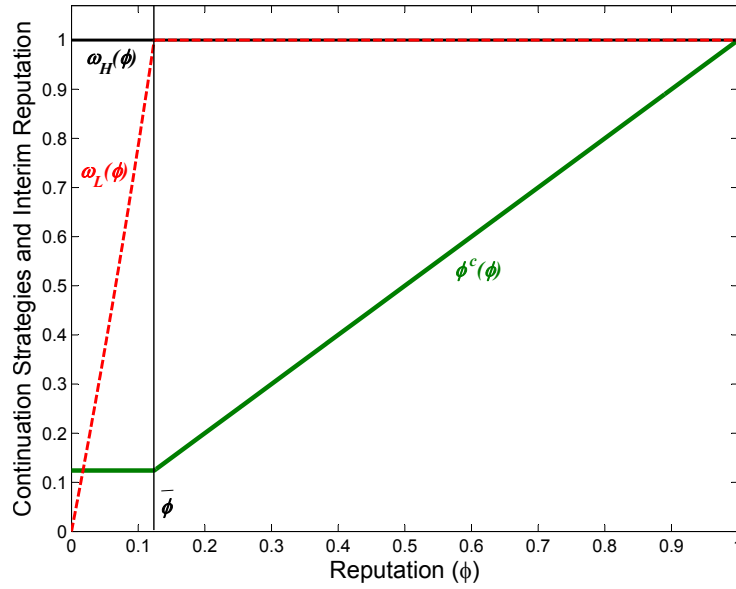
for all $\phi \in (0, \bar{\phi})$, and $\omega_L(0) = 0$, and, finally,

(c) the normalized value function for high-quality firms $V_H(\phi)$ is strictly positive for $\phi > 0$ and equal to zero for $\phi = 0$, whereas, for low-quality firms $V_L(\phi) = 0$ for $\phi \leq \bar{\phi}$ and $V_L(\phi) > 0$ otherwise.

In each of our three cases of imperfectly informative signals, this first step reduces to a fixed point problem of finding an exit threshold $\bar{\phi}$ such that the associated buyers' interim beliefs defined in (a) imply normalized value functions such that $\bar{\phi}$ is the greatest lower bound on the set of ϕ such that $V_L(\phi) > 0$ as in (c), thus ensuring that the continuation strategies specified in (b) are a best response to the buyers' interim beliefs.¹³ In the Appendix we provide closed-form solutions for the value functions and exit thresholds for all three of our signal structures in the continuous time limit as $\Delta \rightarrow 0$. Figures 1 and 2 illustrate this first step of equilibrium construction. These figures are generated based on Brownian diffusion and $\Delta \rightarrow 0$. Figures 3 and 4 show the value functions for the bad and good news cases, respectively, again as $\Delta \rightarrow 0$.

¹³At $\phi = 0$, we have assumed that buyers' interim beliefs are absorbing at $\phi^c(0) = 0$ and that both high- and low-quality firms exit. Bayes' rule is not defined at this point.

Figure 1: Continuation Decisions and Interim Reputation

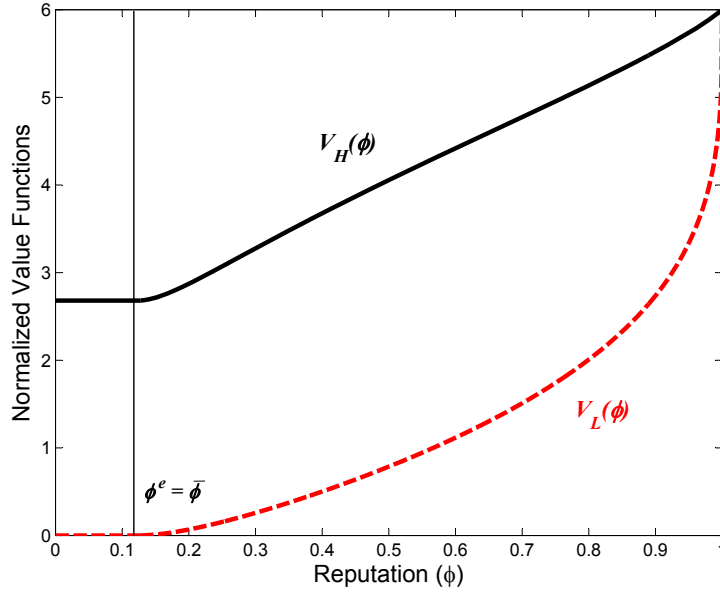


Equilibrium construction step 2: Given a solution to step 1, the reputation at entry is chosen as $\phi^e = \bar{\phi}$. This clearly implies that the zero profit at entry condition for low-quality firms (15) is satisfied. The aggregate production of the experience good Y is chosen such that the normalized value function for high-quality firms, when scaled by the price of the experience good $P = U'(Y)$, satisfies the condition that high-quality firms earn zero profits at entry, i.e., (14) is an equality. The remaining elements of the equilibrium allocation can then be solved from the conditions defining feasibility.

Of course, the normalized value of high-quality firms at $\phi^e = \bar{\phi}$, given by $V_H(\phi^e) > 0$, is determined by the specifics of the signal structure. Hence, the equilibrium level of production of the experience good Y , and the corresponding marginal utility of that experience good $U'(Y)$ needed to satisfy the zero profit condition on entry for high-quality firms (14), are pinned down by the signal structure as well.

This procedure for computing a steady-state spot market equilibrium with imperfectly informative signals gives us the result that the lemons problem that arises with imperfectly informative signals leads to a reduction in the output of the experience good relative to the full information steady state. We state this result in the next proposition.

Figure 2: Value Functions: Brownian diffusion



Proposition 2 *The steady-state level of the experience good output when signals about firms' quality are not perfectly informative is lower than that in the full information benchmark. That is, $Y < \bar{Y}$.*

Proof Using (18) from the full information benchmark, the first-best level of experience good production is given by $V_H(1)U'(\bar{Y}) = C$. With imperfectly informative signals, we have that the fraction of high-quality firms that enter is equal to the lowest level of reputation sustained by the market; that is, $\phi^e = \bar{\phi} < 1$. From (14), the output of the experience good is given by $V_H(\bar{\phi})U'(Y) = C$. Since $V_H(\bar{\phi}) < V_H(1)$, then $Y < \bar{Y}$. Q.E.D.

Intuitively, although reputation mitigates the lemons problem and allows for some positive production of the experience good (relative to the no information benchmark), the need for high-quality firms to endure lower profits after entry as they accumulate a good reputation constrains efficient production. Hence, in a steady-state spot market equilibrium with imperfectly informative signals, the time that high-quality firms spend accumulating a reputation to distinguish themselves from low-quality firms leads to a social cost relative to the full information benchmark. The next section focuses on the potential of regulation to mitigate this social cost.

In the next proposition, we show that regardless of which of the three specific infor-

Figure 3: Value Functions: Bad News

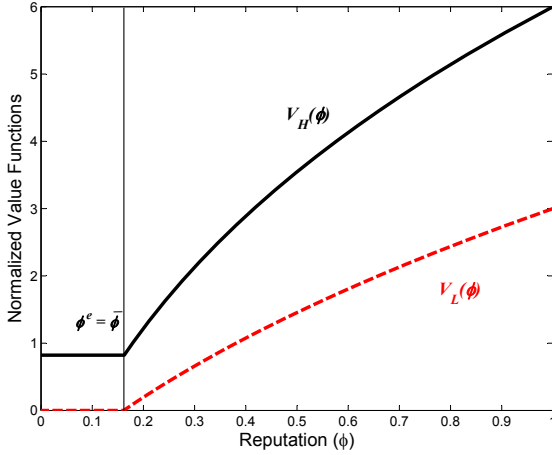
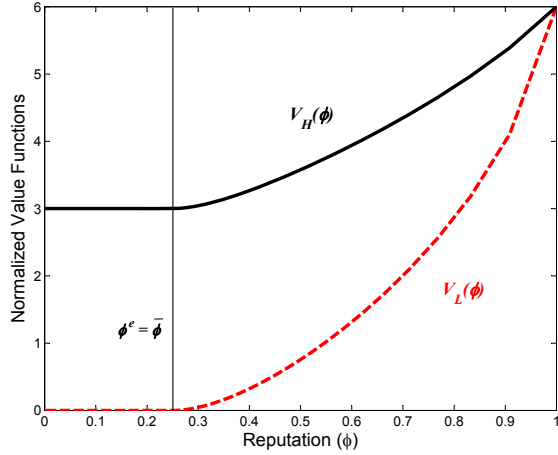


Figure 4: Value Functions: Good News



mation structures we consider, the steady-state spot market equilibrium converges to the benchmark without information as the precision of signals goes to zero and converges to the benchmark with perfect information as the precision of signals goes to infinity. Hence, as the effectiveness of learning improves, the equilibrium ranges from complete market shutdown to the unconstrained first best.

Proposition 3 *In the three information structures considered (bad news, good news, and Brownian diffusion), as $\Delta \rightarrow 0$, the spot market equilibrium converges to $Y = 0$ as the precision of signals goes to zero and to the unconstrained first best $Y = \bar{Y}$, as the precision of signals goes to infinity.*

We prove this proposition in the Appendix by direct calculation using the analytical solutions for the value functions in the continuous time limit.

4 Regulation with Imperfectly Informative Signals

We now study the role of regulation in improving welfare relative to a market economy with reputation based on imperfectly informative signals. For this, we return to the case in which a regulator can set a fixed fee $F \geq 0$ on entry of new firms, and a subsidy s to sales of the final experience good. We first extend our procedure for constructing the steady-state spot market equilibrium with imperfectly informative

signals to the case of equilibrium with experience good sales subsidies s and entry taxes F . We then consider the extent to which a regulator can improve on welfare in the *laissez-faire* equilibrium with these two policy instruments.

The first step in our construction of equilibrium summarized as finding an exit threshold $\bar{\phi}$ and corresponding interim beliefs for buyers $\phi^c(\phi)$, continuation strategies for firms $\omega_i(\phi)$, and normalized value functions $V_i(\phi)$ that satisfy conditions (a), (b), and (c) is unchanged by the presence of an entry fee and a subsidy to sales of the experience good. Hence, the regulatory instruments considered here have no impact on buyers' beliefs and firms' continuation strategies.

The second step of equilibrium construction, however, is altered. Specifically, the second step of our equilibrium construction now becomes one of solving the two zero profit on entry conditions (14) and (15) for the endogenous reputation of entrants ϕ^e and output of the experience good Y .

In what follows, it is useful to write these two zero profit conditions for entry (14) and (15) equivalently as follows. First taking the ratio of (15) to (14), we have

$$\frac{V_L(\phi^e)}{V_H(\phi^e)} = \frac{F}{C + F}. \quad (21)$$

Second, taking the difference between (14) and (15), we have

$$(V_H(\phi^e) - V_L(\phi^e)) sU'(Y) = C. \quad (22)$$

As these equations make clear, the reputation of entrants $\phi^e \geq \bar{\phi}$ depends only on F . Subsidies to the sale of the experience good do not differentially affect the entry decisions of high- and low-quality firms. In contrast, total production of the experience good Y depends on ϕ^e (hence on F) and on s .

Note also that the relationship between entry taxes F and the reputation of entrants ϕ^e depends on how the ratio of the value functions V_L and V_H varies with ϕ . In particular, we clearly have that the right-hand side of (21) is strictly increasing from 0 to 1 as F increases from 0 to ∞ . We also have, by construction, that the ratio of the value functions on the left-hand side of (21) is equal to zero at $\phi^e = \bar{\phi}$ and equal to 1 at $\phi^e = 1$. In the Appendix we prove that for the three signal structures we consider, in the continuous time limit, the ratio of the value functions $V_L(\phi^e)/V_H(\phi^e)$ is strictly

increasing in ϕ^e from 0 at $\phi^e = \bar{\phi}$ to 1 at $\phi^e = 1$,¹⁴ so that we have that the equation (21) implicitly defines the reputation of entrants as a function of the tax on entry denoted $\phi^e(F)$. Moreover, this result implies that it is feasible for a regulator to implement any initial reputation of entrants $\phi^e \in [\bar{\phi}, 1)$ desired by an appropriate choice of entry costs F . Thus, in terms of solving for the equilibrium outcome as a function of the regulatory instruments F and s , we have that (21) gives the reputation of entrants ϕ^e as a function of the entry cost F and then (22) gives the equilibrium level of production of the experience good Y as a function of $\phi^e(F)$ and s .

4.1 Regulation When Final Sales Are Observable

We assume that a regulator who observes final sales of the experience good in the market¹⁵ can choose both the subsidy to the sale of the experience good s and the tax on entrants F . Our main result here is that a regulator with access to these two policy instruments can implement a steady-state equilibrium outcome with welfare for the representative household that is arbitrarily close to welfare in the full information first-best outcome. Specifically, let \bar{Y} and \bar{N} denote the full information optimal steady-state levels of consumption of the experience and numeraire good. We then have the following proposition.

Proposition 4 *Optimal regulation with policies based on transactions.*

When a regulator observes transactions, it is possible to find a combination of F and s that implements a steady-state allocation with $Y = \bar{Y}$ and $N = \bar{N} - \epsilon$ for any $\epsilon > 0$.

The proof of this proposition is in the appendix. The intuition for the proof is as follows. The regulator can set F large enough to make ϕ^e arbitrarily close to 1. Then, it is possible to find a subsidy to target the optimal \bar{Y}

$$s = \frac{C}{U'(\bar{Y}) [V_H(\phi^e) - V_L(\phi^e)]}.$$

¹⁴In the bad news case this ratio of value functions converges to a number less than one.

¹⁵Here we are strictly interpreting the goods in our model as intermediate and final goods transacted in the market. In the next section, we consider a broader interpretation of our model in which trade in these goods is not carried out through observable market transactions. Under this broader interpretation, the production function for the final experience good is instead an aggregator in utility.

The corresponding value of $\phi^e m^e \Delta$ needed to produce \bar{Y} is slightly higher than in the full information first best because there is a small fraction of low-quality active firms in steady state detracting from the output of the experience good. This additional expenditure of the numeraire good required to pay for the extra entry of high-quality firms can be made arbitrarily small by setting ϕ^e arbitrarily close to 1.

We interpret this proposition as indicating that the lemons problem in this economy is one of commitment rather than one of information. The lemons problem arises because the competitive market prices based on the spot gains to trade between a buyer and a seller do not offer sufficient rewards to reputation to ensure high-quality. There is a welfare gain to be achieved here if buyers are able to commit to pay prices that reward good reputation or punish poor reputation over and above the incentives provided by spot market prices.

In some environments, it may be possible to achieve such commitment through long-term contracts between buyers and sellers. If a contract between a buyer and seller with prices based on reputation can be enforced, then the two parties can, with an appropriate choice of parameters F and s , design an incentive contract guaranteeing that most sellers entering into the contract are high-quality. Here we interpret the relationships between the buyers and sellers of intermediate goods as one-shot or short-lived, and hence long-term contracts are not feasible. In this case, regulation is a substitute for missing private capabilities to commit, and can replicate the commitment allocation.

Remark on budget balance: We have not assumed budget balance in which taxes on entry by new intermediate good producers exactly compensate subsidies to experience good producers. Instead, we have assumed that the regulator has access to lump sum taxes to finance subsidies. Budget balance would impose restrictions on how closely one can approximate the unconstrained first best. The government revenues from entry taxes are $F m^e \Delta$ per period, whereas, the government expenditures on subsidies are $(s - 1)U'(Y)Y\Delta$ per period. Revenues have an upper bound (when $\phi^e \rightarrow 1$ and $Y = \bar{Y}$) of $V_L(1) \frac{1 - \exp(-\delta\Delta)}{\Delta} \frac{\bar{Y}}{y(1)} \Delta$, which is finite. Contrarily, required subsidies explode to infinity as $\phi^e \rightarrow 1$ from equation (22). This implies that budget balance restricts the possibilities for simple regulation of the kind considered here to approximate the first-best allocation.

Remark on non-Markov transfers: One can also see immediately that our assumption that subsidies are based on transactions rather than the full history of signals for each firm is restrictive. The standard result that a reputation of $\phi = 1$ is an absorbing state implies that $V_L(1) = V_H(1)$, so it is impossible to have only high-quality firms entering. We can get arbitrarily close to having only high-quality firms enter, but not all the way there.

In contrast, if we allowed the regulator to make transfers based on the full history of signals of quality associated with each firm, then, for a wide range of stochastic signal structures, the regulator could implement an allocation with exactly $\phi^e = 1$. This result follows if the distribution of signal histories for high- and low-quality firms differs sufficiently such that, over time, arbitrarily precise statistical tests of firm quality can be performed given long enough realized signal histories. A transfer scheme that back-loads payments to firms and conditions them on this statistical test of signal histories can then reward the investment of a high-quality firm (with an entry cost $F > 0$) and at the same time deter entry by low-quality firms by leaving them with strictly negative expected profits upon entry. (Note that this can be achieved under budget balance as well). This is not possible with transfers that are Markov in reputation because buyers ignore further signals of quality once $\phi = 1$. Still, the cost in terms of welfare of having such a simple subsidy scheme is negligible compared to the more complicated possibility of having subsidies as a function of the whole history of signals.

4.2 Regulation When Final Sales Are Not Observable

We now assume that the regulator does not observe, and hence cannot subsidize, sales of either of the experience good or intermediate goods. Instead, we assume that the regulator takes as given that active firms are paid the spot market prices $p(\phi) = y(\phi)U'(Y)$ and that the regulator can only use fixed regulatory entry costs F (rebated lump sum to consumers) to influence welfare given by equation (1).

In this case, a regulatory entry cost F still complements reputation by driving up the average quality and reputation ϕ^e of entering firms. But this benefit of regulation is not free: producers of the intermediate good have to be paid higher prices to allow them to recoup this regulatory entry cost. This force can lead to a reduction of the

equilibrium production of the experience good. Thus, a regulator potentially faces a tradeoff between quality and the equilibrium level of production of the experience good in choosing regulatory entry costs. Here, we show that in our three leading information structures, this tradeoff is always resolved in favor of some positive (but finite) level of entry regulations: for small entry fees, the gains from improving average quality always dominate the welfare considerations. Moreover, we derive a formula for the welfare-maximizing entry fee in the limiting case as $r \rightarrow 0$, and show that this fee is increasing in the marginal impact of regulation on average quality, and decreasing in the marginal impact of regulation on market size.

Conceptually, we think of the problem of designing optimal regulation as a Ramsey Policy problem with commitment: starting from some initial measure of reputations across incumbent high- and low-quality firms $\nu_{i0}(\phi)$, the Ramsey planner commits to a sequence of entry fees $\{F_t\}$. Given this sequence, the market outcome is then determined as a (possibly non-stationary) spot market equilibrium, defining a sequence of measures of entering firms m_t^e , a sequence of entry reputations ϕ_t^e , aggregate output levels Y_t and a sequence of measures of reputations across high- and low-quality firms $\nu_{it}(\phi)$ which lead to an allocation that is feasible, and consistent with the entering firms' free entry and incentive conditions. We can formulate this as a Ramsey planning problem in which the planner selects a sequence $\{F_t, Y_t, m_t^e, \phi_t^e\}$, subject to the restrictions imposed by feasibility and by the incentive conditions for high- and low-quality entrants. Letting $V_i(t, \phi)$ denote the value of a type $i \in \{H, L\}$ firm with reputation ϕ in period t , these incentive conditions can be stated as $F_t + C = V_H(t, \phi_t^e)$ and $F_t = V_L(t, \phi_t^e)$ for high and low quality firms, respectively.

In general the solution to this problem is non-stationary and a simple method for solving this problem of optimal regulation design is not available. The issue with stationarity is readily seen when considering the tradeoff the planner faces in selecting the measure m_t^e of entrants at t . The marginal cost of firm creation $C\phi_t^e$ is traded off against two forces: (i) the marginal utility gains from more aggregate output in future periods, and (ii) the impact on future entry incentives. For (i), the greater utility from a larger output in future periods is equal to the expected value of an entrant at period t , this is $\phi_t^e V_H(t, \phi_t^e) + (1 - \phi_t^e) V_L(t, \phi_t^e)$, which is equal to $F_t + \phi_t^e C$. Combining this benefit with the aforementioned cost of firm creation, it is clear that the optimal entry fee F_t must equal the costs coming from (ii).

Now, how does the measure m_t^e of firms entering in period t affect incentives in other

periods? An increase in m_t^e increases output at $t + s$, for all $s > 0$, thus lowering all future prices P_{t+s} for the aggregate experience good. The price at $t + s$ in turn affects the value of any firm that entered in prior periods, $t + s'$. This effect is stronger for high quality firms than for low quality firms. Hence more entry in period t raises output in subsequent periods, but lowers incentives for entry and investment in quality in all prior periods. These incentive effects are captured by the terms under (ii).

When designing optimal entry policies, the Ramsey planner takes entry decisions of pre-existing firms as sunk, and only considers the impact of date t entry on incentives at any $t' \geq 0$. Over time, the proportion of incumbent firms whose incentives the planner cannot affect through the sequence $\{F_t\}$ gradually declines through exit. All this suggests that a planner has a stronger incentive to commit to future regulation than to distort entry today. It also implies that the planner's optimal regulatory policy cannot be time-consistent.

To avoid these non-stationarity issues, we introduce two important additional qualifications in our analysis. First, we focus on the limit of optimal regulation F_t as $t \rightarrow \infty$, and assume that it converges to a constant limit point. In this limit, the planner fully internalizes the impact of regulation on all future and past entrants, which allows us to abstract from the source of non-stationarity that we highlighted above.¹⁶ Second, we focus on the case with a small discount factor r . In the limiting case as $r \rightarrow 0$, the solution to the planners' problem implies the maximization of average per-period welfare, while by continuity, this limiting case offers an approximation of the optimal policy when r is sufficiently small.¹⁷ At a steady-state allocation average per-period welfare is given by

$$W(Y, \phi^e) = U(Y) + 1 - C \frac{Y}{A(\phi^e)}$$

where

$$A(\phi^e) = \frac{\phi^e \tilde{V}_H(\phi^e) + (1 - \phi^e) \tilde{V}_L(\phi^e)}{\phi^e}.$$

¹⁶This non-stationarity and time-inconsistency issue is reminiscent of the related commitment issues in the literature on optimal monetary or fiscal policy design. The limit that we focus on here is also referred to as the optimal policy from a "time-less perspective" (see Woodford (1999, 2000 and 2003)).

¹⁷An alternative interpretation of this limit is to assume that from the start policy has to be time-independent, i.e. to impose the planner can only select among sequences with $F_t = F$ for some $F \geq 0$. The optimal choice of F then balances the maximization of the steady-state welfare levels against costs resulting along the transition. When $r \rightarrow 0$, all the weight is placed on the steady-state welfare.

However, imposing stationarity on the policy rule as an assumption completely abstracts from the issues of non-stationarity and time-consistency, which are interesting in their own right.

Here, $\tilde{V}_i(\phi^e)$ defines the value function for firms with quality $i \in \{L, H\}$, when imposing $r = 0$ and steady-state equilibrium continuation strategies and interim beliefs. The numerator of $A(\phi^e)$ measures steady-state output because the average across high- and low-quality entrants of the expected discounted value of profits with an interest rate of zero (output since marginal cost is zero) is equal to the integral across the cross section of profits in the steady state.¹⁸ Hence, $A(\phi^e)$ measures the aggregate productivity of the experience good sector (i.e., the amount of numeraire good needed per unit of Y in steady-state), which is a function of entry productivity ϕ^e .

Therefore, the planner's problem at the limit can be stated as

$$\begin{aligned} & \max_{Y, \phi^e} W(Y, \phi^e) \text{ s.t.} \\ C &= U'(Y) (V_H(\phi^e) - V_L(\phi^e)) \end{aligned}$$

and the optimal entry fee solves $F = U'(Y) V_L(\phi^e)$. The marginal effect of the marginal quality of entrants, ϕ^e , on steady-state welfare can then be decomposed into an indirect effect on the equilibrium level of output and a direct effect on quality holding the level of output fixed, as follows:

$$\frac{dW(Y, \phi^e)}{d\phi^e} = W_Y(Y, \phi^e) \frac{dY}{d\phi^e} + W_{\phi^e}(Y, \phi^e)$$

where

$$\begin{aligned} W_Y(Y, \phi^e) &= U'(Y) - C \frac{1}{A(\phi^e)} = C \left(\frac{1}{V_H(\phi^e) - V_L(\phi^e)} - \frac{1}{A(\phi^e)} \right) \\ W_{\phi^e}(Y, \phi^e) &= CY \frac{A'(\phi^e)}{A^2(\phi^e)} \\ \frac{dY}{d\phi^e} &= - \frac{U'(Y) V'_H(\phi^e) - V'_L(\phi^e)}{U''(Y) V_H(\phi^e) - V_L(\phi^e)}. \end{aligned}$$

Now, we can prove the following result

Proposition 5 *Optimal regulation with policies not based on transactions, only entry.*

For sufficiently low r , the optimal level of entry costs F is always positive, in all three information scenarios.

¹⁸Note that the computation of the value functions when $r = 0$ takes the impact of endogenous and exogenous exit on the cross section of output into account.

Proof We prove the result by showing that $\frac{dW(Y, \phi^e)}{d\phi^e} > 0$ in the limit as $r \rightarrow 0$, for ϕ^e sufficiently close to $\bar{\phi}$. The result then applies for r sufficiently small because of continuity. When $r \rightarrow 0$, $V_i(\phi) \rightarrow \tilde{V}_i(\phi)$, and $W_Y(Y, \phi^e)$ converges to

$$W_Y(Y, \phi^e) = C \left(\frac{1}{D(\phi^e)} - \frac{1}{V_L(\phi^e)/\phi^e + D(\phi^e)} \right)$$

c. This derivative is positive for any $\phi^e > \bar{\phi}$, but equals zero at $\phi^e = \bar{\phi}$. Thus, the marginal impact on welfare of increasing entry holding entrant quality fixed is zero when there is no entry fee ($\phi^e = \bar{\phi}$) and strictly positive if there is an entry fee (so $\phi^e > \bar{\phi}$).

Next, consider the direct marginal impact on welfare from increasing entrant quality holding fixed the level of production of the experience good. This impact is given by $W_{\phi^e}(Y, \phi^e)$ as follows:

$$W_{\phi^e}(Y, \phi^e) = CY \frac{\tilde{V}'_H(\phi^e) - \tilde{V}'_L(\phi^e) + \tilde{V}'_L(\phi^e)/\phi^e - \tilde{V}_L(\phi^e)/(\phi^e)^2}{A^2(\phi^e)}$$

As $r \rightarrow 0$, this converges to

$$W_{\phi^e}(Y, \phi^e) = CY \frac{D'(\phi^e) + V'_L(\phi^e)/\phi^e - V_L(\phi^e)/(\phi^e)^2}{A^2(\phi^e)}$$

For $\phi^e = \bar{\phi}$, we have

$$W_{\phi^e}(Y, \bar{\phi}) = CY \frac{V'_H(\bar{\phi}) + V'_L(\bar{\phi})(1 - \bar{\phi})/\bar{\phi}}{A^2(\bar{\phi})} \geq 0,$$

so the direct effect is non-negative and it is strictly positive whenever either $V'_H(\bar{\phi}) > 0$ or $V'_L(\bar{\phi}) > 0$, i.e. when either the reflecting barrier condition for high types or the smooth-pasting condition for low types is not applicable.

Finally, consider $\frac{dY}{d\phi^e}$. Clearly, the limit of this expression as $r \rightarrow 0$ depends entirely on the properties of $V'_H(\bar{\phi})$ and $V'_L(\bar{\phi})$ (when both are zero, the limit of $\frac{dY}{d\phi^e}$ is zero).

Therefore, at the limit, since $W_Y(Y, \bar{\phi}) = 0$, it follows that the sign of $\frac{dW(Y, \phi^e)}{d\phi^e}$ only depends on $W_{\phi^e}(Y, \bar{\phi})$. If either $V'_H(\bar{\phi}) > 0$ or $V'_L(\bar{\phi}) > 0$, the latter is strictly positive, which establishes the result for signal processes for which either of these derivatives is strictly positive, as is the case in the bad news scenario.

Suppose now that $V'_H(\bar{\phi}) = V'_L(\bar{\phi}) = 0$, as is the case in the Brownian diffusion and good news scenarios. This also implies that $W_Y(Y, \phi^e) = W_{\phi^e}(Y, \bar{\phi}) = \frac{dY}{d\phi^e} = 0$, hence the first-order impact on steady-state welfare of adding a regulatory entry cost is zero at $\phi^e = \bar{\phi}$. We therefore need to check the second-order condition. Here, notice that the second-order terms coming from $W_Y(Y, \phi^e) \frac{dY}{d\phi^e}$ must be zero, and we are therefore only concerned with the second order terms coming from $W_{\phi^e}(Y, \bar{\phi})$. Those are simply determined by the sign of $W_{\phi^e\phi^e}(Y, \bar{\phi})$. We prove that $W_{\phi^e}(Y, \phi^e) > 0$ for ϕ^e close to $\bar{\phi}$ to establish that these second-order terms must be positive. We are thus interested in the sign of $D'(\phi^e) + V'_L(\phi^e)/\phi^e - V_L(\phi^e)/(\phi^e)^2$. Using L'Hopital's Rule, we have

$$\lim_{\phi \rightarrow \bar{\phi}} \frac{(\phi - \bar{\phi}) V'_L(\phi)}{V_L(\phi)} = \lim_{\phi \rightarrow \bar{\phi}} \frac{V'_L(\phi) + (\phi - \bar{\phi}) V''_L(\phi)}{V'_L(\phi)} = 1 + \lim_{\phi \rightarrow \bar{\phi}} (\phi - \bar{\phi}) \frac{V''_L(\phi)}{V'_L(\phi)} \geq 1,$$

which implies that for any $\varepsilon > 0$, there exists ϕ^e , such that

$$V_L(\phi) \leq V'_L(\phi) (\phi - \bar{\phi} - \varepsilon)$$

for all $\phi \in [\bar{\phi}, \phi^e]$. But then,

$$\begin{aligned} D'(\phi^e) + V'_L(\phi^e)/\phi^e - V_L(\phi^e)/(\phi^e)^2 &> V'_L(\phi^e) \left[\frac{1 - \phi^e}{\phi^e} - \frac{\phi^e - \bar{\phi} - \varepsilon}{(\phi^e)^2} \right] \\ &\geq V'_L(\phi^e) \left[\frac{\bar{\phi}}{(\phi^e)^2} - 1 + \frac{\varepsilon}{(\phi^e)^2} \right]. \end{aligned}$$

If ϕ^e is chosen sufficiently close to $\bar{\phi}$, then this expression is strictly positive, which implies that optimally $\phi^e > \bar{\phi}$. Finally, from the discussion about equation (21), $\phi > \bar{\phi}$ if and only if $F > 0$. Q.E.D.

The result shows that there is always a strict welfare gain from a small positive intervention. Hence the laissez-faire outcome is not constraint efficient. A simple intuition for this result is that the market does not internalize the trade-off (from the incentive condition) between ϕ^e and Y : in fact, if ϕ^e was exogenously given by the laissez-faire outcome ($\phi^e = \bar{\phi}$), then the quantity that solves the planners' problem sets

$$U'(Y) = C \frac{1}{A(\phi^e)}.$$

At the *laissez-faire* outcome, $A(\bar{\phi}) = V_H(\bar{\phi})$, so the incentive condition for firms is exactly fulfilled. The next Proposition explicitly characterizes the optimal level of entry costs F in the limit as $r \rightarrow 0$.

Proposition 6 *Optimal entry fees.*

In the limit as $r \rightarrow 0$, the optimal regulation is characterized as follows:

$$F = \phi^e C \frac{\frac{A'(\phi^e)\phi^e}{A(\phi^e)}}{\frac{dY}{d\phi^e} \frac{\phi^e}{Y}}$$

Proof Here we work directly by setting the first-order condition equal to zero:

$$0 = \frac{dW(Y, \phi^e)}{d\phi^e} = W_Y(Y, \phi^e) \frac{dY}{d\phi^e} + W_{\phi^e}(Y, \phi^e).$$

Multiplying this condition by ϕ^e/Y and rearranging terms, we find

$$\begin{aligned} -\frac{dY}{d\phi^e} \frac{\phi^e}{Y} &= \frac{W_{\phi^e}(Y, \phi^e) \phi^e}{W_Y(Y, \phi^e) Y} = \frac{D(\phi^e)}{D(\phi^e) - A(\phi^e)} \frac{A'(\phi^e) \phi^e}{A(\phi^e)} \\ &= \frac{\phi^e D(\phi^e)}{V_L(\phi^e)} \frac{A'(\phi^e) \phi^e}{A(\phi^e)} = -\frac{\phi^e C}{F} \frac{A'(\phi^e) \phi^e}{A(\phi^e)}. \end{aligned}$$

Q.E.D.

Thus, the optimal entry fee scales $\phi^e C$, the resources required to create a new firm with entry reputation ϕ^e , by the ratio of elasticities of the aggregate productivity, $A(\phi^e)$, and the market size, Y , with respect to entry reputation, ϕ^e . This formula summarizes the tradeoff between enhancing quality (captured by $\frac{A'(\phi^e)\phi^e}{A(\phi^e)}$) and losing market size (captured by $\frac{dY}{d\phi^e} \frac{\phi^e}{Y}$). Entry fees are higher, the more responsive aggregate productivity in the experience good sector is to entry reputation, and the less responsive aggregate market share is to such a change in entry reputation.

Now, we can further decompose the two elasticities on the right-hand side:

$$\frac{dY}{d\phi^e} \frac{\phi^e}{Y} = -\frac{U'(Y)}{U''(Y)Y} \frac{D'(\phi^e) \phi^e}{D(\phi^e)},$$

where $D(\phi^e) = V_H(\phi^e) - V_L(\phi^e)$. Thus, the market-size elasticity depends on the elasticity of the market price for experience goods to the entry reputation, $\frac{D'(\phi^e)\phi^e}{D(\phi^e)}$, and

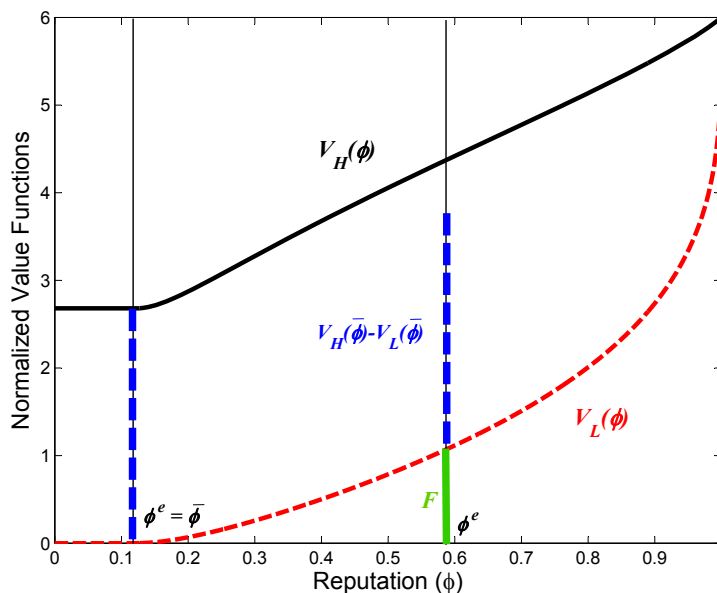
the demand elasticity for experience goods, $\frac{U'(Y)}{U''(Y)Y} < 0$. The first elasticity measures the impact on the market price for experience goods that is required to offset the impact on incentives for quality investment resulting from a change in entry reputation ϕ^e , and is determined endogenously as a property of the equilibrium value functions. The second elasticity then translates the price change required to maintain incentives into a resulting change in aggregate output, and is entirely determined from the representative household preferences. When $V'_H(\phi^e) > V'_L(\phi^e)$, $\frac{dY}{d\phi^e} > 0$ and an increase in ϕ^e increases both average quality and output. In contrast, when $V'_H(\phi^e) < V'_L(\phi^e)$, $\frac{dY}{d\phi^e} < 0$ and an increase in ϕ^e increases average quality, but reduces output.

Next, consider the elasticity of productivity to entry reputation:

$$\frac{A'(\phi^e)\phi^e}{A(\phi^e)} = \frac{D'(\phi^e)\phi^e + V'_L(\phi^e) - V_L(\phi^e)/\phi^e}{D(\phi^e) + V_L(\phi^e)/\phi^e} \geq 0$$

In the limiting case where $\phi^e = \bar{\phi}$, both these elasticities are zero.

Figure 5: Entry Reputation with Entry Fees



To develop some intuition for these propositions consider Figure 5, which illustrates the computation of equilibrium with entry regulation. The value functions shown in this figure and the associated continuation strategies and interim beliefs are the same as those shown in Figures 1 and 2 because an entry cost does not affect the normalized value functions and continuation strategies for firms that have already

entered. Recall that the choice of F directly determines the entry reputation ϕ^e , while the aggregate level of production is then determined from the incentive condition for investing in quality,

$$(V_H(\phi^e) - V_L(\phi^e)) U'(Y) = C. \quad (23)$$

The steady-state scale of production of the experience good Y is thus determined by the difference of the value functions $V_H(\phi^e) - V_L(\phi^e)$, and is a function of just the entry reputation ϕ^e . Whenever this difference in value functions is increasing in ϕ^e , $V'_H(\phi^e) > V'_L(\phi^e)$, an increase in entry reputation brought about by an increase in regulation also leads to an increase in aggregate output through increased entry of more high quality firms. The planner thus faces no tradeoff between quality and aggregate output and always finds further increases in regulation optimal. When instead $V_H(\phi^e) - V_L(\phi^e)$ is decreasing in ϕ^e , the tradeoff between enhancing average quality and aggregate output is present.¹⁹

It follows immediately that in the case of bad news and the Brownian case, the regulator does not face a direct conflict between the objectives of increasing quality at entry ϕ^e and increasing production of the experience good Y , at least for small entry fees. In the case of bad news, the difference $V_H(\phi) - V_L(\phi)$ is increasing in ϕ for any $\phi > \bar{\phi}$, so a regulator who increases F increases ϕ^e and Y simultaneously, and then wants to increase F to drive ϕ^e arbitrarily close to 1. Therefore, in the bad news case, a policy of increasing the entry cost F to drive the average quality of entrants ϕ^e towards one is always welfare improving. When the signal structure follows a Brownian diffusion, the difference $V_H(\phi) - V_L(\phi)$ is increasing in ϕ for ϕ close to $\bar{\phi}$, hence inducing an increase in both ϕ^e and Y , guaranteeing a higher welfare from a higher F , at least initially. But the regulator faces a direct conflict between the objectives of increasing quality at entry ϕ^e and increasing production of the experience good Y once the reputation at entry has achieved some level ϕ^* strictly above the exit threshold $\bar{\phi}$. Finally, in the case of good news, the difference $V_H(\phi) - V_L(\phi)$ is decreasing in ϕ for any $\phi > \bar{\phi}$, and the regulator faces a direct conflict between the objectives of increasing quality ϕ^e and increasing production of the experience good Y . Still, as proven in Proposition 5, there is a level of reputation at entry $\phi^e > \bar{\phi}$ for which welfare is maximized, and hence even in this case the optimal entry fee is strictly positive. This results because

¹⁹While the entry of high quality firms also carries a cost in terms of reduced consumption of the numeraire good, the marginal welfare gain from increased steady-state output, for given ϕ^e , is strictly positive: $U'(Y) - C/A(\phi^e) > U'(Y) - C/D(\phi^e) = 0$

for small entry regulation, the loss in market size is second-order relative to the gains in average quality.

Remark on the welfare impact of regulation: The magnitude of the impact of regulation on welfare is non-monotonic in the precision of signals. When the precision of the signals goes to zero, entry costs do not increase Y much, since the difference between value functions is negligible. However, since the production of the experience good is very small, the marginal welfare gain can still be important. At the other extreme, when the precision of the signal goes to infinity, there is not much room for improvement on the market outcome to be achieved through regulation, since this outcome is already close to the unconstrained first best. This suggests that regulatory policies are more effective in improving the outcome of a market with spot prices when the precision of signals is intermediate.

Remarks on other regulatory tools: Naturally, a regulator can use other regulatory tools, in addition to entry costs, to increase welfare if he or she has access to intermediate levels of information. For example, if a regulator can observe that a firm remains active, then a regulator can also offer operation subsidies or impose taxes on active firms. Such subsidies or taxes may be helpful if they discourage low-quality firms from continuing operations, or disproportionately reward high-quality firms that remain active. Depending on this trade-off, operational taxes or subsidies may be helpful in increasing welfare even further than can be done with regulatory entry costs.

It may also be possible to consider policies that subsidize variables more likely to be experienced by high-quality firms, such as age, or that punish variables more likely to be experienced by low-quality firms, such as exit. However, such variables are only imperfect signals of reputations, and their incidence may be influenced through the design of the tax policy (e.g. when low-quality firms decide to stay around for longer to benefit from age-contingent subsidies). These tools are likely not as effective as subsidies to the experience good that are transferred to intermediate goods producers reflecting their reputation with precision, or other regulatory interventions that are not as easily manipulable. More generally, the regulator can use a wide array of policy combinations to impact welfare. Our model offers a simple framework for analyzing the impact of such policies.

5 Conclusions

We have provided a tractable model for analyzing the interaction of regulation and reputation on spot markets with a lemons problem and with imperfect signals about sellers' quality. We have argued that the lemons problem in this environment is a problem of commitment and not a problem of information. The lemons problem can essentially be eliminated if buyers can commit to offer sellers incentives strong enough to invest in high-quality so as to improve their reputation. When a regulator can design taxes and subsidies contingent on sellers' reputation, a simple taxing scheme can provide the commitment required to mitigate the lemons problem.

Even if a regulator does not have the ability to tax or subsidize sellers contingent on their reputation, that regulator still has the ability to improve welfare by mitigating the lemons problem in a spot market equilibrium by imposing a positive fixed entry cost that is then rebated lump sum to households. Although the regulator potentially faces a trade-off between increasing the average quality of entering sellers and restricting the overall volume of production, we show under our three signal structures that this trade-off is resolved in favor of increasing quality, at least for small entry costs.

An important next step in understanding optimal regulation in the presence of reputation concerns is considering moral hazard problems at each moment. We have assumed that quality is fixed as the result of a one-time investment decision. A large literature examines outcomes when sellers must maintain ongoing investments to preserve quality.²⁰ We anticipate that our first main result will extend to this setting: the problem of moral hazard arises because buyers cannot commit to pay sellers prices contingent on reputation that are high enough to preserve the incentives to invest in quality. We conjecture, then, that a regulator with sufficient flexibility to design transfers contingent on reputation would be able to mitigate both the lemons problem and the moral hazard problem associated with investments to maintain quality. We are not able to derive these results formally, as the required transfer schemes are likely to be non-linear in reputation and thus outside the scope of what we can solve at this time.

²⁰See, for example, Marvel and McCafferty (1984), Maksimovic and Titman (1991), and, more recently, Board and Meyer-ter Vehn (2010).

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A Appendix.

A.1 Proof Lemma 1

We use standard recursive techniques to prove this lemma. Let B represent the set of bounded real functions with domain $[0, 1]$ with the sup norm. We can specify the perpetuity value of $y(1)\Delta$ as an upper bound on functions in this space and zero as a lower bound on functions in this space. This space is a complete normed metric space. Use the Bellman equation (11) to define operators $T_i : B \rightarrow B$ for $i \in \{H, L\}$ in the standard manner. Since functions $f \in B$ are bounded between the perpetuity value of $y(1)\Delta$ and 0, and $T_i(f)$ satisfies the same bounds, then it is also in B .

We now prove that T_i satisfy Blackwell’s sufficient conditions for a contraction (monotonicity and discounting), which guarantees there is a unique solution to the recursive equations (11) above. We let V_i denote the unique fixed point of T_i .

a. Monotonicity: For any $f, h \in B(0, 1)$, with $f(\phi) \geq h(\phi)$ for all $\phi \in (0, 1)$ then $T_i(f)(\phi) \geq T_i(h)(\phi)$ for all $\phi \in (0, 1)$. This is immediate.

b. Discounting: There is some constant $\varrho \in (0, 1)$ such that $(T_i)(f)(\phi) + \varrho a \geq (T_i)(f + a)(\phi)$, for all constants $a \geq 0$. Direct computation for all $a \geq 0$ gives

$$\begin{aligned} T_i(f + a)(\phi) &= \max_{\omega} \omega \left[y(\phi^c)\Delta + \exp(-r\Delta) \left(\alpha_i(\Delta) f(\phi^g(\phi^c) + a) + (1 - \alpha_i(\Delta)) f(\phi^b(\phi^c) + a) \right) \right] \\ &\leq T_i(f)(\phi) + \exp(-r\Delta)a, \end{aligned}$$

which gives the result since $\exp(-r\Delta) \in (0, 1)$. Note here in the last inequality we are allowing the firm to continue (choose $\omega = \exp(-\delta\Delta)$) for the $a > 0$ separately from the choice of ω in $T_i(f)$, and that is what generates the inequality.

A.2 Proof Proposition 1

In what follows we characterize equilibria under the two “reasonable” restrictions on buyers’ interim beliefs ϕ^c defined in Assumptions 1 and 2, and then we show that

the equilibrium characterized in Proposition 1 is unique when the model converges to continuous time ($\Delta \rightarrow 0$).

Using the Bellman equations developed in Lemma 1, Assumption 1 implies that value functions $V_i(\phi)$ are weakly increasing (since operators T_i map weakly increasing functions to weakly increasing functions under this restriction) and $V_H(\phi) \geq V_L(\phi)$ (since $T_H(V_L)(\phi) \geq T_L(V_L)(\phi) = V_L(\phi)$, because $\alpha_H(\Delta) > \alpha_L(\Delta)$ and $V_L(\phi)$ is weakly increasing in ϕ).

This first assumption, however, is not enough to rule out multiplicity of equilibrium generated by off-equilibrium beliefs, simply because the beliefs $\phi^c(\phi) = 0$ for all ϕ satisfy monotonic updating, as do beliefs that set $\phi^c(\phi) = 0$ over regions $[0, k]$. This is the reason we introduce Assumption 2. If Bayes' rule can be used, then $\phi^c(\phi) \geq \phi$ iff $\omega^H(\phi, \Delta) \geq \omega^L(\phi, \Delta)$. This assumption itself is stronger in that we require it of beliefs even when Bayes' rule cannot be used.

The combination of these two assumptions implies that there exists a level of reputation $\bar{\phi} < 1$ such that both $V_H(\phi)$ and $V_L(\phi)$ are strictly greater than zero for all $\phi > \bar{\phi}$. This result follows from the observation that $V_i(\phi) \geq \max(0, y(\phi)\Delta)$ when Assumption 2 is satisfied and we have $y(\phi) > 0$ for sufficiently high ϕ (a firm can always operate for one period, earning $y(\phi^c(\phi))\Delta$ and then exit). Assumption 2 together with Assumption 1 (weakly increasing value functions) implies that the region for which $V_L(\phi) > 0$ is an interval that we denote by $(\bar{\phi}, 1]$ and the region for which $V_H(\phi) > 0$ is an interval that we denote by $(\underline{\phi}, 1]$, with $\underline{\phi} \leq \bar{\phi}$.

We next prove that equilibrium beliefs, value functions, and exit strategies in all equilibria with beliefs that satisfy Assumptions 1 and 2 have the following form.

Lemma 2 *Characterization of equilibrium beliefs, value functions, and exit strategies*

Under assumptions 1 and 2 on buyers' beliefs, the equilibrium beliefs, value functions, and continuation strategies have the following form. There exists a value of $\bar{\phi} \in (0, 1)$, which is pinned down uniquely independently of the specification of beliefs $\phi^c(\phi)$ for $\phi < \bar{\phi}$, such that $V_L(\phi) = 0$ for $\phi \leq \bar{\phi}$ and $V_L(\phi) > 0$ for $\phi > \bar{\phi}$. There exists a value of $\underline{\phi} \in [0, \bar{\phi})$ such that $V_H(\phi) = 0$ for $\phi \leq \underline{\phi}$, $V_H(\phi)$ is constant and strictly positive on $(\underline{\phi}, \bar{\phi}]$ and $V_H(\phi)$ is strictly increasing for $\phi > \bar{\phi}$. The associated interim beliefs are $\phi^c(\phi) = \phi$ for $\phi \geq \bar{\phi}$, $\phi^c(\phi) = \bar{\phi}$ for $\phi \in (\underline{\phi}, \bar{\phi})$, and $\phi^c(\phi)$ is any function bounded above by $\underline{\phi}$ and that satisfies assumptions 1 and 2 on $\phi \in [0, \underline{\phi}]$. The continuation strategies have the form $\omega^H(\phi, \Delta) = \exp(-\delta\Delta)$ for all $\phi > \underline{\phi}$, $\omega^H(\phi, \Delta) = 0$ for all $\phi \leq \underline{\phi}$, $\omega^L(\phi, \Delta) = \exp(-\delta\Delta)$ for $\phi > \bar{\phi}$, $\omega^L(\phi, \Delta)$ is set greater than zero so that $\phi^c(\phi) = \bar{\phi}$ for $\phi \in (\underline{\phi}, \bar{\phi}]$ and $\omega^L(\phi, \Delta) = 0$ for all $\phi \leq \underline{\phi}$.

Proof We have already shown that given Assumption 2, there exist ϕ sufficiently large such that $V_i(\phi) > 0$. Given Assumption 1, the value functions V_i are non-decreasing with $V_H(\phi) \geq V_L(\phi)$ so that the regions for which V_i are strictly positive

are intervals $(\underline{\phi}, 1]$ and $(\bar{\phi}, 1]$ for high- and low-quality firms respectively, with $\underline{\phi} \leq \bar{\phi}$. The equilibrium requirement that $\omega^i(\phi, \Delta) = \exp(-\delta\Delta)$ whenever $V_i(\phi) > 0$ implies that these must be the continuation strategies for $\phi > \bar{\phi}$ and Bayes' rule then implies that $\phi^c(\phi) = \phi$ for $\phi > \bar{\phi}$. Assumptions 1 and 2 then imply that $\phi^c(\bar{\phi}) = \bar{\phi}$ as well.

Observe next that $V_H(\bar{\phi}) > 0$. To see this, observe that $\phi^g(\bar{\phi}) > \bar{\phi}$ and $\phi^b(\bar{\phi}) < \bar{\phi}$ which implies that $V_L(\phi^g(\bar{\phi})) > 0$ and $V_L(\phi^b(\bar{\phi})) = 0$. This implies that $T_H(V_L)(\bar{\phi}) > T_L(V_L)(\bar{\phi}) \geq 0$, which gives the result.

Note that it is not possible to have an equilibrium with $\bar{\phi} = 0$ as, in this case, $\phi^c(\phi) = \phi$ for all $\phi \in [0, 1]$ and we have already assumed parameters such that $V_i^c(\phi) < 0$ for sufficiently low ϕ under these interim beliefs. Also note that the value of $\bar{\phi}$ is the pinned down uniquely independently of the specification of beliefs $\phi^c(\phi)$ for $\phi < \bar{\phi}$. This last result follows from the fact that $V_L(\phi) = 0$ for all $\phi < \bar{\phi}$. To be specific, note that the most optimistic beliefs that satisfy Assumptions 1 and 2 and the result that $\phi^c(\phi) = \phi$ for all $\phi \geq \bar{\phi}$ have $\phi^c(\phi) = \bar{\phi}$ for all $\phi \in (0, \bar{\phi})$. Likewise, the most pessimistic beliefs that satisfy Assumptions 1 and 2 have $\phi^c(\phi) = \phi$ for all ϕ . It is straightforward to verify that if V_L is the fixed point of the operator T_L with beliefs $\phi^c(\phi) = \phi$ for all $\phi \geq \bar{\phi}$ and $\phi^c(\phi) = \bar{\phi}$ for all $\phi \in (0, \bar{\phi})$ and satisfies $V_L(\phi) = 0$ for all $\phi \leq \bar{\phi}$, then that same V_L is the fixed point of the operator T_L with beliefs $\phi^c(\phi) = \phi$ for all ϕ . Hence, $\bar{\phi}$ is independent of the specification of beliefs for $\phi < \bar{\phi}$.

For $\phi \in (\underline{\phi}, \bar{\phi})$ we have $V_H(\phi) > 0$ and hence $\omega^H(\phi, \Delta) = \exp(-\delta\Delta)$. Monotonicity of updating thus requires that $\omega^L(\phi, \Delta) > 0$ for ϕ in this region. The definition of $\bar{\phi}$ implies that $V_L(\phi) = 0$ in this region. Hence, we must have $V_L^c(\phi) = 0$ in this region. The strict monotonicity of $y(\phi)$ and the weak monotonicity of $V_L(\phi)$ implies that we must have $\phi^c(\phi)$ constant for ϕ in this region. The only constant that satisfies assumption 2 is $\phi^c(\phi) = \bar{\phi}$, which pins down the continuation strategy $\omega^L(\phi, \Delta)$ in this region.

What remains is to characterize equilibrium behavior for $\phi \leq \underline{\phi}$. In this region, we have $V_i(\phi) = 0$. Note that since $V_H^c(\bar{\phi}) > 0$, we must have $\phi^c(\phi) < \bar{\phi}$ in this region. The strict monotonicity of $y(\phi)$ and weak monotonicity of $V_L(\phi)$ then implies that $V_L^c(\phi^c(\phi)) < 0$ in this region and thus $\omega^L(\phi, \Delta) = 0$ in this region. To ensure that $V_H^c(\bar{\phi}) < 0$, we must then have $\phi \leq \underline{\phi}$ in this region as well. Q.E.D.

This ends the proof of Lemma 2. Now we argue that there is a value of Δ low enough such that there is a unique equilibrium in which $\underline{\phi} \rightarrow 0$. By doing so, we will have proved part (ii) of Proposition 1, as this result will imply that $\phi^c(\phi^b(\bar{\phi})) = \bar{\phi}$.

First, note that Assumption 2 imposes $\phi^c \geq \phi$, then the most pessimistic beliefs consistent with such assumption are $\phi^c = \phi$. In this case, the value of $\underline{\phi}$ is also pinned down uniquely independently of the specification of beliefs $\phi^c(\phi)$ for $\phi < \underline{\phi}$. As in the case of $\bar{\phi}$, this result follows from the fact that $V_H(\phi) = 0$ for all $\phi < \underline{\phi}$. By Assumption 1 $V_H(\phi) \geq V_L(\phi)$. Since $\alpha_H(\Delta) > \alpha_L(\Delta)$, then $\underline{\phi} < \bar{\phi}$ for all $\Delta > 0$. Furthermore, $\underline{\phi} < \bar{\phi}$ also holds strictly in the continuous time limit, as $\Delta \rightarrow 0$, (we show this later using

analytical value functions in continuous time, but intuitively this comes from a higher drift in reputation for high-quality firms also in the limit). This implies there exists a Δ small enough such that, $\underline{\phi} < \phi^b(\bar{\phi}) < \bar{\phi}$. Assuming $\phi^e(\phi^b(\bar{\phi})) = \phi^b(\bar{\phi})$, $V_H(\phi^b(\bar{\phi})) > 0$ and $V_L(\phi^b(\bar{\phi})) < 0$. However, this implies $\omega^H(\phi^b(\bar{\phi})) = 1$ and $\omega^L(\phi^b(\bar{\phi})) = 0$, which are inconsistent with pessimistic beliefs $\phi^e(\phi^b(\bar{\phi})) = \phi^b(\bar{\phi})$. This implies that for values of Δ low enough the equilibrium interim beliefs and continuation strategies described in point (ii) of Proposition 1 are unique.

To complete the proof of Proposition 1, note that points (i) and (iii) follow from the free entry requirement that $V_L(\phi^e) = 0$. Point (iv) of Proposition 1 is implied by subtracting the free entry requirement for low-quality firms from that for high-quality firms.

A.3 Value Functions in Continuous Time and Their Properties

Here we obtain analytical solutions for the value functions $V_i(\phi)$, under bad news, good news, and Brownian diffusion, for general payment functions $q(\phi)$. We also show the properties described in the text hold when the function $q(\phi)$ is linear in ϕ , as we assume is the case with spot prices where $q(\phi) = y(\phi)$. In this section, for notational simplicity we denote

$$y(\phi) = a_1\phi - a_0,$$

where $a_1 = y(1) - y(0) > 0$ and $a_0 = -y(0) > 0$. For simplicity we also define $\hat{r} = r + \delta$.

A.3.1 Bad News

In this case $dS_t \in \{0, 1\}$, which means there is either a signal or no signal at each t . The bad news case is defined by $Pr(dS_t = 1|H) = 0$ and $Pr(dS_t = 1|L) = \lambda dt$, which means there is a positive Poisson arrival only for low-quality firms. When a signal arrives, the firm is revealed to be of low quality and hence the public belief about its quality drops to $\phi = 0$. With this reputation, the firm would never be able to sell its output at a non-negative price. Thus, following this event, it is optimal for the firm to cease production and exit as quickly as possible.

It is convenient to use a transformed variable $l = (1 - \phi)/\phi : [0, 1] \rightarrow (\infty, 0]$ to summarize the reputation level of a firm. The evolution of l is determined by

$$\frac{dl_t}{dt} = \left[\frac{Pr(dS_t|L) - Pr(dS_t|H)}{Pr(dS_t|H)} \right] l_t.$$

When bad news arrives (i.e., $dS_t = 1$)

$$\frac{dl_t}{dt} = \left[\frac{\lambda dt - 0}{0} \right] l_t = \infty,$$

and reputation jumps immediately to $l = \infty$. Since $\phi = \frac{1}{1+l}$, this means reputation drops immediately to $\phi = 0$.

While there are no news (i.e., $dS_t = 0$), reputation increases. From the Poisson distribution, the probability that a high-quality firm does not generate news for an interval of time of length t is $e^{-\lambda t}$. Then, after a time interval of length t of no news, accumulating the change in reputation

$$l_t = \left[\frac{Pr(S_t = 0|L)}{Pr(S_t = 0|H)} \right] l_0 = \left[\frac{e^{-\lambda t}}{1} \right] l_0 = e^{-\lambda t} l_0,$$

where $l_0 = \frac{1-\phi^e}{\phi^e}$. This means l_t is decreasing (reputation is increasing) over time at a rate $\lambda \in [0, \infty)$. While there is no news, the evolution of reputation for firms with high and low-quality is the same. After bad news, a firm exits and obtains zero thereafter. Then, the value functions for both types only differ in their discount factor.

Lemma 3 *Value functions for general profit functions and bad news*

A value function for a low-quality firm with reputation l , for a general $q(l)$, is

$$\hat{V}_L(l) = \int_{\tau=0}^{\infty} e^{-(\hat{r}+\lambda)\tau} q(e^{-\lambda\tau}l) d\tau,$$

and the value function for a high-quality firm with reputation l is

$$\hat{V}_H(l) = \int_{\tau=0}^{\infty} e^{-\hat{r}\tau} q(e^{-\lambda\tau}l) d\tau.$$

Solving explicitly the integrals for the case of linear payoffs and no marginal costs, $q(\phi) = y(\phi) = a_1\phi - a_0$ (hence $q(l) = \frac{a_1}{1+l} - a_0$),

$$\hat{V}_L(l) = \frac{1}{\hat{r} + \lambda} [a_1 \Upsilon_{m_{\hat{r}+\lambda}}(-l) - a_0], \quad (24)$$

$$\hat{V}_H(l) = \frac{1}{\hat{r}} [a_1 \Upsilon_{m_{\hat{r}}}(-l) - a_0], \quad (25)$$

where $\Upsilon_m(-l) = {}_2F_1(1, m; m+1, -l)$ is an hypergeometric function, and

$$m_{\hat{r}} = \frac{\hat{r}}{\lambda} > 0 \quad \text{and} \quad m_{\hat{r}+\lambda} = \frac{\hat{r} + \lambda}{\lambda} = 1 + m_{\hat{r}}.$$

The hypergeometric function has well-defined properties when $m > 0$. In particular, it is monotonically increasing in ϕ (from 0 to 1) and monotonically decreasing in m .

$$\Upsilon_m \left(-\frac{1-\phi}{\phi} \right) : [0, 1] \rightarrow [0, 1] \quad \text{and} \quad \frac{\partial \Upsilon_m(\cdot)}{\partial m} < 0.$$

Now we denote $V_i(\phi) = \hat{V}_i(l)$ for all ϕ and $i \in \{L, H\}$. Since $\lim_{\phi \rightarrow 0} V_L(\phi) = -\frac{a_0}{\hat{r}+\lambda} < 0$ with no exit, there is a $\phi = \bar{\phi}$ such that $V_L(\bar{\phi}) = 0$. Hence $\bar{\phi}$ is the highest reputation at which low-quality firms are indifferent between exiting or not. As discussed above, exiting strategies imply that in equilibrium, no firm has a reputation below $\bar{\phi}$. Value functions in the range $[\bar{\phi}, 1]$ are

$$\begin{aligned} V_L(\phi) &: [\bar{\phi}, 1] \rightarrow [0, \frac{a_1 - a_0}{\hat{r} + \lambda}] \\ V_H(\phi) &: [\bar{\phi}, 1] \rightarrow [V_H(\bar{\phi}), \frac{a_1 - a_0}{\hat{r}}], \end{aligned}$$

where $V_H(\bar{\phi}) = \frac{1}{\hat{r}} \left[a_1 \Upsilon_{m_{\hat{r}}} \left(-\frac{1-\bar{\phi}}{\bar{\phi}} \right) - a_0 \right] > 0$ (since $m_{\hat{r}} < m_{\hat{r}+\lambda}$).

Lemma 4 *Derivatives with linear payoffs*

For all $\phi \in [\bar{\phi}, 1]$ and all $\hat{r} \geq 0$

$$V'_L(\phi) > 0 \quad \text{and} \quad V'_H(\phi) > 0$$

First recall that derivatives of hypergeometric functions are

$$\frac{\partial \Upsilon_{m_{\hat{r}}}(-l)}{\partial l} = \Upsilon'_{m_{\hat{r}}}(-l) = m_{\hat{r}} l \left[\frac{l}{1+l} - \Upsilon_{m_{\hat{r}}}(-l) \right] < 0.$$

Then

$$\hat{V}'_L(l) = \frac{a_1 l}{\lambda} \left[\frac{l}{1+l} - \Upsilon_{m_{\hat{r}+\lambda}}(-l) \right], \quad (26)$$

$$\hat{V}'_H(l) = \frac{a_1 l}{\lambda} \left[\frac{l}{1+l} - \Upsilon_{m_{\hat{r}}}(-l) \right], \quad (27)$$

which are negative for all $[0, \bar{l}]$. In particular, at \bar{l} , $\hat{V}_L(\bar{l}) = 0$, then $\Upsilon_{m_{\hat{r}+\lambda}}(-\bar{l}) = \frac{a_0}{a_1}$.

The lemma follows from

$$V'_i(\phi) = V'_i(l) \frac{\partial l}{\partial \phi} = -\frac{V'_i(l)}{\phi^2}.$$

Applying the properties of hypergeometric functions, the ratio $V_L(\phi)/V_H(\phi)$ is monotonically increasing, from 0 to $\frac{\hat{r}}{\hat{r}+\lambda}$, and the difference $V_H(\phi) - V_L(\phi)$ is also monotonically increasing, from $V_H(\bar{\phi})$ to $\frac{\lambda}{\hat{r}+\lambda} V_H(1)$.

A.3.2 Good News

In this case $Pr(dS_t = 1|H) = \lambda dt$ and $Pr(dS_t = 1|L) = 0$. When a signal arrives, the firm is revealed to be of high-quality, and hence the public belief ϕ regarding this firm jumps up to $\phi = 1$. After good news, the firm maintains a reputation of $\phi = 1$ until it exits exogenously.

Again, we use the variable $l = (1 - \phi)/\phi$. When good news arrives (i.e., $dS_t = 1$)

$$\frac{dl}{dt} = \left[\frac{0 - \lambda dt}{\lambda dt} \right] l_t = -l_t,$$

and reputation jumps immediately to $l = 0$, or $\phi = 1$.

While there is no news (i.e., $dS_t = 0$), reputation decreases. After a time interval of length t of no news, accumulating the change in reputation

$$l_t = \left[\frac{Pr(S_t = 0|L)}{Pr(S_t = 0|H)} \right] l_0 = \left[\frac{1}{e^{-\lambda t}} \right] l_0 = e^{\lambda t} l_0,$$

which means l_t is increasing (reputation is decreasing) over time at a rate λ .

Denoting $q(l(1))$ the payoffs for a firm with $\phi = 1$, the value function for a firm that has experienced good news is,

$$V(l(1)) = \frac{q(l(1))}{\hat{r}}.$$

There is a key difference between good news and bad news. Under bad news, reputation only increases, which means endogenous exit never occurs unless a bad signal is revealed. Under good news, reputation continuously decreases and low-quality firms that hit $\bar{\phi}$ will exit at a rate λ such that reputation never drifts below $\bar{\phi}$.

Lemma 5 *Value functions for general profit functions and good news*

The value function for a low-quality firm with reputation l is

$$\hat{V}_L(l) = \int_{\tau=0}^{T(l)} e^{-\hat{r}\tau} q(e^{\lambda\tau} l) d\tau. \quad (28)$$

The value function for a high-quality firm with reputation l is

$$\hat{V}_H(l) = \int_{\tau=0}^{T(l)} e^{-(\hat{r}+\lambda)\tau} \left[q(e^{\lambda\tau} l) + \lambda \frac{q(l(1))}{\hat{r}} \right] d\tau + \int_{\tau=T(l)}^{\infty} e^{-(\hat{r}+\lambda)(\tau-T(l))} \lambda \frac{q(l(1))}{\hat{r}} d\tau, \quad (29)$$

where $T(l)$ is the time required for l to increase up to $\bar{l} = \frac{1-\bar{\phi}}{\bar{\phi}}$.

$$T(l) = \frac{\log(\bar{l}/l)}{\lambda} > 0. \quad (30)$$

In the case of linear payoffs and no marginal costs, the reputation at which low-quality firms are willing to exit is given by $q(\bar{l}) = \frac{a_1}{1+\bar{l}} - a_0 = 0$. In this case, \bar{l} is given by the reputation below which profits are negative. Then $\bar{l} = \frac{a_1 - a_0}{a_0}$ and $T(l)$ is given following equation (30). The value functions are

$$\hat{V}_L(l) = \frac{1}{\hat{r}} \left[a_1 \left(1 - \Upsilon_{m_{\hat{r}}} \left(-\frac{1}{l} \right) \right) - a_0 \right] - \frac{e^{-\hat{r}T(l)}}{\hat{r}} \left[a_1 \left(1 - \Upsilon_{m_{\hat{r}}} \left(-\frac{a_0}{a_1 - a_0} \right) \right) - a_0 \right], \quad (31)$$

$$\hat{V}_H(l) = \frac{1}{\hat{r} + \lambda} \left[a_1 \left(1 - \Upsilon_{m_{\hat{r} + \lambda}} \left(-\frac{1}{l} \right) \right) - a_0 + \lambda \frac{a_1 - a_0}{\hat{r}} \right] - \frac{e^{-(\hat{r} + \lambda)T(l)}}{\hat{r} + \lambda} \left[a_1 \left(1 - \Upsilon_{m_{\hat{r} + \lambda}} \left(-\frac{a_0}{a_1 - a_0} \right) \right) - a_0 \right]. \quad (32)$$

Now we denote $V_i(\phi) = \hat{V}_i(l)$ for all ϕ and $i \in \{L, H\}$. Since $T(l(1)) = \infty$, using the previously discussed properties of the hypergeometric functions,

$$V_L(\phi) : [\bar{\phi}, 1] \rightarrow [0, \frac{a_1 - a_0}{\hat{r}}],$$

$$V_H(\phi) : [\bar{\phi}, 1] \rightarrow [\frac{\lambda}{\hat{r} + \lambda} \frac{a_1 - a_0}{\hat{r}}, \frac{a_1 - a_0}{\hat{r}}].$$

Lemma 6 *Derivatives with linear payoffs*

For all $\hat{r} \geq 0$,

$$V'_L(\phi) > 0 \quad \text{and} \quad V'_H(\phi) > 0 \quad \text{for all } \phi \in (\bar{\phi}, 1]$$

$$V'_L(\bar{\phi}) = V'_H(\bar{\phi}) = 0$$

$$V''_L(\bar{\phi}) = V''_H(\bar{\phi}) > 0.$$

First recall that derivatives of hypergeometric functions are

$$\frac{\partial \Upsilon_{m_{\hat{r}}}(-1/l)}{\partial l} = \Upsilon'_{m_{\hat{r}}}(-1/l) = m_{\hat{r}} l \left[\frac{l}{1+l} - \Upsilon_{m_{\hat{r}}}(-1/l) \right] \left(-\frac{1}{l^2} \right)$$

and define

$$X_{m_{\hat{r}}} = (a_1 - a_0) - a_1 \Upsilon_{m_{\hat{r}}} \left(-\frac{1}{l} \right).$$

Taking derivatives

$$V'_L(l) = -\frac{a_1}{\hat{r}} \Upsilon'_{m_{\hat{r}}} \left(-\frac{1}{l} \right) + \frac{\hat{r} T'^{-\hat{r}T(l)}}{\hat{r}} X_{m_{\hat{r}}}$$

and plugging in the expressions above for $\Upsilon'_{m_{\hat{r}}}(-\frac{1}{l})$ and $T'(l)$, we have that

$$V'_L(l) = \frac{a_1}{\lambda l} \left[\frac{l}{1+l} - \Upsilon_{m_{\hat{r}}}(-\frac{1}{l}) \right] - \frac{e^{-\hat{r}T(l)}}{\lambda l} \left[a_1 \left(1 - \Upsilon_{m_{\hat{r}}}(-\frac{1}{l}) \right) - a_0 \right]$$

and similarly for high-quality firms

$$V'_H(l) = \frac{a_1}{\lambda l} \left[\frac{l}{1+l} - \Upsilon_{m_{\hat{r}+\lambda}}(-\frac{1}{l}) \right] - \frac{e^{-(\hat{r}+\lambda)T(l)}}{\lambda l} \left[a_1 \left(1 - \Upsilon_{m_{\hat{r}+\lambda}}(-\frac{1}{l}) \right) - a_0 \right].$$

These derivatives are negative for $l \in [0, \bar{l})$ and exactly zero at $\bar{l} = \frac{a_1 - a_0}{a_0}$ (this is the point $\bar{\phi}$ at which profits are zero, $a_1\bar{\phi} - a_0 = 0$), regardless of the \hat{r} used. Since

$$V'_i(\phi) = V'_i(l) \frac{\partial l}{\partial \phi} = -\frac{V'_i(l)}{\phi^2}.$$

Then the first derivatives of value function are positive for $\phi \in (\bar{\phi}, 1]$ and

$$V'_L(\bar{\phi}) = V'_H(\bar{\phi}) = 0 \quad \text{for all } \hat{r}. \quad (33)$$

Now we take second derivatives, for low-quality firms

$$V''_L(l) = -\frac{a_1}{\lambda(1+l)^2} \left[1 - \frac{\hat{r}(1+l)}{\lambda l} \right] + \frac{\lambda - \hat{r}}{\lambda^2 l^2} \left[a_1 \Upsilon_{m_{\hat{r}}}(-\frac{1}{l}) + e^{-\hat{r}T(l)} X_{m_{\hat{r}}} \right]$$

and similarly for high-quality firms

$$V''_H(l) = -\frac{a_1}{\lambda(1+l)^2} \left[1 - \frac{(\hat{r} + \lambda)(1+l)}{\lambda l} \right] - \frac{\hat{r}}{\lambda^2 l^2} \left[a_1 \Upsilon_{m_{\hat{r}+\lambda}}(-\frac{1}{l}) + e^{-(\hat{r}+\lambda)T(l)} X_{m_{\hat{r}+\lambda}} \right].$$

Evaluating these expressions at \bar{l} for low-quality firms, for all \hat{r}

$$V''_L(\bar{l}) = \frac{\lambda a_0^3}{\lambda^2 a_1 (a_1 - a_0)}$$

and similarly for high-quality firms

$$V''_L(\bar{l}) = V''_H(\bar{l}) = \frac{a_0}{\lambda \bar{l} (1 + \bar{l})} > 0.$$

Again, using transformed derivatives to compute derivatives with respect to ϕ

$$V''_i(\phi) = \frac{V''_i(l)}{\phi^4} + \frac{2V'_i(l)}{\phi^3}.$$

This implies that, at $\bar{\phi}$,

$$V''_L(\bar{\phi}) = V''_H(\bar{\phi}) > 0 \quad \text{for all } \hat{r}. \quad (34)$$

Finally, applying the properties of hypergeometric functions, the ratio $V_L(\phi)/V_H(\phi)$ is monotonically increasing, from 0 to 1, and the difference $V_H(\phi) - V_L(\phi)$ is monotonically decreasing, from $\frac{\lambda}{\hat{r}+\lambda} \frac{a_1 - a_0}{\hat{r}}$ to 0.

A.3.3 Brownian diffusion

Assume now the signal process follows a Brownian diffusion

$$dS_t = \mu_i dt + \sigma dZ_t,$$

where $i = \{L, H\}$, drifts depend on the firm's type $\mu_H > \mu_L$ and the noise σ is the same for both types.

The following proposition shows that reputation, for both high- and low-quality firms, also follows a Brownian diffusion process when based purely on signals. As discussed in Proposition 1, given the equilibrium exit rates, this is also the updating rule for all $\phi > \bar{\phi}$, whereas the updating for all $\phi \leq \bar{\phi}$ follows an immediate jump up to $\bar{\phi}$.

Lemma 7 *Reputation process based on Brownian diffusion signals.*

The reputation process high-quality firms expect is a submartingale

$$d\phi_t^H = \frac{\lambda^2(\phi_t)}{\phi_t} dt + \lambda(\phi_t) dZ_t, \quad (35)$$

and the reputation process low-quality firms expect is a supermartingale

$$d\phi_t^L = -\frac{\lambda^2(\phi_t)}{(1-\phi_t)} dt + \lambda(\phi_t) dZ_t, \quad (36)$$

where $\lambda(\phi_t) = \phi_t(1-\phi_t)\zeta$ and $\zeta = \frac{\mu_H - \mu_L}{\sigma}$ is the signal-to-noise ratio.

Proof The activities of the two types of firms induce two different probability measures over the paths of the signal S_t . Fix a prior ϕ^e and assume exogenous exit. Then reputation evolves following the equation:

$$\phi_t = \frac{\phi^e Pr(S_t|H)}{\phi^e Pr(S_t|H) + (1-\phi^e) Pr(S_t|L)}$$

or

$$\phi_t = \frac{\phi^e \xi_t}{\phi^e \xi_t + (1-\phi^e)}, \quad (37)$$

where ξ_t is the ratio between the likelihood that a path $S_s : s \in [0, t]$ arises from type H and the likelihood that it arises from type L . As in Faingold and Sannikov (2011), from Girsanov's Theorem, this ratio follows a Brownian diffusion characterized by $\mu_\xi = 0$ and $\sigma_\xi = \xi_t \zeta$,

$$d\xi_t = \xi_t \zeta dZ_s^L, \quad (38)$$

where $\zeta = \frac{\mu_H - \mu_L}{\sigma}$ and $dZ_s^L = \frac{dS_t - \mu_L dt}{\sigma}$ is a Brownian diffusion under the probability measure generated by type L .²¹

By Ito's formula,

$$d\phi = [\phi' \mu_\xi + \frac{1}{2} \phi'' \zeta^2] dt + \phi' \sigma_\xi dZ_s^L$$

$$d\phi_t = -\frac{1}{2} \frac{2\phi^{e2}(1 - \phi^e)}{(\phi^e \xi_t + (1 - \phi^e))^3} \xi_t^2 \zeta^2 dt + \frac{\phi^e(1 - \phi^e)}{(\phi^e \xi_t + (1 - \phi^e))^2} \xi_t \zeta dZ_s^L,$$

and from equation (37) we can express it in terms of ϕ_t rather than ϕ^e

$$d\phi_t = -\phi_t^2(1 - \phi_t)\zeta^2 dt + \phi_t(1 - \phi_t)\zeta dZ_s^L$$

$$d\phi_t = \phi_t(1 - \phi_t)\zeta[dZ_s^L - \phi_t \zeta dt]$$

replacing by the definition of dZ_s^L ,

$$d\phi_t = \lambda(\phi_t) dZ_t^\phi, \quad (39)$$

where $dZ_t^\phi = \frac{1}{\sigma}[dS_t - (\phi_t \mu_H + (1 - \phi_t)\mu_L)dt]$ and

$$\lambda(\phi_t) = \phi_t(1 - \phi_t) \frac{\mu_H - \mu_L}{\sigma}. \quad (40)$$

Conversely, suppose that ϕ_t is a process that solves equation (39). Define ξ_t using equation (37),

$$d\xi_t = -\frac{1 - \phi^e}{\phi^e} \frac{\phi_t}{1 - \phi_t}$$

By applying Ito's formula again, ξ_t satisfies equation (38). This implies ξ_t is the ratio between the likelihood that a path $S_s : s \in [0, t]$ arises from type H and the likelihood it arises from type L . Hence, ϕ_t is determined by Bayes' rule.

Finally, consider that different types will have different paths, that in expectation will move their reputation. Replacing dS_t^i in dZ_t^ϕ in equation (39) for the two different types of firms, deliver equations (35) and (36). Q.E.D.

Four clear properties arise from inspecting equations (35) and (36). First, high-quality firms expect a positive drift in their evolution of reputation, whereas low-quality firms expect a negative drift. Second, when reputation ϕ_t is either 0 or 1, drifts and volatilities are zero, which means at those points reputation do not change, for both high- and low-quality firms. Third, reputation varies more at intermediate levels of

²¹It is also possible to solve the problem defining $\xi_t = \frac{Pr(S_t|L)}{Pr(S_t|H)}$ such that $\phi_t = \frac{\phi^e}{\phi^e + (1 - \phi^e)\xi_t}$, where $d\xi_t = \xi_t \zeta dZ_s^H$.

ϕ_t , and volatilities are larger. Finally, the drift for high-quality firms is higher than for low-quality firms for bad reputations and lower for good reputations, since ϕ_t is in the denominator of the drift for high-quality firms, whereas $(1 - \phi_t)$ is in the denominator of the drift for low-quality firms.

Lemma 8 *The ordinary differential equations that characterize the value functions for high and low-quality firms are*

$$\hat{r}\rho V_L(\phi) = \rho q(\phi) - \phi^2(1 - \phi)V_L'(\phi) + \frac{1}{2}\phi^2(1 - \phi)^2 V_L''(\phi), \quad (41)$$

$$\hat{r}\rho V_H(\phi) = \rho q(\phi) + \phi(1 - \phi)^2 V_H'(\phi) + \frac{1}{2}\phi^2(1 - \phi)^2 V_H''(\phi), \quad (42)$$

where

$$\rho = \frac{\sigma^2}{(\mu_H - \mu_L)^2}. \quad (43)$$

Proof First, we prove the following lemma.

Lemma 9 *Define Ψ the space of progressively measurable processes ψ_t for all $t \geq 0$ with $E[\int_0^T \psi_t^2 dt] < \infty$ for all $0 < T < \infty$. A bounded process W_t^i for all $t \geq 0$ is the continuation value for type $i = \{H, L\}$ if and only if, for some process ψ_t^i in Ψ , we have*

$$dW_t^i = \hat{r}[W_t^i - q(\phi_t)]dt + \psi_t^i dZ_t. \quad (44)$$

Proof The flow continuation value W_t^i for type i is the expected payoff at time t ,

$$W_t^i = \hat{r}E_t^i \left[\int_t^\infty e^{-\hat{r}(\tau-t)} q(\phi_\tau) d\tau \right].$$

Denote U_t^i the discounted sum of payoffs for type i conditional on the public information available at time t ,

$$U_t^i = \hat{r}E_t^i \left[\int_0^\infty e^{-\hat{r}\tau} q(\phi) d\tau \right] = \int_0^t e^{-\hat{r}\tau} \hat{r}q(\phi_\tau) d\tau + W_t^i. \quad (45)$$

Since U_t^i is a martingale, by the Martingale Representation Theorem, there exists a process ψ_t^i in Ψ such that

$$dU_t^i = e^{-\hat{r}t} \psi_t^i dZ_t. \quad (46)$$

Differentiating (45) with respect to time

$$dU_t^i = e^{-\hat{r}t} \hat{r}q(\phi_t) dt - \hat{r}e^{-\hat{r}t} W_t^i dt + e^{-\hat{r}t} dW_t^i. \quad (47)$$

Combining (46) and (47), we can obtain (44).

Q.E.D.

In a Markovian equilibrium, we know $W_i^i = V_i(\phi_t)$. Since this continuation value depends on the reputation, which follows a Brownian diffusion, using Ito's Lemma,

$$dV_i(\phi) = \left[\mu_{i,\phi} V_i'(\phi) + \frac{1}{2} \sigma_\phi^2 V_i''(\phi) \right] dt + \sigma_\phi V_i'(\phi) dZ, \quad (48)$$

where $\mu_{H,\phi} = \frac{\lambda^2(\phi)}{\phi}$, $\mu_{L,\phi} = -\frac{\lambda^2(\phi)}{(1-\phi)}$ and $\sigma_\phi = \lambda(\phi)$ from Proposition 7.

Matching drifts of equations (44) and (48) for each type i yields the linear second-order differential equation that characterizes continuation values $V_H(\phi)$ and $V_L(\phi)$,

$$\frac{1}{2} \lambda^2(\phi) V_L''(\phi) - \frac{\lambda^2(\phi)}{(1-\phi)} V_L'(\phi) - \hat{r} V_L(\phi) + q(\phi) = 0 \quad (49)$$

and

$$\frac{1}{2} \lambda^2(\phi) V_H''(\phi) + \frac{\lambda^2(\phi)}{\phi} V_H'(\phi) - \hat{r} V_H(\phi) + q(\phi) = 0. \quad (50)$$

Using the definition for $\lambda(\phi)$ from equation (40) we obtain the second-order differential equations in the proposition. Q.E.D.

Solving these ordinary differential equations (ODEs), we can obtain the value functions for high- and low-quality firms. Imposing that these value functions are non-negative introduces endogenous exit, which regulates the reputation process.

Lemma 10 *Value functions for general profit functions and Brownian diffusion*

The value function of low-quality firms with reputation l is

$$\hat{V}_L(l) = K \left\{ \int_0^1 \theta^{\gamma-1} q(\theta l) d\theta - \int_{x/l}^1 \theta^{-\gamma} q(\theta l) d\theta \right\}. \quad (51)$$

The value function of high-quality firms with reputation l is

$$\hat{V}_H(l) = K \left\{ \int_0^1 \theta^{\gamma-2} q(\theta l) d\theta - \int_{\psi/l}^1 \theta^{-\gamma-1} q(\theta l) d\theta + \frac{q(0)}{\gamma} \left(\frac{\psi}{l} \right)^{-\gamma} \right\}, \quad (52)$$

where $\theta = l'/l$,

$$\gamma = \frac{1}{2} + \sqrt{\frac{1}{4} + 2\hat{r}\rho} \quad \text{and} \quad K = \frac{\rho}{\sqrt{\frac{1}{4} + 2\hat{r}\rho}}.$$

Proof We first solve the previous ODEs imposing the boundary conditions at $\phi = 1$ and $\phi = \bar{\phi}$. After simplifying the expressions we take derivatives, which will be useful later to characterize the properties of these value functions.

Changing variables to $l = (1 - \phi) / \phi$ and defining $\hat{V}(l) = V(\phi)$, the ODEs above can be written as

$$\begin{aligned}\hat{r}\rho\hat{V}_L(l) &= \rho q(l) + l\hat{V}'_L(l) + \frac{1}{2}l^2\hat{V}''_L(l) \\ \hat{r}\rho\hat{V}_H(l) &= \rho q(l) + \frac{1}{2}l^2\hat{V}''_H(l).\end{aligned}$$

1. Solving the ODEs

1.a) Solving for $\hat{V}_L(l)$: We conjecture a solution of the form

$$\hat{V}_L(l) = K \left[l^{-\gamma} \int_{x_1}^l l'^{\gamma} \frac{q(l')}{l'} dl' - l^{\gamma-1} \int_{x_2}^l l'^{1-\gamma} \frac{q(l')}{l'} dl' \right]$$

for some parameters γ and K . With this, we have

$$\hat{V}'_L(l) = K \left[(-\gamma) l^{-\gamma-1} \int_{x_1}^l l'^{\gamma} \frac{q(l')}{l'} dl' - (\gamma-1) l^{\gamma-2} \int_{x_2}^l l'^{1-\gamma} \frac{q(l')}{l'} dl' \right]$$

$$\begin{aligned}\hat{V}''_L(l) &= K \left[(-\gamma)(-\gamma-1) l^{-\gamma-2} \int_{x_1}^l l'^{\gamma} \frac{q(l')}{l'} dl' - (\gamma-1)(\gamma-2) l^{\gamma-3} \int_{x_2}^l l'^{1-\gamma} \frac{q(l')}{l'} dl' \right] \\ &\quad + K(1-2\gamma) \frac{q(l)}{l^2}\end{aligned}$$

$$l\hat{V}'_L(l) + \frac{1}{2}l^2\hat{V}''_L(l) = \frac{\gamma(\gamma-1)}{2}\hat{V}_L(l) + K \left(\frac{1-2\gamma}{2} \right) q(l),$$

which solves the ODE when $2\hat{r}\rho = \gamma(\gamma-1)$ and $K(1-2\gamma) = -2\rho$, or

$$\gamma = \frac{1}{2} + \sqrt{\frac{1}{4} + 2\hat{r}\rho} \quad \text{and} \quad K = \frac{\rho}{\sqrt{\frac{1}{4} + 2\hat{r}\rho}}.$$

Recall $\gamma(\rho) : [0, \infty] \rightarrow [1, \infty]$ and $K(\rho) > 0$. The parameters χ_1 and χ_2 will be determined later from boundary conditions.

1.b) Solving for $\hat{V}_H(l)$: Define: $\Delta_H(l) = q(0) - \hat{V}_H(l)$, $\bar{q}(l) = q(0) - q(l)$. Notice $\bar{q}(l)$ is increasing in l .

Rewriting the ODE for the high type as

$$\rho \Delta_H(l) = \rho \bar{q}(l) + \frac{1}{2} l^2 \Delta_H''(l).$$

Proceeding as above, we conjecture a solution of the form

$$\Delta_H(l) = K \left[l^{1-\gamma} \int_{\psi_1}^l l'^{\gamma-1} \frac{\bar{q}(l')}{l'} dl' + l^\gamma \int_l^{\psi_2} l'^{-\gamma} \frac{\bar{q}(l')}{l'} dl' \right]$$

for the same parameters γ and K defined previously. With this, we have

$$\begin{aligned} \Delta_H'(l) &= K \left[(1-\gamma) l^{-\gamma} \int_{\psi_1}^l l'^{\gamma-1} \frac{\bar{q}(l')}{l'} dl' + \gamma l^{\gamma-1} \int_l^{\psi_2} l'^{-\gamma} \frac{\bar{q}(l')}{l'} dl' \right] \\ \Delta_H''(l) &= K \left[-\gamma(1-\gamma) l^{-\gamma-1} \int_{\psi_1}^l l'^{\gamma-1} \frac{\bar{q}(l')}{l'} dl' + \gamma(\gamma-1) l^{\gamma-2} \int_l^{\psi_2} l'^{-\gamma} \frac{\bar{q}(l')}{l'} dl' \right] \\ &\quad + K(1-2\gamma) \frac{\bar{q}(l)}{l^2} \\ \frac{1}{2} l^2 \Delta_H''(l) &= \frac{\gamma(\gamma-1)}{2} \Delta_H(l) + K \left(\frac{1-2\gamma}{2} \right) q(l) \end{aligned}$$

that fulfill the ODE by construction with the parameters γ and K defined above. The parameters ψ_1 and ψ_2 will be determined later also from boundary conditions.

2. Dealing with the boundary conditions at $l = 0$.

Notice that we need $\lim_{l \rightarrow 0} \hat{V}_L(l) = \lim_{l \rightarrow 0} q(l) = q(0)$, and $\lim_{l \rightarrow 0} \Delta_H(l) = \lim_{l \rightarrow 0} \bar{q}(l) = \lim_{l \rightarrow 0} q(l) - q(0) = 0$. The two limiting properties hold if and only if $\chi_1 = 0$ and $\psi_1 = 0$ (we then relabel $\chi_2 = \chi$ and $\psi_2 = \psi$).

We will proceed with the proof for the high type. The proof for the low type is related. Using Lipschitz continuity of $\bar{q}(l)$, assuming $\bar{q}(l) \leq \Lambda l$, and $\psi_2 \leq \infty$:

$$\begin{aligned} \Delta_H(l) &= K \left[l^{1-\gamma} \int_{\psi_1}^l l'^{\gamma-1} \frac{\bar{q}(l')}{l'} dl' + l^\gamma \int_l^{\psi_2} l'^{-\gamma} \frac{\bar{q}(l')}{l'} dl' \right] \\ &\leq \Lambda K \left[l^{1-\gamma} \int_{\psi_1}^l l'^{\gamma-1} dl' + l^\gamma \int_l^{\psi_2} l'^{-\gamma} dl' \right] \\ &= \Lambda K \left[l^{1-\gamma} \left(\frac{l^\gamma}{\gamma} - \frac{\psi_1^\gamma}{\gamma} \right) + l^\gamma \left(\frac{\psi_2^{1-\gamma}}{1-\gamma} - \frac{l^{1-\gamma}}{1-\gamma} \right) \right] \\ &= \Lambda K \left[l \left(\frac{1}{\gamma} - \frac{1}{1-\gamma} \right) \right] \\ &= \Lambda l \end{aligned}$$

if and only if $\psi_1 = 0$ and assuming $\psi_2 = \infty$. Hence, $\lim_{l \rightarrow 0} \Delta_H(l) = 0$ if and only if $\psi_1 = 0$. A similar analysis delivers $\lim_{l \rightarrow 0} \hat{V}_L(l) = q(0)$ if and only if $\chi_1 = 0$.

3. Simplifying the Value Functions.

Changing variables inside the integrals: $\theta = l'/l$, so $ld\theta = dl'$ and the limits of integration. We start from obtaining $V_H(l)$.

$$\Delta_H(l) = K \left\{ \int_0^1 \theta^{\gamma-2} \bar{q}(\theta l) d\theta + \int_1^{\psi/l} \theta^{-\gamma-1} \bar{q}(\theta l) d\theta \right\}$$

Since $\bar{q}(\theta l) = q(0) - q(\theta l)$ and $\hat{V}_H(l) = q(0) - \Delta_H(l)$

$$\begin{aligned} \hat{V}_H(l) &= q(0) \left(1 - K \int_0^1 \theta^{\gamma-2} d\theta - K \int_1^{\psi/l} \theta^{-\gamma-1} d\theta \right) \\ &\quad + K \left\{ \int_0^1 \theta^{\gamma-2} q(\theta l) d\theta + \int_1^{\psi/l} \theta^{-\gamma-1} q(\theta l) d\theta \right\} \\ \hat{V}_H(l) &= q(0) \left[\frac{K}{\gamma} \left(\frac{\psi}{l} \right)^{-\gamma} \right] + K \left\{ \int_0^1 \theta^{\gamma-2} q(\theta l) d\theta + \int_1^{\psi/l} \theta^{-\gamma-1} q(\theta l) d\theta \right\}. \end{aligned}$$

Hence,

$$\hat{V}_H(l) = K \left\{ \int_0^1 \theta^{\gamma-2} q(\theta l) d\theta - \int_{\psi/l}^1 \theta^{-\gamma-1} q(\theta l) d\theta + \frac{q(0)}{\gamma} \left(\frac{\psi}{l} \right)^{-\gamma} \right\}. \quad (53)$$

Similarly, the low type's value function can be written as

$$\hat{V}_L(l) = K \left\{ \int_0^1 \theta^{\gamma-1} q(\theta l) d\theta - \int_{\chi/l}^1 \theta^{-\gamma} q(\theta l) d\theta \right\}. \quad (54)$$

In reduced form,

$$\hat{V}_L(l) = K[B_L(l) - A_L(l)] \quad (55)$$

$$\hat{V}_H(l) = K[B_H(l) - A_H(l)], \quad (56)$$

where

$$B_L(l) = \int_0^1 \theta^{\gamma-1} q(\theta l) d\theta \quad \text{and} \quad A_L(l) = \int_{\chi/l}^1 \theta^{-\gamma} q(\theta l) d\theta$$

$$B_H(l) = \int_0^1 \theta^{\gamma-2} q(\theta l) d\theta \quad \text{and} \quad A_H(l) = \int_{\psi/l}^1 \theta^{-\gamma-1} q(\theta l) d\theta - \frac{q(0)}{\gamma} \left(\frac{\psi}{l}\right)^{-\gamma}.$$

4. Taking Derivatives.

Taking derivatives of $\hat{V}_L(l)$ components and multiplying by l ,

$$l \frac{\partial A_L(l)}{\partial l} = \int_{\chi/l}^1 \theta^{-\gamma} q'(\theta l) \theta l d\theta - \left(\frac{\chi}{l}\right)^{-\gamma} q(\chi) \left(-\frac{\chi}{l^2}\right) l.$$

Integrating the first term by parts,

$$\begin{aligned} \int_{\chi/l}^1 \theta^{1-\gamma} q'(\theta l) l d\theta &= \theta^{1-\gamma} q(\theta l) \Big|_{\chi/l}^1 - \int_{\chi/l}^1 (1-\gamma) \theta^{-\gamma} q(\theta l) d\theta \\ &= q(l) - \left(\frac{\chi}{l}\right)^{1-\gamma} q(\chi) - (1-\gamma) \int_{\chi/l}^1 \theta^{-\gamma} q(\theta l) d\theta. \end{aligned}$$

Then,

$$l \frac{\partial A_L(l)}{\partial l} = q(l) - (1-\gamma) \int_{\chi/l}^1 \theta^{-\gamma} q(\theta l) d\theta = q(l) - (1-\gamma) A_L(l).$$

Similarly,

$$\begin{aligned} l \frac{\partial A_H(l)}{\partial l} &= q(l) + \gamma \int_{\psi/l}^1 \theta^{-\gamma-1} q(\theta l) d\theta - \frac{q(0)}{\gamma} (-\gamma) \left(\frac{\psi}{l}\right)^{-\gamma-1} \left(-\frac{\psi}{l^2}\right) l \\ &= q(l) + \gamma \int_{\psi/l}^1 \theta^{-\gamma-1} q(\theta l) d\theta - \gamma \frac{q(0)}{\gamma} \left(\frac{\psi}{l}\right)^{-\gamma} = q(l) + \gamma A_H(l) \end{aligned}$$

$$l \frac{\partial B_L(l)}{\partial l} = q(l) - \gamma \int_0^1 \theta^{\gamma-1} q(\theta l) d\theta = q(l) - \gamma B_L(l)$$

$$l \frac{\partial B_H(l)}{\partial l} = q(l) - (\gamma-1) \int_0^1 \theta^{\gamma-2} q(\theta l) d\theta = q(l) - (\gamma-1) B_H(l).$$

The derivatives can then be simplified as follows:

$$l \hat{V}'_L(l) = K[-\gamma B_L(l) + (1-\gamma) A_L(l)] \quad (57)$$

$$l \hat{V}'_H(l) = K[(1-\gamma) B_H(l) - \gamma A_H(l)]. \quad (58)$$

5. Smooth Pasting Conditions.

Boundary conditions (value matching and smooth pasting for low and high types) must be satisfied at \bar{l} . These conditions jointly determine \bar{l} , χ and ψ :²²

$$\hat{V}_L(\bar{l}) = \hat{V}'_L(\bar{l}) = \hat{V}'_H(\bar{l}) = 0.$$

Using the formal expressions of the value functions and derivatives,

$$\begin{aligned}\hat{V}_L(\bar{l}) &= 0 \Rightarrow \int_{\chi/\bar{l}}^1 \theta^{-\gamma} q(\theta\bar{l}) d\theta = \int_0^1 \theta^{\gamma-1} q(\theta\bar{l}) d\theta, \\ \bar{l}\hat{V}'_L(\bar{l}) &= 0 \Rightarrow (1-\gamma) \int_{\chi/\bar{l}}^1 \theta^{-\gamma} q(\theta\bar{l}) d\theta = \gamma \int_0^1 \theta^{\gamma-1} q(\theta\bar{l}) d\theta.\end{aligned}$$

Combining the two conditions, we find the equation that pins down \bar{l} :

$$\int_0^1 \theta^{\gamma-1} q(\theta\bar{l}) d\theta = 0 \tag{59}$$

and the equation that pins down χ :

$$\int_{\chi/\bar{l}}^1 \theta^{-\gamma} q(\theta\bar{l}) d\theta = 0 \quad \Rightarrow \quad \chi = \bar{l}. \tag{60}$$

Finally, the condition that pins down ψ is

$$(1-\gamma) \int_0^1 \theta^{\gamma-2} q(\theta\bar{l}) d\theta = \gamma \left[\int_{\psi/\bar{l}}^1 \theta^{-\gamma-1} q(\theta\bar{l}) d\theta - \frac{q(0)}{\gamma} \left(\frac{\psi}{\bar{l}}\right)^{-\gamma} \right]. \tag{61}$$

These expressions completely characterized value functions and the reputation at which low-quality firms exit. Q.E.D.

Here we can make a digression. In proving Proposition 1, we show that $\phi < \bar{\phi}$. Here we can see this property also holds in continuous time. The condition that pins down

²²Value matching and smooth pasting conditions for low-quality firms arise from optimal exiting decisions, and the smooth pasting condition for high-quality firms arises from the belief process that is reflecting at $\bar{\phi}$.

$\bar{\phi}$ is the one shown in equation (59). Similarly, under the pessimistic beliefs $\phi^e = \phi$, $\underline{\phi} = \frac{1}{1+l}$ can be pinned down from the conditions

$$\begin{aligned}\hat{V}_H(l) &= 0 \Rightarrow A_H(l) = B_H(l) \\ l\hat{V}'_H(l) &= 0 \Rightarrow \gamma A_H(l) = (1 - \gamma)B_H(l).\end{aligned}$$

This implies $A_H(l) = 0$, since the condition that pins down \bar{l} is $A_L(\bar{l}) = 0$. Then,

$$\int_0^1 \theta^{\gamma-2} q(\theta l) d\theta = \int_0^1 \theta \theta^{\gamma-2} q(\theta \bar{l}) d\theta = 0.$$

Since $\theta \in [0, 1]$, then $q(\theta l) < q(\theta \bar{l})$ for all θ . Hence, $\underline{\phi} < \bar{\phi}$ strictly.

Lemma 11 *Ratio and Differences of Value Functions for linear payoffs*

The ratio of value functions $V_L(\phi)/V_H(\phi)$ is monotonically increasing in ϕ . The difference between value functions $V_H(\phi) - V_L(\phi)$ is increasing for low reputation levels and decreasing for high reputation levels.

Proof First we prove the ratio of value functions $V_L(\phi)/V_H(\phi)$ is monotonically increasing in ϕ . Then we prove the difference between value functions $V_H(\phi) - V_L(\phi)$ is increasing for low reputation levels and decreasing for high reputation levels.

1. *Increasing Ratio $V_L(\phi)/V_H(\phi)$*

The ratio $\frac{V_L(\phi^e)}{V_H(\phi^e)}$ is an increasing function of ϕ^e if and only if $\frac{V_L(l_0)}{V_H(l_0)}$ is a decreasing function of l_0 . This is because $l_0 = \frac{1-\phi^e}{\phi^e}$, then $V'_i(\phi^e) = V'_i(l_0) \frac{\partial l_0}{\partial \phi^e} = -\frac{V'_i(l_0)}{\phi^{e2}}$ and $\frac{\partial(V_L(\phi^e)/V_H(\phi^e))}{\partial \phi^e} = -\frac{1}{\phi^{e2}} \frac{\partial(V_L(l_0)/V_H(l_0))}{\partial l_0}$.

First, we define the domain and image of the function. The lowest possible reputation in the market is \bar{l} , where $\hat{V}_L(\bar{l}) = 0$ and $\hat{V}_H(\bar{l}) > 0$. We also know that $\hat{V}_L(1) = \hat{V}_H(1) > 0$. Finally, $0 < \hat{V}_L(l) < \hat{V}_H(l)$ for all other $l_0 \in [0, \bar{l}]$. This implies $\frac{\hat{V}_L(l_0)}{\hat{V}_H(l_0)}$ is a mapping from $l_0 = [0, \bar{l}]$ to $[1, 0]$.

We show the ratio $\frac{\hat{V}_L(l)}{\hat{V}_H(l)}$ is monotonically decreasing in $l \in [0, \bar{l}]$. This is the case if

$$\frac{l\hat{V}'_L(l)}{\hat{V}_L(l)} < \frac{l\hat{V}'_H(l)}{\hat{V}_H(l)}.$$

Using the simplified expressions for the value functions, after some algebra, dropping the argument l , this condition implies

$$B_H \left[\left(1 - \gamma \frac{A_H}{B_H} \right) (B_L - A_L) + (2\gamma - 1)A_L \right] > A_H [\gamma(B_L - A_L) + (2\gamma - 1)A_L]. \quad (62)$$

We show the left-hand side of (62) is positive and the right-hand side of (62) is negative for all $l \in [0, \bar{l}]$; hence, the condition is always satisfied and the ratio of value functions decreasing in that range.

1.a. $B_H(l) > 0$ for all $l \in [0, \bar{l}]$

First, we develop the integrals $B_L(l)$ and $B_H(l)$.

Recall the profit function is linear in ϕ , ($y(\phi) = a_1\phi - a_0$) and $\phi = \frac{1}{1+\theta l}$,

$$q(\theta l) = \frac{a_1}{1 + \theta l} - a_0$$

and consider the general solution to the following integral (see Abramowitz and Stegun (1972)),

$$\int \theta^m \left(\frac{a_1}{1 + \theta l} - a_0 \right) d\theta = a_1 \theta^{m+1} \Phi(-\theta l, 1, m + 1) - \frac{\theta^{m+1}}{m + 1} a_0,$$

where $\Phi(-\theta l, 1, m + 1)$ is a Hurwitz Lerch zeta-function.

Applying this result to B_L ,

$$B_L(l) = \int_0^1 \theta^{\gamma-1} \left(\frac{a_1}{1 + \theta l} - a_0 \right) d\theta = \left[a_1 \theta^\gamma \Phi(-\theta l, 1, \gamma) - \frac{\theta^\gamma}{\gamma} a_0 \right]_0^1$$

$$B_L(l) = a_1 \Phi(-l, 1, \gamma) - \frac{a_0}{\gamma}$$

and similarly,

$$B_H = a_1 \Phi(-l, 1, \gamma - 1) - \frac{a_0}{\gamma - 1}.$$

Our strategy is to prove first $B_L(l) > 0$ for all $l \in [0, \bar{l}]$ and then to prove $B_H(l) > B_L(l)$ for all $l \in [0, \bar{l}]$.

Important properties of Herwitz Lerch zeta functions for the parameters we are considering ($\gamma \geq 1$) are (see Laurincikas and Garunkstis (2003)):

•

- $\Phi(0, 1, \gamma) = \frac{1}{\gamma}$
- $\frac{\partial \Phi(-l, 1, \gamma)}{\partial l} = \frac{1}{l} \left[\frac{1}{l+1} - \gamma \Phi(-l, 1, \gamma) \right] < 0$
- $(\gamma - 1)\Phi(-l, 1, \gamma - 1) > \gamma \Phi(-l, 1, \gamma)$.

By construction, $B_L(\bar{l}) = 0$, hence $\Phi(\bar{l}, 1, \gamma) = \frac{a_0}{\gamma a_1}$. Given the properties above,

$$B_L(l) : [0, \bar{l}] \rightarrow \left[\frac{a_1 - a_0}{\gamma}, 0 \right].$$

Furthermore, $B_L(l)$ is monotonically decreasing in the range

$B_H(l) > B_L(l)$ for all $l \in [0, \bar{l}]$ if

$$\gamma(\gamma - 1)[\Phi(-l, 1, \gamma - 1) - \Phi(-l, 1, \gamma)] > \frac{a_0}{a_1}.$$

Considering the third property above,

$$(\gamma - 1)\Phi(-l, 1, \gamma - 1) > \Phi(-l, 1, \gamma) + (\gamma - 1)\Phi(-l, 1, \gamma) > \frac{a_0}{\gamma a_1} + (\gamma - 1)\Phi(-l, 1, \gamma),$$

and hence, $B_H(l) > 0$ for all $l \in [0, \bar{l}]$.

1.b. $A_H(l) < 0$ for all $l \in [0, \bar{l}]$

We develop the integral $A_L(l)$ and $A_H(l)$ following the steps above:

$$A_L(l) = \int_{\chi/l}^1 \theta^{-\gamma} \left(\frac{a_1}{1 + \theta l} - a_0 \right) d\theta = \left[a_1 \theta^{1-\gamma} \Phi(-\theta l, 1, 1 - \gamma) - \frac{\theta^{1-\gamma}}{1 - \gamma} a_0 \right]_{\chi/l}^1$$

$$A_L(l) = a_1 [\Phi(-l, 1, 1 - \gamma) - (\chi/l)^{1-\gamma} \Phi(-\chi, 1, 1 - \gamma)] + \frac{a_0}{\gamma - 1} (1 - (\chi/l)^{1-\gamma})$$

and,

$$A_H(l) = a_1 [\Phi(-l, 1, -\gamma) - (\psi/l)^{-\gamma} \Phi(-\psi, 1, -\gamma)] + \frac{a_0}{\gamma} - \frac{a_1}{\gamma} (\psi/l)^{-\gamma}.$$

Consider $A_H(0) = A_H(\psi) = -\frac{a_1 - a_0}{\gamma} < 0$. We show that, if the function grows, the maximum is still negative. That is, we prove that $A_H(\hat{l}) < 0$, where $\hat{l} = \operatorname{argmax} A_H(l)$ (hence $\frac{\partial A_H(l)}{\partial l} \Big|_{l=\hat{l}} = 0$):

$$\frac{\partial A_H(l)}{\partial l} = \frac{a_1}{l} \left[\left(\frac{1}{1+l} + \gamma \Phi(-l, 1, -\gamma) \right) - \gamma (l/\psi)^\gamma \Phi(-\psi, 1, -\gamma) \right] - \frac{a_1}{l} (l/\psi)^\gamma.$$

The condition satisfied at $l \frac{\partial A_H(l)}{\partial l} = 0$ is

$$[\Phi(-l, 1, -\gamma) - (\psi/l)^{-\gamma} \Phi(-\psi, 1, -\gamma)] = \frac{1}{\gamma} (l/\psi)^\gamma - \frac{1}{1+l}.$$

Evaluating $A_H(\hat{l})$ considering that condition

$$a_1 \left[\frac{1}{\gamma} (l/\psi)^\gamma - \frac{1}{1+l} \right] + \frac{a_0}{\gamma-1} (1 - (\chi/l)^{1-\gamma}) < 0$$

since

$$\gamma a_1 \frac{1}{1+l} > a_0.$$

Hence, $A_H(l) < 0$ for all $l \in [0, \bar{l}]$.

Finally, just for completeness, $A_L(0) = -\frac{a_1 - a_0}{\gamma - 1} < 0$, $A_L(\chi) = 0$ because $\chi/l = 1$ and $A_L(\bar{l}) = 0$ by construction. It can be further shown that $A_L(l) < 0$ for all $l \in (0, \chi)$ and $A_L(l) > 0$ for all $l \in (\chi, \bar{l})$.

1.c. $\gamma(B_L(l) - A_L(l)) + (2\gamma - 1)A_L(l) > 0$ for all $l \in [0, \bar{l}]$

Recall $\gamma(B_L - A_L) + (2\gamma - 1)A_L = \gamma B_L + (\gamma - 1)A_L = -\frac{l\hat{W}'_L(l)}{K}$.

By construction $\gamma B_L + (\gamma - 1)A_L = 0$ at $l = 0$ and $l = \bar{l}$.

For $l \in (\chi, \bar{l})$, since $A_L(l) \geq 0$ and $B_L(l) > 0$, $\gamma B_L + (\gamma - 1)A_L > 0$. In particular, at \bar{l} , $A_L(\chi) = 0$ and $\gamma B_L(\chi) > 0$.

As shown above, for $l \in [0, \chi]$, $B_L(l) > 0$ monotonically increasing and $A_L(l) < 0$ monotonically increasing. This implies $\gamma B_L + (\gamma - 1)A_L$ goes monotonically from 0 at $l = 0$ to $\gamma B_L(\bar{l}) > 0$ and hence positive in the whole range.

1.d. $\left[\left(1 - \gamma \frac{A_H(l)}{B_H(l)} \right) (B_L(l) - A_L(l)) + (2\gamma - 1)A_L(l) \right] > 0$ for all $l \in [0, \bar{l}]$

First, recall $(\gamma - 1)B_H + \gamma A_H = -\frac{l\hat{W}'_H(l)}{K}$. Hence, as in the point above, $(\gamma - 1)B_H + \gamma A_H = 0$ at $l = 0$ and $l = \bar{l}$ by construction, which we can rewrite as $1 - \gamma \frac{A_H(0)}{B_H(0)} = 1 - \gamma \frac{A_H(\bar{l})}{B_H(\bar{l})} = \gamma$. Hence at these two extreme points, the term in the left-hand side is zero, the same as the one in the right-hand side.

More generally $(\gamma - 1)B_H + \gamma A_H > 0$ (and then $1 < 1 - \gamma \frac{A_H(l)}{B_H(l)} < \gamma$). Since $A_L(\chi) = 0$, $\left(1 - \gamma \frac{A_H(l)}{B_H(l)}\right) B_L(l) > 0$. This part of the proof follows from the same monotonicity arguments above.

2. Non-monotonic Difference $V_H(\phi) - V_L(\phi)$

First, $\hat{V}'_L(\bar{\phi}) = \hat{V}'_H(\bar{\phi}) = 0$ by construction and $\hat{V}'_L(1) = \hat{V}'_H(1) = 0$, from the expressions above. Second $\hat{V}'_L(\phi)$ and $\hat{V}'_H(\phi)$ are positive for all $\phi \in (\bar{\phi}, 1)$. Third, these derivatives are single peaked and the reputation that maximizes $\hat{V}'_H(\phi)$ is lower than the reputation that maximizes $\hat{V}'_L(\phi)$. Finally, $\hat{V}''_H(\bar{\phi}) > \hat{V}''_L(\bar{\phi})$ and $\hat{V}''_H(1) < \hat{V}''_L(1)$, which means the two derivatives cross only one time, at ϕ^* . These properties arise from inspection of the derivatives of linear profits value functions and from properties of the hypergeometric functions that characterize them. Q.E.D.

A.4 Proof Proposition 3

We split the proof in two parts.

1) As the precision of signals go to zero.

In this case, to prove the steady-state spot market equilibrium converges to the benchmark without information ($Y \rightarrow 0$), it is enough to prove that $V_H(\phi) \rightarrow V_L(\phi)$ for all ϕ . This is because, from equation (23), $Y \rightarrow 0$ as $V_H(\phi^e) \rightarrow V_L(\phi^e)$.

In the bad and good news cases, the precision of signals is zero when $\lambda = 0$, hence there are no news about the true type of the firm. It is trivial to see, from Propositions 3 and 5, that $V_L(\phi) = V_H(\phi) = q(\phi)/\hat{r}$ for all ϕ when $\lambda = 0$.

In the Brownian diffusion case, the precision of signals is zero when the signal to noise ratio $\frac{\mu_H - \mu_L}{\sigma} = 0$, and then $\rho = \infty$. From the ODEs (41) and (42), $V_H(\phi) = V_L(\phi) = q(\phi)/\hat{r}$.

2) As the precision of signals go to infinity.

In this case, to prove that the steady-state spot market equilibrium converges to the unconstrained first best benchmark with perfect information ($Y \rightarrow \bar{Y}$), it is enough to prove that $V_H(\phi) \rightarrow V_H(1)$ and $V_L(\phi) \rightarrow 0$ for all $\phi > 0$, as precision goes to infinity. This is because, from Proposition 1, $V_H(\phi^e)U'(Y) = C$. As precision goes to infinity, low-quality firms exit fast, and the reputation at entry does not matter to determine the average quality of firms in steady-state (that is, $\frac{m(1)}{m(0)+m(1)} \rightarrow 1$ regardless of $\bar{\phi} > 0$).

In the bad and good news cases, the precision of signals is infinity when $\lambda = \infty$, hence news about the true type of the firm arrive immediately. In this case, low-quality firms spend almost no time with a reputation different than 0. From Propositions 3 and 5,

it is straightforward to check that $V_H(\phi) = q(1)/r$ and $V_L(\phi) = 0$ for $\lambda = \infty$ and all $\phi > 0$. Even when $\bar{\phi} < 1$, since all low-quality firms almost instantaneously leave the market when $\lambda \rightarrow \infty$, effectively $\frac{m(1)}{m(0)+m(1)} \rightarrow 1$ in the market.

In the Brownian diffusion case, the precision of signals is infinite when the signal-to-noise ratio $\frac{\mu_H - \mu_L}{\sigma} = \infty$. Then $\rho = 0$ and $\gamma = 1$. From evaluating equation (42) at l with $\rho = 0$, $V_H''(l) = 0$ for all l . Combining this result with equations (61) and the definition of $V_H'(l)$ in the Appendix, $V_H'(\bar{l}) = 0$. This implies that $V_H(\bar{\phi}) = V_H(1)$, and then the production of the experience good is \bar{Y} . Furthermore, even when $\bar{\phi} < 1$, since all low-quality firms almost instantaneously leave the market, effectively $\frac{m(1)}{m(0)+m(1)} \rightarrow 1$ in the market.

A.5 Proof Proposition 4

Specifically, under full information, the steady-state measure of high-quality firms is $\bar{m}_H = \bar{Y}/y(1)$, so the rate of entry of high-quality firms is $\phi^e \bar{m}^e = \frac{(1 - \exp(-\delta\Delta))}{\Delta} \frac{\bar{Y}}{y(1)}$ and the associated consumption of the numeraire good is $\bar{N} = 1 - \frac{(1 - \exp(-\delta\Delta))}{\Delta} \frac{\bar{Y}}{y(1)} C$. In continuous time, $\bar{N} = 1 - \delta \frac{\bar{Y}}{y(1)} C$.

In the equilibrium with regulation, a fraction $(1 - \phi^e)$ of entering firms are low-quality in steady state, since we proved $\phi^e < 1$ in equilibrium. Hence, there is a fraction of all active firms that is low-quality, where this fraction is positive. Denote the equilibrium steady-state ratio of low- to high-quality firms by \hat{m}_L/\hat{m}_H , also positive. From the resource constraint for the experience good (2) in steady state, to produce output \bar{Y} there must be a stock of \hat{m}_H high-quality firms given by

$$\hat{m}_H = \frac{\bar{Y}}{y(1) + y(0)\hat{m}_L/\hat{m}_H} > \bar{m}_H,$$

and a steady-state entry rate of high-quality firms of $\phi^e m^e = \frac{(1 - \exp(-\delta\Delta))}{\Delta} \hat{m}_H$ is required to maintain that production. As a result of this required elevated rate of entry of high-quality firms, consumption of numeraire good is $N = 1 - \frac{(1 - \exp(-\delta\Delta))}{\Delta} \hat{m}_H C < \bar{N}$. The gap between N and \bar{N} can be made arbitrarily small by choosing F to set ϕ^e as close to one as is required to drive \hat{m}_L/\hat{m}_H sufficiently close to zero and hence drive \hat{m}_H sufficiently close to \bar{m}_H .