

Notes on Auctions[†]

ECON 201B - Game Theory

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1 Describing an auction game

In these notes we will discuss auctions as an extension of Bayesian Games. We will consider auctions for a single indivisible object (say a painting, a car, a franchise, etc). Potential buyers differ in their valuations of the object and the seller does not know these valuations. Hence, since each player has private information about the true value of the object to him (or about private signals of the true value, in the case of common values objects), this problem has basically the same structure that any Bayesian Game.

In these notes we will focus on symmetric auctions (where participants are ex-ante identical), independent signals and symmetric equilibria.

More formally, consider I bidders. The type of the i th bidder is his true private valuation v_i , drawn from a common distribution on the interval $[0,1]$. The cumulative distribution function F_i is smooth and strictly increasing, such that $F(v)$ is the probability that a given bidder's true value is v or less.

A pure strategy for a player i is the bidding function $f_i : [0, 1] \rightarrow b_i \in \mathbb{R}_+$, where $f_i(v)$ is player i 's bid if his true value is v . We will assume f_i is a continuous and differentiable function (this assumption only helps the exposition but in fact nothing in the definition of equilibrium requires that strategies be differentiable, continuous or even measurable).

⁰† These notes were prepared as a back up material for TA session. If you have any questions or comments, or notice any errors or typos, please drop me a line at guilord@ucla.edu

We will analyze first price, second price and third price auctions, generalizing the results to 4th and lower-price auctions. We will also discuss the Revenue Equivalence concept.

2 First Price Auction

In a first price auction, the bidder who offers a highest price gets the good and pays his bid. The other bidders do not pay anything.

Given the assumptions we can define $\Pr(v, y) = \int_0^y F(t) dt = G(y)$

Define also a bid function f that maps values v to bids b , such that f is continuously differentiable and strictly increasing $f' > 0$. Also consider $f(0) = 0$ and $f(1) = M$. This is, when the value of the bidder is 0 he will bid 0 while when it is 1, he will bid M .

The expected utility for the bidder is

$$Eu_i(b|v_i, v_{-i}) = E(v - b | b \text{ wins}) = E(v_i - b | b \geq f(v_{-i}))$$

We will change variables defining $w = f_{(b)}^{-1} \implies b = f(w)$. Hence we can redefine $E(v_i - f(w_i) | f(w_i) > f(v_{-i}))$, being w_i the value the individual i will pretend to have

Because we assume f is strictly increasing, we can express the previous expression as:

$$\begin{aligned} E(v_i - f(w_i) | w_i > v_{-i}) &= (v_i - f(w_i)) \Pr(w_i > v_{-i}) \\ &= (v_i - f(w_i)) G_{(w)}^{I-1} \end{aligned}$$

where $G_{(w)}^{I-1}$ is the probability i wins (the product of probabilities the other $(I - 1)$ bidders value the good by less than w).

Hence, the bidder would like to solve the following problem

$$\max_w Eu(f_w | v; f_{-i}) = (v - f(w)) G_{(w)}^{I-1}$$

$$FOC: \frac{\partial}{\partial w} Eu(f_w | v; f_{-i}) \Big|_{w=v} = 0$$

By equilibrium condition we further know that $f(w) = f(v)$, and, which is the same, $w = v$. Hence

$$\frac{\partial}{\partial w}(v - f(w))G_{(w)}^{I-1} \Big|_{w=v} = (-f'_{(w)})G_{(w)}^{I-1} + [v - f(w)](I-1)G_{(w)}^{I-2}G'_{(w)} = 0$$

Replacing in the FOC.

$$(-f'_{(v)})G_{(v)}^{I-1} + [v - f(v)](I-1)G_{(v)}^{I-2}G'_{(v)} = 0$$

$$v(I-1)G_{(v)}^{I-2}G'_{(v)} = f'_{(v)}G_{(v)}^{I-1} + f(v)(I-1)G_{(v)}^{I-2}G'_{(v)}$$

being the right hand side of the equation just $[f_{(v)}G_{(v)}^{I-1}]'$, then

$$[f_{(v)}G_{(v)}^{I-1}]' = v(I-1)G_{(v)}^{I-2}G'_{(v)}$$

Integrating both sides between 0 and \bar{v} (the value for the bidder),

$$f_{(\bar{v})}G_{(\bar{v})}^{I-1} - f_{(0)}G_{(0)}^{I-1} = \int_0^{\bar{v}} v(I-1)G_{(v)}^{I-2}G'_{(v)}dv \quad (\text{and } f_{(0)} = 0)$$

$$f_{(\bar{v})} = \frac{(I-1)}{G_{(\bar{v})}^{I-1}} \int_0^{\bar{v}} vG_{(v)}^{I-2}G'_{(v)}dv$$

Particular case: Uniform Distribution

Consider v is distributed uniformly between $[0,1]$. In this particular case, $G_{(y)} = \Pr(v < y) = y$

$$f_{(\bar{v})} = \frac{(I-1)}{\bar{v}^{I-1}} \int_0^{\bar{v}} v.v^{I-2}.dv$$

In solving this integral recall there is a constant of integration. To eliminate it we need a boundary condition. In our case this condition is given

by the fact that no player would bid more than M or less than 0. Hence the constant of integration is 0 and

$$f(\bar{v}) = \frac{(I-1)\bar{v}^I}{\bar{v}^{I-1}I} \implies f(\bar{v}) = \frac{I-1}{I}\bar{v}$$

Hence,

$$f_{(\bar{v})}^{1^{st}} = (1 - \frac{1}{I})\bar{v} \tag{1}$$

This means that the bid of each person will depend on his or her own valuation \bar{v} and on the number of bidders I . It's interesting to note that the more the quantity of competing bidders, the less the difference between each bidder's real value and the bid offered.

Expected Revenue for the seller

To obtain the expected revenue to the seller, we need to integrate over all participants with value less than the value of the winner. In the case of the winner we need to integrate over 0 and 1. This multiple integral should be multiplied by the number of bidders I to have all possible winners combinations (it does not matter the order of the losers, only who is the winner). Without loss of generality consider bidder 1 is the winner.

We use the general formulation considering that the equilibrium bid is given by $f(\bar{v}) = (1 - \frac{1}{I})\bar{v}$

$$\text{Expected Revenue} \quad ER = I \int_{v_1 > v_{-1}} \dots \int [(1 - \frac{1}{I})v_1] dv_I . dv_{I-1} . \dots . dv_1$$

The limits of the integration can be written as

$$ER = I \int_0^1 \int_0^{v_1} \dots \int_0^{v_1} [(\frac{I-1}{I})v_1] dv_I . dv_{I-1} . \dots . dv_1$$

$$ER = I \int_0^1 \int_0^{v_1} \dots \int_0^{v_1} [(\frac{I-1}{I})v_1 v_I |_0^{v_1}] dv_{I-1} . \dots . dv_1$$

$$ER = I \int_0^1 \int_0^{v_1} \dots \int_0^{v_1} [(\frac{I-1}{I})v_1^2] dv_{I-1} . \dots . dv_1$$

and following this pattern until we reach v_1 , we have

$$ER = I \int_0^1 \left(\frac{I-1}{I}\right) v_1^I dv_1 = I \left(\frac{I-1}{I}\right) \frac{v_1^{I+1}}{I+1} \Big|_0^1$$

Hence,

$$ER = \frac{I-1}{I+1} \tag{2}$$

Hence, when 2 buyers participate the expected revenue to the seller is $\frac{1}{3}$, when 3 bidders participate the expected revenue is $\frac{1}{2}$, when 4 bidders participate the expected revenue is $\frac{3}{5}$, and so on. As can be seen, as $I \rightarrow \infty$, $ER \rightarrow 1$, which means the seller almost gets the maximum possible value players have. This is because, when the number of participants increase, both the bidders' bid are close to their real value and the probability a buyer draw a value arbitrarily close to 1, is almost one.

3 Second Price Auction

In this type of auctions, the highest bidder (call him i) wins the object and pays the second highest bid (such that the winner gets a utility $v_i - \max_{j \neq i} b_j$ while other bidders pay nothing having a utility of 0). If several bidders win the auction, the good is allocated randomly among them. In any case this case has zero probability of occurrence since we are dealing with continuous functions.

We will show that in this environment each player i will bid his or her own valuation. This will be a (weakly) dominance equilibrium.

Define the highest bid as b^* . We will analyze all the possible ordering among own bids b_i , own true values v_i and the highest bid by others b^*

We will compare the strategy of bidding any b_i , against the strategy of bidding the own true value v_i .

		<i>i</i> Bids v_i		<i>i</i> Bids b_i	
		Result	Utility	Result	Utility
1	$b_i < v_i < b^*$	Lose	0	Lose	0
2	$v_i < b_i < b^*$	Lose	0	Lose	0
3	$b_i < b^* < v_i$	Win	$v_i - b^* > 0$	Lose	0
4	$v_i < b^* < b_i$	Lose	0	Win	$v_i - b^* < 0$
5	$b^* < b_i < v_i$	Win	$v_i - b^* > 0$	Win	$v_i - b^* > 0$
6	$b^* < v_i < b_i$	Win	$v_i - b^* > 0$	Win	$v_i - b^* > 0$

Hence, bidding the true value v_i weakly dominates bidding any other value b_i .

$$f_{(\bar{v})}^{2nd} = \bar{v} \quad (3)$$

This is true for all bidders, such that everybody bids the own valuation.

Expected Revenue for the seller

To obtain the revenue expected by the seller, we need to integrate over all players with a value smaller than the value of the winner. We have to integrate over the runner up bid (in equilibrium the runner up true value). Without loss of generality we will call this payment v_2 .

This multiple integral should be multiplied by the number of all possible combinations of winners and runner ups, that is by $I(I - 1)$. (It does not matter the order of the losers from the third place on. In order to obtain the payments only matters who is the winner and who is the runner up).

$$\text{Expected Revenue} \quad ER = I(I - 1) \int_{v_1 > v_2} \int_{v_2 > v_{-1,-2}} \dots \int v_2 \cdot dv_I \cdot dv_{I-1} \dots dv_1$$

The limits of the integration can be written as

$$ER = I(I - 1) \int_0^1 \int_0^{v_1} \int_0^{v_2} \dots \int_0^{v_2} v_2 \cdot dv_I \cdot dv_{I-1} \dots dv_1$$

$$ER = I(I-1) \int_0^1 \int_0^{v_1} \int_0^{v_2} \dots \int_0^{v_2} [v_2 v_I |_0^{v_2}] dv_{I-1} \dots dv_1$$

$$ER = I(I-1) \int_0^1 \int_0^{v_1} \int_0^{v_2} \dots \int_0^{v_2} [v_2^2] dv_{I-1} \dots dv_1$$

$$ER = I(I-1) \int_0^1 \int_0^{v_1} \int_0^{v_2} \dots \int_0^{v_2} [v_2^2 v_{I-1} |_0^{v_1}] dv_{I-2} \dots dv_1$$

$$ER = I(I-1) \int_0^1 \int_0^{v_1} \int_0^{v_2} \dots \int_0^{v_2} [v_2^3 |_0^{v_1}] dv_{I-2} \dots dv_1$$

Following this calculation exercise up to v_2 , we have

$$ER = I(I-1) \int_0^1 \int_0^{v_1} [v_2^{I-1}] dv_2 \cdot dv_1 = I(I-1) \int_0^1 \frac{v_2^I}{I} \Big|_0^{v_1} dv_1 = I(I-1) \int_0^1 \frac{v_1^I}{I} \cdot dv_1$$

$$ER = I(I-1) \frac{v_1^{I+1}}{I(I+1)} \Big|_0^1$$

Hence,

$$ER = \frac{I-1}{I+1} \tag{4}$$

Also in this type of actions, when 2 buyers participate the expected revenue to the seller is $\frac{1}{3}$, when 3 bidders participate the expected revenue is $\frac{1}{2}$, when 4 bidders participate the expected revenue is $\frac{3}{5}$ and so on. As can be seen, as $I \rightarrow \infty$, $ER \rightarrow 1$, which means the seller almost gets the maximum possible value players can have.

4 Third Price Auction

By the third price auction we mean the highest bid wins but pays the third highest price.

Without loss of generality, we will assume an order in the valuations, $v_1 > v_2 > \dots > v_I$, being v_1 the highest valuation and v_I the lowest one. The idea of the ordering is very important to obtain the probabilities of winning and the expected payments from winning.

To obtain the symmetric equilibrium of the third price auction we will follow the same method used to obtain first price auction equilibrium.

In the first price auction we maximize the expected revenue of a bid, given by the difference between the own value and the own bid, conditional on winning. However, in the third price auction this is not so simple because the expected value to maximize is the difference between the own value and the expected value of the third price, conditional on being the third bid and conditional on the bidder winning the auction.

In the third price auction, the expected utility for the bidder is,

$$Eu_i(b_i|v_i, v_{-i}) = E_i(v_i - E(b_3|b_2 < b_i)|b_i \text{ wins}) = E_i(v_i - b_3|b_i \geq f_{(v_{-i})})$$

being b_i the own bid and $E(b_3|b_2 < b_i)$ the expected value of the highest third price that is necessary to pay in case of winning the bid (i.e. conditional on the fact that the own bid is greater than the second highest price).

Now on, we will refer to the first bidder as the winner, without loss of generality.

We can redefine $E_1(v_1 - E(f_{(w_3)}|f_{(w_2)} < f_{(w_1)})|f_{(w_2)} < f_{(w_1)})$, being w_i the value the individual i will pretend to have. Subscripts 1, 2 and 3 refers to the position of the bid in the auction. Bidder 1 only wins with the highest bid $f_{(w_1)}$, in which case he would pay the third highest bid $f_{(w_3)}$.

Since we are assuming f is strict increasing, we can express the previous expression as

$$\begin{aligned} E_1(v_1 - E(f_{(w_3)}|f_{(w_2)} < f_{(w_1)})|w_2 < w_1) \\ E_1 &= [v_1 - E(f_{(w_3)}|f_{(w_2)} < f_{(w_1)})] \Pr(w_2 < w_1) \\ E_1 &= [v_1 - E(f_{(w_3)}|f_{(w_2)} < f_{(w_1)})] G_{(w_1)}^{I-1} \end{aligned}$$

Naturally, previously to any maximization, it is important to develop the expression $E(f_{(w_3)}|f_{(w_2)} < f_{(w_1)})$, which is the expected payment the winner has to face

$$E(f_{(w_3)}|f_{(w_2)} < f_{(w_1)}) = \int_0^{w_1} f_{(w_3)} \cdot h_{(w_3|w_2 < w_1)} dw_3 \quad (5)$$

Recall w is the variable over which we are integrating and $h_{(w_3|w_2 < w_1)}$ is a conditional density function of an order statistic. To solve it consider that by the definition of a conditional distribution

$$h_{(w_3|w_2 < w_1)} = h_{3,1(w_3, w_1)} / h_{1(w_1)} \quad (6)$$

Recall that each w_i is distributed with a density function g , then the joint density function of w_1, w_2, \dots, w_I , such that $0 < w_1 < w_2 < \dots < w_I < 1$, can be written as $h(w_1, w_2, \dots, w_I) = (I!)g_{(w_1)}g_{(w_2)}\dots g_{(w_I)}$. It is possible to express it as the product of individual densities since they are independent. We also need to multiply this by the factorial of the number of individuals (I), which shows the possible arrangements in which w 's can be ordered.

Now, we can obtain the marginal distribution of w_1 ,

$$\begin{aligned} h_{1(w_1)} &= \int_0^{w_1} \int_0^{w_2} \int_0^{w_3} \dots \int_0^{w_{I-1}} (I!)g_{(w_1)}g_{(w_2)}\dots g_{(w_I)}dw_I\dots dw_2 \\ &= \int_0^{w_1} \int_0^{w_2} \int_0^{w_3} \dots \int_0^{w_{I-2}} (I!) \left[\int_0^{w_{I-1}} g_{(w_I)}dw_I \right] g_{(w_1)}g_{(w_2)}\dots g_{(w_{I-1})}dw_{I-1}\dots dw_2 \\ &= \int_0^{w_1} \int_0^{w_2} \int_0^{w_3} \dots \int_0^{w_{I-2}} (I!)G_{(w_{I-1})} \cdot g_{(w_1)}g_{(w_2)}\dots g_{(w_{I-1})}dw_{I-1}\dots dw_2 \end{aligned}$$

In the next step, $\int_0^{w_{I-2}} G_{(w_{I-1})} \cdot g_{(w_{I-1})}dw_{I-1} = \left. \frac{[G_{(w_{I-1})}]^2}{2} \right|_0^{w_{I-2}} = \frac{[G_{(w_{I-2})}]^2}{2}$
(because $G_{(0)} = 0$)

and so on

$$\begin{aligned} h_{1(w_1)} &= (I!) \int_0^{w_1} \frac{[G_{(w_2)}]^{I-2}}{(I-2)!} g_{(w_1)}g_{(w_2)}dw_2 \\ h_{1(w_1)} &= (I!) \frac{[G_{(w_1)}]^{I-1}}{(I-1)!} g_{(w_1)} \end{aligned} \quad (7)$$

We can calculate any marginal density function of a k^{th} order statistic doing

$$h_k(w_k) = \int_{w_2}^{w_1} \int_{w_3}^{w_1} \dots \int_{w_k}^{w_1} \int_0^{w_k} \dots \int_0^{w_{I-1}} (I!) g(w_1) g(w_2) \dots g(w_I) dw_I \dots dw_{k+1} dw_{k-1} \dots dw_1$$

Hence, the result of the integral up to the $(k-1)$ th variable is equal to the case for the first order statistic, but as if it were k bidders. So,

$$h_k(w_k) = \left[(I!) \frac{[G(w_k)]^{k-1}}{(k-1)!} g(w_k) \right] \int_{w_2}^{w_1} \int_{w_3}^{w_1} \dots \int_{w_k}^{w_1} g(w_1) g(w_2) \dots g(w_{k-1}) dw_{k-1} \dots dw_1$$

and

$$\begin{aligned} \int_{w_2}^{w_1} g(w_1) dw_1 &= G(w_1) \Big|_{w_2}^{w_1} = G(w_1) - G(w_2) \\ h_k(w_k) &= \left[(I!) \frac{[G(w_k)]^{k-1}}{(k-1)!} g(w_k) \right] \int_{w_3}^{w_1} \int_{w_4}^{w_1} \dots \int_{w_k}^{w_1} [G(w_1) - G(w_2)] g(w_2) \dots g(w_{k-1}) dw_{k-1} \dots dw_2 \\ \int_{w_3}^{w_1} [G(w_1) - G(w_2)] g(w_2) dw_2 &= - \left. \frac{[G(w_1) - G(w_2)]^2}{2} \right|_{w_3}^{w_1} = - \frac{[G(w_1) - G(w_1) - G(w_1) - G(w_3)]^2}{2} \\ &= \frac{[G(w_1) - G(w_3)]^2}{2} \end{aligned}$$

Doing the same until finishing the integrals, we obtain finally the k^{th} order statistic.

$$h_k(w_k) = \frac{(I!)}{(I-1)!(k-1)!} [G(w_k)]^{I-1} [G(w_1) - G(w_k)]^{k-1} g(w_k) \quad (8)$$

Now, combining both results we can finally obtain the joint density of any two order statistic, i and j such that $w_i > w_j$.

Now we are in conditions to finally obtain the expected payment to face in case of winning.

$$h_{j,i}(w_j, w_i) = \int_{w_2}^{w_1} \int_{w_3}^{w_1} \dots \int_{w_i}^{w_1} \int_{w_{i-1}}^{w_i} \dots \int_{w_j}^{w_{j+1}} \int_{w_{j-1}}^{w_j} \dots \int_0^{w_{I-1}} (I!) g(w_1) \dots g(w_I) dw_I \dots dw_{i+1} dw_{i-1} \dots dw_{j+1} dw_{j-1} \dots dw_1$$

Operating similarly, we can obtain,

$$h_{j,i}(w_j, w_i) = \frac{(I!)}{(i-1)!(j-i-1)!(I-j)!} [G(w_j)]^{I-j} [G(w_i) - G(w_j)]^{j-i-1} [G(w_1) - G(w_i)]^{i-1} g(w_j)g(w_i)$$

For our specific case, where $i = 1$ and $j = 3$.

$$h_{3,1}(w_3, w_1) = \frac{(I!)}{(I-3)!} [G(w_3)]^{I-3} [G(w_1) - G(w_3)] g(w_3)g(w_1)$$

Now, applying (6) to obtain the conditional density function,

$$h_{(w_3|w_2 < w_1)} = \frac{h_{3,1}(w_3, w_1)}{h_1(w_1)} = \frac{\frac{(I!)}{(I-3)!} [G(w_3)]^{I-3} [G(w_1) - G(w_3)] g(w_3)g(w_1)}{\frac{(I!)}{(I-1)!} [G(w_1)]^{I-1} g(w_1)}$$

$$h_{(w_3|w_2 < w_1)} = (I-1)(I-2) \frac{[G(w_3)]^{I-3} [G(w_1) - G(w_3)] g(w_3)}{[G(w_1)]^{I-1}} \quad (9)$$

Plugging (9) in (5) we have the expected amount the bidder should pay in case he wins the auction,

$$E(f(w_3)|f(w_2) < f(w_1)) = \int_0^{w_1} f(w) \cdot (I-1)(I-2) \frac{[G(w_3)]^{I-3} [G(w_1) - G(w_3)] g(w_3)}{[G(w_1)]^{I-1}} dw \quad (10)$$

and the expected utility for the bidder in case of winning is

$$E_1(v_1 - E(f(w_3)|f(w_2) < f(w_1)) | w_2 < w_1) = [v_1 - E(f(w_3)|f(w_2) < f(w_1))] G_{(w_1)}^{I-1}$$

$$E_1(\cdot) = v_1 G_{(w_1)}^{I-1} - G_{(w_1)}^{I-1} E(f(w_3)|f(w_2) < f(w_1))$$

$$E_1(\cdot) = v_1 G_{(w_1)}^{I-1} - G_{(w_1)}^{I-1} \int_0^{w_1} f(w_3) \cdot (I-1)(I-2) \frac{[G(w_3)]^{I-3} [G(w_1) - G(w_3)] g(w_3)}{[G(w_1)]^{I-1}} dw_3$$

Rearranging and canceling terms

$E_1(\cdot) = v_1 G_{(w_1)}^{I-1} - \int_0^{w_1} f(w_3) (I-1) [G_{(w_1)} - G_{(w_3)}] \left[(I-2) [G_{(w_3)}]^{I-3} g_{(w_3)} dw_3 \right]$
 knowing $\left[(I-2) [G_{(w_3)}]^{I-3} g_{(w_3)} dw_3 \right] = dG_{(w_3)}^{I-2}$, we can express.

$$E_1(\cdot) = v_1 G_{(w_1)}^{I-1} - (I-1) \int_0^{w_1} f(w_3) [G_{(w_1)} - G_{(w_3)}] dG_{(w_3)}^{I-2} \quad (11)$$

Finally, this is the expression bidder 1 has to maximize with respect to w_1 , the control variable or the valuation the bidder will pretend to have. Recall in equilibrium, $w_1 = v_1$

FONC

$$\begin{aligned}
 \{w_1\} \quad & v_1 (I-1) G_{(w_1)}^{I-2} G'_{(w_1)} + (I-1) f(w_1) G_{(w_1)} dG_{(w_1)}^{I-2} - (I-1) f(w_1) G_{(w_1)} dG_{(w_1)}^{I-2} \\
 & - (I-1) G'_{(w_1)} \int_0^{w_1} f(w_3) dG_{(w_3)}^{I-2} = 0 \\
 \implies & v_1 (I-1) G_{(v_1)}^{I-2} G'_{(v_1)} = (I-1) G'_{(v_1)} \int_0^{v_1} f(w_3) dG_{(w_3)}^{I-2}
 \end{aligned}$$

dividing by $(I-1)G'_{(v_1)}$,

$$v_1 G_{(v_1)}^{I-2} = \int_0^{v_1} f(w_3) dG_{(w_3)}^{I-2}$$

differentiating with respect to v_1

$$G_{(v_1)}^{I-2} + v_1 \frac{dG_{(v_1)}^{I-2}}{dv_1} = f(v_1) \frac{dG_{(v_1)}^{I-2}}{dv_1}$$

and considering $\frac{dG_{(v_1)}^{I-2}}{dv_1} = (I-2)G_{(v_1)}^{I-3}G'_{(v_1)}$

$$f(v_1) = v_1 + \frac{G_{(v_1)}^{I-2}}{(I-2)G_{(v_1)}^{I-3}G'_{(v_1)}}$$

$$f_{(v_1)}^{3rd} = v_1 + \frac{G_{(v_1)}}{(I-2)G'_{(v_1)}} \quad (12)$$

Because the equilibrium is symmetric, then this is the optimal strategy for all bidders.

Particular case: Uniform Distribution

Consider values are distributed uniformly between 0 and 1. In this case $G_{(v_1)} = v_1$ and $G'_{(v_1)} = 1$. Hence,

$$f_{(v_1)} = \frac{I-1}{I-2}v_1$$

Compare this result with first price auctions symmetric equilibrium strategies ($f_{(v_1)} = \frac{I-1}{I}v_1$) and with the second price symmetric equilibrium strategies ($f_{(v_1)} = \frac{I-1}{I-1}v_1 = v_1$).

Generalizing, a set of conclusions can be obtained from here:

First, we can extrapolate the results to any lower-price auctions with symmetric equilibrium strategies for the case in which values are distributed uniformly between 0 and 1. In general,

$$f_{(v_1)}^{k^{st}} = \frac{I-1}{I-k+1}v_1 \quad (13)$$

being k the bid the winner should pay in case of winning (by having the highest bid). In fact we could check this result is an equilibrium for any lower-price strategy. As can be seen, a lower-price auction (an increase in k) leads to higher bidding.

Second, for third and lower-price auctions, bids are higher than own valuations. This is the opposite to shade valuations, as is the case in first price auctions.

Third, as in first price auction, bids converges to the true value when the number of bidders I increases. The difference is that while in first price auctions the convergence is from below, in lower price auctions the convergence is from above.

Finally, it's possible to show also that expected revenue to the seller is the same no matter the kind of auction we are considering. It does not matter if we are talking about a first, second, third or any lower-price auction, always the expected utility for the seller will be the same. This property is called Revenue Equivalence.

5 First-Price Auctions with a Reservation Value

In many first-price auctions (such as the ones you can play in E-Bay), the seller is entitled to set a reservation value V and commits to not sell the item unless the winning price exceeds V . We will try to find in this case the symmetric equilibria to see how it changes with respect to the standard case without reservation values.

As we will see it does not imply that every bidder must hold a bid surpassing this threshold (a bidder with a reservation value lower than the implicit in the seller's reservation value will bid below this bound leading his probability of winning to zero - autoexclusion scenario).

We will assume that the seller reveals his true private value so the reserve price announced to the bidders is in fact the true reservation value ($p_{(\underline{V})} = \underline{V}$)

As an announced seller's reservation value (\underline{V}) exist, the offer by the i^{th} bidder has to fulfill the following restriction,

$$b_i = \begin{cases} f(v_i) & v_i \geq f_{(p_{(\underline{V})})}^{-1} \\ < p_{(\underline{V})} & v_i \leq f_{(p_{(\underline{V})})}^{-1} \end{cases}$$

The expected utility of the bidder is given by

$$Eu_i(b|v_i, v_{-i}) = \begin{cases} E(v - b|b \text{ wins}) = E(v_i - b|b \geq f_{(v_{-i})}) & v_i \geq f_{(p_{(\underline{V})})}^{-1} \\ 0 & v_i \leq f_{(p_{(\underline{V})})}^{-1} \end{cases}$$

An important feature is the reservation value is announced beforehand, and therefore it is not necessary to compute the probability of beating this value (as it is necessary when computing the probability of beating other participants' bids). Otherwise the seller would be considered as an additional bidder whose private reserved valuation (transformed into a "seller bid" or reserve price) should be beat and the conditioned expectation of the utility of the bidder should recognize this fact.

Using the same tools that the standard case without reservation value we get,

$$[f(v)G_{(v)}^{I-1}]' = v.(I-1)G_{(v)}^{I-2}.G'_{(v)}$$

Integrating both sides between \underline{V} (the public value for the seller) and \bar{v} (the private value for the bidder) we have

$$\int_{\underline{V}}^{\bar{v}} [f(v)G_{(v)}^{I-1}]' = \int_{\underline{V}}^{\bar{v}} v.(I-1)G_{(v)}^{I-2}.G'_{(v)}$$

$$f(\bar{v})G_{(\bar{v})}^{I-1} - f(\underline{V})G_{(\underline{V})}^{I-1} = \int_{\underline{V}}^{\bar{v}} v.(I-1)G_{(v)}^{I-2}.G'_{(v)}dv \quad (\text{and } f(\underline{V}) = \underline{V})$$

$$f(\bar{v}) = \underline{V} \frac{G_{(\underline{V})}^{I-1}}{G_{(\bar{v})}^{I-1}} + \frac{(I-1)}{G_{(\bar{v})}^{I-1}} \int_{\underline{V}}^{\bar{v}} v.G_{(v)}^{I-2}.G'_{(v)}dv$$

Particular case: Uniform Distribution

Consider v is distributed as a uniform between $[0,1]$.

$$f(\bar{v}) = \underline{V} \frac{\underline{V}^{I-1}}{\bar{v}^{I-1}} + \frac{(I-1)}{\bar{v}^{I-1}} \int_{\underline{V}}^{\bar{v}} v.v^{I-2}.dv$$

$$f(\bar{v}) = \frac{\underline{V}^I}{\bar{v}^{I-1}} + \frac{(I-1)}{\bar{v}^{I-1}} \left(\frac{v^I}{I} \Big|_{\underline{V}}^{\bar{v}} \right)$$

$$f(\bar{v}) = \frac{\underline{V}^I}{\bar{v}^{I-1}} + \frac{(I-1)}{I.\bar{v}^{I-1}} (\bar{v}^I - \underline{V}^I)$$

$$f(\bar{v}) = \frac{\underline{V}^I}{\bar{v}^{I-1}} \left(1 - \frac{I-1}{I} \right) + \frac{(I-1)}{I.\bar{v}^{I-1}} (\bar{v}^I)$$

$$f_{(\bar{v})}^{1^{st}-w/RV} = \left(\frac{V}{\bar{v}}\right)^I \left(\frac{\bar{v}}{I}\right) + \left(1 - \frac{1}{I}\right) \bar{v} \quad (14)$$

This result is consistent with a Bayesian Nash symmetric equilibrium where each player's bid depends on his own valuation (\bar{v}), on the valuation of the seller (V) and on the number of bidders (I). (in the limit, as $I \rightarrow \infty$, $f_{(\bar{v})} \rightarrow \bar{v}$).

Remember also that in the simplest case, when there is no reservation value for the seller, the optimal bid was

$$f_{(\bar{v})} = \left(1 - \frac{1}{I}\right) \bar{v}$$

which is the case by setting $V = 0$ (i.e. no reservation price for the seller) in equation (14).

Expected Revenue for the seller when reservation value is V

To obtain the expected revenue to the seller, we need to integrate over all players with value below the value of the winner and, for the winner between 0 and V .

Setting $v_1 = \bar{v}$ without loss of generality. Using the general formulation, $f_{(\bar{v})} = \left(\frac{V}{\bar{v}}\right)^I \left(\frac{\bar{v}}{I}\right) + \left(1 - \frac{1}{I}\right) \bar{v}$, the expected revenue is,

$$\begin{aligned} ER &= I \int \dots \int_{v_1 > v_{-1}} \left[\left(\frac{V}{\bar{v}}\right)^I \left(\frac{\bar{v}}{I}\right) + \left(1 - \frac{1}{I}\right) \bar{v} \right] dv_I . dv_{I-1} . \dots . dv_1 \\ ER &= I \int_{\underline{V}}^{\bar{v}} \int_0^{\bar{v}} \dots \int_0^{\bar{v}} \left[\frac{\bar{v}}{I} \left(\frac{V}{\bar{v}}\right)^I + \left(1 - \frac{1}{I}\right) \bar{v} \right] dv_I . dv_{I-1} . dv_{I-2} . \dots . d\bar{v} \\ ER &= I \int_{\underline{V}}^{\bar{v}} \left[\left(\frac{\bar{v}}{I} \left(\frac{V}{\bar{v}}\right)^I + \left(1 - \frac{1}{I}\right) \bar{v} \right) \left(\int_0^{\bar{v}} \dots \int_0^{\bar{v}} dv_I . dv_{I-1} . dv_{I-2} . \dots . d_2 \right) \right] . d\bar{v} \end{aligned}$$

Hence,

$$ER = I \int_{\underline{V}}^{\bar{v}} \left[\left(\frac{\bar{v}}{I} \left(\frac{V}{\bar{v}}\right)^I (\bar{v}^{I-1}) + \left(1 - \frac{1}{I}\right) \bar{v} \right) (\bar{v}^{I-1}) \right] . d\bar{v}$$

$$\begin{aligned}
ER &= \int_{\underline{V}}^1 (\underline{V}^I) d\bar{v} + (I-1) \int_{\underline{V}}^1 \bar{v}^I .d\bar{v} \\
ER &= (\underline{V}^I) \bar{v} \Big|_{\underline{V}}^1 + (I-1) \frac{\bar{v}^{I+1}}{I+1} \Big|_{\underline{V}}^1 \\
ER &= \underline{V}^I - \underline{V}^{I+1} + (I-1) \frac{1 - \underline{V}^{I+1}}{(I+1)} \\
ER &= \underline{V}^I \left(1 - \frac{2\underline{V}^I}{(I+1)} \right) + \frac{I-1}{I+1} \tag{15}
\end{aligned}$$

We can obtain the expected revenue of first price auction without reservation value (2) by replacing in (15) $\underline{V} = 0$

Now, a natural question arises. Which is the value of \underline{V} that yields the greatest expected revenue for the seller?

$$\begin{aligned}
\frac{\partial ER}{\partial \underline{V}} &= 0 \\
I\underline{V}^{I-1} \left(1 - \frac{2\underline{V}^I}{(I+1)} \right) + \underline{V}^I \left(-\frac{2I}{(I+1)} \right) &= 0 \\
I\underline{V}^{-1} - \frac{2I^2}{(I+1)} - \frac{2I}{(I+1)} &= 0
\end{aligned}$$

$$\underline{V}^* = 1/2$$

Just to verify this candidate is indeed a a maximum

$$\begin{aligned}
\frac{\partial^2 ER}{(\partial \underline{V})^2} \Big|_{\underline{V}=1/2} &< 0 \\
\frac{\partial^2 ER}{(\partial \underline{V})^2} &= \frac{\partial}{\partial \underline{V}} \left[I\underline{V}^{I-1} \left(1 - \frac{2\underline{V}^I}{(I+1)} \right) + \underline{V}^I \left(-\frac{2I}{(I+1)} \right) \right] \\
&\equiv (I-1)(2) - 2I \text{ vs } 0 \\
&\equiv -2 < 0
\end{aligned}$$

$$\frac{\partial^2 ER}{(\partial \underline{V})^2} \Big|_{\underline{V}=1/2} < 0$$

Hence the reserve value of 1/2 for the seller is the maximizer of his expected revenue in this particular case.