# A Walrasian Theory of Sovereign Debt Auctions with Asymmetric Information\*

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#### Abstract

Sovereign bonds are highly divisible, usually of uncertain quality, and auctioned in large lots to a large number of potential investors. This leads us to assume that no individual bidder can affect the bond price, and to develop a tractable Walrasian theory of Treasury auctions in which investors are asymmetrically informed about the quality of the bond. We characterize the price of the bond for different degrees of asymmetric information, both under discriminatory-price (DP) and uniform-price (UP) protocols. We show that there is a trade-off between the level and volatility of sovereign bonds spreads, but only when information is sufficiently asymmetric. By endogenizing information acquisition we show that (i) DP protocols may display multiple informational equilibria, (ii) UP protocols are less likely to induce information acquisition and (iii) the cost of information determines how the trade-off is resolved.

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# 1 Introduction

Governments finance their fiscal needs by selling bonds in sovereign debt auctions: a large volume of these bonds is usually sold at once to a large number of investors, and investors are free to try and buy as many units of bonds as they can afford.<sup>1</sup> Based on the demand for the bonds, these auctions determine the price of newly issued sovereign bonds and the corresponding revenues for the government (or the future debt burden that a given revenue implies). These results play a critical role in governments' cost of financing deficits, the implementation of monetary policy and even the possibility to navigate succesfully internal and external macroeconomic shocks.

Understanding the evolution of sovereign bond prices at auction is challenging. Figure 1 show interest rate data for 91-day Cetes, issued by the Mexican government. These bonds were domestically denominated, sold in very small denominations and large lots, to a wide variety of investors, using auctions that alternated between discriminating (shaded in the figure) and uniform price protocols.<sup>2</sup> During the long period displayed in the figure, annual interest rates on these bonds displayed periods of high turbulence (coincident to events such as the Latin American debt crisis of the 1980s and the "Tequila Crisis" in 1995) and periods of prolonged stability (the 2000s). This long period includes a wide range of shocks, some public, some private, some learnable, some not. Some of these shocks affected common factors, like the probability of default, some of those shocks may have affected the private valuation of the government's bonds, like liquidity shocks. It is however difficult to pinpoint the evolution of these prices to a particular set of "fundamentals" as a given shock seems to affect prices at some times, but not at others.

Aguiar et al. (2016b) show that emerging market interest rate spreads (against LIBOR) vary substantially across both countries and time, suggesting as an important component

<sup>&</sup>lt;sup>1</sup>Malvey, Archibald, and Flynn (1995) report that the U.S. Treasury typically receives 75-85 competitive bids or tenders, many of which come from the 37 primary deals. They also receive 850-900 noncompetitive tenders through the book-entry system and another 19,000 through TREASURY DIRECT.

<sup>&</sup>lt;sup>2</sup>Cetes are zero-coupon bonds which investors can obtain directly online by using Cetesdirecto since 2010. Cetes remain among the most important public debt instruments in Mexico. In 2001, for example, Cetes alone represented 25% of all government securities, and were auctioned 180 times to 3,581 participating bidders. Mexico has switched the auction protocol in October 5, 2017 to uniform price auction after more than two decades of using discriminatory-price auctions.





in their determination the uncertainty about the likelihood of default or renegotiation. While these spreads are partially and occasionally accounted for by country fundamentals, like debt-to-output ratios or the growth rate of output, the high (on average) spreads on emerging market debt relative to actual defaults suggest that the pricing of these bonds include a substantial, but usually unaccounted for, risk premium.

Why sometimes a given shock to a "fundamental" seems to matter and sometimes not? What are the "fundamentals" that affect the valuation of a bond? In this paper we depart from the standard presumption that fundamentals are always in the information set of investors and explore how prices are determined if investors have to acquire at a cost information about fundamentals that are not public or if they have to process at a cost information that is public. This departure introduces the possibility that investors are asymmetric in their information sets, which introduces several challenges in determining the resulting prices of different auction protocols.

To analyze these auctions, we propose a novel model of auctions with three characteristics: (i) the good being auctioned is perfectly divisible, (ii) the number of bidders is large, and (iii) there is both common uncertainty about the good quality and about the mass of investors who participate in the auction. Given these three characteristics, the price-quantity strategic aspects of standard auction theory become less relevant, and a price-taking, or Walrasian, analysis emerges as a good approximation.<sup>3</sup> We will show that *Walrasian auctions* are particularly tractable and allow for an analysis of the role of information on equilibrium prices and, once we include information acquisition, the role of each auction protocol in determining the amount and asymmetry of information.

As with Cetes in Mexico, sovereign debt auctions worldwide are generally conducted using one of two formats - uniform-price auctions and discriminating-price auctions with discriminating-price auctions being slightly more prevalent and uniform-price auctions being the standard method used in the United States.<sup>4</sup> Given the prevalence of both protocols, and the frequency at which some countries switch between protocols (as is clear for Mexico), we consider both types of auction protocols and examine their impact, both for governments and investors.

A key aspect of our model is information asymmetry, some investors know more about the bond than others. We are able to characterize the equilibria of our auction model under both auction formats for different degrees of asymmetry (different ratio between informed and uninformed investors). This success comes despite the fact that we deviate from the standard CARA preferences with normal shocks, and that there is always a degree of asymmetric information under which prices do not perfectly reveal the quality of the bond ex-post.<sup>5</sup>

Introducing costly information acquisition (or similarly, costly processing of public information) in sovereign debt markets allows us to uncover a new source of multiplicity. We show that discriminatory-price auctions have the potential to generate multiple informational equilibrium regimes with different degrees of asymmetric information about

<sup>&</sup>lt;sup>3</sup>There are papers in the auction literature that yield price-taking as the number of bidders get large. A recent example is Fudenberg, Mobius, and Szeidl (2007), who show that the equilibria of large double auctions with correlated private values are essentially fully revealing and approximate price-taking behavior. Another is Reny and Perry (2006) who show a similar result when bidders have affiliated values and prices are on a fine grid.

<sup>&</sup>lt;sup>4</sup>The heterogeneity of treasury auction formats is well-documented. For example, the U.S. switched to a uniform-price format from a discriminating-price format in the 1970s, while Canada and Germany use the price-discriminating format. Bartolini and Cottarelli (2001) study a sample in which 39 out of 42 countries use discriminatory price auctions. Brenner, Galai, and Sade (2009) analyze a sample of 48 countries, out of which 24 use discriminatory-price auctions, 9 use uniform-price auctions and the rest use either both or an hybrid between the two.

<sup>&</sup>lt;sup>5</sup>It is well known that proving the existence of an equilibrium when prices are not fully revealing is very difficult (see Allen and Jordan (1998)), and that this challenge is usually tackled assuming CARA-normal cases because this combination yields a tractable linear price function.

the quality of sovereign bonds, while uniform-price auctions always display a unique equilibrium degree of asymmetric information. Still, the amount of information induced by a uniform-price auctions is never higher than the one induced by a discriminatory-price auction. The sovereign bond literature has considered other sources of multiplicity in sovereign debt markets, ranging from self-fulfilling beliefs on the part of investors to endogeneity in the default probability based on consistent beliefs, but it has neglected modeling how inference play a role in determining bond spreads and affecting incentives to acquire and process information. Even though we will discuss how these other sources of *price multiplicity* manifest in our setting, our focus will be in exploring the novel *informational regime multiplicity.*<sup>6</sup>

Our paper fills an important gap in the sovereign debt literature which has usually neglected the specifics on how the government sells bonds and the role of investors' information on the determination of prices. As we focus on that challenge, we have neglected some of the assumptions in other papers. First, most papers typically focus on the strategic decision of the government; something we completely neglect to focus on the auction mechanics and investors choices.<sup>7</sup> Second, most of the literature generates a clear mapping between the bond quality and its price by assuming investors to be risk neutral and then requiring that the return, adjusted for the probability of default, equals the risk-free rate. We depart from these settings dramatically by assuming risk aversion and a specific auction protocol. Finally, while there has been some attention to the impact of the timing of decisions and of debt maturity in sovereign markets (see Aguiar et al. (2016a)), the actual mechanics of how sovereign bonds are sold in reality through auctions and their impact on observed prices has been ignored. Our paper focuses on the neglected role of information acquisition when the auction is explicitly modeled.

There has been a recent effort to empirically document the implication of the different auction protocols and of the information sharing across dealers on the revenue of govern-

<sup>&</sup>lt;sup>6</sup>There are two main approaches to the role of self-fulfilling beliefs in this literature; the first is Calvo (1988) (with successor papers Lorenzoni and Werning (2013) or Ayres et al. (2016)) and the second is Cole and Kehoe (2000) (with successor papers Aguiar et al. (2015)) and Aguiar et al. (2017)).

<sup>&</sup>lt;sup>7</sup>See for example Eaton and Gersovitz (1981), the review articles by Aguiar and Amador (2013) and Aguiar et al. (2016b), and the recent quantitative literature by Aguiar and Gopinath (2006), Arellano (2008), Chatterjee and Eyigungor (2012), Bocola and Dovis (2016).

ments. For the former, see the survey by Hortaçsu (2011). For the latter, see Boyarchenko, Lucca, and Veldkamp (2017). As our setting is very different, we discuss our paper's relationship with this literature once we have presented our model. There we also discuss the relationship to general equilibrium theory and to the auctions literature more broadly.

# 2 Model with Exogenous Information Asymmetry

### 2.1 Environment

This is a two-period model featuring a measure one of ex-ante identical risk-averse investors and a government. The government is modeled mechanically: it needs to raise *D* units of the numeraire good in period one by auctioning a bond that promises repayment in period two.<sup>8</sup> This bond is risky because it constitutes a claim to one real unit in period two only if the government does not default. If the government defaults, then investors cannot recover any of the investment.

The probability of default,  $\kappa$ , is random and takes on two values,  $\kappa_g$  if the state is  $\theta = g$ and  $\kappa_b$  if the state is  $\theta = b$ . We assume that  $\kappa_g < \kappa_b$  and that the ex-ante probability of each state is given by f(g) and f(b) respectively, with f(g) + f(b) = 1. Since the default probability determines the expected repayment of the bond, we refer to the realization of  $\kappa$  as the *quality shock*, where the bond with  $\kappa_g$  is a *good quality bond* and the one with  $\kappa_b$  is a *bad quality bond*.<sup>9</sup>

The government sells these bonds in an auction in period one. If the amount of money raised at auction falls short of D, then we assume that the government simply defaults on any bonds that it sold in period one (we can take this to also mean that they defaulted on the bonds coming due in period one).

The objective of investors is to maximize their expected, strictly concave, flow utility functions *U* over their second period consumption. Each investor has wealth *W* in period

<sup>&</sup>lt;sup>8</sup>The supply of bonds being auctioned to raise D is therefore pinned down by the realized bids. This is meant to capture the impact of revenue needs on the government's cost of financing.

<sup>&</sup>lt;sup>9</sup>It is straightforward to think of the  $\kappa_{\theta}$  realizations being themselves governed by an *aggregate public* shock  $\nu$  at the beginning of the period. Then information would not be about  $\kappa$  but about  $\nu$ , but the analysis that follows remains the same.

one and can either invest in a risk-free bond (storage) or the risky bond being auctioned by the government. In addition to the quality shock that determines the probability of default, there is a *demand shock*, which we model as a random share of investors who show up to the auction. We denote the random fraction of the potential investors who do not make it to the auction by  $\eta$ . Those that do not make it to the auction have no choice but to invest all of their wealth in the risk-free bond and eat the proceeds in the second period. The investors who do make it to the auction have the option to bid and invest a fraction of their wealth in the risky government bond, storing the residual.

Two natural interpretations of  $\eta$  are: (i) it governs the fraction of investors who suffer a liquidity shock and end up with wealth 0 and hence have nothing to invest, and (ii) it governs the fraction of investors who have a more favorable outside investment opportunity and therefore choose not to bid on the risky bond at the auction.<sup>10</sup> Viewed in this way, the shock to demand coming through  $\eta$  can be thought of as a *correlated private value shock* in the context of the auction literature, while the shock to the quality of the bond coming through  $\theta$  as a *common value shock*.

We assume that  $\eta$  is continuously distributed on the interval  $\mathcal{H} = [0, \eta_M]$  according to a continuous density function  $g(\eta)$  that is nonzero everywhere on the interior of the interval, with  $\eta_M < 1$ . We will refer to  $s = (\theta, \eta)$  as the state of the world and the set of states is denoted by  $\mathcal{S} = \{g, b\} \times \mathcal{H}$ .

There will be two types of investors at the auction: those who are informed about  $\theta$  and those who are not. We denote by  $i \in \{I, U\}$  the type of investor and use  $n \in [0, 1]$  to denote the share of investors who are *informed* (*I*), with 1 - n denoting the share who are *uninformed* (*U*). The fraction *n* determines the *degree of asymmetric information* in the sense that measures the relative mass of investors with superior information about the quality of the bond. Because informed (uninformed) investors are otherwise identical, they behave the same and we can refer to a representative informed (uninformed) investor. No investor is informed about  $\eta$ , which means that all investors face uncertainty about the minimum price at which they can buy the bond conditional on their information (or lack

<sup>&</sup>lt;sup>10</sup>This shock to the demand for the bond can be also interpreted as a shock to its supply, or the amount of funds that the government needs to raise at the auction in period one,  $D\psi$  where  $\psi = 1/(1 - \eta)$ . We will also use this alternative interpretation later in our numerical illustration.

thereof) about  $\theta$ . Consistent with our mechanical modeling of the government we assume that it observes neither  $\theta$  nor  $\eta$  before the auction.

### 2.2 Auction

At the auction, investors can submit multiple bids. Each bid is a price and quantity pair  $\{P, B\}$  representing a commitment to purchase B units of the bond either at price P should the government decide to execute the bid. The government treats each bid independently, sorts all bids from the highest to the lowest bid price, and accepts all bids in descending order to the highest bid price at which the amount D is raised. We refer to this highest possible "lowest" accepted price as the *marginal price*  $\overline{P}$ , and to bids above the marginal price as *bids in the money*.

The price that an investor has to pay when a bid is accepted depends on the auction protocol. We consider two protocols that are widely used in a large volume auctions of a common good, not only sovereign bonds. In the first, the government sells bonds using a *discriminatory-price* (DP) auction (bonds are sold at the bid price, or "pay as you bid"). In the second, the government sells bonds using a *uniform-price* (UP) auction (all accepted bids are executed at the lowest accepted, or marginal, price). We assume that the government and the investors take the auction protocol as given. If the marginal price does not exactly clear the auction (i.e. the auction generates revenue strictly greater than *D* at the marginal price) then only a fraction of the bids at the marginal price are accepted and bonds are rationed pro-rata among investors. As we will show, rationing does not happen in equilibrium.

Investors lack commitment in two important dimensions. First, they cannot commit to honor any inter-temporal contracts. We will take this to mean that they cannot borrow at the risk-free rate, nor can they make negative bids at the auction. Investors must therefore bid nonnegative quantities ( $B \ge 0$ ) and can spend no more than their wealth W on bonds. Second, they cannot commit to credibly share their information about  $\theta$ . We will take this to mean that there is no market for information.

A unit of the bond is a claim to a real unit of the numeraire good in period two. As

this claim either pays 1 or 0, the range of possible prices is  $P \in [0, 1]$ . Since investors will typically find it optimal to submit multiple bids, we start by taking the investors' strategy to be a bid function  $B^{I}(P|\theta)$  for the informed and  $B^{U}(P)$  for the uninformed.

In a UP auction, if  $\overline{P}(s)$  is the marginal price in state *s*, then the amount that the government raises in state *s* is

$$(1-\eta)\left[\int_{\bar{P}(s)}^{1}\left[nB^{I}(P|\theta(s))+(1-n)B^{U}(P)\right]dP\right]\bar{P}(s),$$

where  $\theta(s)$  is the quality shock that corresponds to state *s*. This expression is simply the marginal price  $\overline{P}(s)$  multiplied by the accepted number of bids given this marginal price. The amount the government raises is increasing in the marginal price  $\overline{P}(s)$ , but the number of accepted bids is decreasing in the marginal price. As the auction clears when the demand equals *D* there may be multiple price points at which the government raises the necessary amount *D*. However, the auction protocol specifies that the government will always use the highest such price. This restriction implies that a government that needs to raise more funds will have to accept a lower marginal price.

In a DP auction, if P(s) is the marginal price in state s, then the amount that the government raises in state s is

$$(1-\eta)\left[\int_{\bar{P}(s)}^{1}\left[nB^{I}(P|\theta(s))+(1-n)B^{U}(P)\right]PdP\right].$$

In contrast to the UP auction, this amount is always declining in the marginal price P(s) since the paid prices are fixed at the bidding price while the number of accepted bids is decreasing in the marginal price. As the marginal price is determined when this demand equalizes D it also implies that a government that needs to raise more funds will also face a lower marginal price.

# 2.3 Bidding

We assume that investors have rational expectations: the set of marginal prices, their probabilities and the states associated with them are all common knowledge before sub-

mitting the bids. After the auction has been performed and the realization of the marginal price has been revealed, informed and uninformed investors can make inferences with respect to the state. For the informed investor this is straightforward since they know  $\theta$  and can infer  $\eta$  by inverting the price schedule. For the uninformed this is somewhat more complicated. If the price  $\bar{P}(s)$  is uniquely generated by a quality shock, then they too can infer the state perfectly. If there are two states  $(g, \eta_g)$  and  $(b, \eta_b)$  with a common price, then they will still be able to update their beliefs about the set of possible states and their probabilities from observing the price. *However*, this ex-post information is of limited use since all of the investors must choose their bids *prior* to observing the marginal price that realizes upon the auction.

At the time they make their bids, the informed investors know  $\theta$  and the probability distribution over  $\eta$ , while the uninformed investors know the probabilities distributions over both  $\theta$  and  $\eta$ . They can also compute the bidding strategies of the other investors so they know how the realized state will determine the marginal price at the auction. With this information the can make inferences about the set of states and their probabilities in the event they are able (or not able) to buy at a bid price of *P*.

In the DP auction it is a strictly dominating strategy to bid only at the possible marginal prices  $\bar{P}(s)$ . Otherwise, if the bid is made at a price slightly above  $\bar{P}(s)$  the same bid is accepted under the same conditions but the investor pays a higher price. In the UP auction it is a weakly dominating strategy to do so. In light of this, we restrict our agents to only bid at marginal prices. With this restriction, we no longer need to think of our agents having a bidding strategy for each possible price but instead think of them as choosing how many bonds to bid only at the marginal price for each state. Since bids only happen at marginal prices, we drop the notation  $\bar{P}$  and just refer to P. Also, for this reason, we switch to a simpler and starker specification of prices and bids.

**Definition 1.** For each state  $s = (\theta, \eta) \in S$ , the marginal price is denoted  $P(\theta, \eta)$  and the set of marginal prices by  $\mathcal{P}$ . An action for the uninformed investors is a function  $B^U(\theta, \eta)$  which denotes the number of units bid at the marginal price  $P(\theta, \eta)$ . An action for the informed investors is a function  $B^I(\theta, \eta | \hat{\theta})$  which denotes their bids at the various possible states when the realized  $\theta$  is  $\hat{\theta}$ .

**Remark 1.** For any pair of states  $(g, \eta_g)$  and  $(b, \eta_b)$  such that  $P(g, \eta_g) = P(b, \eta_b)$ , bids at these states are perfect substitutes since they will be accepted and rejected in identical circumstances across realized states. Thus, the investor just chooses the total quantity  $B(g, \eta_g) + B(b, \eta_b)$  to bid at the price  $P = P(g, \eta_g) = P(b, \eta_b)$ .

**Remark 2.** This stark specification allows us to directly compare our auction to a competitive equilibrium. It is also particularly helpful when the set of  $\eta$ 's is finite, and so is the set of possible marginal prices. In our original specification of actions as bids on the set of all potential prices  $P \in [0, 1]$  this would mean that the bid function would be positive only at a finite set of points corresponding to those marginal prices. But even when  $\eta$  is continuous, the set of marginal prices is a strict subset of the set of potential prices.

# **3** Auction Equilibrium

# 3.1 Definition

We start defining the problem of the uninformed investor. If the government ends up defaulting in the second period, the uninformed investor simply consumes the unit payoff from his risk-free bonds, which we denote by  $B_{RF}^U(s)$ . If the government does not default, then the investor also consumes the unit payoff from his total purchases of the risky bond, which we denote by  $\mathcal{B}_{R}^U(s)$ . The expected payoff to an uninformed investor is given by

$$\sum_{\theta \in \{g,b\}} \int_{\eta} \left\{ U(B_{RF}^{U}([\theta,\eta]))\kappa_{\theta} + U\left(B_{RF}^{U}([\theta,\eta]) + \mathcal{B}_{R}^{U}([\theta,\eta])\right)(1-\kappa_{\theta}) \right\} f(\theta)g(\eta)d\eta$$
(1)

where we are summing over the conditional payoffs in each possible state  $(\theta, \eta)$ , weighted by the probability of that state. The total risky bonds purchased by an uninformed bidder in each state,  $\mathcal{B}_{R}^{U}(s)$ , is

$$\mathcal{B}_{R}^{U}(s) = \sum_{s': P(s') \ge P(s)} B^{U}(s').$$
<sup>(2)</sup>

The total expenditures on these risky bonds determines the amount invested in the risk-free bond. As the price of risk-free bonds is equal to 1, the unit payoff of risk-free bonds

is the same as the expense on those bonds. Then,

UP auction : 
$$B_{RF}^U(s) = W - \left[\sum_{s':P(s') \ge P(s)} B^U(s')\right] P(s),$$
 (3)

DP auction : 
$$B_{RF}^{U}(s) = W - \left[\sum_{s': P(s') \ge P(s)} B^{U}(s') P(s')\right].$$
 (4)

As investors cannot short-sell or borrow, they face the following nonegativity constraints

$$B^{U}(s) \ge 0 \text{ and } B^{U}_{RF}(s) \ge 0 \qquad \forall s \in \mathcal{S}.$$
 (5)

The problem of an uninformed investor is to choose  $B^U(s)$  for all  $s \in S$  to maximize (1) subject to (2), (3) and (5) for the UP auction, and (2), (4) and (5) for the DP auction.

The problem of an informed investor, given that he knows the realized quality shock  $\hat{\theta} \in \{g, b\}$ , is higher dimensional as he has to determine bids  $B^{I}(s, \hat{\theta})$  for each state  $s \in S$  conditional on the realized  $\hat{\theta}$ . Trivially, just as no investor should bid at prices that are not marginal conditional of his information, the informed investor should only bid at prices  $P(\theta, \eta)$  and not at prices  $P(\theta', \eta)$  ( $\theta' \neq \theta$ ) given that he knows the quality shock is  $\theta$  in state  $s = [\theta, \eta]$ . Then we can focus on investors choosing  $B^{I}([\theta, \eta], \theta)$  so as to maximize

$$\int_{\eta} \left\{ U(B_{RF}^{I}([\theta,\eta],\theta))\kappa_{\theta} + U\left(B_{RF}^{I}([\theta,\eta],\theta) + \mathcal{B}_{R}^{I}([\theta,\eta],\theta)\right)(1-\kappa_{\theta}) \right\} g(\eta)d\eta \qquad \forall \theta \in \{g,b\},$$
(6)

where the total purchases of the risky bond for each realized  $\theta$  is

$$\mathcal{B}_{R}^{I}(s,\theta) = \sum_{s':P(s') \ge P(s)} B^{I}(s',\theta) \qquad \forall \theta \in \{g,b\},$$
(7)

the total, auction-specific, purchases of the risk-free bond for each realized  $\theta$  is,

UP auction : 
$$B_{RF}^{I}(s,\theta) = W - \left[\sum_{s':P(s') \ge P(s)} B^{I}(s',\theta)\right] P(s); \forall s \in \mathcal{S} \text{ and } \forall \theta \in \{g,b\} (8)$$

DP auction : 
$$B_{RF}^{I}(s,\theta) = W - \left[\sum_{s':P(s') \ge P(s)} B^{I}(s',\theta)P(s')\right]; \forall s \in \mathcal{S} \text{ and } \forall \theta \in \{g,b\}(9)$$

and the nonegativity constraints are

$$B^{I}(s,\theta) \ge 0 \text{ and } B^{I}_{RF}(s,\theta) \ge 0 \qquad \forall s \in \mathcal{S} \text{ and } \forall \theta \in \{g,b\}.$$
 (10)

The problem of an informed investor is to choose  $B^{I}(s,\theta)$  for all  $s \in S$  and each realized  $\theta \in \{g,b\}$  to maximize (6) subject to (7), (8) and (10) for the UP auction, and (7), (9) and (10) for the DP auction.

**Bid overhang constraint.** Because the marginal price P(s) is defined to be the *highest* price such that demand is enough to cover the government's revenue needs, bids and prices must also satisfy an additional constraint that we call the *bid-overhang constraint*. This constraint requires that there *cannot* exist a state  $\tilde{s}$  such that  $P(\tilde{s}) > P(s)$ , and at the marginal price  $P(\tilde{s})$ , there is enough demand to generate the government's revenue needs in state s. As per our auction protocol, even though P(s) may satisfy clearing in state s it cannot be an equilibrium price since the government would *also* be able to raise those funds at a higher price, namely  $P(\tilde{s})$ .

**Definition 2.** Formally, the bid-overhang constraint is the requirement that

for any 
$$s \in S$$
 and for all  $\tilde{s} \in S : P(\tilde{s}) > P(s)$ , (11)  

$$UP : (1 - \eta(s)) \begin{cases} (1 - n) \left[ \sum_{s':P(s') \ge P(\tilde{s})} B^{U}(s') \right] \\ +n \left[ \sum_{s':P(s') \ge P(\tilde{s})} B^{I}(s', \theta(s)) \right] \end{cases} P(\tilde{s}) < D$$

$$DP : (1 - \eta(s)) \begin{cases} (1 - n) \left[ \sum_{s':P(s') \ge P(\tilde{s})} B^{U}(s')P(s') \right] \\ +n \left[ \sum_{s':P(s') \ge P(\tilde{s})} B^{I}(s', \theta(s))P(s') \right] \end{cases} < D.$$

Notice that P(s) is obtained from auction clearing in state s when evaluating the de-

mand at P(s) and the same for  $P(\tilde{s})$ . If  $P(\tilde{s}) > P(s)$ , what this constraint implies is that there cannot be excess demand in state *s* when the demand is evaluated at  $P(\tilde{s})$ .

One can see from inspection that this constraint cannot bind in the DP auction since the price at which a bid is executed is fixed by the bid price. Therefore, in the DP auction the demand in state s when evaluated at  $P(\tilde{s})$  is always smaller than when evaluated at P(s) when  $P(\tilde{s}) > P(s)$ . The bid-overhand constraint, however, can and does bind in the UP auction. The reason is that in the UP auction the demand in state s when evaluated at  $P(\tilde{s})$  can be larger than when evaluated at P(s) when  $P(\tilde{s}) > P(s)$ , as all bidders pay a larger price,  $P(\tilde{s})$ , for all their bids.

**Definition 3.** An equilibrium of our auction model is defined as a function  $P : S \to [0, 1]$ , and bidding fuctions  $B^U : S \to [0, \infty)$  and  $B^I : S \times \{g, b\} \to [0, \infty)$ , such that

- 1. each type of investor's bid function solves their problem,
- 2. the auction clearing condition is satisfied for all  $s \in S$ , and
- *3. the bid-overhand constraint is satisfied at each*  $s \in S$ *.*

**Remark 3.** Formulating an equilibrium in this stark fashion, where bids are defined as functions directly of the state, is isomorphic to a more standard formulation where bids are defined as functions of prices. The price functions are the same, as  $B^U(P(s)) = B^U(s)$  and 0 elsewhere, while  $B^I(P(s), \theta) = B^I(s, \theta)$  and 0 elsewhere. The main difference is that the standard formulation defines the bid function over all potential prices rather than marginal prices only. Even though the two formulations are formally identical, the standard formulation is poorly behaved computationally as the bid function is discontinuous around marginal prices given no investor bids on non-marginal prices.

**Proposition 1.** For both the UP and the DP auctions the price function  $P(\theta, \eta)$  is decreasing in  $\eta$ . Hence a bid made at a price  $P(\theta, \eta)$  is in-the-money for all  $\hat{\eta} \ge \eta$  given  $\theta$ , and if there exists an  $\bar{\eta}$ , given  $\bar{\theta} \ne \theta$ , such that  $P(\bar{\theta}, \bar{\eta}) = P(\theta, \eta)$ , then it is in-the-money for all  $\hat{\eta} \ge \bar{\eta}$  given  $\bar{\theta}$ . Because the price schedule conditional on  $\theta$  is bounded and monotonic, it follows that it is both continuous and differentiable almost everywhere.

*Proof.* For the DP auction this follows directly from the auction clearing condition. For the UP auction it follows from the bid-overhang constraint.  $\Box$ 

### 3.2 Comparison of Auction Equilibrium and Competitive Equilibrium

Now we have defined an auction equilibrium, we can compare its structure with that of a competitive equilibrium. By construction it is evident that a DP auction cannot be a competitive equilibrium, as in the former risky bonds sell at multiple prices displaying a dispersion of prices that is absent in the later. In contrast, because risky bonds are always sold at a common price in the UP auction it is in many cases isomorphic to a standard competitive equilibrium with heterogeneous information. We establish this link here.

In the UP auction, the uninformed investor problem is choosing  $B^U(s)$  to maximize (1) subject to (2), (3) and the nonegativity constraint (5). In the competitive equilibrium his problem is identical with the only difference that the nonnegativity constraint applies to the total purchases of the risky and the risk-free bond at each state *s*, and then we replace constraint (5) with

$$\mathcal{B}_{R}^{U}(s) \ge 0 \text{ and } B_{RF}^{U}(s) \ge 0 \ \forall s \in \mathcal{S}.$$
 (12)

The same considerations hold for the informed investor.

We can now relate the constraints (5) and (12) that differentiate a UP auction equilibrium from a competitive equilibrium. To go from constraint (5) to (12) we can construct the associated total risky bond purchases just by summing over the in-the-money bids. This implies that as long as constraint (5) does not bind in any state *s*, then the same is true for constraint (12).

To go the other way, from constraint (12) to (5), we can construct the associated stateby-state bids given the total bond purchases in a competitive equilibrium,  $\{B_R^I(s,\theta), B_R^U(s)\}$ using the difference between the risky bond purchases at *s* and those at the next highest price *s'*; this is

$$B^{U}(s) = \mathcal{B}_{R}^{U}(s) - \mathcal{B}_{R}^{U}(s'), \qquad (13)$$

where  $P(s') = \min \{P(s'') > P(s)\}$  for all  $s'' \in S$ . The same object for the informed is constructed using their conditional total purchases  $\mathcal{B}_R^I(s, \theta(s))$ . It is not immediate that

constraint (12) not binding implies that constraint (5) will not bind in some state s.

If the state-by-state bids are nonnegative, then the nonnegativity constraint on bids for the UP auction is not violated and there is an associated competitive equilibrium. Inversely, if the state-by-state bids are such that the nonnegativity constraint on bids for the UP auction is violated, there is an associated competitive equilibrium only when the total risky bond purchases also bind and are zero. This discussion implies that there is no associated competitive equilibrium when there are states for which the nonnegativity constraint does not bind for total purchases in the competitive equilibrium, but do bind in particular states in the UP auction. We summarize this discussion as follows.

**Proposition 2.** A UP auction equilibrium  $\{P(s), B^{I}(s, \theta), B^{U}(s)\}$  in which the nonnegativity constraint on risky bond purchases only binds when total risky bond purchases are 0 for all  $s \in S$  has an associated competitive equilibrium  $\{P(s), B^{U}_{CE}(s), B^{I}_{CE}(s, \theta)\}$ . Any competitive equilibrium  $\{P(s), B^{U}_{CE}(s), B^{I}_{CE}(s, \theta)\}$ . Any competitive equilibrium  $\{P(s), B^{I}_{CE}(s, \theta), B^{U}_{CE}(s)\}$  in which the associated bids  $\{B^{I}(s, \theta), B^{U}(s)\}$  constructed using (13) are nonnegative and satisfy the bid-overhang constraint (11) is an auction equilibrium.

Even assuming the differences in nonegativity constraints are not operational such that the investors' problems deliver the same allocation, and then the auction equilibrium has an associated competitive equilibrium, there is another condition that is unique to the auction protocol and may break the mapping between the two equilibria: the bidoverhang constraint. This aggregate constraint leads to the set of UP auction equilibria being a subset of the set of competitive equilibria. In the numerical examples, we will illustrate how this can happen.

## 3.3 Investor's Optimal Bidding

We turn next to explicitly characterizing the optimal bidding choices for the investors in our two auction protocols.

### 3.3.1 Informed Investors

The informed investors' problem is the simplest because they know the true quality shock to the bond, which corresponds to one of the components of the state s,  $\theta(s)$ . As informed

investors do not have to infer the quality of the bond, their first-order condition for their bids,  $B^{I}([\theta^*, \eta^*])$  for each state  $s = [\theta^*, \eta^*]$  at a UP auction is given by

$$\int_{\eta} \left\{ \begin{array}{c} -U'(B_{RF}^{I}([\theta^{*},\eta]))\kappa_{\theta^{*}}P([\theta^{*},\eta]) \\ +U'\begin{pmatrix} B_{RF}^{I}([\theta^{*},\eta]) \\ +\mathcal{B}_{R}^{I}([\theta^{*},\eta]) \end{pmatrix} (1-\kappa_{\theta^{*}})(1-P([\theta^{*},\eta])) \end{array} \right\} \mathcal{I}\left\{ P\left([\theta^{*},\eta^{*}]\right) \ge P([\theta^{*},\eta])\right\} g(\eta)d\eta \\ -\chi^{I}([\theta^{*},\eta^{*}]) = 0,$$

where  $\chi^{I}(s)$  is the multiplier on the nonnegativity constraint, and  $\mathcal{I} \{\cdot\}$  is an indicator function. In this condition, the price  $P([\theta^*, \eta])$  at which the investor buys the risky bond plays two roles. On the one hand, it determines the set of states in which his purchase is in-the-money through the indicator function. On the other hand, it also determines the price at which the investor purchases his bid since the investor pays the marginal price  $P([\theta^*, \eta])$  on all possible demand shocks  $\eta$  in which his bid is in the money.

The first-order condition for the DP auction is

$$\int_{\eta} \left\{ \begin{array}{c} -U'(B_{RF}^{I}([\theta^{*},\eta]))\kappa_{\theta^{*}}P([\theta^{*},\eta^{*}]) \\ +U'\begin{pmatrix} B_{RF}^{I}([\theta^{*},\eta]) \\ +\mathcal{B}_{R}^{I}([\theta^{*},\eta]) \end{pmatrix} (1-\kappa_{\theta^{*}})(1-P([\theta^{*},\eta^{*}])) \\ -\chi^{I}([\theta^{*},\eta^{*}]) = 0, \end{array} \right\} \mathcal{I}\left\{ P\left( [\theta^{*},\eta^{*}] \right) \geq P([\theta^{*},\eta]) \right\} g(\eta) d\eta$$

In contrast to the UP auction, in the DP auction the price  $P([\theta^*, \eta])$  only determines the set of states the investor is in the money, as the investor always pays the price of the bid,  $P([\theta^*, \eta^*])$  when it is in the money.

#### 3.3.2 Uninformed Investors

Next we turn to an uninformed investors' problem. An important problem to the uninformed investor is inferring the quality of the bond he expects to receive conditional on a given bid being executed. In order to rewrite the problem in terms of the believed probability of default conditional on observing a marginal price, we have to specify first how the inference problem is resolved.

**Inference:** Given that bids are executed depending on the realization of the marginal price and that the quality of a bond is fully pinned down by its default probability, the inference problem is equivalent to computing the expected default probability of a bond given the realization of a marginal price. We denote this conditional expected default probability by  $\tilde{\kappa}$ . For the informed,  $\tilde{\kappa}(P(\theta, \eta)|\theta) = \kappa_{\theta}$  because they know the true  $\theta$ . For the uninformed there are two cases:

- 1. For any  $(\theta, \eta)$  such that  $\nexists (\theta', \eta')$  with  $P(\theta, \eta) = P(\theta', \eta')$ , then  $\tilde{\kappa}(P(\theta, \eta)) = \kappa_{\theta}$ .
- 2. If there are two states  $(\theta, \eta)$  and  $(\theta', \eta')$  such that  $P(\theta', \eta') = P(\theta, \eta)$  and  $\theta' \neq \theta$  the solution to the uninformed investor's inference problem is as follows.

Given  $P(\theta, \eta)$ , define  $\eta = \phi(P|\theta)$ , where  $\phi$  is the inverse function of the price with respect to  $\eta$ .<sup>11</sup> The probability *h* of an interval of prices  $\mathbf{P} \subset \mathcal{P}$  conditional on  $\theta$  is

$$h(\mathbf{P}|\theta) = \int_{\{\eta: P(\theta,\eta) \in \mathbf{P}\}} g(\eta) d\eta = \int_{\tilde{P} \in \mathbf{P}} g(\phi(\tilde{P}|\theta)) \frac{\partial \phi(P|\theta)}{\partial \tilde{P}} d\tilde{P}$$

Note that the slope of the inverse function with respect to the price determines the size of the set of  $\eta$ 's that are associated with the prices in **P** (given  $\theta$ ). The unconditional probability of the set of prices is then given by

$$h(\mathbf{P}) = \sum_{\theta} f(\theta) h(\mathbf{P}|\theta),$$

and the probability of  $\theta$  conditional on a price in **P** is simply  $f(\theta)h(\mathbf{P}|\theta)/h(\mathbf{P})$ . We can define the probability density function of a particular price  $P \in \mathbf{P}$ , given  $\theta$ , by shrinking the set  $\mathbf{P} \to P$ , and observing *h* in the limit, or

$$Pr(P|\theta) = \lim_{\mathbf{P} \to P} \frac{h(\mathbf{P}|\theta)}{\Delta(\mathbf{P})},$$

<sup>&</sup>lt;sup>11</sup>Note from proposition 1 that  $P(\theta, \eta)$  is continuous almost everywhere and since rationing does not occur in equilibrium, strictly monotonic, thus it is invertible almost everywhere. See Rudin (1964, p. 90).

where  $\Delta(\mathbf{P})$  is the length of the price interval. This then leads to the inferred default probability

$$\widetilde{\kappa}(P) = \frac{\sum_{\theta} f(\theta) Pr(P|\theta) \kappa_{\theta}}{\sum_{\theta} f(\theta) Pr(P|\theta)}.$$

**Uninformed Investor's Problem:** Given this inference problem, and regrouping states according to marginal prices, the uninformed investor's payoff can be expressed in terms of the expected probability of default as

$$\sum_{\theta \in \{g,b\}} \int_{\eta} \left\{ \begin{array}{c} U(B_{RF}^{U}([\theta,\eta]))\widetilde{\kappa}(P([\theta,\eta])) \\ +U\left(B_{RF}^{U}([\theta,\eta]) + \mathcal{B}_{R}^{U}([\theta,\eta])\right)\left(1 - \widetilde{\kappa}(P([\theta,\eta]))\right) \end{array} \right\} f(\theta)g(\eta)d\eta.$$

The uninformed investor's f.o.c. for  $B^U([\theta^*, \eta^*])$  at a UP auction is given by

$$\sum_{\theta \in \{g,b\}} \int_{\eta} \left\{ \begin{array}{c} -U'(B_{RF}^{U}([\theta,\eta]))\widetilde{\kappa}(P([\theta,\eta])P([\theta,\eta])) \\ +U'\begin{pmatrix} B_{RF}^{U}([\theta,\eta]) \\ +\mathcal{B}_{R}^{U}([\theta,\eta]) \end{pmatrix} (1-\widetilde{\kappa}(P([\theta,\eta]))(1-P([\theta,\eta])) \\ \mathcal{I}\left\{P\left([\theta^{*},\eta^{*}]\right) \geq P([\theta,\eta])\right\}f(\theta)g(\eta)d\eta - \chi^{U}([\theta^{*},\eta^{*}]) = 0. \end{array} \right\}$$

The uninformed investor's f.o.c. for  $B^U(s)$  at a DP auction is given by

$$\sum_{\theta \in \{g,b\}} \int_{\eta} \left\{ \begin{array}{c} -U'(B_{RF}^{U}([\theta,\eta]))\widetilde{\kappa}(P([\theta,\eta])P([\theta^{*},\eta^{*}])) \\ +U'\begin{pmatrix} B_{RF}^{U}([\theta,\eta]) \\ +\mathcal{B}_{R}^{U}([\theta,\eta]) \end{pmatrix} (1-\widetilde{\kappa}(P([\theta,\eta]))(1-P([\theta^{*},\eta^{*}])) \\ \mathcal{I}\left\{P\left([\theta^{*},\eta^{*}]\right) \ge P([\theta,\eta])\right\}f(\theta)g(\eta)d\eta - \chi^{U}([\theta^{*},\eta^{*}]) = 0. \end{array} \right\}$$

Then, the same considerations about the impact of the marginal price on the decision of bidding that we made about informed investors hold for uninformed investors, but using expected probabilities of default instead of realized ones.

# 4 Characterization of UP Auction Equilibrium

We start characterizing the equilibrium for the uniform-price auction. In particular, we show how equilibrium prices depend on the fraction of informed investors, n, so we can later compute sovereign bond yields, the government's debt burden and the investors' payoffs for a given n. Accordingly, we will use  $P^{UP}(s;n)$  to denote the price function for each state and degree of asymmetric information in the UP auction. When there is no risk of confusion, we will simply write P(s). In a similar fashion, we will use  $B^{UP,U}(s;n)$  for the bids of the uninformed and  $B^{UP,I}(s,\theta;n)$  for the bids of the informed, but use the simpler notation  $B^{U}(s)$  and  $B^{I}(s,\theta)$  when there is no risk of confusion.

## 4.1 Symmetric Information Benchmarks

We begin by considering the two symmetric information benchmarks: *a symmetric ignorance equilibrium* in which no investor is informed (n = 0), and *a symmetric information equilibrium* in which all investors are informed (n = 1).

#### 4.1.1 Symmetric Ignorance

Since there are no informed investors in the symmetric ignorance equilibrium, marginal prices cannot depend upon  $\theta$ , and we must have  $P(g, \eta) = P(b, \eta)$  for all  $\eta \in \mathcal{H}$ . Hence, we can simplify our notation and write  $P(\eta)$  for prices and  $B(\eta)$  for bond purchases.

Because  $P(\eta)$  is declining in  $\eta$ , the in-the-money set for bids  $B(\eta)$  is  $[\eta, \eta_M]$ , this is, the investor gets to buy  $B(\eta)$  whenever the demand shock is in the set  $[\eta, \eta_M]$ . It follows also that when the demand shock is  $\eta$ , all bids  $B(\hat{\eta})$  for  $\hat{\eta} \in [0, \eta]$  are in-the-money. Then, the auction clearing condition can simply be stated as

$$(1-\eta)\left[\int_0^{\eta} B(\hat{\eta})d\hat{\eta}\right]P(\eta) = D.$$
(14)

It follows from the clearing condition that  $B(\eta) > 0$ , and hence the short-sale constraint cannot bind for any  $\eta$ . Because prices cannot convey any information about  $\theta$ , it follows that  $\tilde{\kappa}(P) = \kappa^U = f(g)\kappa_g + f(b)\kappa_b$ , which is the ex-ante probability of default for all P. Since the bid  $B(\eta^*)$  in state  $\eta^*$  for the uninformed investor (all investors in this case) is in the money when  $P(\eta) < P(\eta^*)$  (or when  $\eta > \eta^*$ ), his fist-order condition is simply

$$\int_{\eta^*}^{\eta_M} \left\{ \begin{array}{c} -U'(B_{RF}(\eta))\kappa^U P(\eta) \\ +U'\left(B_{RF}(\eta) + \left[\int_0^{\eta} B(\hat{\eta})d\hat{\eta}\right]\right)(1-\kappa^U)(1-P(\eta)) \end{array} \right\} g(\eta)d\eta = 0.$$

From the auction clearing condition, it follows that

$$\left[\int_0^{\eta} B(\hat{\eta}) d\hat{\eta}\right] P(\eta) = \frac{D}{1-\eta}$$

and from the budget constraint the residual purchase of risk-free bonds is

$$B_{RF}(\eta) = W - \frac{D}{1-\eta}.$$

It is important to highlight here that the solution to the system of first order conditions (one for each  $\eta$  value) is block recursive. As the term in brackets should be 0 for any interval  $[\eta^*, \eta_M]$ , it follows that it is also 0 at each  $\eta$ . Since the term in brackets can be rewritten for each  $\eta$  just as a function of  $P(\eta)$ , this implies that for each  $\eta$ , the price  $P(\eta)$ is given by the solution to the following equation

$$U'\left(W - \frac{D}{1-\eta}\right)\kappa^{U}P(\eta) = U'\left(W - \frac{D}{1-\eta} + \frac{1}{P(\eta)}\frac{D}{1-\eta}\right)(1-\kappa^{U})(1-P(\eta))$$
(15)

What are the gains to be a single atomistic informed investor in this equilibrium? Information implies facing the same first order condition (15) but using  $\kappa_{\theta}$  instead. When the bond is of good quality the informed investor would bid more that other investors because  $\kappa_g < \kappa^U$  and the marginal benefit of bidding the same than uninformed investors (the right hand side of the equation) is larger than the marginal cost (the left hand side). The opposite happens when the bond is of bad quality. There are gains from being able to choose how much to bid more accurately.

#### 4.1.2 Symmetric Information

We now consider the symmetric information equilibrium in which all investors are informed (n = 1). Since all investors bid contingent on  $\theta$ , the equilibrium can be determined conditional on the realized quality shock  $\theta$ . The logic is just as in the symmetric ignorance case, only now the default probability is  $\kappa_{\theta}$ , instead of  $\kappa^{U}$ , when the state is  $\theta$ . Thus the first order condition for an informed investor of biding in state ( $\theta$ ,  $\eta$ ) is similar to that above, and we can again appeal to the auction clearing condition and to the budget constraint to get a simple condition which we can solve for the price  $P(\theta, \eta)$ ,

$$U'\left(W - \frac{D}{1-\eta}\right)\kappa_{\theta}P(\theta,\eta) = U'\left(W - \frac{D}{1-\eta} + \frac{1}{P(\theta,\eta)}\frac{D}{1-\eta}\right)(1-\kappa_{\theta})(1-P(\theta,\eta))$$

What are the costs of being a single atomistic uninformed investor in this equilibrium? As we will show next, there may be no cost if the uninformed investor is able to perfectly replicate what informed investors do.

#### 4.1.3 Replication

We know that, in a competitive equilibrium, if prices are fully revealing there are no gains from information. A similar condition holds in the UP auction under somewhat more stringent conditions.

**Proposition 3.** In a UP auction the uninformed will be able to replicate the total bids of the informed state-by-state, and hence their ex-ante payoff, if:

- 1. Each marginal price is associated with a unique state in S.
- 2. When we order the marginal prices and the associated total bids of the informed from the highest to the lowest marginal price, the total bids of the informed are also weakly ranked from lowest to highest.

*Proof.* The first condition ensures that the uninformed can compute the state associated with each marginal price, and even though not observing the marginal price ex-ante, being able to bid at those prices accordingly. The second condition ensures that the no

short-sale constraint does not bind, so the uninformed investor can optimize in each state with respect to total bond purchases, knowing the default probability at which these will be executed. When this is true, the uninformed can replicate the payoff of the informed by matching their total bond purchases to those of the informed in each state *s*.

**Corollary 4.** For any equilibrium with n > 0,  $P(g, \eta_M; n) > P(b, 0; n)$  and  $\mathcal{B}_R^I([g, \eta_M], g; n) < \mathcal{B}_R^I([b, 0], b; n)$  are sufficient conditions for the second condition of Proposition 3. The reason is that along a price schedule corresponding to a given  $\theta(s)$ , prices are declining in  $\eta$  and total bids are increasing, fulfilling the second condition. The only situation in which the condition may fail is when comparing price schedules across  $\theta(s)$ .

Note that the uninformed will not make the same *marginal bids* as the informed in each state, but they will buy the same total amount as the informed in each state, achieving perfect portfolio replication. The reason is all of the bids by uninformed investors on the  $\theta = g$  schedule are accepted when  $\theta = b$ . This is not the case for the informed investors, whose bids are conditioned on  $\theta$ . Thus, the uninformed make the same marginal bids as the informed at prices on the *high-quality schedule* (the schedule for  $\theta = g$ ), but make less marginal bids than the informed at prices on the *low-quality schedule* (the schedule for  $\theta = b$ ) in order to achieve the same *total* bids on both price schedules than informed investors.

### 4.2 Asymmetric Information - Example with Log Preferences

Now we characterize the equilibrium with asymmetric information. To highlight the forces that determine how prices change as *n* changes in the most transparent way, we specialize our model by assuming log preferences for investors. This assumption allows us to solve the first order conditions in a clean closed form expression. Nothing in the characterization, however, hinged on anything beyond concavity.

If the short sale constraint does not bind, then an investor  $i \in \{I, U\}$  has a marginal condition (15) at each distinct price P(s) which is given by

$$\frac{\tilde{\kappa}_i(s)P(s)}{B_{RF}^i(s)} = \frac{(1-\tilde{\kappa}_i(s))(1-P(s))}{B_{RF}^i(s) + \mathcal{B}_R^i(s)}$$

where  $\tilde{\kappa}_i(s)$  is the investor *i*'s belief as to the probability of default on a bond purchased at price P(s),  $B_{RF}^i$  is his total purchases of the risk-free bond and  $\mathcal{B}_R^i(s) = \int_{s':P(s') \ge P(s)} B^i(s)$ of the risky bond. Since the total purchases of risk-free bonds are determined as a residual,  $B_{RF}^i(s) = W - P(s)\mathcal{B}_R^i(s)$ , using this condition in the first order equation we can solve in closed form for the total expenditures in risky bonds,

$$P(s)\mathcal{B}_{R}^{i}(s) = \left[1 - \frac{\tilde{\kappa}_{i}(s)}{1 - P(s)}\right]W.$$
(16)

From this expression we can see that total expenditures on the risky bond are declining smoothly in both the price P(s) and the expected probability of default  $\tilde{\kappa}_i(s)$ , and increasing in wealth W.

In the presence of both informed and uninformed investors, auction clearing is

$$(1-\eta)[nP(s)\mathcal{B}_R^I(s) + (1-n)P(s)\mathcal{B}_R^U(s)] = D.$$

We can simplify the exposition changing variables to

$$\psi = 1/(1 - \eta)$$
 such that  $\psi \in [1, \psi_M]$ ,

where  $\psi$  represents the *supply of bonds per investor* who makes it to the auction (if  $\eta = 0.5$ , for instance, only half of available investors participate in the auction and the supply of bonds per investor duplicates,  $\psi = 2$ ). Given this change now the state can be expressed as  $s = [\theta, \psi]$ .

We also define as  $\hat{\kappa}_{\theta}$  the average belief among all investors (informed and not) about the default probability when the true quality of the bond is  $\theta$ . Then

$$\widehat{\kappa}_{\theta} = n\kappa_{\theta} + (1-n)\widetilde{\kappa}$$
 for  $\theta \in \{g, b\}$ .

Replacing these new elements in equation (16) and auction clearing, we can now write a

simple expression for the price in each state,

$$P(\theta, \psi) = 1 - \frac{\widehat{\kappa}_{\theta}}{1 - \frac{D}{W}\psi}.$$
(17)

This equation provides a simple interpretation of what determine prices. First, we can confirm that the previous benchmarks are just special cases. One the one extreme, when there is symmetric information (everyone is informed),  $\hat{\kappa}$  is equal to either  $\kappa_b$  or  $\kappa_g$  depending on the state, and then, for a given  $\psi$  prices are lower when the bond is bad (high default probability). For this reason we will refer to the *good(bad)-bond price sched-ule* as *high(low)-price schedule* when there is no risk of confusion. On the other extreme, when there is symmetric ignorance (nobody is informed),  $\hat{\kappa} = \kappa^U$ , and prices lie between the two cases. Note that in these informational extremes, expenditures must smoothly increase with  $\psi$  and hence the short-sale constraint can never bind, confirming that the original presumption we use to construct this price holds.

#### 4.2.1 Prices as *n* changes

To display how prices depend on *n*, in what follows we use the parameters in Table 1. These parameters guarantee the two conditions for perfect replication in Proposition 3, this is (i) the price schedules under symmetric information do not overlap and (ii) the no short-sale constraint never binds for uninformed investors at those prices. We change parameters later to relax these implications and prevent perfect replication at the symmetric information benchmark.

Table 1: Benchmark Parameterization

$\kappa_g = 0.15$		$\kappa_b = 0.35$
W = 250		D = 60
$\eta_M = 0.17$	$\implies$	$\psi_M = 1.2$
		$\psi \sim U[1, \Psi_M]$

We are interested in understanding how prices depend on n given parameters and also how parameters affect prices given n. One of the most interesting parameters to think about is the ex-ante probability that the bond is of bad quality, f(b). For this reason we will construct examples for two possible values, f(b) = 0.5 and f(b) = 0.25.

We first compute the equilibrium prices in the two extremes of symmetric information and ignorance, which we depict in panel (a) of Figure 2. Naturally, prices in a symmetric information equilibrium do not depend upon f(b) because the prices are determined based on the realization of  $\theta$ . In contrast, prices in a symmetric ignorance equilibrium do depend on f(b), as it determines  $\kappa^U$ . As f(b) falls, the price schedules (in black) rise from the fully informed with belief  $\kappa_b$  (in red) to the fully informed with belief  $\kappa_g$  (in blue). The corresponding, increasing, bid schedules are in panel (b) of the figure.



Figure 2: Symmetric Benchmarks for UP Auctions

(b) Bid Schedules

(a) Price Schedules

Even though parameters guarantee that prices do not overlap and the no short-sale constraint does not bind for the uninformed when n = 1, perfect replication is not feasible for all n. There is always an n low enough such that the demand of the uninformed when  $\theta = g$  is enough to cover the revenue needs of the government also when  $\theta = b$ .

More precisely, for each *n* there is a value  $\hat{\psi}$  in which the bid-overhang constraint binds and perfect replication is not feasible. As long as  $\hat{\psi} > \psi_M$ , this is not a problem for replication because the bid-overhang constraint does not bind in the relevant space. What determines  $\widehat{\psi}$ ? Consider a state  $[g, \widehat{\psi}]$ . Auction clearing implies that

$$P(g,\widehat{\psi})\mathcal{B}_{R}^{U}([g,\widehat{\psi}]) = D\widehat{\psi}$$

Now consider the state [b, 1], this is the bond of bad quality when the maximum amount of investors participate in the auction. In this case the uninformed can cover all government's revenue needs at price  $P(g, \hat{\psi})$  as long as

$$(1-n)P(g,\widehat{\psi})\mathcal{B}_R^U([g,\widehat{\psi}]) \ge D$$

This implies that the *bid-overhang constraint* binds when  $1 - n \ge \frac{1}{\psi}$ , or equivalently  $n \le \hat{\eta}$ . For all  $n > \eta_M$  the price and total bid schedules are identical to those under symmetric information. But once  $n < \eta_M$  the bid-overhang constraint binds for some value of  $\eta$ , the uninformed investors become the marginal buyer of bonds, then forcing the price being the same in both quality states. As n falls, it binds at lower and lower values of  $\psi < \psi_M$ .

Once the bid-overhang constraint forces price pooling. How prices are determined? Take any two states  $s = [g, \psi_g]$  and  $s' = [b, \psi_b]$  in which the bid-overhang constraint forces a common price, P = P(s) = P(s'), the two auction clearing equations are

$$n\left(\frac{1-\kappa_g-P}{1-P}\right) + (1-n)\left(\frac{1-\tilde{\kappa}-P}{1-P}\right) = \frac{D}{W}\psi_g,\tag{18}$$

$$n \max\left[\left(\frac{1-\kappa_b-P}{1-P}\right), 0\right] + (1-n)\left(\frac{1-\tilde{\kappa}-P}{1-P}\right) = \frac{D}{W}\psi_b.$$
(19)

Note that we have allowed for the fact that the no short-sale constraint can bind on the informed when the quality shock is  $\kappa_b$ . It cannot bind on the uninformed, since then their demand would be zero and would violate the bid-overhang restriction (based on a high demand from uninformed). It cannot bind on the informed either when the quality shock is  $\kappa_g$ , since the informed always demand weakly more than the uninformed when the quality of the bond is good.

The common price, *P*, and the inference of the uninformed,  $\tilde{\kappa}$ , should jointly solve auction clearing conditions (18) and (19). As an illustration of the solution of these equations

we graph prices for n = 0.12, 0.07 and 0.02 in Figure 3. Since the bid-overhang constraint binds whenever  $n < \eta_M$ , it binds in all these three cases and then should display a region of prices that overlap. Since the overlapping prices start at a pair of states  $(g, \psi_g)$  and (b, 1)in which the demand of the uniformed match the total supply in state (b, 1), it is straightforward to see that  $P = P(g, \psi_g)$  and  $\tilde{\kappa} = \kappa_g$  solve both equations, where the demand of the informed in state s = (b, 1) is 0 when  $P > 1 - \kappa_b$  (recall from equation (16 that the informed would want to spend a negative amount when  $1 - P < \kappa_b$  in the bad  $\theta$ -state and then the no short-sale constraint binds). This price P is the highest one that correspond to both states in the figures and the threshold  $1 - \kappa_b$  above which the informed do not bid anything in the bad  $\theta$ -state is represented by the pink-dotted horizontal lines.



Figure 3: Equilibrium Prices when the Bid-Overhang Constraint Binds

When the two schedules share common prices, the no short-sale constraint for the informed introduces the possibility of multiple equilibria. One possibility is that prices are high enough (sustained by optimistic uninformed investors, with low  $\tilde{\kappa}$ ) such that the no short-sale constraint binds for the informed in the bad  $\theta$ -state and then they are not willing to bid any positive amount at those prices (this is,  $P > 1 - \kappa_b$ ). Another possibility, however, is that prices are low enough (sustained by pessimistic uninformed investors, with high  $\tilde{\kappa}$ ) such that the no short-sale constraint does not bind for the informed in the bad  $\theta$ -state and they bid positive amounts.

Notice that the solution for  $\tilde{\kappa}$  not only has to solve equations (18) and (19) but should also be consistent with Bayesian updating. As discussed above,  $\tilde{\kappa}$  is related to the relative slope of the two price schedules. Also, as prices have to be always declining in  $\psi$ , there may be a value  $\psi'$  such that that no short-sale constraints bind for the informed when  $\psi < \psi'$  and they do not bind for  $\psi > \psi'$ . These equilibria, then, may display a jump in the range of overlapping prices. Notice that  $\psi'$  is indeterminate, depending on when the uninformed jumped from optimism to pessimism. The logic behind multiplicity (and also non-existence) is the same that pervades competitive equilibria with heterogeneous information more generally. It arises from the fact that prices are determined by local beliefs about the asset, which can jump discontinuously while being consistent with optimality and updating. We discuss in detail the construction of multiple equilibria and the sources of non-existence in the Appendix.

Panel (a) of Figure 3 shows the unique equilibrium when n = 0.12. This equilibrium is unique as the bid-overhang constraint binds for a relatively high  $\psi$  (recall  $\hat{\eta} = n = 0.12$ and then  $\hat{\psi} = 1/(1 - n) = 1.136$ ), with a high enough price such that there is no level of pessimism by the uninformed that justifies a low enough price when the informed would bid a positive amount in the bad state. In panel (b) we show three possible equilibria for n = 0.07, two with a discontinuous jump in the range of overlapping prices (in dashed and dotted lines). Finally, panel (c) shows that when n = 0.02. As n declines, the bidoverhang constraint forces convergence towards the symmetric ignorance schedule (solid black in the figures).

In Figure 4 we graph the equilibrium beliefs of uninformed investors that sustain the equilibrium prices depicted in Figure 3. The vertical axes of the figure is bounded between  $\kappa_g = 0.15$  and  $\kappa_b = 0.35$ . Here one can see the discontinuous change in beliefs that sustain the jump in the range of overlapping prices. The pink-dotted line is the ex ante probability  $\kappa^U = 0.25$ , and jumps occur from a point below  $\kappa^U$  to a point above  $\kappa^U$ . As *n* declines and prices overlap to converge towards the symmetric ignorance schedules, the uninformed investors beliefs about the expected probability of default for a given price also converge towards  $\kappa^U$ .

**Remark on the nature of multiplicity:** In this particular example this schedule lies everywhere above the threshold  $1 - \kappa_b$  (above which the no short-sale constraint binds). Therefore, there is space between the dotted pink constraint  $1 - \kappa_b$  and the solid black ignorance schedule for the prices in the low-quality schedule to exist without inducing



Figure 4: Impact of Information Acquisition on Uninformed Equilibrium Beliefs

informed investors to bid positive amounts for low-quality bonds. Parameters may be such, however, that there is a region of high  $\eta$ 's for which the symmetric ignorance schedule lies below  $1 - \kappa_b$ . In this situation, the equilibrium without jumps in the range of overlapping prices will cease to exist for low enough n, as the prices in the low-quality schedule will be forced below the symmetric ignorance schedule, and then under  $1 - \kappa_b$ , forcing informed investors to bid positive amounts and prices then displaying jumps.

**Remark on the presumption of perfect replication:** The previous analysis was performed for the case in which the no short-sale constraint for the uninformed does not bind at the symmetric information prices (condition 2 in proposition 3 was satisfied). When this constraint binds on the uninformed, however, they can no longer replicate the total purchases of the informed at the prices corresponding to the symmetric information equilibrium and perfect replication is never feasible, even at n = 1. The reason is that the uninformed' first order condition cannot be solved block recursively, as a bid at a state *s* determines total purchases in all states *s*' in which the no short-sale constraint binds, forcing total bids in those states to be the same. In this case, the bid at *s* is determining bond purchases over a set of states, and this will be reflected in the optimality condition associated with this choice. We discuss this in further detail in the Appendix and present a version of our numerical example in which this situation arises.

# 4.3 Convergence to Symmetric Information and Ignorance

We illustrated above how, as we vary n from 1 to 0, the price schedules move from the symmetric information benchmark (indeed they are the same as the benchmark for all  $n \ge \eta_M$ ) to the symmetric ignorance benchmark. As we discussed, even when all conditions for perfect replication are granted, the bid-overhang constraint is the auction-imposed restriction that prevents replication and forces convergence when  $n < \eta_M$ , as seen in Figure 3. This section generalizes this convergence result.

The following proposition is a characterization of the UP auction equilibrium in the symmetric information case and follows from the same logic as the example.

**Proposition 5.** If in the informed equilibrium the price function satisfies  $P([g, \eta_M]; n = 1) > P([b, 0]; n = 1)$  and the total bids of the informed  $\mathcal{B}^I([g, \eta_M], g, 1) < \mathcal{B}^I([b, 0], b, 1)$  then the informed equilibrium price function and total bids are an equilibrium outcome for any  $n > \eta_M$  and there is complete replication by the uninformed.

We turn now to the convergence to the other extreme, as  $n \rightarrow 0$ ,

**Proposition 6.** As  $n \to 0$  the price schedules  $P([g, \eta]; n)$  and  $P([b, \eta]; n)$ , in both auction protocols, converge to each other (for interior  $\eta$ ).

*Proof.* From Proposition 1, prices are a decreasing function of  $\eta$ . Then

$$P([\theta, \eta]; n) < P([\theta, \eta']; n)$$
 for all  $\eta' > \eta$ .

When  $n \to 0$ , almost all expenditures at the auction, which we denote by  $X^U([\theta, \eta]; n)$ , must be made by the uninformed type. This implies that

$$X^U([\theta,\eta];n) \to D/(1-\eta)$$
 for all  $\theta$ ,

Therefore it must be the case that when  $n \to 0$ ,

$$X^U([\theta,\eta];n) \to X^U([\theta',\eta];n)$$
 for all  $\theta$  and  $\theta'$ .

and then

$$P([\theta, \eta]; n) \to P([\theta', \eta]; n)$$
 for all  $\theta$  and  $\theta'$ .

This implies that, when  $n \to 0$ , prices must be sorted by  $\eta$ . This is, there is always a  $\epsilon$  small enough such that for  $\eta' - \eta = \epsilon$ ,

$$P([\theta,\eta];n) < P([\theta',\eta'];n) < P([\theta,\eta'];n).$$

When  $\eta$  is a continuous interval, this implies that the price schedules must converge at every interior point in which the price schedules are continuous. This proof does not apply at the extremes of the  $\eta$  interval, and then convergence may not (and will not) happen at the extremes  $\eta = 0$  and  $\eta = \eta_M$ .

As can be seen this proof does not rely on any particular property of UP auctions, and as we will highlight later, convergence to the symmetric ignorance price schedule also happens in the DP auction protocol.

# 5 Characterization of DP Auction Equilibrium

We now characterize discriminatory-price auctions. In parallel with the UP auction, we will use  $P^{DP}(s;n)$  to denote the price function for each state and degree of asymmetric information, n, in the DP auction. When there is no risk of confusion, we simply write P(s). Similarly, we will use  $B^{DP,U}(s;n)$  for the bids of the uninformed and  $B^{DP,I}(s,\theta;n)$  for the bids of the informed, but may use the simpler notation  $B^{U}(s)$  and  $B^{I}(s,\theta)$  when there is no risk of confusion.

### 5.1 Symmetric Information Benchmarks

We also start analyzing the two symmetry benchmarks, symmetric ignorance (n = 0) and symmetric information (n = 1).

#### 5.1.1 Symmetric Ignorance

With symmetric ignorance we can again simplify our notation to have  $P(\eta)$  and  $B(\eta)$ , as the equilibrium cannot depend on the realized  $\theta$ . With this change, the auction clearing condition (14) becomes

$$(1-\eta)\int_0^{\eta} B(\hat{\eta})P(\hat{\eta})d\hat{\eta} = D.$$

Note the first important difference between UP and DP auctions: bids in-the-money are not paid at the marginal price  $P(\eta)$  but at the bid price  $P(\hat{\eta})$ , then the clearing condition in the DP auction does not just depend upon the total number of bids, but also on the *prices* at which the individual bids are executed. Because of this, total expenditures *must* be monotonically increasing in  $\eta$ . Auction clearing implies that the bids must always be positive, i.e.  $B(\eta) > 0$  for all  $\eta \in \mathcal{H}$  (as there are less investors, each investor must bid an extra amount). This in turn implies that the short-sale constraint for the uninformed cannot bind for any  $\eta$ , as total bids have to be monotonically increasing in  $\eta$ .

The first-order condition for the uninformed investor at  $\eta^*$  can be expressed as

$$\int_{\eta^*}^{\eta_M} \left\{ \begin{array}{c} -U'(B_{RF}(\eta))\kappa^U P(\eta^*) \\ +U'\left(B_{RF}(\eta) + \left[\int_0^{\eta} B(\hat{\eta})d\hat{\eta}\right]\right)(1-\kappa^U)(1-P(\eta^*)) \end{array} \right\} g(\eta)d\eta = 0.$$
(20)

Notice a second key difference relative to the UP auction: the cumulation of the marginal utilities are determined by the residually determined risk-free bonds, which are computed based on the bid price  $P(\hat{\eta})$  and the bid return  $(1-P(\hat{\eta}))$  rather than by the marginal prices. This means that the system is not block-recursive as in the UP auction. Instead, it must be solved simultaneously.

To solve for the optimal bid function, then, we can resort to the following linear algebra structure. Assume a fine grid on the space of  $\eta$ , { $\eta_0$ , ...,  $\eta_J$ } which is indexed by j, where  $\eta_0 = 0$ ,  $\eta_J = \eta_M$  and where  $\eta$ 's are increasing in j. This is then a discretization of the continuous  $\eta$  that we have assumed so far. Next, define the following set of vectors:

$$\vec{P} = \begin{bmatrix} P(\eta_0) \\ \vdots \\ P(\eta_J) \end{bmatrix}, \ \vec{B}^U = \begin{bmatrix} B^U(\eta_0) \\ \vdots \\ B^U(\eta_J) \end{bmatrix}, \ \left(1 - \vec{P}\right) = \begin{bmatrix} 1 - P(\eta_0) \\ \vdots \\ 1 - P(\eta_J) \end{bmatrix},$$

and the following set of triangular matrices of dimension  $J \times J$ 

$$\mathbf{P} = \begin{cases} P_{ij} = P(\eta_i) \text{ if } i \le j \\ P_{ij} = 0 \text{ o.w.} \end{cases}, \ \mathbf{1} - \mathbf{P} = \begin{cases} 1 - P_{ij} = 1 - P(\eta_i) \text{ if } i \le j \\ 1 - P_{ij} = 0 \text{ o.w.} \end{cases}$$

With this notation, the system of equations can be expressed in vector form as

$$-U'\left(W - \mathbf{P} \times \vec{B}^U\right) \cdot \vec{P} \cdot \kappa^U + U'\left(W + [\mathbf{1} - \mathbf{P}] \times \vec{B}^U\right) \cdot \left[1 - \vec{P}\right] * [1 - \kappa^U] = 0.$$

and the solution of **P** should solve this system of equations.

What are the gains to be a single atomistic informed investor in this equilibrium? As in the UP auction, the gains from information comes from knowning  $\kappa_{\theta}$  instead of using  $\kappa^{U}$  and then choosing how much to bid more accurately.

#### 5.1.2 Symmetric Information

With symmetric information (n = 1), we can solve for the equilibrium separately for each  $\theta$ . Again, the logic is just as in the symmetric ignorance case, but replacing  $\kappa^U$  with the appropriate conditional default probability  $\kappa_{\theta(s)}$ , then proceeding analogously thereafter.

$$-U'\left(W-\mathbf{P}\times\vec{B}^{I}\right)\cdot\vec{P}\cdot\kappa_{\theta(s)}+U'\left(W+[\mathbf{1}-\mathbf{P}]\times\vec{B}^{I}\right)\cdot\left[1-\vec{P}\right]*[1-\kappa_{\theta(s)}]=0.$$

What are the costs of being a single atomistic uninformed investor in this equilibrium? As we will show next, this is a third important difference between the UP and the DP auction. While in the UP auction a single uninformed investor can replicate the bids and payoffs of informed investors when the no short-sale constraint does not bind, in the DP auction the uninformed investor would never be able to replicate the portfolio of informed investors, no matter whether the no short-sale constraints binds or not. In other words, under DP auction there is always a cost of being uninformed.

#### 5.1.3 Replication

To see clearly the lack of replication in DP auctions, consider the case in which P(g, 0) > P(b, 0) and assume the informed investor bids a positive amount of bonds at both prices. If each uninformed bids a positive amount at P(g, 0), when the state is (b, 0) he spends  $P(g, 0)B^U([g, 0]) + P(b, 0)B^U([b, 0])$ , while each informed spends  $P(b, 0)B^I([b, 0])$ . Thus even if  $B^U([g, 0]) + B^U([b, 0]) = B^I([b, 0])$ , and they are both buying the same quantity of bonds, the uninformed pay more. In words, if prices are different the uniformed can never replicate: even though they can bid the same quantity in the different states, they pay different amounts.

Even if P(g, 0) = P(b, 0) the uninformed cannot replicate the informed. This is because the uninformed cannot alter the quantities he bids in response to the quality shock, which they cannot observe. In words, if prices are the same the uniformed cannot replicate because they cannot bid different quantities in the different  $\theta$ -states, as informed do.

The following proposition formalizes this discussion and its generality.

**Proposition 7.** *In a DP auction, the uninformed will never be able to replicate the payoffs and bids of the informed so long as:* 

- 1.  $\kappa_g \neq \kappa_b$  and f(g) and f(b) are both positive.
- 2. the informed investor bids positive amounts for both  $\theta = g$  and  $\theta = b$  for some values of  $\eta$ .

# 5.2 Asymmetric Information - Example with Log Preferences

We also illustrate the solution with asymmetric information under the assumption that investors have log preferences. Define a discrete grid of  $\eta \in \mathcal{H}$ , and prices P(s;n) so that we can order the states  $s_i = [\theta_i, \eta_i]$  by prices in descending order,  $P(s_i) > P(s_{i+1})$ . Based on this notation, we can construct the following set of vectors

$$\vec{P} = \begin{bmatrix} P(s_0) \\ \vdots \\ P(s_J) \end{bmatrix}, \ \vec{B}^U = \begin{bmatrix} B^U(s_0) \\ \vdots \\ B^U(s_J) \end{bmatrix}, \ \left(1 - \vec{P}\right) = \begin{bmatrix} 1 - P(s_0) \\ \vdots \\ 1 - P(s_J) \end{bmatrix}, \ \varkappa = \begin{bmatrix} \kappa(\theta_0) \\ \vdots \\ \kappa(\theta_J) \end{bmatrix}$$

and the following set of triangular matrices

$$\mathbf{P} = \begin{cases} P_{ij} = P(s_i) \text{ if } i \le j \\ P_{ij} = 0 \text{ o.w.} \end{cases}, \ \mathbf{1} - \mathbf{P} = \begin{cases} 1 - P_{ij} = 1 - P(s_i) \text{ if } i \le j \\ 1 - P_{ij} = 0 \text{ o.w.} \end{cases}$$

With this construction, the system of equations to solve for the bond policy of the uninformed with log preferences can be expressed in vector form as

$$\left(W - \mathbf{P} \times \vec{B}^U\right)^{-1} \cdot \vec{P} \cdot \varkappa + \left(W + [\mathbf{1} - \mathbf{P}] \times \vec{B}^U\right)^{-1} \cdot \left[1 - \vec{P}\right] \cdot [1 - \varkappa] \le 0$$

and

 $B^U(s_i) = 0$  when this inequality is slack.

The system of equations to solve for the bond policy of the informed with log preferences can be expressed in vector form as

$$\left(W - \mathbf{P} \times \vec{B}^{I}\right)^{-1} \cdot \vec{P} \cdot \kappa_{\theta(s)} + \left(W + [\mathbf{1} - \mathbf{P}] \times \vec{B}^{I}\right)^{-1} \cdot \left[1 - \vec{P}\right] * [1 - \kappa_{\theta(s)}] = 0.$$

If we let the grid on  $\mathcal{H}$  become arbitrarily fine, then our solution will converge arbitrarily close to the true solution (of course the linear algebra conditions will become infinite dimensional).

A fourth important difference between UP and DP protocols, as we highlight in the next remark, relates to the nature of the inference problem solved by an uninformed investor.

**Remark 4.** While the UP investor is concerned with the default probability at the marginal price of his bid  $(\bar{B}, \bar{P})$ ;  $\tilde{\kappa}(\bar{P})$ , the DP investor is concerned with the default probability of the entire set

of states at which his bid  $(\bar{B}, \bar{P})$  is in the money; i.e.  $E\{\tilde{\kappa}(P)|P \leq \bar{P}\}$ . Since the DP investor buys at the bid price when the bid is in-the-money, he is concerned about all the states in which he buys, not only the marginal one, as is the case for the UP investor.

In what follows we use the same parameters as in the UP auction (Table 1) to illustrate how prices are determined and how they change with n. As we discussed these parameters guarantee that the price schedules under symmetric information do not overlap. We first compute the equilibrium in the two extremes of symmetric information and ignorance, using again two possible values for the probability that the bond is of bad quality, f(b) = 0.5 and f(b) = 0.25. The price functions are in panel (a) of Figure 5. As with the UP auction, the price schedules under symmetric information do not depend upon the ex-ante probability of the bad state, but under symmetric ignorance they do. As f(b) falls, the bad  $\theta$ -state is less likely and the price schedule under symmetric ignorance (in black) rises from the fully informed low-quality price schedule (in red) to the fully informed high-quality price schedule (in blue).

These graphs look very similar to their UP analogs in panel (a) of Figure 2, both in terms of shape and levels. This is not a coincidence and in the Appendix we show formally that average prices should indeed be very similar in the symmetric benchmarks.



Figure 5: Symmetric Benchmarks for DP Auctions

To illustrate how close the UP and the DP auctions are at the symmetric information

benchmarks, in panel (b) of Figure 5 we have graphed the marginal price and bid functions for these two protocols when n = 0 (notice the change in scale on the prices axis with respect to panel (a)). In addition, since there is price dispersion in the DP auction, we have included the *average price in the DP auction* for each value of the shock  $\psi$  (in solid black). In the UP auction, as there is a single price, average and the marginal prices coincide (in solid red). Note that the average and marginal prices in the DP auction also coincide at  $\psi = 1$  since there is no price dispersion in this case (all bids are executed at the highest bidding price). For the rest of  $\psi$  values the average price is higher than the marginal price and the *average price schedule* is flatter.

The reason the average price schedule is flatter in DP that in UP auctions is that many of the purchases in DP auctions are locked in at high prices, and lower marginal prices only apply to residual amounts of additional funds needed to cover demand shocks, while for UP auctions those lower marginal prices apply to all bids, not only residual ones. By construction, when the average price in the DP auction is the same as the marginal (average) price in the UP auction, the total bids per investor are also the same.

Note also that the marginal price in the UP auction when all investors participate (at  $\psi = 1$ ) is higher than in the DP auction because bidders are willing to bid more aggressively knowing that they do have to pay such high price in case of a shortage of investors ( $\psi$  is larger). At the same time, the need to raise more additional revenue per capita in the UP auction as  $\psi$  increases causes the prices to fall faster than in DP until marginal prices eventually cross.

This comparison between UP and DP protocols under symmetric ignorance is revealing of a more general property of DP auctions: the average price at which the government raises funds is less sensitive to demand shocks. In other words, in DP auctions the average price that the government expects to receive for its bonds is less volatile, given a volatility of demand shocks, than UP prices. Even though prices are dispersed under DP auctions, their average is more stable. We now turn to examining how equilibrium prices change as we shrink n from 1 to 0. Some examples are plotted in Figure 6. It is immediately clear that equilibrium prices behave very differently compared to the UP auction, which we illustrated at different values of n in Figure 3.



Even though mathematically very similar, the forces underpinning the evolution of equilibrium prices for different degrees of asymmetric information are quite different. Two of the critical forces in the evolution of prices for the UP auctions are absent in DP

Figure 6: DP Auctions as Information Shrinks

auctions: in a DP protocol there is never replication and the bid-overhang constraint never binds. Adverse selection play a more prominent role in a DP protocol and each one of the examples in in Figure 3 is chosen to highlight a particular force as n declines.

When n is relatively large (n = 0.6 in our example) the low-quality schedule (associated with  $\theta = b$ ) is independent of *n*, while the high-quality schedule (associated with  $\theta = q$ ) declines as *n* shrinks. The reason is that, given the large spread between the two price schedules, uninformed investors bidding at the high-quality schedule would face a very severe adverse selection problem: their bids would always be executed at very high prices when the bond quality is low. To avoid buying bad bonds heavily overpriced, uninformed investors rather do not bid at all at high-quality schedule prices. This has two effects. First, the uninformed know that they only buy bad bonds so conditional on buying, they are thus perfectly informed about the bond quality, and they thus choose the same portfolio as informed investors when  $\theta = b$ . As informed and uninformed investors bid the same quantity when the bond is bad, the low-quality schedule does not change with *n* as long as uninformed investors do not bid anything on the high-quality schedule. Second, precisely because the uninformed do not participate when the bond is good, the informed are disproportionately exposed to the government's default risk when  $\theta = g$ and the high-quality schedule must fall as n decreases because there are less informed investors participating, being more exposed. This does not happen in the UP auction, as when n is large enough there is perfect replication.

When *n* is lower (n = 0.4 in our example), prices on the high-quality schedule have declined enough such that the uninformed investors are less worried about the over price they may end up paying for bad bonds, and begin participating on both schedules. At that point, the bids made at prices on the high-quality schedule mean that there is less extra demand that has to be squeezed out when the bond quality is low. Hence this shift in the bidding of the uninformed both slows down the decline of the high-quality schedule and raises the prices on the low-quality schedule.

When *n* is even lower (n = 0.1 in the figure), the adverse selection effect is so strong (uninformed almost always buy at prices in the high-quality schedule, both when the bond is good and bad) that it forces a large fraction of the high-quality schedule to drop

below the *uninformed* price schedule.

Finally, when *n* is very small (n = 0.02 in the figure), price schedules should start overlapping (as proved in Proposition 6 prices converge to a single, symmetric ignorance, schedule as  $n \rightarrow 0$ ), which relaxes the adverse selection faced by uninformed investors such that the average prices start increasing again until converging to the symmetric ignorance price schedule.

It is important to revisit here the role of the bid-overhang constraint in DP auctions. Even when never binding, once the uninformed start bidding at prices on the high-quality schedule, their overhanging bids impact on the equilibrium prices at the low-quality schedule in a manner somewhat reminiscent of the way the *bid-overhang constraint* affected the UP auction: they drag up the prices on the low-quality schedule. But here, the mechanism is quite different. The uninformed accumulated bids reduce the available supply of bonds for the informed and thereby raise prices. Another key difference is while the bid-overhang constraint restricted the set of equilibria in the UP auction, with DP auctions the overhanging bids act on the nature of the equilibrium. The bid-overhang does not induce a pooling of prices as the inference is not at each price but instead in-themoney set. The bid-overhang operates as soon as uninformed investors start bidding at prices in the high-quality schedule (around 0.4 in the example) and not when they become the marginal investor (at  $n = \eta_M$  as in the UP auction example). Even though the bid-overhang constraint operates differently in both protocols, it is the force that guarantees convergence of price schedules in both as  $n \to 0$ .

**Remark on the effect of a better government:** What if a government's quality improves? In particular, what if the probability that a country is in the bad state, f(b), declines? Interestingly there is not much effect when a large number of investors are informed. In the UP auction, there is perfect replication for  $n > \eta_M$  (the bid-overhang not binding) and the two schedules are independent on f(b) (of course from an ex-ante perspective the likelihood of observing one or the other changes, but conditional on a state, they do not depend on f(b)). In the DP auction, *conditional* on the uninformed not participating on the high-quality schedule (the bid-overhang not operating) prices are also

independent of f(b).

The probability of a bad bond f(b), however, changes the ex-ante expected probability of default  $\kappa^U$ , which is critical in determining the symmetric ignorance schedule towards which prices converge as  $n \to 0$ . Indeed, once the bid-overhang constraint operates, at relatively low n, the parameter f(b) does affect prices. In the UP auction, the point at which the bid-overhang constraint binds is just  $\eta_M$ , independent of f(b). Conditional on binding, f(b) affects the inference problem and then the shape of the overlapping price schedules. In the DP auction, a lower f(b) also reduces the likelihood that the uninformed buys a bad bond at an overprice, speeding up and increasing the participation of the uninformed on the high-quality schedule. This in turn leads to the low price schedule rising for higher values of n: different than the UP auction, the point at which the bid-overhang constraint operates do depend on f(b). This is summarized in the next proposition.

**Proposition 8.** In both auction protocols, as long as  $\kappa_{\theta}$  remain unchanged, prices are affected by f(b) only if the bid-overhang constraint operates, for  $n < \eta_M$  (with  $\eta_M$  independent of f(b)) in the UP protocol and for  $n < \eta^*$  (with  $\eta^*$  decreasing in f(b)) in the DP protocol.

We will expand this discussion in the next section, when we discuss how equilibrium prices translates into bond spreads, the debt burden and the utility for investors, and then how f(b) affects these equilibrium objects.

# 6 Comparison Between UP and DP Protocols

### 6.1 Sovereign Yield Level and Variance Across Auctions

We now examine the implications of the two protocols for the government and how these vary with the share of informed investors in the economy. Even though we have not specified a payoff function for the government, in the following discussion we are based on the premise that governments prefer to sell bonds at high prices (this is, to face low sovereign yields) and that they would rather not face volatility on those prices as a function of demand shocks.

Define the sovereign yield as the promised return on bonds, given by

$$Yield = \frac{1-P}{P}$$

For UP auctions P is simply the marginal price (and also the average price). For DP auctions there is a dispersion of prices and we have to compute the average yield. This average yield captures the risk-neutral component of the government's payoff. Notice that the government's debt burden can be defined as D/P = D(1 + Yield), so the government faces a higher average burden when facing a higher average yield. The risky component of the government's payoff is captured by the variation in the average yield conditional on the quality of the bond. This component captures the exposure of the government to shocks on the demand of the bonds given by how many investors show up in the auction.<sup>12</sup>

The average yield and its average conditional variance for both types of auctions are displayed in Figure 7.



leld

0.45

0.4 0.5 0.0 Number of Informed

0.65

/ield

0.4

0.4 0.5 Number of Infor

Figure 7: Bond Yields and Conditional Variances

Compare first the benchmarks of symmetric information and ignorance (the extremes

<sup>&</sup>lt;sup>12</sup>There is also an ex-ante variance across quality bonds. Because we have made the default probabilities so different in order to allow for perfect replication in the UP auction, the differences across quality shocks swamp the conditional. However, this would not be true when these differences were smaller. For this reason, we have chosen to focus on the behavior of the average conditional variance.

at n = 1 and n = 0 respectively). First, given that the prices and total bids are very similar across auction protocols under symmetry, the average yields are also similar across protocols in these symmetric benchmarks. Second, the average conditional variance of the yield, while small in both cases, is substantially lower under the DP auction protocol. As we discussed when describing Figure 5, in a DP auction a lion share of the bids are executed at the largest bid price. Intuitively, these variances capture the slope of the price schedules conditional on a  $\theta$ -state, and average price schedules are flatter in DP auctions. Third, the average yield is lower under symmetric ignorance.

Now we turn to the asymmetric information cases ( $n \in (0, 1)$ ). When information is asymmetric yields and variances differ widely across protocols. Under a UP protocol, the average yield rises monotonically until the point of perfect replication, and is constant thereafter as the equilibrium becomes invariant with respect to n. In contrast, the average conditional variance of the yield shows a strong non-monotonicity in the region in which the bid-overhang constraint binds. The reason is that a binding bid-overhang constraint forces price pooling, steeping the slope of overlapping price schedules. As the schedules become steeper in a range of  $\eta$ 's, prices become very sensitive to demand shocks.

Under a DP protocol, the average yield is hump-shaped, increasing initially for low levels of n and declining later for high levels of n. When n is low, the uninformed bid at prices in the high-quality schedule and there is an *adverse selection effect* that depresses prices in the high-quality schedule that is not compensated by an increase in prices on the low-quality schedule. This adverse selection effect increases in importance as n grows until the uninformed stop participating in the high-quality schedule. When only the informed bid at prices in the high-quality schedule the information rents are gradually competed away as n grows. This *cannibalization effect* raises prices in the good state and reduces the yield as n approaches 1.

In contrast to UP auctions, under the DP auction, the conditional variance is always small, regardless of n. The main reason is again that a lion share of expenditures is made at the highest bid price, and then the exposure to demand shocks is only residual. Furthermore, as n changes, the convergence of prices does involve forcing a large slope in price schedule, as pooling does in UP auctions. As we have discussed, the changes in inference are based on the set of in-the-money and not on price levels, and then the slope of price schedules is not very sensitive to changes in *n*.

The behavior of the average level and variance of yields display an interesting riskreturn trade-off for the government. When information is symmetric both protocols deliver the same prices but the DP auction is more beneficial in terms of lower exposure to demand shocks. When n is relatively low average yields are also similar (raising with nsimilarly) but UP auction is much worse in terms of exposure to demand shocks. When nis relatively larger (when n is in a region in which the UP auction allows for replication), the UP auction still displays a larger conditional variance, but a much smaller average yield. This is a relevant trade-off because it shows that different levels of asymmetric information may induce a combination of these forces that make either the UP or DP auction preferred by a government.

In Figure 7 we also graph the average levels and conditional variances of the yields for two different values of f(b). When the ex-ante probability that the bond is of low quality is smaller (f(b) = 0.25 instead of f(b) = 0.5) there is a decline in the average yield and its variance for all fractions of informed investors. This is quite natural as the bond is less likely to default, increasing prices. More interestingly is that for DP auctions, the peak in the average yields that correspond to the *n* below which the bid-overhang constraint operates is higher the lower is f(b), as the uninformed investors are less afraid of participating at prices in the high-quality schedule. In the figure, the peak in average yields happens at around n = 0.5 when f(b) = 0.5 and at n = 0.75 when f(b) = 0.25.

### 6.2 Investors' Payoffs and Information Acquisition

We now show the implications of the two protocols for investors. We first compute their expected utility for different values of n and plot these results for both types of auctions in the top panels of Figure 8.

In the symmetric ignorance case, both protocols deliver a very similar utility to informed and uninformed investors. As prices and bids are very similar across the two protocols, both the payoffs for the uninformed (the investors that effectively participate



#### Figure 8: Payoffs to Informed and Uninformed

when n = 0) and the payoffs for a single atomistic informed investor who would target his bidding more effectively are similar.

This is not the case under symmetric information though. In that case both protocols deliver a very similar utility for informed investors (again because prices and bids are very similar) but not for the uninformed. A single atomistic uninformed investor would have very different payoffs in the two protocols. In the UP auction, there is perfect replication, and then the uninformed would enjoy the same payoffs than informed investors. In the DP auction, however, it is costly to be uninformed as they only bid at prices in the low-quality schedule, obtaining lower payoffs than the informed, who participate in both schedules.

In the asymmetric information case the payoffs to being uninformed is almost invariant to the degree of asymmetry in the market, however it is slightly increasing with n in the UP auction and decreasing with n in the DP auction. The main difference between these two protocols is given by the payoffs to being informed when information is asymmetric. In the UP auction, the informed investors always lose as there are more informed investors participating in the auction, until there is perfect replication and then they are indifferent about n. In contrast, in the DP auction the informed investors initially gain from more informed investors when they are relatively scarce (low n), as they contribute to increase the difference between price schedules and the adverse selection problem for uninformed, which depresses prices in average. While in this range of *n* the uninformed gain from lower prices but lose from more adverse selection, the informed just gain from lower prices. In contrast, informed investors loses from more informed investors when they are relatively abundant. The reason is that the uninformed stop participating in the high-quality schedule and more informed investors just increases prices in the highquality schedule.

In these figures we also show the comparison with two different levels of f(b). A lower probability of facing a bad bond increases prices, and investors have a lower payoff from higher prices. This is why at each state investors are worse off when f(b) = 0.25. While the effect of f(b) just scales down prices for each n in the UP auction (as we discussed f(b) does not change the point at which the bid-overhang constraint binds in UP auctions), in DP auctions a lower f(b) implies that uninformed participate earlier in the high-quality schedule, then the non-monotonicity pattern is different. Also notice that, as good bonds are more likely, the cost for the uniformed of not participating in the high-quality schedule when n = 1 is higher (the payoffs decline more for the uniformed than for the informed in the symmetric information benchmark).

#### 6.2.1 Endogenizing Information Acquisition

We now endogenize the share of informed investors by allowing for information acquisition. All investors are initially uninformed. After learning whether they will make it to the auction, investors can learn the true value of  $\theta$  by paying a *utility cost* of K. An investor will choose to become informed so long as the differential benefit of doing so justifies the cost. Otherwise, everyone will choose to be uninformed. Similarly, an investor will choose to stay uninformed if the benefit from doing so is weakly better than becoming informed, otherwise everyone will be informed. Denote by  $V^U(n)$  the expected utility of an uninformed investor, and denote by  $V^I(n)$  the expected utility of an informed investor, as depicted in the top panel of Figure 8. The following optimality conditions determine the equilibrium level of n

$$V^{I}(n) - K \geq V^{U}(n) \text{ if } n > 0$$
(21)

$$V^{I}(n) - K \leq V^{U}(n) \text{ if } n < 1.$$
 (22)

Both of these equations must hold simultaneously in an interior equilibrium in which  $n \in (0, 1)$ . This requires that both conditions hold with equality. We are now ready to define an equilibrium with information acquisition.

**Definition 4.** For both the UP and DP auctions, an equilibrium of the model with endogenous information acquisition consists of the measure of informed traders  $n^*$ , a price schedule P(s), a bid schedule for the uninformed  $B^U(s)$ , and a bid schedule for the informed  $B^I(s,\theta)$ . The bid schedules must be solutions to the investors' problems given P and satisfy no short-selling constraints. The bids and price schedules must satisfy deb-overhang constraints and auction clearing for all  $(\theta, \eta)$ , and  $n^*$  must satisfy the information acquisition criterion in (21) and (22).

Based on Figure 8 we can construct the payoff differentials (or gaps) between informed and uninformed,  $V^{I}(n) - V^{U}(n)$ , which are depicted in Figure 9. One can see that there are large differences in the payoff gaps of the two protocols, which has important consequences for equilibrium information acquisition. As the equilibrium  $n^*$  is determined by equalizing the gap with the information acquisition cost K, just from inspecting the figure the next proposition follows.

**Proposition 9.** There are values  $\overline{K} > \widehat{K} > \underline{K} > 0$  such that, when  $K > \overline{K}$  symmetric ignorance is the unique equilibrium in both auction protocols. When  $\overline{K} > K > \widehat{K}$  symmetric ignorance is the unique equilibrium in UP auctions but there are two stable equilibrium in DP auctions, one with symmetric ignorance and the other with asymmetric information, characterized by  $n^*$ decreasing in K. When  $\widehat{K} > K > \underline{K}$ , asymmetric information is the unique equilibrium in both auction protocols, with  $n_{UP}^* < n_{DP}^*$ . Finally, when  $K < \underline{K}$ , asymmetric information is the unique equilibrium in UP auctions and symmetric information is the unique equilibrium in DP auctions. Symmetric information is never an equilibrium in UP auctions.

The *K* thresholds in the proposition can be illustrated in Figure 9. When f(b) = 0.5,

Figure 9: Payoff Differentials Between Informed and Uninformed

#### (a) UP Auction

(b) DP Auction



 $\overline{K} = 0.041$ , the value of the maximum gap in a DP auction,  $\widehat{K} = 0.018$ , the value of the gap (both in UP and DP auctions) at n = 0 and  $\underline{K} = 0.005$ , the value of the gap for DP auctions at n = 1. This proposition leads to the next corollary

**Corollary 10.** The fraction of informed investors in DP auctions is never smaller than in UP auctions.

#### Comparison with Grossman and Stiglitz (1980):

A classical question in general equilibrium is "where information comes from?" As Grossman and Stiglitz (1976) pointed out, agents have no incentive to look at their private information since all of that information is already encapsulated in the price. But, as their quantity choices do not reflect their private information, then how does this information get aggregated into prices? This difficulty manifests itself as a nonexistence problem if acquiring the information is costly; the Grossman-Stiglitz paradox. When information is costly and prices are fully revealing, then no individual wants to acquire information.

This nonexistence problem is captured in our UP auction setting when the price schedules do not overlap with n = 1 and we eliminate the bid-overhang constraint. In such a case perfect replication in UP auctions can be implemented for all n > 0. This implies that the gains of information are 0 for all n > 0 and positive for n = 0. When the cost of information is positive, K > 0 it is clear that there is nonexistence problem if K is smaller than the information gains when n = 0. In words, when n = 0 investors want to acquire information as prices are noninformative, so n > 0. But if n > 0 it is preferred not to acquire information, so n = 0.

The bid-overhang constraint imposed by the auction protocol impedes perfect replication for low levels of n, eliminating the discontinuity of information gains at n = 0, and thus inducing equilibrium existence. Grossman and Stiglitz (1980) proposed a solution by adding a second source of noise to prevent the price system being invertible.<sup>13</sup> In contrast we do not need to impose enough noise to prevent prices from be fully revealing, but instead the auction protocol endogenizes the fraction of informed investors, n, for which prices are not fully revealing, and then the equilibrium with costly information acquisition lies endogenously on that range.

**Remark on Adding Dealers to the Environment:** There are usually two types of participants at sovereign debt auctions: *individual investors* who are bidding on their own, and *dealers* who are both bidding on their own and executing orders for other individual investors. Because they are generally taking very large positions, dealers have very strong incentives to acquire information, hence it is natural to think of them as being informed about the quality shock  $\theta$ . It is also natural to think of these dealers as having information about the size of the demand shock coming through their order flow. We can think of this information being akin observing a noisy signal about  $\eta$ . <sup>14</sup>

It would also be straightforward to endow the dealer with market power. Assume there is a single large dealer (with a larger W) with perfect information about the quality shock  $\theta$  and who can perfectly infer  $\eta$  from order flows, then knowing the state [ $\theta$ ,  $\eta$ ]. The rest of investors would remain relatively small (price-takers) as modeled above. While

<sup>&</sup>lt;sup>13</sup>The existence of equilibria when the shocks are continuous and hence states can have the same price is well known to be problematic, see Allen and Jordan (1998) for a survey of the existence literature. The combination of CARA preferences and jointly normal shocks was key to the construction of an equilibrium in Grossman and Stiglitz (1980), though recently Breon-Drish (2015) has developed a characterization that drops joint normality.

<sup>&</sup>lt;sup>14</sup>A particularly tractable way to incorporate this signal would be to partition the set of possible  $\eta$ 's,  $[0, \eta_M]$  into two subsets,  $[0, \eta_M/2]$  and  $[\eta_M/2, \eta_M]$ , and assume that this signal informs the dealers as to which partition the realized  $\eta$  is in. In this case dealers would find it optimal to only bid on the prices coming from the relevant partition.

investors would take as given the bids of other investors' and also the bid of the dealer, choosing their own bids for each possible marginal price, the dealer would choose his bid considering its effect on the residual supply and then on the equilibrium price. In other words, the investors would face a flat residual supply (as in our setting) while the dealer would not. In equilibrium these strategies and prices are jointly determined.

# 7 Discussion of Relationship to Literature

Our work is motivated by the complex dynamics displayed by the prices at which governments sell their bonds in auctions, particularly in emerging economies. We have already discussed the contribution of our model to the sovereign debt literature that focuses on understanding and measuring the impact of these dynamics on the welfare of countries. In this section we discuss the relationship between our work and other several important topics and approaches to the question.

The first concerns the foundations of general equilibrium theory (GE) and the question of "where do prices come from?" The second concerns the question of "where the information encoded in prices gets aggregated?". The third is about auction theory when there are a large number of bidders for a perfectly divisible good with uncertain common value, and the specific empirical application to the auction of sovereign bonds .

With respect to the question "where do prices come from?," the price vector in GE is an endogenous object that is not chosen by anyone, yet determined by the accumulated actions of individuals who cannot affect prices. To get around the conundrum, Walras made up his fictional "auctioneer" that matches total supply and total demand in a market of perfect competition (perfect information and no transaction costs). But this has long been considered a thought experiment that did not adequately address the issue.<sup>15</sup>

One response to the price problem has been the market games literature, which seeks to provide a fuller description of the environment and in which all endogenous objects are selected by the agents (including prices) based upon noncooperative game theory.<sup>16</sup>

<sup>&</sup>lt;sup>15</sup>See Hahn (1989) for a discussion.

<sup>&</sup>lt;sup>16</sup>See Gale (2000) for a survey of this literature and for a discussion of the alternative cooperative approach.

Examples of this market game approach include Rubinstein and Wolinsky (1985)'s sequential bargaining model in which buyers and sellers are paired up under complete information each period.<sup>17</sup>

This problem is more severe when the prices are simultaneously clearing the market and aggregating information as in Lucas (1972), Radner (1979) and Grossman and Stiglitz (1980). This leads to the complementary question "where does the information in prices come from?" as agents "need to know" both the price function *and* the realized price in order to make their inferences and determine their individual demands.<sup>18</sup>

As we discussed in the section in which we endogenize information, another layer of complication is "where information comes from?" the seminal Grossman-Stiglitz paradox under which fully revealing information prices are logically impossible. But even if it exists a fully revealing equilibrium, there is an implementability problem, as it may not be possible to find a trading mechanism that induces it. Jackson (1991) proposes a resolution when the number of agents is finite and there is no price-taking, as they internalize that the extent of information in prices depends upon the demand schedule they submit. Kyle (1989) study proposes a non-competitive rational expectations model in which agents submit limit orders (demand schedules, as in our case) taking into account their effect on the equilibrium price. Golosov, Lorenzoni, and Tsyvinski (2014) propose a decentralized approach which features a sequence of bilateral meetings with take-it-orleave-it offers.<sup>19</sup> Finally, Vives (2014) and Gaballo and Ordonez (2017) propose settings with large centralized markets in which the valuation of each trader has both common and private value components, and the costly signal bundles information about these two components, such that prices can be fully revealing and yet there are incentives to acquire information.

Our paper speaks to both of these problematic aspects of GE by using the structure

<sup>&</sup>lt;sup>17</sup>Gale (1987) shows that these sequential bargaining models converge to a common price equilibrium as the number of agents gets large.

<sup>&</sup>lt;sup>18</sup>Dubey, Geanakoplos, and Shubik (1987) consider the Nash equilibrium of a sequential trading game with incomplete information where traders make quantity offers to buy and sell and the price is determined by the ratio of the total buy versus sell offers. Here information revelation occurs largely in one-step through the vector of different prices for the different goods.

<sup>&</sup>lt;sup>19</sup>A related contribution is Albagli, Hellwig, and Tsyvinski (2014) who develop a dynamic REE with dispersed information in which information enters nonlinearly into prices.

of an auction to answer where prices come from and by obtaining the conditions for informational gains in two different auction protocols to answer how information gets into prices. In particular, our model features a specific order of moves. First, investors submit their bids (where each bid is a price-quantity pair). Second, a specific protocol is used to select the bids which are accepted and the prices at which they are executed. Information revelation occurs after the marginal price at the auction is revealed. This information revelation may be complete, as in REE. A related paper which takes a similar auction-based approach to micro found REE is Milgrom (1981). He considers, however, an auction in which bidders are restricted in the units to buy and where the price is not clouded by a demand (or supply) shocks. Our paper relax both of these aspects.

Our paper also contributes to the auction literature on multi-unit auctions, in particular as it is applied to goods such as sovereign bonds. The backbone of the auction literature is based on the selling of a single object, either to bidders with independent private values<sup>20</sup> or to bidders with correlated values,<sup>21</sup> studying the strategic considerations when choosing the bid price. To capture goods such as treasury bonds, however, the literature extended these models to multi-unit auctions, facing the challenge of solving an equilibrium that involves bidders with a double dimensional strategic problem when submitting a combination of quantities and prices.<sup>22</sup>

Recent work tackles these questions from an empirical perspective. Hortaçsu and McAdams (2010) develop a model, based on Wilson (1979), of a multi-unit discriminatoryprice auction to a finite set of potential risk-neutral bidders with symmetric and independent private values, using data from Turkish Treasury auctions to estimate those bidders' private values. In their model the symmetric equilibrium bidding functions depend on how an individual bid changes the probability distribution of the market clearing price, a complicated object to characterize analytically. Given this theoretical difficulty, they construct a non-parametric consistent estimator of the distribution exploiting a resampling technique.<sup>23</sup>

<sup>&</sup>lt;sup>20</sup>See Vickrey (1962), Harris and Raviv (1981), Myerson (1979) and Maskin and Riley (1985).

<sup>&</sup>lt;sup>21</sup>See Milgrom and Weber (1982) and McAfee and McMillan (1987).

<sup>&</sup>lt;sup>22</sup>See Wilson (1979), Engelbrecht-Wiggans and Kahn (1998), Perry and Reny (1999), Kagel and Levin (2001) and McAdams (2006).

<sup>&</sup>lt;sup>23</sup>Kastl (2011) extended Wilson model, which is based on continuous and differentiable functions, to

The key difficulty the empirical literature confronts is that "common value" and "independent value" auctions are (nearly) observationally equivalent even when one assumes risk-neutrality.<sup>24</sup> This is because it is hard to distinguish the extent to which bidders adjust their bids to take advantage of market power vs. inferences they make from their bid determining the marginal price (the so-called "shading factor"). Risk aversion, which we deem necessary to understand may countries' auction data, would only complicate the already complicated identification further. Our assumption of price-taking offers a potentially useful empirical as well as theoretical simplification relative to the alternative.

Several differences separate our modeling assumptions from this body of work. First, and most important, our model is based on the presumption that bidders' valuations of the auctioned treasury bond are perfectly correlated (common value) instead of independent (private value). Laffont and Vuong (1996) argue that common value models are observationally equivalent to an affiliated private value model unless there are exogenous variations that allow for identification. Hortacsu and Kastl (2012) use Canadian treasury auctions to test whether common values or private values are a better representation of the motivations to buy treasury bonds. In Canada, some bidders (dealers) are allowed to observe the bonds of another set of bidders (costumers) when preparing their own bids. In a private value auction, observing the bid of a costumer only gives information about the competition the dealer faces (and then the probability of winning the auction) but not about the fundamental value of the bond. In this case the dealer would not revise the bid if this was higher than the observed competing bid. In a common value price auction, however, observing a costumer's bid induce learning not only of competition but also of the fundamental, leading to a revision of the intended bid also when the bid was higher that the observed competing bid. Since Canadian auctions are discriminatory, this testable implication is not as straightforward, but they propose a correction, concluding that there is no evidence in their data that dealers learn about fundamentals from cos-

more realistic discrete-step functions, showing that in such case only upper and lower bounds on private valuations can be identified, which he does by using the previously methodology on Czech bills auctions.

<sup>&</sup>lt;sup>24</sup>Laffont and Vuong (1996) argue that common value models are observationally equivalent to an affiliated private value model unless there are exogenous variations that allow for identification. Hortaçsu and Kastl (2012) proposes an identification based on dealers observations of their costumers bids using discriminatory Canadian treasury auctions, while Hortaçsu, Kastl, and Zhang (2017) use uniform U.S. treasury auctions.

tumers. This is not prima facie evidence that dealers follow private values, but instead that they may have superior information than costumers about the common value.<sup>25</sup>

The second relevant difference is that we assume investors are risk averse, not risk neutral. Following Wilson (1979), this departure is indeed relevant for the interpretation of the shading factor. The third difference is our modeling of several correlated shocks, departing heavily form the assumptions of independent realizations across bidders. The quality shock, the demand shocks and the signal that all informed investors receive are perfectly correlated, which implies that bid shading only happens for uninformed investors in response to the possibility of adverse selection and not to competitive forces.

Closer to our setting, Boyarchenko, Lucca, and Veldkamp (2017) study an auction environment with risk averse investors that are asymmetrically informed about the common value of a bond. They consider some investors have superior information and have market power, calibrate the model to U.S. Treasury auctions and show that information sharing across investors increase government revenues as investors are willing to invest more as they become better informed. By focusing on the assumption that investors are price-takers we focus on the effect of asymmetric information on prices instead of on the effect of strategic considerations on prices.

# 8 Conclusion

We develop a rich model of sovereign debt auctions which features an (implicit) public shock, a quality shock about which there can be heterogeneous information, and a correlated private shock which determines auction participation. Our numerical example illustrates how our model can speak to the kind of data we see in sovereign debt auctions like that for Mexican bonds in figure 1. During crisis periods, when the range of potential default probabilities increase, the model generates a sharp rise and highly volatile interest rates, as those we see during Mexican "Tequila Crisis" of 1995. When the level of a country's indebtedness increase there is a substantial pressure for information acquisition

<sup>&</sup>lt;sup>25</sup>With a similar methodology applied to uniform-price auctions of U.S. Treasury bills, Hortaçsu, Kastl, and Zhang (2017) estimate that the informational advantage of primary dealers leads them to higher yield bids as a response to a larger ability to bid-shade their bids.

and information asymmetry (specially under discriminatory-price auctions), decreasing participation of uninformed investors under the threat of adverse selection and increasing interest rates even when the quality shock is good. This reduction in participation is also reminiscent of the decline in bids and failed auctions that we commonly observe during crises. At the same time, under either symmetric information or symmetric ignorance when there is little heterogeneity, the additional risk premia associated with demand shocks is small. This illustrates how the model can also accommodate the relatively low volatility of the Mexican Cetes interest rates in recent years.

Our model also provides interesting insights into the impact of different auction protocols on the extent of information acquisition. There is a trade-off between the two protocols, with both behaving similarly in terms of payoffs and yields when information is symmetric, though the discriminatory-price auction does offer lower variation in the yields in response to demand shocks. These symmetric cases can be thought of as occurring during tranquil periods when there is no information to be obtained about the government's future likelihood of default, or during eventful periods in which this information is public. However, during turbulent periods in which there are large incentives to obtain information, the adverse selection effect in discriminatory-price auctions can lead to a wide dispersion in realized auction prices and higher average yields as uninformed investors are reluctant to participate. These results contribute to the wide discussion, dating back at least to Friedman (1960), of whether sovereign debt auctions should be conducted with a uniform-price or a price-discriminating protocol.<sup>26</sup> Our results suggest that the benefit of choosing a discriminatory or a uniform price auction depends on the extent of the incentives to acquire private information.

In a follow-up paper (Cole, Neuhann, and Ordonez (2016)) we examine the implications of discriminatory-price auctions within a two-country setting. We use the insights developed here to discuss spillovers of information across countries and the role of secondary markets. We show that the sources of complementarities inherent to discriminatory-

<sup>&</sup>lt;sup>26</sup>Friedman proposed (pp 64-65) that the U.S. Treasury abandons its previous price-discriminating practice and make all awards at the stopout price instead of at differing prices down through that price. The U.S. Treasury finally adopted this uniform-price protocol for all auctions of 2-year and 5-year notes on September 3, 1992. An excellent summary of this discussion is Chari and Weber (1992). Earlier discussions about Friedman's proposal include Goldstein (1962), Friedman (1963), Rieber (1964) and Friedman (1964).

price auctions extend from cross-states to cross-bonds and make debt crises contagious even in the absence of other linkages.

The model that we develop is applicable to a number of other important circumstances, including auctions of liquidity infusion by central banks, electricity, emission permits, gas, oil, and mineral rights. The key requirement is that the auction involves a "thick" enough market for a homogenous divisible good of uncertain quality so as to make the price-taking assumption a close approximation to reality. Our model also provides a potential mechanism to micro found competitive equilibria for the case of the uniform-price protocol and to break the circularity inherent in having prices and quantities determined simultaneously.

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# A More Details on UP Auctions

# A.1 Non-existence and Multiplicity

Here we discuss more formally multiplicity and discuss how nonexistence may arise. As these two features are related we discuss them in the same section.

First, recall that the auction clearing condition when the quality of the bond is  $\theta = g$  (equation (18)) is,

$$n\left(\frac{1-\kappa_g-P}{1-P}\right) + (1-n)\left(\frac{1-\tilde{\kappa}-P}{1-P}\right) = \frac{D}{W}\psi_g.$$

When the bid-overhang constraint does not bind, then there is perfect replication given our set of parameters. When the bid-overhang constraint binds, the price *P* that solves for auction clearing when  $\theta = g$  is the same as the one that solves auction clearing when  $\theta = b$ . There may be, however, two versions of auction clearing when  $\theta = b$ , as in clear from equation (equation (19)),

1. The short-sale binds on the informed in the bad state  $(P > 1 - \kappa_b)$ 

$$(1-n)\left(\frac{1-\tilde{\kappa}-P}{1-P}\right) = \frac{D}{W}\psi_b.$$
(23)

2. The short-sale constraint does not bind on the informed in the bad state ( $P < 1 - \kappa_b$ )

$$n\left(\frac{1-\kappa_b-P}{1-P}\right) + (1-n)\left(\frac{1-\tilde{\kappa}-P}{1-P}\right) = \frac{D}{W}\psi_b.$$
(24)

As the price *P* is the same in the good and the bad  $\theta$ -state when the bid-overhang constraint binds, we can substract the auction auction clearing of the two  $\theta$ -states in both states cases, which gives us

1. The short-sale binds on the informed in the bad state  $(P > 1 - \kappa_b)$ 

$$n\left(1 - \frac{\kappa_g}{1 - P}\right) = \frac{D}{W}(\psi_g - \psi_b)$$

2. The short-sale constraint does not bind on the informed in the bad state ( $P < 1 - \kappa_b$ )

$$n\left(\frac{\kappa_b - \kappa_g}{1 - P}\right) = \frac{D}{W}(\psi_g - \psi_b).$$

In this construction there is a monotonic relation between prices P and the supply gap  $\psi_g - \psi_b$ . Further, as the demand by the uninformed investors in both states is the same, the gap has to be covered solely by the informed investors. This is the reason why the beliefs of the uninformed about the likelihood of each state,  $\tilde{\kappa}$ , disappears from these equations and the reason why n multiplies this extra demand.

We can now obtain analytical expressions for the prices in both cases

1. The short-sale binds on the informed in the bad state  $(P > 1 - \kappa_b)$ 

$$P = 1 - \frac{\kappa_g}{1 - \frac{D}{nW}(\psi_g - \psi_b)}$$
(25)

2. The short-sale constraint does not bind on the informed in the bad state ( $P < 1 - \kappa_b$ )

$$P = 1 - \frac{\kappa_b - \kappa_g}{\frac{D}{nW}(\psi_g - \psi_b)}$$
(26)

Notice that in the first case (when the short sale constraint binds) the price is decreasing in the gap  $\psi_g - \psi_b$ , while in the second case (when the short sale constraint does not bind) the price is increasing in the gap  $\psi_g - \psi_b$ . This is illustrated by the black and blue lines respectively in the first panel of Figure A.1, which is constructed by fixing  $\psi_b$  and assuming n = 0.2.

#### Figure A.1: Prices and beliefs with and without SS constraints



An important property to notice is that there is a unique gap level for which the price is the same, with and without the short-selling constraint binding. This is when

$$(\psi_g - \psi_b)^* = \frac{\kappa_b - \kappa_g}{\kappa_b - \frac{D}{nW}}$$

At this gap, when the two price functions of the panel (a) of Figure 3 cross,  $P = 1 - \kappa_b$ , represented by the red horizontal line. As the decreasing (black) function is constructed under the presumption that the short sale constraint binds ( $P > 1 - \kappa_b$ ), this is consistent only when the gap is smaller than  $(\psi_g - \psi_b)^*$ . Similarly, the increasing (blue) function is constructed under the presumption that the short sale constraint does not bind ( $P < 1 - \kappa_b$ ), which is also consistent only when the gap is smaller than  $(\psi_g - \psi_b)^*$ . This is, no gap above  $(\psi_g - \psi_b)^*$  can be rationalized by any price: either the price is too high if we assume the short-sale constraint does not bind (in which case it would), or too low if we assume that the short-sale constraint binds (in which case it would not).

We can now use one of the auction clearing conditions (either for the good or bad  $\theta$ state) to back out the inference parameter  $\tilde{\kappa}$  for the uninformed that rationalizes the price corresponding to a given supply gap. Regardless of the short-sale constraint binding or not, there is a negative relation between the price and the inference  $\tilde{\kappa}$ . This implies that, when the short selling constraint binds, as the supply gap increases the price declines, only consistent with an increasing  $\tilde{\kappa}$  (more pessimism about the state (the black function in panel (b) of Figure Figure 3). Similarly, when the short selling constraint does not bind, as the supply gap increases the price also increases, only consistent with a decreasing  $\tilde{\kappa}$ (more optimism about the state (the blue function in panel (b) of Figure Figure 3). As this inference is bounded between  $\kappa_b$  and  $\kappa_g$ , as shown in the figure, there is a constrained range of inferences that is consistent with an equilibrium.

With these relations that arise from auction clearing we can now discuss the forces behind the construction of the equilibria displayed in the text. Since the bid-overhang constraint always binds first in the high price schedule when  $\eta = 1 - \frac{1}{\psi} = n$ , denote as  $\psi_n = \frac{1}{1-n}$  the value of  $\psi$  for which the bid-overhang constraint binds first such that  $\psi_b = 1$ (or  $\eta = 0$ ). The price *P* that is common to the two states  $[g, \phi_n]$  and [b, 1] and covers the gap  $\psi_n - 1$  is always consistent with the short sale constraint binding in state *b*. To see this note that *P* is determined in such situation by

$$P = 1 - \frac{\kappa_g}{1 - \frac{D}{nW}(\psi_n - 1)} = 1 - \frac{\kappa_g}{1 - \frac{D}{(1 - n)W}}$$

as  $\psi_n - 1 = \frac{1}{1-n} - 1 = \frac{n}{1-n}$ . Notice that this price simultaneously solves auction clearing in states *b* and *g* when  $\tilde{\kappa} = \kappa_g$ , regardless of n.<sup>27</sup>

Since the short sale constraint for the informed in the bad state (case 1) is what determines prices at the point in which the bid-overhang constraint and prices are continuous, prices walk down the decreasing (black) function in panel (a) of the figure, determining  $\tilde{\kappa}$  and then the consistent supply gap  $\psi_g - \psi_b$ , according to the equation for bayesian updating in the text.

Notice, however, that as long as the blue and black functions in panel (b) of the figure are inside the range of plausible  $\tilde{\kappa}$ , for each supply gap to the left of  $(\psi_g - \psi_b)^*$  there are two consistent prices, and two consistent  $\tilde{\kappa}$ , one in which the short-sale constraint for the informed in the bad state binds and another in which it does not. As explained in the text, one price is high and consistent with pessimism by the uninformed and the other is low and consistent with optimism by the uninformed.

The fact that for the same supply gap there may be two consistent prices imply that the prices may show a discontinuous jump from one function to another. In other words, the price would run downwards following the black function of panel (a), jump at some point

<sup>&</sup>lt;sup>27</sup>If *n* is sufficiently large, the common price that is consistent with the bid-overhang constraint binding for the first time may be consistent with the short-sale constraint not binding, displaying an immediate discontinuous jump down. To see this, replace a common price *P* in states (b, 1) and  $(g, \psi_n)$  from equation (26) in the auction clearing consistent with no short selling constraint from equation (24), the uninformed investors' inference that make those two conditions consistent is  $\tilde{\kappa} = \frac{W-D}{D}(\kappa_b - \kappa_g) - \frac{n}{1-n}\kappa_b$ . The expected probability of default can never be higher than the maximum one feasible,  $\tilde{\kappa} \leq \kappa_b$ , which only happens if  $n \geq 1 - \frac{\kappa_b}{\kappa_b - \kappa_g} \frac{D}{W-D}$ .

to the blue function and keep going down on that function. Consistently, this implies that initially the uninformed become more pessimistic (going up the black function in panel b), then jumping to the blue function and keep becoming more pessimistic.

It is trivial to see that there is a continuous range in which this jump may occur, so there is an indeterminate number of equilibria characterized by different point at which prices discontinuously change. All these equilibria are characterized by a single discontinuous decline in prices, as prices cannot increase in equilibrium.

The possible nonexistence arises from the constraint that beliefs are consistent with Bayesian updating. A given  $\tilde{\kappa}$  that is consistent with an equilibrium price for a gap  $\psi_g - \psi_b$  determines the slope of the price function that is consistent with such inference. For instance, if  $\tilde{\kappa}$  is low, this implies that the price applies for a relatively large mass of  $\psi$  points in the good schedule compared to the bad schedule, as discussed in the main text. Then,  $\tilde{\kappa}$  also determines the speed at which the gap  $\psi_g - \psi_b$  changes along the schedules. When the gap is forced to the right of the point in which the two lines cross in panel (a), there is no price that clears the auction.

In short, as long as prices for all states are consistent with a gap smaller than  $(\psi_g - \psi_b)^*$  there is multiplicity. When for some states prices are forced to be consistent with a gap larger than  $(\psi_q - \psi_b)^*$  there is non-existence.

### A.2 Prices when the Short-Sale Constraint Binds on the Uninformed

To illustrate how prices change in a situation in which the short-sale constraint binds for the uninformed investors in the prices that characterizes the symmetric information situation, we use default probabilities that are closer to each other, so that those price schedules are not very far apart. More precisely, in the example below we use  $\kappa_g = 0.24$ and  $\kappa_b = 0.29$ . This change still guarantees that prices in the symmetric information case do not overlap, but now the uninformed wants to bid more at the price corresponding to the state  $[g, \psi_M]$  than what they want to bid at the price corresponding to [b, 1], which is the sufficient condition that breaks condition ii) for perfect replication in proposition 3.

When there is no overlap in the price schedule, the binding region is particularly simple. Binding will occur over intervals of the form  $[\psi_g, \psi_M]$  on the  $\theta = g$  schedule and  $[1, \psi_b]$  on the  $\theta = b$  schedule. In this case, the total risky bond purchases will be equal in this range, and the f.o.c. ignoring the short-sale constraint hold at the endpoints; i.e.  $B^U(g, \theta_g) = B^U(b, \theta_b)$ . In contrast to the previous analysis, the f.o.c. does not hold state by state, but the integral of the f.o.c. over the ranges will equal 0, as  $s = (g, \psi_g)$  is the point at which the short-sale constraint starts binding. It is easy to see that starting from any  $s = (g, \psi_g + \epsilon)$  the integral will turn negative, and extending the integral beyond  $(b, \eta_b)$  will turn it positive. Note that, as discussed in section 3.2, when this occurs, an auction equilibrium of the UP model no longer has an associated competitive equilibrium, as this is a case of nonnegativity constraints binding for bids in a particular range but not binding for total purchases on that range.

In Figure A.2 we plot the price functions for n = 0.8, 0.6 and 0.4. As is clear from the figures, the lack of perfect replication has an effect on the price schedules. Because the bids of the uninformed do not adjust to the price over the range that the short-sale binds, it follows that all of the adjustment to match the change in per capita supply must be

done by the informed. As they shrink in number, this requires a larger change in the price to induce them change their risk exposure and cover those extra bonds. In the high price schedule the uninformed bid a fixed-amount starting from the point  $(g, \psi g)$  in which the short-sale constraint binds. The need for extra bonds has to be compensated by informed investors, which depresses prices in equilibrium in the (blue) region  $[\psi_g, \psi_M]$ . In contrast, in the low price schedule, uninformed are bidding more than informed until the point  $(b, \psi b)$  at which the short-sale constraint stop binding. The extra demand inflates the prices in the (red) region  $[1, \psi_b]$ .



Figure A.2: Prices with Binding Short-Sales Constraint on the Uninformed

Eventually, when *n* becomes small enough the price at the bottom of the high-price schedule will fall below the binding region for the short-sale constraint of the uninformed, and this price will now be the same as a top point on the low-price schedule where outcomes must be determined point-by-point along with the inference parameter  $\tilde{\kappa}$  of the uninformed, just as in the discussion above.

Finally, notice that in all these cases, the number of informed investors is above the threshold for the bid-overhang constraint to start binding. Indeed, the threshold at which the bid-overhang constraint starts binding when the short-sale constraint binds on the uninformed is not  $\eta_M$  anymore, but smaller. Intuitively, total bids of uninformed investors at the bottom of the high-price schedule are depressed relative to the informed bidders, and then it is more difficult for the uninformed investors alone to cover the revenue needs of the government in the state [*b*, 1].

# **B** Almost Equal Revenue

Here we show that under the two auctions, with symmetric ignorance and symmetric information, the average yields and investors' payoffs are the same. Take a fine grid on the  $\eta$ 's indexed by j = 1, ..., J. Define the f.o.c. kernel for a bond  $B_i$  in state j (where  $j \ge i$ ) under the the UP auction as

$$X_{ij}^{UP} = \frac{(1-\kappa)[1-P_j]}{W - \frac{D}{1-\eta_j} + \frac{D}{1-\eta_j}\frac{1}{P_j}} - \frac{\kappa^U P_j}{W - \frac{D}{1-\eta_j}},$$

which does not depend on *i*, and then  $X_{ij}^{UP} = X_j^{UP}$ . Since  $\eta$  is distributed uninformly, the f.o.c. for a bond  $B_i$  can be expressed as

$$\sum_{j=i}^{J} X_j^{UP} = 0 \text{ for all } i,$$

from where it follows that

$$X_j^{UP} = 0$$
 for all  $j$ .

Similarly, define the f.o.c. kernel for a bond  $B_i$  in state j (where  $j \ge i$ ) under the the DP auction as

$$X_{ij}^{DP} = \frac{(1-\kappa)[1-P_i]}{W - \frac{D}{1-\eta_j} + \frac{D}{1-\eta_j}\frac{1}{\tilde{P}_j}} - \frac{\kappa^U P_i}{W - \frac{D}{1-\eta_j}},$$

where  $\tilde{P}_j$  is the average price that satisfies the condition

$$\tilde{P}_j * \mathcal{B}_{R,j} = \frac{D}{1 - \eta_j},$$

where  $\mathcal{B}_{R,j}$  is the number of risky bonds sold in state  $\eta_j$ . This average price must be equal to the marginal price when j = 1, will decline more slowly than the marginal price as j increases. The first-order condition for bond  $B_i$  can then be expressed as

$$\sum_{j=i}^{J} X_{ij}^{DP} = 0 \text{ for all } i.$$

Note that if  $P_j$  does not decline very much so  $P_1$  is close to  $P_J$ , then  $\tilde{P}_j \simeq P_j \simeq P_{j'}$ . In this case, the condition becomes very close to that in the UP auction, and

$$X_{ij}^{DP} \simeq 0.$$

In figure 2 we compare prices in the symmetric ignorance case with the "average price" in the DP auction. What we see is that while the marginal prices are fairly flat, with the DP price schedule being flatter than the UP schedule, the "average price" paid schedule in a DP auction is even flatter. Given this, it is unsurprising in light of the above discussion that the average yield is also close, and that the conditional variance of the yield is much lower under the DP protocol.