

Lecture Notes

Macroeconomics - ECON 510a, Fall 2010, Yale University

A Neo-Classical Benchmark Economy

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1 The Neo-Classical Benchmark

In these notes, we discuss a dynamic stochastic general equilibrium (DSGE) economy with complete markets. This model serves as a useful benchmark and point of departure for many questions in macroeconomics. It also serves as a simple first pass at a positive model of business cycles or asset pricing in closed- and open-economy contexts. You should familiarize yourself with all the salient features and results of this model. On the quantitative side, much of the modern research on quantitative business cycle models is based on departures from this benchmark that seek to improve the model's ability to account for the data. On the theoretical side, this model provides a useful backdrop for much of the recent work on the effects of frictions for macroeconomic outcomes, the role of heterogeneity, and the optimal design of public policy.

1.1 Definitions and Environment

Time and Uncertainty: We use a discrete time, infinite horizon economy, where time is denoted by $t = 0, 1, 2, \dots$. Uncertainty is encoded in a set of dated histories $\{s^t\}$

that are ordered into an infinite tree, where s^t denotes the set of all events up to date t . Each event has a single predecessor s^{t-1} , and a positive finite set of successors s^{t+1} , denoting by $s^{t+1} \succ s^t$ the successor relation. We define probabilities $\pi(s^t) > 0$ such that the sum of the probabilities in each branch equals the probability of s^t . Formally, $\pi(s^t) = \sum_{s^{t+1} \succ s^t} \pi(s^{t+1})$.

Agents, Preferences and Technologies: There is a single consumption good, which at each date can be consumed or converted into capital. There are J classes of households, each represented by a measure μ_j of agents. The households rank consumption and labor supply sequences according to the following von-Neumann Morgenstern utility function:

$$U(\{c^j(s^t); n^j(s^t)\}) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) u(c^j(s^t); n^j(s^t))$$

The consumption good $c^j(s^t)$ is produced by a representative firm, using efficiency units of labor $N(s^t)$ and capital $K(s^t)$ as inputs (note that since this is a representative firm, that firm can also be interpreted as the aggregate production function, justifying our use of capital letters). The total number of efficiency units of labor at s^t is

$$N(s^t) = \sum_{j=1}^J \mu_j \theta^j(s^t) n^j(s^t)$$

The variables $\theta^j(s^t)$ encode variations in the productivity of different households, and may be interpreted as idiosyncratic shocks. Different types thus differ according to their labor productivity. The aggregate output of the consumption good is

$$Y(s^t) = A(s^t) f(K(s^{t-1}), N(s^t))$$

where $K(s^{t-1})$ is the current capital stock, which was set at the end of the previous period. $A(s^t)$ denotes the aggregate total factor productivity (TFP) level.¹ The capital

¹Total Factor Productivity (TFP) is the portion of output not explained by the amount of inputs used in production. As such, its level is determined by how efficiently and intensely the inputs are combined in production. This variable will be key in our study of Real Business Cycle.

stock depreciates at a rate δ , so the aggregate resource constraint is

$$\sum_{j=1}^J \mu_j c^j(s^t) + K(s^t) - (1 - \delta)K(s^{t-1}) \leq A(s^t)f(K(s^{t-1}), N(s^t)), \quad \forall s^t$$

Remarks

- The restriction to finite sets of events at each date simplifies substantially the analysis for our purposes. It implies that we can work with sums to appropriately define expectations. On a few occasions it will be useful to introduce continuous random variables, typically in order to make use of specific functional form assumptions.
- This class, future macro courses and most of the macro literature you will read allow in one way or another for shocks to tastes, endowments, depreciation of capital, discounting, technology and monetary policy (e.g. sunspots). These shocks can be either aggregate or household specific (or both). The interpretation of these shocks, as well as their stochastic structure, are defined on a case-by-case basis, depending on the context and the questions that are being tackled.

For our purposes, however, it suffices to have a single source of cross-sectional and a single source of aggregate uncertainty, for now. These are productivity shocks.

- Notice that we haven't imposed any recursion. It is straight-forward to do so, however, by assuming that s^t takes on a first-order Markov structure over a finite set of states. We will come back to this point later in these notes.

1.2 A Planner's Problem

We approach the analysis of this economic model by first positing it as a planning problem. Suppose a planner attaches weights ψ_j to the different types of agents utilities. That is, the planner ranks economy-wide allocations according to

$$\sum_{j=1}^J \psi_j \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) u(c^j(s^t); n^j(s^t))$$

We can then solve for the optimal allocation of resources, i.e. the optimal sequence $\{c^j(s^t); n^j(s^t), K(\cdot)\}$ by maximizing the planners objective subject to the aggregate resource constraint. We solve this problem by setting up a Lagrangian

$$\begin{aligned} \mathcal{L} = & \sum_{j=1}^J \psi_j \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) u(c^j(s^t); n^j(s^t)) + \\ & \sum_{t=0}^{\infty} \sum_{s^t} \lambda(s^t) [A(s^t) f(K(s^{t-1}), N(s^t)) + (1 - \delta)K(s^{t-1}) - K(s^t) - \sum_{j=1}^J \mu_j c^j(s^t)] \end{aligned}$$

Taking first-order conditions,

$$\begin{aligned} \{c^j(s^t)\} : & \quad \psi_j \beta^t \pi(s^t) u_c(c^j(s^t); n^j(s^t)) = \mu_j \lambda(s^t) \\ \{n^j(s^t)\} : & \quad -\psi_j \beta^t \pi(s^t) u_n(c^j(s^t); n^j(s^t)) = \mu_j \lambda(s^t) A(s^t) f_n(K(s^{t-1}), N(s^t)) \theta^j(s^t) \\ \{K(s^t)\} : & \quad \lambda(s^t) = \sum_{s^{t+1} \succ s^t} \lambda(s^{t+1}) \{A(s^{t+1}) f_K(K(s^t), N(s^{t+1})) + 1 - \delta\} \end{aligned}$$

Several important results follow immediately from these first order conditions:

Main Properties of the Efficient Allocation

1. **The efficient allocation equates each household's marginal rate of substitution between consumption and labor to his marginal product.** From FOCs 1 and 2.

$$\theta^j(s^t) A(s^t) f_n(K(s^{t-1}), N(s^t)) = - \frac{u_n(c^j(s^t); n^j(s^t))}{u_c(c^j(s^t); n^j(s^t))}$$

Thus at the level of any individual, the planner's solution optimally trades off the costs and gains of additional work effort. If there is no heterogeneity in skills, then there should also be no heterogeneity in the marginal rates of substitution and then no difference in the allocation of labor and consumption.

2. **The efficient allocation equates the inter-temporal marginal rate of substitution across all households.** From FOCs 1 evaluated at periods 0 and t .

$$\beta^t \pi(s^t) \frac{u_c(c^j(s^t); n^j(s^t))}{u_c(c^j(s^0); n^j(s^0))} = \frac{\lambda(s^t)}{\lambda(s^0)}, \quad \forall j, s^t$$

Since $\beta^0 = 1$, $\pi(s^0) = 1$ and we assume the ratio ψ_j/μ_j constant over time. Thus, any difference in individual allocations are due to differences between the plan-

ning weights ψ_j and the population weights μ_j , which affects the marginal utility of consumption or effort by the same proportion at all periods. Intuitively then, any initial difference in planning weights have a permanent effect on allocations, but do not affect the individuals inter-temporal margins. The ratio ψ_j/μ_j only affects the overall fraction of resources of this economy to which type j household have access, but has no other effects on the allocation of these resources over time.

This result changes when the household's productivity depend on unobservable efforts of households. This introduces efficiency constraints to redistribution objectives.

3. **The efficient allocation provides full insurance against idiosyncratic uncertainty:** Marginal utilities of consumption do not depend on the histories of idiosyncratic shocks $\theta^j(s^t)$.
4. **Efficient investment levels satisfy the inter-temporal Euler Equation for each household.** From FOCs 1 and 3.

$$u_c(c^j(s^t); n^j(s^t)) = \sum_{s^{t+1} \succ s^t} \beta \pi(s^{t+1}|s^t) u_c(c^j(s^{t+1}); n^j(s^{t+1})) \{A(s^{t+1}) f_K(K(s^t), N(s^{t+1})) + 1 - \delta\}$$

Remarks

- The variable $\lambda(s^t)$ represents the shadow value of aggregate consumption in different states of the world. You see from the first FOC how this variable summarizes the scarcity and valuation of resources across time, states of nature and individuals. Intuitively, $\lambda(s^t)$ is the cost to having aggregate capital in the economy, which should equalize the benefits of that capital, which accrues from future production.
- So far we have assumed that the economy is closed. We can consider a SMALL open-economy version of this economy by assuming that the social planner can trade consumption with an outside world at a fixed set of state prices $q(s^t)$. In that case, the planner maximizes his objective, subject to an inter-temporal

budget constraint.

$$\sum_{s^t} q(s^t) [A(s^t) f(K(s^{t-1}), N(s^t)) + (1 - \delta)K(s^{t-1}) - K(s^t) - \sum_{j=1}^J \mu_j c^j(s^t)] \geq 0$$

where we are implicitly assuming that the present value of the country's total wealth is finite. With this assumption, after denoting by λ the multiplier on the inter-temporal aggregate resource constraint, we find exactly the same first order conditions, but with $\lambda(s^t) = \lambda q(s^t)$. Thus, the shadow values are now determined by the exogenous state-prices, given by the rest of the world. The difference between the open- and closed-economy models is therefore that in the former the shadow-values are exogenous parameters, while in the latter they are endogenously determined to clear the resource constraint in every state and date.

- If in addition the per-period utility function is additively separable between consumption and leisure, then consumption is entirely pinned down from the state-price process, and completely separated from the work-effort decisions. Check why this is the case! (Hint: Analyze the second result above)

1.3 Decentralization

Next, we show how the solution to this planning problem can be decentralized, and interpreted as the solution to a competitive equilibrium in a dynamic economy with complete contingent markets. This will offer us a concrete, positive, interpretation of the allocations in the planner solution in terms of a competitive market outcome. It completes the foundation for comparing the models implications with the data by offering a clear theoretical counterpart for the price variables, in particular wages, interest rates, and asset prices.

We describe an Arrow-Debreu environment, where agents trade a complete set of claims to consumption whose delivery is contingent on a particular realization of the state of the world. Completeness means that consumption claims can be purchased that are contingent on any realization of the state of the world. In the case of additive utility, these Pareto-efficient allocations do not depend on the history of the economy

(except through the size of the aggregate endowment of course!). There is alternative way of decentralizing Pareto-efficient allocations by allowing agents to trade Arrow securities sequentially. In this decentralization markets re-open each period and agents re-trade each period. We will describe the sequential trade equilibrium later. In this part, we focus on a decentralization where all trade happens at period 0.

Later in the lecture notes we will introduce frictions in the financial markets, but we will not exogenously restrict the menu of traded assets. Instead, we will focus our attention on endogenous restrictions on the portfolios of agents that flow from their inability to commit to contracts. Limited commitment introduces history-dependence in the constrained efficient allocations.

1.4 Environment

Consider a competitive production and exchange economy, in which in each period there is a spot labor market and a complete contingent set of securities. The representative firm pays dividends, which we define in the following way

$$D(s^t) = A(s^t)f(K(s^{t-1}), N(s^t)) + (1 - \delta)K(s^{t-1}) - K(s^t) - \sum_{j=1}^J \mu_j n^j(s^t) w^j(s^t)$$

for each state s^t , where $w^j(s^t)$ is the wage for type j household that provides labor $n^j(s^t)$. The firm determines the use of capital and labor so as to maximize the net present values of future dividends

$$\sum_{t=0}^{\infty} \sum_{s^t} q(s^t) D(s^t)$$

where $q(s^t)$ is the price of consumption at state s^t , relative to s^0 .

Let χ_j denote the share of dividends paid to a type j household (such that $\sum_{j=1}^J \mu_j \chi_j = 1$). We also allow for an initial allocation of wealth $\{a^j(s^0)\}$ for different types, where $\sum_{j=1}^J \mu_j a^j(s^0) = 0$

Definition 1 A distribution of wealth is a vector $\vec{a}_t(s^t) = \{a_t^j(s^t)\}_{j=1}^J$ that satisfies the condition $\sum_j \mu_j a_t^j(s^t) = 0$ at each period t .

Definition 2 A competitive equilibrium is a sequence of allocations $\{c^j(s^t); n^j(s^t); K(s^t); \}$ and a sequence of state prices and wages $\{q(s^t); w^j(s^t)\}$, such that given an initial distribution of wealth $\vec{a}_0(s^0)$, households maximize expected utility and firms maximize expected dividends, and all markets clear.

Characterization of the competitive equilibrium For any set of planner weights $\{\psi_j\}$, there exists a distribution of initial wealth levels $\{a^j(s^0)\}$, for which the resulting optimal allocation $\{c^j(s^t); n^j(s^t); K(s^t); \}$ can be obtained as a competitive equilibrium with equilibrium wages and state prices equal to $q(s^0) = 1$ and

$$q(s^t) = \frac{\lambda(s^t)}{\lambda(s^0)}$$

(later we will be more precise and we will denote $q_t^\tau(s^t)$ the claim traded at period τ for some goods delivered at period t under the history s^t . Since here we're assuming trade only at period 0, we just denote $q(s^t)$ for simplicity).

These are the prices that equalize FOCs from the decentralized situation and the FOCs from the planner's problem. Anyways, this result is an immediate consequence of the second welfare theorem, for a convex competitive economy with complete markets. However, we will discuss this point later.

The firm's problem is

$$\max_{\{n^j(s^t), K(s^t)\}} \sum_{t=0}^{\infty} \sum_{s^t} q(s^t) D(s^t)$$

The first order conditions for the firms problem are given by

$$\begin{aligned} \{n^j(s^t)\} & : \quad w^j(s^t) = \theta^j(s^t) A(s^t) f_n(K(s^{t-1}), N(s^t)) \\ \{K(s^t)\} & : \quad q(s^t) = \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \{A(s^{t+1}) f_K(K(s^t), N(s^{t+1})) + 1 - \delta\} \end{aligned}$$

which are automatically satisfied by the planning allocation, under our proposed equilibrium prices.

Each household j maximizes the expected discounted utility subject to its inter-temporal budget constraint:

$$a_0^j(s^0) + \sum_{t=0}^{\infty} \sum_{s^t} q(s^t) (\chi_j D(s^t) + w^j(s^t) n^j(s^t)) \geq \sum_{t=0}^{\infty} \sum_{s^t} q(s^t) c^j(s^t)$$

the household's first order conditions and budget constraints are exactly satisfied. To see this, and considering we can write the maximization problem for the household j (taking as exogenous the dividends $D(s^t)$) as

$$\max_{\{c^j(s^t), n^j(s^t)\}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) u(c^j(s^t); n^j(s^t)) + \lambda_j q(s^t) [\chi_j D(s^t) + w^j(s^t) n^j(s^t) - c^j(s^t)] - \lambda_j a_0^j(s^0)$$

The type j households first order conditions are

$$\begin{aligned} \{c^j(s^t)\} & : \quad \beta^t \pi(s^t) u_c(c^j(s^t); n^j(s^t)) = q(s^t) \lambda_j \\ \{n^j(s^t)\} & : \quad -\beta^t \pi(s^t) u_n(c^j(s^t); n^j(s^t)) = w^j(s^t) q(s^t) \lambda_j \end{aligned}$$

and hence

$$\begin{aligned} q(s^t) &= \beta^t \pi(s^t) \frac{u_c(c^j(s^t); n^j(s^t))}{u_c(c^j(s^0); n^j(s^0))} \\ w^j(s^t) &= -\frac{u_n(c^j(s^t); n^j(s^t))}{u_c(c^j(s^t); n^j(s^t))} \end{aligned}$$

as in the social planner's solution. Finally, notice that these allocations clear the labor market and aggregate resource constraints, and therefore constitute a competitive equilibrium of the decentralized economy.

1.5 Implications and Applications

1.5.1 Risk sharing and insurance

The second and third properties of optimal allocations described above lead to stark implications about risk-sharing. First, individual marginal utilities should not respond to household-specific income or productivity fluctuations, but only fluctuate with aggregate conditions, which are summarized by $\lambda(s^t)$. Second, allocations across households should equalize their intertemporal marginal rates of substitution, regardless of their productivity shocks.

Household incomes should affect consumption only through the total wealth level (Permanent Income Hypothesis). With household level data on incomes and consumption, it is in principle possible to test this implication directly.

To do so, we need some additional assumptions. In the simplest form of the test, let us assume that preferences are additively separable between consumption c and labor supply n . Then, marginal utility directly maps into household-level consumption. Suppose now that we have a household panel of consumption and income observations. The second property then implies that, if we regress household consumption c on household income and some control variables for aggregate conditions (to proxy for $\lambda(s^t)$), the perfect risk-sharing hypothesis holds that household-level income data hold no additional explanatory power for household level consumption. Townsend (1994) conducts a test along these lines using data collected from three rural villages in India. His test rejects the null hypothesis of full risk-sharing. However, the amount of risk sharing observed is remarkable, which would be an indication that individual income shocks are pretty well absorbed in such an environment.

This test (in its initial forms) relied on two strong assumptions: first, the additive separability assumption which allowed us to map consumption into marginal utilities, and second, that household preferences were homogeneous within the panel. More recent work enriches the existing tests to allow for unobserved household-level heterogeneity (in particular in risk aversion). Second, complementarities in preferences may generate a positive relation between household level income and consumption that would otherwise be suggestive of lack of risk-sharing. For instance, after retirement, if separability held, even though labor becomes zero, consumption should remain the same. In U.S. data this is not the case: we usually see a drop in consumption of about 20 – 30% at retirement, which is suggestive of non-separability in preferences (substituting home-cooked meals for restaurants, less need for clothing, transportation and other work-related consumption expenditures, etc. after retirement). Another important branch of the micro literature on consumption and risk-sharing at the household level is the question to what extent departures from the full risk-sharing benchmark can be accounted for by information, agency and other frictions.

1.5.2 Real Business Cycle (RBC)

We can use the model to think about implications for business cycles. If we set a single type j , we have a representative household, and our model corresponds to the standard one-sector Real Business Cycle world. In such an environment, wages co-

move positively with productivity and consumption is less volatile than output and investment over the business cycle. The interesting issue in such an environment is how to get the volatility of labor right. We will discuss these details in next lectures.

1.5.3 International RBC

Assume $j = 2$ and interpret θ_j as a country specific shock. The question is how should consumption, labor and income co-move across countries. Even with two capital accumulation equations, i.e., say, if capital is fixed per country so that it cannot be freely moved, consumption across both countries should co-move a lot. In fact, it should co-move a lot more than income. In reality, income comoves more than consumption across countries. In this setting, a planner sends capital to the country where TFP is highest, so that we should also observe a strongly negative co-movement in K . However, in the data we observe that patterns of investment are positively correlated across countries. As a reference, check Backus, Kehoe and Kydland (1993) to see what the real exchange rate implications are, as well as the relative price of goods in different countries. In data we also observe home-bias, i.e. investors tend to invest much more inside their home country, which points toward stating that there is little evidence of international risk sharing.

1.5.4 Asset Pricing

Since we have completely determined the state-prices for Arrow securities, we can use these state-prices to price any other financial asset as a redundant security. These securities have no specific economic role on top of the Arrow securities, but they can be constructed to have interpretable empirical counter-parts, and we can therefore use our model to draw implications for all sorts of financial asset returns. We will discuss more about this and the so called "Equity Premium Puzzle" (which analyze the consumption based pricing kernel) in next lectures.

1.5.5 Corporate Finance

A noticeable feature of the complete markets economy and its decentralization is the complete absence of financial considerations from the analysis. The financial struc-

ture of corporations determines how the cash-flows generated by productive activities are divided into the returns of different types of financial claims, such as debt or equity. We will show next that we can simply price these securities as redundant securities by decomposing their valuations into those of the underlying combination of Arrow securities. These securities are also redundant for the firms purposes: as long as the firm chooses labor demand and investments to maximize the net present value of future cash flows, $\sum_{s^t} q(s^t)D(s^t)$, it does not matter how these cash flows are divided into financial assets and sold to households.

This striking result is called the Modigliani-Miller Theorem, which states that under the existence of complete contingent markets and no distortions, financial structure is irrelevant for real decisions. In effect, we have divided the firms problem into a first step of maximization of expected dividends, and a second step of pricing of these dividends in the form of equity shares and debt. Much of the modern corporate finance literature departs from this theorem and discusses how departures from this benchmark result (due to informational frictions, tax reasons, or other departures from the Neo-classical benchmark) may influence the firms optimal financing choices, their financial structure, and the incentives of their managers.

2 Other Definitions of Equilibrium

2.1 A Primer on Pricing

In the previous section we assumed agents can trade claims that are consumption contingent on all states of the world s^t , and that all trading occurs at time 0, after observing the state s^0 . However, we also want to study if this assumption is restrictive or sequential markets that open at each t achieve the same results. This is then a good point to stop and introduce some key concepts on pricing, such as the SDF (Stochastic Discount Factor).

For completeness, let's denote $q_t^0(s^t)$ the price of a claim traded at period 0 for consumption at period t when the state of the world is s^t . As can be seen, this is the same as above but being more general about when a claim is traded and when it is paid off.

- When markets are complete, we can price an asset by breaking it up into its

state-dependent components and pricing those with the relevant state-price deflator $\frac{q_t^0(s^t)}{\pi_t(s^t)}$

- The price of an asset with claim $d_t(s^t)$ in different periods t and states s^t is

$$p_0^0(s^0) = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) d_t(s^t)$$

If not, there would be an arbitrage opportunity!!!

- The price of a riskless console, which always pays 1, regardless of the state s^t .

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)$$

- The price of a tail asset, which pays $d_t(s^t)$ starting at period τ

$$p_\tau^0(s^\tau) = \sum_{t=\tau}^{\infty} \sum_{s^t | s^\tau} q_t^0(s^t) d_t(s^t)$$

To convert this into units of s^τ consumption goods

$$p_\tau^\tau(s^\tau) \equiv \frac{p_\tau^0(s^\tau)}{q_\tau^0(s^\tau)} = \sum_{t=\tau}^{\infty} \sum_{s^t | s^\tau} \frac{q_t^0(s^t)}{q_\tau^0(s^\tau)} d_t(s^t)$$

but we know that

$$\frac{q_t^0(s^t)}{q_\tau^0(s^\tau)} = q_t^\tau(s^t)$$

Then, the price of a tail asset is,

$$p_\tau^\tau(s^\tau) = \sum_{t=\tau}^{\infty} \sum_{s^t | s^\tau} q_t^\tau(s^t) d_t(s^t)$$

Why is this important? Because $q_t^\tau(s^t)$ are the prices that would obtain if markets were reopened at time τ in this economy!

Proposition 1 *Starting from the time t wealth distribution implied by the original Arrow-Debreu equilibrium, if markets were reopened at time t , no trading will occur. All households*

choose the tails of their consumption plans.

- Pricing one period returns

$$q_{\tau+1}^{\tau}(s^{\tau+1}) = \beta \pi(s_{\tau+1}|s^{\tau}) \frac{u_c(c_{\tau+1}^j(s^{\tau+1}), n_{\tau+1}^j(s^{\tau+1}))}{u_c(c_{\tau}^j(s^{\tau}), n_{\tau}^j(s^{\tau}))}$$

- Then the price of a claim to dividends only in the next period is

$$p_{\tau}^{\tau}(s^{\tau}) = \sum_{s^{\tau+1}|s^{\tau}} q_{\tau+1}^{\tau}(s^{\tau+1}) d_{\tau+1}(s^{\tau+1})$$

$$p_{\tau}^{\tau}(s^{\tau}) = E_{\tau} \left[\beta \frac{u_c(c_{\tau+1}^j(s^{\tau+1}), n_{\tau+1}^j(s^{\tau+1}))}{u_c(c_{\tau}^j(s^{\tau}), n_{\tau}^j(s^{\tau}))} d_{\tau+1}(s^{\tau+1}) \right]$$

- Let $R_{\tau+1}(s^{\tau+1}) \equiv \frac{d_{\tau+1}(s^{\tau+1})}{p_{\tau}^{\tau}(s^{\tau})}$ denote the return. Then, for any asset

$$1 = E_{\tau} \left[\beta \frac{u_c(c_{\tau+1}^j(s^{\tau+1}), n_{\tau+1}^j(s^{\tau+1}))}{u_c(c_{\tau}^j(s^{\tau}), n_{\tau}^j(s^{\tau}))} R_{\tau+1}(s^{\tau+1}) \right]$$

or

$$1 = E_{\tau} [m_{\tau+1} R_{\tau+1}(s^{\tau+1})]$$

where $m_{\tau+1} \equiv \beta \frac{u_c(c_{\tau+1}^j(s^{\tau+1}), n_{\tau+1}^j(s^{\tau+1}))}{u_c(c_{\tau}^j(s^{\tau}), n_{\tau}^j(s^{\tau}))}$ is the **stochastic discount factor (SDF)**

Example: A power utility function

Assume the utility function is additive on consumption and labor $u(c_t^j, n_t^j) = \frac{c_t^j}{1-\gamma} - v(n_t^j)$, where γ is the constant relative coefficient of risk aversion.

In this case, $u_c(c_t^j, n_t^j) = (c_t^j)^{-\gamma}$, then,

$$m_{t+1} = \beta \left[\frac{c_{t+1}^1}{c_t^1} \right]^{-\gamma} = \dots = \beta \left[\frac{c_{t+1}^J}{c_t^J} \right]^{-\gamma} = \beta \left[\frac{\sum_{j=1}^J c_{t+1}^j}{\sum_{j=1}^J c_t^j} \right]^{-\gamma}$$

The SDF only depends on aggregate consumption growth; all other properties of the equilibrium allocations have no bearing on the SDF. In the lecture about asset pricing we will discuss the properties of US aggregate consumption growth and its troubling implications for asset pricing.

2.2 Sequential Competitive Equilibrium

There is another way of decentralizing Pareto-efficient allocations by allowing markets to re-open in each period. In this sequential trading environment, we have to resort to borrowing constraints to keep agents from running Ponzi schemes. We chose to use natural borrowing constraints. These constraints require the debt of a household to be smaller than what it could pay back if its consumption were zero from that period onwards in all future states. If we impose an Inada condition on the utility function, these borrowing constraints will never bind, because the marginal utility of consumption exploded as consumption tend to zero!

In the following assume the household j just have a given endowment stream. This is, there is no working decisions nor production decisions, just an endowment technology, where the value of the endowment of household j at time t under state s^t is just $\omega_t^j(s^t)$. In this case, the natural debt limit is the value of claim to household j 's endowment. It is the maximum amount the household j can repay. In this part I assume households do not choose labor supply (workers supply labor inelastically, such that $n_t^j(s^t) = 1$ for all j, t and s^t) since it simplifies the exposition about the equivalence of the sequential and the recursive representation of the equilibrium. In case of having labor supply decision this debt limit is not straightforward since there is no way the household can commit to keep working to repay debts. This has many complications, some of which we will discuss when studying limited commitment and private information.

Define

$$\Gamma_t^j(s^t) = \sum_{\tau=t} \sum_{s^\tau|s^t} q_\tau^t(s^\tau) [c_\tau^j(s^\tau) - \omega_\tau^j(s^\tau)]$$

(from feasibility constraints at each period t , $\sum_{j=1}^J c_t^j(s^t) \leq \sum_{j=1}^J \omega_t^j(s^t)$, which implies $\sum_{j=1}^J \Gamma_t^j(s^t) = 0$).

The natural debt limit by a household j is then given by

$$A_t^j(s^t) = \sum_{\tau=t} \sum_{s^\tau|s^t} q_\tau^t(s^\tau) \omega_\tau^j(s^\tau)$$

since the household j cannot borrow more than what it can promise to pay. At time $t - 1$ household j cannot promise to repay more in any state of the world tomorrow

than the value of its endowment and dividend income stream $A_t^j(s^t)$.

Let $Q(s_{t+1}|s^t)$ be the price of one unit of consumption delivered contingent on the realization of s_{t+1} . The household faces a sequence of budget constraints,

$$c_t^j(s^t) + \sum_{s_{t+1}|s^t} Q(s_{t+1}|s^t) a_{t+1}^j(s_{t+1}|s^t) \leq \omega_t^j(s^t) + a_t^j(s^t)$$

At time t the household chooses $\{c_t^j(s^t), a_{t+1}^j(s_{t+1}|s^t)\}$.

We impose a state-by-state borrowing constraint

$$-a_{t+1}^j(s_{t+1}|s^t) \leq A_{t+1}^j(s^{t+1})$$

because of Inada condition, natural debt limit will not be binding, since that would mean the household has to commit to consume 0 forever after in its future.

The household j 's problem is then to maximize the expected discounted utility subject to the budget constraint and the natural debt limit above. This can be written as the following lagrangian

$$\begin{aligned} \mathcal{L} = & \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) u(c^j(s^t)) + \\ & \lambda_t^j(s^t) [\omega_t^j(s^t) + a_t^j(s^t) - c_t^j(s^t) - \sum_{s_{t+1}|s^t} Q(s_{t+1}|s^t) a_{t+1}^j(s_{t+1}|s^t)] \\ & \eta_t^j(s_{t+1}|s^t) [a_{t+1}^j(s_{t+1}|s^t) + A_{t+1}^j(s^{t+1})] \end{aligned}$$

First order conditions are

$$\begin{aligned} \{c^j(s^t)\} & : \quad \beta^t \pi(s^t) u_c(c^j(s^t)) = \lambda_t^j(s^t) \\ \{a_{t+1}^j(s_{t+1}|s^t)\} & : \quad \lambda_{t+1}^j(s^{t+1}|s^t) = \lambda_t^j(s^t) Q(s_{t+1}|s^t) + \eta_t^j(s_{t+1}|s^t) \end{aligned}$$

As discussed, the natural borrowing limit will not be binding, and hence $\eta_t^j(s_{t+1}|s^t) = 0$. This implies

$$Q(s_{t+1}|s^t) = \beta \pi(s_{t+1}|s^t) \frac{u_c(c_{t+1}^j(s^{t+1}))}{u_c(c_t^j(s^t))} = \pi(s_{t+1}|s^t) m_{t+1}$$

Definition 3 A *sequential trading* competitive equilibrium in a pure trading economy is a sequence of allocations $\{c^j(s^t)\}$ and pricing kernels $\{Q_{t+1}(s_{t+1}|s^t)\}$, such that given an initial distribution of wealth $\vec{a}_0(s^0)$, households maximize expected utility, and all markets clear.

2.2.1 Equivalence of Allocations

The first order conditions for the Arrow-Debreu in our problem are

$$\begin{aligned} q_{t+1}^0(s^t) &= \beta^t \pi(s^t) \frac{u_c(c^j(s^t))}{u_c(c^j(s^0))} \\ q_t^0(s^{t+1}) &= \beta^{t+1} \pi(s^{t+1}) \frac{u_c(c^j(s^{t+1}))}{u_c(c^j(s^0))} \end{aligned}$$

Guess that for given Arrow-Debreu prices $\{q_t^0(s^t)\}$, we can recover $Q_t(s_{t+1}|s^t)$.

$$q_{t+1}^0(s^{t+1}) = q_t^0(s^t) Q_t(s_{t+1}|s^t)$$

If this recursion is fulfilled,

$$Q_t(s_{t+1}|s^t) = \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)} = q_{t+1}^t(s^{t+1}) = \beta \pi(s^{t+1}|s^t) \frac{u_c(c^j(s^{t+1}))}{u_c(c^j(s^t))}$$

We also need

$$a_{t+1}^j(s_{t+1}|s^t) = \sum_{\tau=t+1} \sum_{s^\tau|s^{t+1}} \frac{q(s^\tau)}{q(s^{t+1})} [c_\tau^j(s^\tau) - \omega_\tau^j(s^\tau)]$$

which is the same FOC as in the case of trade at time-0 decentralized equilibrium.

Finally, we only need to additionally guarantee:

- 1) A state-by-state borrowing constraint $a_{t+1}^j(s_{t+1}) \geq \Gamma_{t+1}^j(s^{t+1})$ should be fulfilled.
- 2) The initial wealth of the sequential-trading equilibrium is such that the sequential-trading competitive equilibrium duplicates the Arrow-Debreu competitive equilibrium allocation. This initial wealth vector $\vec{a}_0(s^0)$ of the sequential trading economy should be chosen to be the null vector. This means that each household must rely on its own endowment stream to finance consumption, in the same way that house-

holds are constrained to finance their purchases for the infinite future at time 0 in the Arrow-Debreu economy.

To see why this is the case, guess

$$a_{t+1}^j(s_{t+1}|s^t) = \Gamma_{t+1}^j(s^{t+1}) = \sum_{\tau=t+1} \sum_{s^\tau|s^{t+1}} q_\tau^{t+1}(s^\tau) [c_\tau^j(s^\tau) - \omega_\tau^j(s^\tau)]$$

adding over all states with multiplying by the pricing kernels

$$\sum_{s_{t+1}} a_{t+1}^j(s_{t+1}|s^t) Q_t(s_{t+1}|s^t) = \sum_{s^{t+1}|s^t} \Gamma_{t+1}^j(s^{t+1}) Q_t(s_{t+1}|s^t)$$

Considering the definitions of $\Gamma_{t+1}^j(s^{t+1})$ and $Q_t(s_{t+1}|s^t)$,

$$\sum_{s_{t+1}} a_{t+1}^j(s_{t+1}|s^t) Q_t(s_{t+1}|s^t) = \sum_{\tau=t+1} \sum_{s^\tau|s^{t+1}} \frac{q_\tau^0(s^\tau)}{q_{t+1}^0(s^{t+1})} [c_\tau^j(s^\tau) - \omega_\tau^j(s^\tau)] \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)}$$

Then

$$\sum_{s_{t+1}} a_{t+1}^j(s_{t+1}|s^t) Q_t(s_{t+1}|s^t) = \sum_{\tau=t+1} \sum_{s^\tau|s^{t+1}} q_\tau^t(s^\tau) [c_\tau^j(s^\tau) - \omega_\tau^j(s^\tau)]$$

Considering the budget constraint measured at $t = 0$,

$$c_0^j(s^0) + \sum_{s_1|s^0} Q(s_1|s^0) a_1^j(s_1|s^0) \leq \omega_0^j(s^0) + a_0^j(s^0)$$

and plugging the definition of $\sum_{s_1|s^0} Q(s_1|s^0) a_1^j(s_1|s^0)$ from above,

$$\sum_{t=0} \sum_{s^t|s^0} q_t^0(s^t) [c_t^j(s^t) - \omega_t^j(s^t)] \leq \omega_0^j(s^0) - c_0^j(s^0) + a_0^j(s^0)$$

Then, with the restriction of $a_{t+1}^j(s_{t+1}) \geq \Gamma_{t+1}^j(s^{t+1})$ and $a_0^j(s^0) = 0$, the budget constraint from the Arrow-Debreu equilibrium is fulfilled.

You'll be asked to show these two conditions are guaranteed in a Problem Set.

2.3 Recursive Competitive Equilibrium

The most obvious problem with the previous formulation is the level of generality of history dependence given by s^t . We would like to work in an environment with a limited number of states that summarize the effects of past events and current information. This can be achieved with some exogenous assumption about the exogenous process that defines the states that allows for a recursive formulation of the sequential-trading equilibrium.

We can write a recursive competitive equilibrium, in which the equilibrium allocations are only a function of the current state, this is $c_t^j(s^t) = c^j(s^t)$ and $Q_t(s_{t+1}|s^t) = Q(s_{t+1}|s^t)$, under the two following assumption

Assumption 1 *Markovian structure on π*

$$\pi(s^t|s^0) = \pi(s^t|s_{t-1})\pi(s_{t-1}|s_{t-2})\dots\pi(s_1|s_0)$$

Given this Markov property, the conditional probability $\pi(s^t|s^\tau)$ for $t > \tau$ only depends on the realized state s_τ at time τ , not on the history before τ . This is because

$$\pi(s^t|s^\tau) = \pi(s^t|s_{t-1})\pi(s_{t-1}|s_{t-2})\dots\pi(s_1|s_\tau)$$

and hence it can be written mathematically as you studied during the first part of the course and as I sketch below. We can define policy functions for households that define consumption, labor, investments and future assets,

$$\begin{aligned} c^j &= h^j(a^j, s) \\ a^{j'}(s') &= g^j(s'|a^j, s) \end{aligned}$$

The Bellman equation for household j 's problem is

$$V^j(a, s) = \max_{c^j, \{a^{j'}(s')\}} \left\{ u(c) + \beta \sum_{s'} V^j(a'(s'), s') \pi(s'|s) \right\}$$

subject to,

$$c + \sum_{s'} a'(s')Q(s'|s) \leq \omega^j(s) + a$$

$$c \geq 0$$

and

$$-a'(s') \leq A^j(s')$$

Definition 4 A *recursive competitive equilibrium* is a set of decision rules $\{h^j(a, s); g^j(a, s, s')\}$, pricing kernels $\{Q(s'|s)\}$, sets of value functions $\{V^j(a, s)\}_{j=1}^J$ such that given an initial distribution of wealth $\vec{a}_0(s^0)$, households maximize expected utility, and all markets clear.

3 Welfare Theorems

This section briefly discusses the First and Second Welfare Theorems. The First Welfare Theorem establishes that competitive equilibrium is Pareto optimal (the "Invisible Hand"). The Second Welfare Theorem establishes that any Pareto optimum can, for suitable chosen prices, be supported as a competitive equilibrium.

In this section we follow Stokey and Lucas (1989), chapter 15. To show the assumptions under which these two theorems hold it is worthwhile to interpret any economic problem using Debreu's Theory of Value.

Debreu showed commodities and prices are dual concepts. As Debreu wrote "A commodity is characterized by its physical properties, the date at which it will be available, and the location at which it will be available. The price of a commodity is the amount which has to be paid **now** for the (future) availability of one unit of that commodity"

In other words, a loaf of bread in Minneapolis is a different commodity than a loaf of bread in New York. Furthermore, that bread is a different commodity than the same one but that will be sold in a month. In fact a loaf of bread to be sold in New York in month when the economy is booming is a different commodity than the same one if the economy is in recession. Commodities can be both goods or services. Again, citing Debreu "A commodity is a good or a service completely specified physically, temporally and spatially. It is assumed that there is only a finite number of distinguishable L commodities....The space $S = R^L$ will be called the **commodity space**"

Since all these are different commodities, we can attach different prices to them. If we can attach one price to all possible commodities, we say we have complete markets.

"The price of a commodity h (an element of R^L) may be positive (scarce commodity), null (free commodity) or negative (noxious commodity)." Debreu finally defined the price system as "the L -tuple $p = (p_1, \dots, p_h, \dots, p_L)$; it can clearly be represented by a point of R^L . The value of an action a relative to the price system p is $\sum_{h=1}^L p_h a_h$, this is the inner product $p \cdot a$ ".

Now, we will define the production and consumption sets. Assume there are n producers. Given a production y_i for each producer, the sum $y = \sum_{i=1}^n y_i$ is called total production, where y described the **net** result of the activity of all producers together, after considering the transfers of products between them (i.e., the final product of producer i used as input by producer j). This vector can also have positive and negative coordinates. A positive coordinate represents outputs not transferred to the production sector; a negative coordinate represent inputs not transferred from the production sector. The production set Y is defined as $Y = \sum_{i=1}^n Y_i$, where $y_i \in Y_i$, and defined the production possibilities of the whole economy.

When defining the consumption set (assume there are m total consumers, who can be households), Debreu assumed, as a convention, that the inputs of the j^{th} consumer are represented by positive numbers (their food, clothes and shelter, for example), while his output (labor, capital, time, for example) by negative numbers. The consumption set X is defined as $X = \sum_{j=1}^m X_j$, where $x_j \in X_j$, and defined the consumption possibilities of the whole economy.

Based on these definitions, together with preferences given by utility functions defined over these commodity sets, Stokey and Lucas define the conditions under which the two welfare theorems will be fulfilled.

The First Welfare Theorem requires no assumptions on technology (besides the ones required for the existence of an equilibrium) and only a nonsatiation condition for consumers. This basically means that, no matter what is the consumption of consumer j (in X_j), there is another one (in X_j) which the consumer j prefers, but may not be feasible. For the proof of the First Welfare Theorem, see Stokey and Lucas, pp 453.

The First Welfare Theorem holds if

1. Nonsatiation for consumers

The Second Welfare Theorem requires additional conditions since we depart from Pareto efficiency and want to prove its allocations can be implemented as a competitive equilibrium, which is a more complex object given the existence of prices.

The Second Welfare Theorem holds if

1. Consumption set X_j is convex, for all j .
2. Production set Y is convex.
3. Preferences are Quasi-concave: For each j , if $x, x' \in X_j$, $u_j(x) > u_j(x')$ and $\theta \in (0, 1)$, then $u_j(\theta x + (1 - \theta)x') > u_j(x')$.
4. Preferences are continuous: For each j , $u_j : X_j \rightarrow \mathbb{R}$ is continuous.
5. Either the production set Y has an interior point or the commodity set S is finite dimensional.

Again, for the proof, read Stokey and Lucas, pp. 455.

4 A Clean Comparison Between Competitive Equilibrium under Sequential and Recursive Analysis

Just to summarize the discussion above in simple terms, I'll discuss the social planning problem and the competitive equilibrium, reducing notation by eliminating reference to the state s^t and considering all those states as different commodities, (consistent with the Debreu's Theory of Value) and considering the case of a representative agent, such that $k_t = K_t$.

4.1 Main Properties

Social Planning

1. NO PRICES, JUST ALLOCATIONS.
2. Objective function: $\sum_j \psi_j u^j$

3. Constraint: Resource constraint.

Competitive Equilibrium

1. There are prices and markets.
2. Each agent is a price taker.
3. Each agent has his own objective function.
4. Each agent has his own resource constraint.
5. At equilibrium: markets clear and prices are determined.

4.2 Under Sequential Analysis

Sequential Social Planning

Find $\{c_t, n_t, k_t\}_{t=0}^{\infty}$ which solve

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, n_t)$$

subject to (assuming $A_t = 1$ for all t and a given k_0)

$$c_t + k_{t+1} - (1 - \delta)k_t = f(k_t, n_t)$$

Sequential Competitive Equilibrium

1. Allocation for households $\{c_t, n_t^s, k_t^s\}_{t=0}^{\infty}$.
2. Allocation for firms $\{y_t, n_t^d, k_t^d\}_{t=0}^{\infty}$.
3. Set of prices $\{w_t, q_t\}_{t=0}^{\infty}$

such that

1. Given 3, the allocations in 1 solve the household's problem

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, n_t)$$

subject to (assuming a given k_0)

$$c_t + i_t = w_t n_t + q_t k_t^s + D_t$$

$$k_{t+1}^s = (1 - \delta)k_t^s + i_t$$

2. Given 3, the allocations in 2 solve the firm's problem

$$\max D_t = y_t - w_t n_t^d - q_t k_t^d$$

subject to $y_t = f(k_t^d, n_t^d)$

3. Markets clear.

$$k_t^s = k_t^d$$

$$n_t^s = n_t^d$$

$$c_t + i_t = y_t$$

4.3 Under Recursive Analysis

Recursive Social Planning

Find $V(k)$ that solves the Bellman equation

$$V(k) = \max_{k', n} \{u[f(k, n) + (1 - \delta)k - k', n] + \beta V(k')\}$$

Recursive Competitive Equilibrium

1. A value function $V(k, K)$.
2. A set of individual decision rules $n(k, K); k'(k, K)$
3. A set of firms' decisions rules $n^f(K); k'(K)$

4. Pricing functions, $w(K), q(K)$.
5. A law of motion for $K, G(K)$.

such that

1. Given 4 and 5, values in 1 and policy functions in 2 solve the dynamic programming for households.

$$V(k, K) = \max \{u(c, n) + \beta V(k', K')\}$$

subject to $k' = G(K)$ and

$$c + k' - (1 - \delta)k = w(K)n + r(K)k$$

2. Given 4, policy functions in 3 solve the static problem for firms.

$$\max f(k^f, n^f) - w(K)n^f(K) - q(K)k^f(K)$$

3. Markets clear

$$n^f(K) = n(k, K)$$

$$k^f(K) = K = k$$

4. Perceptions are correct $k'(K, K) = G(K)$