Leverage dynamics and credit quality*

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Abstract

We study a dynamic model of collateralized credit markets with asymmetric information, which allows for a rich set of signaling strategies based on the path of debt and repayment. Whether credit history reveals private information about credit quality depends on the degree of uncertainty in collateral value. When uncertainty is low, good borrowers fully and costlessly separate by deleveraging, that is borrowing a sufficiently high amount such that subsequent repayment reveals the presence of unobservable income. When uncertainty is higher, good borrowers pay an adverse selection cost through a higher interest rate because bad borrowers could default, and asymmetric information is not always resolved.

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1. Introduction

How can leverage signal credit quality? Ross (1977) and Leland and Pyle (1977) develop static models of credit markets with asymmetric information about borrowers' profitability and deadweight costs of default. Because good borrowers have a lower probability of default for a given loan amount, they can signal their type through higher leverage in a separating equilibrium. A large empirical literature that followed finds that leverage is negatively related to profitability in the cross-section of firms (Titman and Wessels, 1988; Rajan and Zingales, 1995), which is opposite of the simple theoretical prediction. In a comprehensive review of the literature, Schmid Klein et al. (2002) conclude that there must be other confounding factors that cause the empirical correlation to be opposite of the theoretical prediction.

This paper considers an alternative possibility, that the world is dynamic. A lender can potentially infer a borrower's credit quality based on the entire repayment history, not just the current loan balance. Indeed, repayment history is the most important of five factors that determine the FICO score (FICO, 2015), a leading measure of consumer credit quality, and the only factor that determines the PAYDEX score (Dun and Bradstreet, 2017), a leading measure of credit quality for small businesses.

We study how the lender's perception of credit quality evolves based on the history of borrowing, repayment, and default in a dynamic model of credit markets with asymmetric information. Under what conditions can credit history resolve asymmetric information at no cost? Under what conditions does information revelation involve costly default? Under what conditions does information revelation not even happen? These questions are important insofar as informational frictions could reduce efficiency in credit markets and cause misallocation of real resources to borrowers with worse investment opportunities. We show that the borrower's ability to signal credit quality through debt and repayment and the cost of such signaling strategies depend critically on the degree of uncertainty in collateral value.

We develop a three-period model of credit markets in which borrowers do not have immediate investment needs and borrow purely for signaling reasons, following Ross (1977). There are two types of borrowers, good and bad. Both types of borrowers have a pledgeable asset that can be used as collateral and generates observable income, whose value is subject to uncertainty. Only good borrowers have a non-pledgeable asset that cannot be used as collateral and generates unobservable income. Borrowers maximize net worth, as perceived by the lender or outside investors, which is increasing in *reputation* (i.e., the perceived probability that the borrower is good).[1] The lender is risk neutral and prices debt to break even,

[1] Reputation is publicly observable in our model so that we are only concerned with "directly placed debt" in the language of Diamond (1991).

conditional on reputation. Reputation is updated through Bayes' rule, based on repayment versus default and rollover choices (i.e., the new loan amount conditional on repayment).

Good borrowers have an incentive to signal through a strategic path of debt and repayment that reveals the presence of unobservable income. Bad borrowers have an incentive to mimic the path of debt, if possible, to delay or prevent information revelation. When uncertainty in collateral value is low, good borrowers fully separate by borrowing a sufficiently high amount and subsequently repaying with unobservable income. Bad borrowers, who do not have unobservable income, must roll over more debt to repay. Therefore, the ability to deleverage signals that the borrower is good. Importantly, signaling through deleveraging is costless because it does not involve default in equilibrium.

The effectiveness of costless separation through deleveraging depends critically on the degree of uncertainty in collateral value. When uncertainty in collateral value is higher, full separation is no longer possible through deleveraging alone. A loan amount that is necessary for separation through deleveraging when the collateral value rises forces bad borrowers to default when the collateral value falls. Although good borrowers do not default, they bear an ex-ante cost of adverse selection through a higher interest rate that reflects the possibility that bad borrowers default. Because we assume no deadweight costs of default in contrast to Ross (1977), the higher interest rate arises from adverse selection only. Bad borrowers benefit at the cost of good borrowers through pure cross-subsidization. In choosing the optimal loan amount, good borrowers must trade off the ex-post benefit of separation against the ex-ante cost of paying a higher interest rate. This tradeoff depends on uncertainty in collateral value because the benefit of separation is constant, while the interest rate for a given loan amount increases with uncertainty as default becomes more likely.

For intermediate uncertainty in collateral value, the benefit of separation outweighs the higher interest cost, so there is *costly full separation* in equilibrium. Good borrowers borrow a sufficiently high amount to fully separate by deleveraging if the collateral value subsequently rises. However, if the collateral value falls instead, bad borrowers do not have sufficient collateral and are forced to default. Thus, there is full information revelation with bad borrowers rolling over more debt in good states and defaulting in bad states.

For high uncertainty in collateral value, the higher interest cost outweighs the benefit of separation, so full separation is no longer optimal. Good borrowers borrow a relatively low amount such that, if the collateral value subsequently rises, even bad borrowers repay by rolling over a low amount, so there is no information revelation. If the collateral value falls instead, good borrowers fully separate by deleveraging. Thus, there is *costless partial separation* in equilibrium, only in bad states.

In summary, the cost of asymmetric information rises with uncertainty in collateral value.

When uncertainty in collateral value is low, good borrowers can costlessly signal their type by deleveraging, and asymmetric information is fully resolved. When uncertainty in collateral value is intermediate, the equilibrium entails bad borrowers defaulting in bad states. Although asymmetric information is fully resolved, good borrowers must bear an adverse selection cost through a higher interest rate. When uncertainty in collateral value is high, good borrowers borrow a conservative amount such that asymmetric information is not always resolved. In this case, there are realized paths of collateral value such that both types repay or default.

An important implication of our results is that credit history is a less precise signal of credit quality in environments with high uncertainty. For example, uncertainty could be higher in recessions or for collateral that is difficult to value. In such environments, asymmetric information becomes a more relevant friction that credit history cannot fully resolve, which leads to a higher probability of default and a higher interest rate.

Because the intended purpose of our model is to provide general insights about leverage dynamics and credit quality, it is not tailored to any particular credit market. However, we motivate our model with two examples of short-term credit markets with private information and uncertainty in collateral value. In consumer credit markets, a new borrower without a credit history may put a balance on a credit card to establish a credit history. A successful history of balances followed by repayment would improve the FICO score, which may be useful for a future durable-good purchase or capital investment. Although credit cards are a type of unsecured debt, creditors have "collateral" in the form of expected recovery value. The pledgeable asset in our model corresponds to the part of future income allocated to repayment of unsecured creditors in a chapter 13 bankruptcy.

In a banking context, asset-backed commercial paper (Acharya et al., 2013) and repurchase agreements (Gorton and Metrick, 2012) are examples of short-term collateralized debt. Banks borrow extensively through these short-term (sometimes overnight) contracts that are collateralized by asset-backed securities. These contracts must be rolled over frequently because of their short maturity, which provides frequent signaling opportunities through changes in debt and repayment. Investors' beliefs about the bank's credit risk is priced into contract terms such as repo rates and haircuts on collateral. In extreme scenarios, a bank may not be able to roll over contracts if its perceived credit risk is too high.

1.1. Related literature

This paper is part of a long tradition of studying asymmetric information in credit markets and the potential role that financing strategies play in revealing private information (Ross, 1977; Myers and Majluf, 1984). Some notable extensions of Ross (1977) and Leland and Pyle

(1977) include managers with different objective functions (Heinkel, 1982), projects with different mean returns (Blazenko, 1987; John, 1987), and projects with different variance of returns (Brick et al., 1998). This paper also contributes to the literature on reputation in credit markets (Diamond, 1989).

This paper complements a more recent effort to extend models of credit markets with asymmetric information to a dynamic setting (Hennessy et al., 2010; Morellec and Schürhoff, 2011; Strebulaev et al., 2016). These papers essentially reduce the optimal choice of debt to a static problem by assuming that private information is short-lived or that debt is a one-time choice in a real options framework. In contrast, this paper allows for a richer set of signaling strategies through the path of debt and repayment. We also abstract from optimal investment choice to isolate the pure signaling motive for debt. Thus, the only role of debt in our model is to signal credit quality, not to fund investment.

This paper is related to a recent literature that studies signaling in dynamic models with persistent asymmetric information. Sannikov (2007) finds that an increasing credit line is an optimal contract in a dynamic principal-agent model with asymmetric information and moral hazard. Geelen (2017) extends Diamond (1993) and studies dynamics of debt maturity as a signal of default probability. Borrowers do not have unobservable income in his model, however, which precludes the set of signaling strategies that we consider. Bond and Zhong (2016) study a dynamic model of equity issuance and repurchase under asymmetric information. Guerrieri and Shimer (2014) study the frequency of trade as a signal of asset quality in an exchange economy. Guerrieri and Shimer (2018) and Chang (2018) extend this analysis to the case of multi-dimensional private information and show the limitations of signaling in dynamic settings.

The remainder of the paper is organized as follows. Section 2 presents a dynamic model of credit markets with asymmetric information and uncertainty in collateral value. Section 3 discusses some important properties of the equilibrium that we will use to prove our main results. Section 4 presents our main results on how uncertainty in collateral value determines information revelation through deleveraging or default. Section 5 discusses the robustness of our results to incorporating long-term debt, hidden savings, and alternative distributional assumptions about uncertainty in collateral value. Section 6 concludes.

2. A dynamic model of credit markets

We present a dynamic model of credit markets with asymmetric information and uncertainty in collateral value. The dynamic model allows for a rich set of signaling strategies through the path of debt and repayment, which could resolve asymmetric information. Uncertainty in collateral value affects the precision of the signaling strategies and whether separation entails default in equilibrium.

2.1. Pledgeable and non-pledgeable assets

There are two types of assets, pledgeable and non-pledgeable, which generate stochastic income streams. The pledgeable asset can be used as collateral in credit transactions, whereas the non-pledgeable asset cannot be used as collateral. A pledgeable asset can be thought of as a tangible and observable asset such as financial assets, land, physical structure, or equipment. A non-pledgeable asset can be thought of as an intangible and unobservable asset such as human capital, innovative ability, managerial skill, or organizational structure.

The pledgeable asset generates observable income X_t in each period t, which follows a martingale (i.e., $\mathbb{E}_t[X_{t+s}] = X_t$). Let $R > 1$ denote the gross riskless interest rate, which satisfies $R^2(R-1) < 1$. The value of the pledgeable asset is the present value of its income:

$$V_t = \frac{\mathbb{E}_t[X_{t+1} + V_{t+1}]}{R} = \sum_{s=1}^{\infty} \frac{\mathbb{E}_t[X_{t+s}]}{R^s} = \frac{X_t}{R-1}.$$

The non-pledgeable asset generates unobservable income Y_t in each period t, which also follows a martingale (i.e., $\mathbb{E}_t[Y_{t+s}] = Y_t$). All income is perishable and must be immediately consumed or used to repay debt. In Section 5, we show that are our main results are robust to an extension in which unobservable income can be saved.

2.2. Borrowers with private information

There are two types of risk-neutral borrowers, "good" and "bad". Both types of borrowers are endowed with a unit of the pledgeable asset. Only good borrowers are also endowed with a unit of the non-pledgeable asset. Whether a given borrower is good or bad is private information to the borrower, which arises from the fact that income from the non-pledgeable asset is unobservable. By definition, the non-pledgeable asset cannot be used as collateral, which rules out a separating equilibrium based on good borrowers signaling through higher collateral value (Bester, 1985; Besanko and Thakor, 1987; Martin, 2009).

Each borrower is in the credit market for at most three periods, which we denote as $t \in \{1, 2, 3\}$. Let F_0 be the face value of existing debt that matures in period 1. Let $\pi_0 \in (0, 1)$ be the lender's perceived probability that the borrower is good, which we refer to as reputation, at the beginning of period 1.[2] More generally, the borrower enters each

[2] We assume that π_0 is exogenous for simplicity, but it can be endogenized as the mass of borrowers who choose to invest in the non-pledgeable asset (Atkeson et al., 2015).

period t with maturing debt F_{t-1} and reputation π_{t-1}.

The borrower receives income $X_t + Y_t$ if good or X_t if bad. The borrower can either repay the face value of maturing debt or default. Let $D_{i,t}$ denote an endogenous default boundary such that it is feasible and optimal for a borrower of type $i \in \{g, b\}$ (i.e., good or bad) to repay if $F_{t-1} \leq D_{i,t}$ and to default otherwise. The borrower can repay using his income as well as the proceeds from rolling over one-period debt with face value $F_{i,t}$ at the equilibrium price P_t.[3] Conditional on repayment, the lender updates reputation to π_t. Note that not only repayment, but also the face value of new debt, could serve as signals for the updating of reputation. Conditional on default, the lender takes possession of the collateral (i.e., the pledgeable asset and its income in period t) and updates reputation to $\widehat{\pi}_t$. For simplicity, there are no deadweight costs of default.

The borrower essentially faces the same problem in period 3, which is the terminal period. The only difference is that instead of rolling over debt, he can sell the pledgeable asset at market value to repay. Therefore, only repayment can serve as a signal for the updating of reputation in period 3.

Following Ross (1977), we assume that the borrower maximizes net worth, as perceived by the lender (or outside investors with knowledge of only reputation). The value of the non-pledgeable asset in period 3 is $W_3 = \frac{\pi_3 Y_3}{R-1}$ in case of repayment and $\widehat{W}_3 = \frac{\widehat{\pi}_3 Y_3}{R-1}$ in case of default. That is, the value of the non-pledgeable asset is equal to the probability that the borrower has the non-pledgeable asset times its value conditional on ownership. Let $\mathbb{1}_g(i)$ be an indicator function that is equal to one if the borrower is good and zero otherwise. The net worth for a type i borrower in period 3 is

$$J_{i,3} = \begin{cases} X_3 + \mathbb{1}_g(i) Y_3 + V_3 + W_3 - F_2 & \text{if } D_{i,3} \geq F_2 \\ \mathbb{1}_g(i) Y_3 + \widehat{W}_3 & \text{if } D_{i,3} < F_2 \end{cases}. \tag{1}$$

In case of repayment, net worth is income plus the terminal value of both types of assets minus the face value of maturing debt. In case of default, net worth is the terminal value of the non-pledgeable asset and (for a good borrower) its income.

In our model, the borrower's objective is to maximize reputation in the terminal period. In reality, borrowers care about reputation only indirectly through the fact that borrowing capacity (and the ability to make a durable-good purchase or capital investment) increases in reputation. Thus, our model describes a new borrower without a credit history, who wants to establish a credit score in the first two periods in order to borrow and make a durable-good

[3] In Section 5, we show that the assumption of one-period debt is not restrictive. The reason is that good borrowers (at least weakly) prefer one-period debt to two-period debt in period 1.

purchase or capital investment in the terminal period. The assumption that the borrower cares directly about reputation is a reduced-form representation of the fact that borrowing capacity in the terminal period would naturally be increasing in reputation.

We define the borrower's net worth in period $t \in \{1, 2\}$ recursively as

$$J_{i,t} = \begin{cases} X_t + \mathbb{1}_g(i) Y_t + P_t F_t - F_{t-1} + \frac{\mathbb{E}_t[J_{i,t+1}]}{R} & \text{if } D_{i,t} \geq F_{t-1} \\ \mathbb{1}_g(i) Y_t + \widehat{W}_{i,t} & \text{if } D_{i,t} < F_{t-1} \end{cases}, \tag{2}$$

where

$$\widehat{W}_{i,t} = \sum_{s=1}^{3-t} \frac{\mathbb{1}_g(i) \mathbb{E}_t[Y_{t+s}]}{R^s} + \frac{\mathbb{E}_t[\widehat{\pi}_3 Y_3]}{R^{3-t}(R-1)} = \frac{((R^{3-t}-1)\mathbb{1}_g(i) + \widehat{\pi}_t) Y_t}{R^{3-t}(R-1)}. \tag{3}$$

In case of repayment, net worth is income plus the net proceeds from rolling over debt plus the borrower's continuation value. In case of default, net worth is the terminal value of the non-pledgeable asset and (for a good borrower) its income through period 3. Note that once the borrower defaults in period 2, there is no further updating of reputation so that $\widehat{\pi}_3 = \widehat{\pi}_2$.

2.3. Lender

The representative lender is risk neutral and earns an expected gross return R on each debt. The lender does not know whether a given borrower is good or bad. However, the lender updates reputation based on repayment versus default and the new loan amount conditional on repayment.

We assume throughout the paper that $F_0 \leq X_1$ to rule out a trivial outcome of immediate default in period 1. As we discussed above, there is no refinancing in period 3. The lender updates reputation based on the new loan amount in period 1, the new loan amount and the default decision in period 2, and the default decision in period 3. Three periods is the minimum number necessary to capture deleveraging strategies in which the borrower increases the loan amount in period 1 and subsequently decreases it in period 2.

Conditional on repayment in period $t \in \{1, 2\}$, the lender updates reputation through Bayes' rule:

$$\pi_t = \left(1 + \frac{(1 - \pi_{t-1}) \Pr(\{D_{b,t} \geq F_{t-1}\} \cap \{F_{b,t} = F_t\})}{\pi_{t-1} \Pr(\{D_{g,t} \geq F_{t-1}\} \cap \{F_{g,t} = F_t\})}\right)^{-1}. \tag{4}$$

This formula accounts for the fact that not only repayment, but also the face value of new debt F_t, potentially reveals borrower type. The fact that reputation depends on repayment history and debt outstanding is consistent with the determinants of FICO score.[4] Conditional on default in period t, reputation is

$$\widehat{\pi}_t = \left(1 + \frac{(1-\pi_{t-1})\Pr(D_{b,t} < F_{t-1})}{\pi_{t-1}\Pr(D_{g,t} < F_{t-1})}\right)^{-1}. \tag{5}$$

Because there is no refinancing in period 3, the lender updates reputation based on repayment alone. Conditional on repayment in period 3, the terminal reputation is

$$\pi_3 = \left(1 + \frac{(1-\pi_2)\Pr(D_{b,3} \geq F_2)}{\pi_2\Pr(D_{g,3} \geq F_2)}\right)^{-1}. \tag{6}$$

Since a borrower's type is time invariant, his actions are either fully revealing or not at all, conditional on the realized collateral value. Therefore, reputation either remains the same or updates fully to one or zero for good and bad borrowers, respectively. Shocks to the borrower type could prevent the full updating of reputation, but such an extension would unnecessarily complicate the analysis and the exposition.

To complete the model, we must make auxiliary assumptions about beliefs off the equilibrium path. We assume that a borrower who repays is believed to be good if no borrower is expected to repay in equilibrium. Similarly, a borrower who defaults is believed to be bad if no borrower is expected to default in equilibrium. These restrictions on off-equilibrium beliefs arise naturally from the intuitive criterion (Cho and Kreps, 1987). We state our assumptions more formally as follows.

Assumption 1. *The lender's off-equilibrium beliefs are given by*

$$\pi_t = 1 \text{ if } \Pr(\{D_{g,t} \geq F_{t-1}\} \cap \{F_{g,t} = F_t\}) = \Pr(\{D_{b,t} \geq F_{t-1}\} \cap \{F_{b,t} = F_t\}) = 0$$

for all F_t *and*

$$\widehat{\pi}_t = 0 \text{ if } \Pr(D_{g,t} < F_{t-1}) = \Pr(D_{b,t} < F_{t-1}) = 0.$$

In period $t \in \{1, 2\}$, the lender's break-even condition determines the equilibrium price of debt P_t, given face value F_t and reputation π_t:

$$P_t F_t = \pi_t C_{g,t} + (1-\pi_t) C_{b,t}, \tag{7}$$

[4] According to FICO (2015), repayment history determines 35%, and debt outstanding determines 30% of the FICO score.

where

$$C_{i,t} = \frac{\Pr(D_{i,t+1} \geq F_t)F_t + \Pr(D_{i,t+1} < F_t)\mathbb{E}_t[X_{t+1} + V_{t+1} | D_{i,t+1} < F_t]}{R}. \tag{8}$$

That is, the lender breaks even if the value of debt is equal to the expected repayment discounted at R. The expected repayment is equal to the probability that the borrower is good multiplied by good borrowers' expected repayment plus the probability that the borrower is bad multiplied by bad borrowers' expected repayment.[5]

2.4. Model summary

The borrower can signal through the new loan amount in periods 1 and 2 and through repayment in periods 2 and 3. We summarize the model as follows.

Period 1. The borrower starts with face value of debt $F_0 \leq X_1$ and reputation π_0.

(a) The borrower receives income $X_1 + Y_1$ if good and X_1 if bad.

(b) The borrower takes out new debt with face value F_1 at the equilibrium price P_1. The lender updates reputation to π_1.

Period 2. The borrower enters with face value of debt F_1 and reputation π_1.

(a) The borrower receives income $X_2 + Y_2$ if good and X_2 if bad.

(b) The borrower decides whether or not to repay F_1.

- In case of repayment, the borrower takes out new debt with face value F_2 at the equilibrium price P_2. The lender updates reputation to π_2.
- In case of default, the lender takes possession of the pledgeable asset (i.e., $X_2 + V_2$) and updates reputation to $\widehat{\pi}_2$. The borrower's terminal value is the non-pledgeable asset and its income (i.e., $\mathbb{1}_g(i)Y_2 + \widehat{W}_{i,2}$).

Period 3. In case of repayment in period 2, the borrower enters with face value of debt F_2 and reputation π_2.

(a) The borrower receives income $X_3 + Y_3$ if good and X_3 if bad.

(b) The borrower decides whether or not to repay F_2.

- In case of repayment, the lender updates reputation to π_3.

[5] In a separating equilibrium, equation (7) holds for a good borrower with $\pi_t = 1$ and separately for a bad borrower with $\pi_t = 0$.

- In case of default, the lender takes possession of the pledgeable asset (i.e., $X_3 + V_3$) and updates reputation to $\widehat{\pi}_3$. The borrower's terminal value is the non-pledgeable asset and its income (i.e., $\mathbb{1}_g(i)Y_3 + \widehat{W}_3$).

3. Properties of the equilibrium

We first characterize some important properties of the equilibrium that do not depend on additional parametric assumptions. We will use the lemmas in this section to prove our main results in Section 4.

3.1. Borrowers' maximization problem

In period 3, a type i borrower can repay if $X_3 + \mathbb{1}_g(i)Y_3 + V_3 \geq F_2$. That is, he can repay if his income plus the value of the pledgeable asset exceeds the face value of maturing debt. Moreover, equation (1) implies that it is optimal for the borrower to repay if $X_3 + V_3 + W_3 - \widehat{W}_3 \geq F_2$. Combining feasibility and optimality, the default boundary in period 3 is

$$D_{i,3} = X_3 + V_3 + \min\left\{\mathbb{1}_g(i)Y_3, W_3 - \widehat{W}_3\right\}. \tag{9}$$

In period $t \in \{1, 2\}$, a type i borrower can repay if

$$X_t + \mathbb{1}_g(i)Y_t + \max_{F_t} \; P_t F_t \geq F_{t-1}. \tag{10}$$

That is, the borrower can repay if his income plus the maximum amount that he can borrow exceeds the face value of maturing debt. The following lemma establishes the condition under which repayment is optimal, which implies the default boundary when combined with feasibility.

Lemma 1. *In period $t \in \{1, 2\}$, the borrower's net worth is*

$$J_{i,t} = \begin{cases} X_t + \mathbb{1}_g(i)Y_t + V_t + W_{i,t} - F_{t-1} & \text{if } D_{i,t} \geq F_{t-1} \\ \mathbb{1}_g(i)Y_t + \widehat{W}_{i,t} & \text{if } D_{i,t} < F_{t-1} \end{cases}, \tag{11}$$

where the value of the non-pledgeable asset conditional on repayment is

$$\begin{aligned} W_{i,t} = &-(\mathbb{1}_g(i) - \pi_t)(C_{g,t} - C_{b,t}) + \frac{\mathbb{1}_g(i)Y_t + \Pr(D_{i,t+1} \geq F_t)\mathbb{E}_t[W_{i,t+1}|D_{i,t+1} \geq F_t]}{R} \\ &+ \frac{\Pr(D_{i,t+1} < F_t)\mathbb{E}_t\left[\widehat{W}_{i,t+1}|D_{i,t+1} < F_t\right]}{R}. \end{aligned} \tag{12}$$

The default boundary is

$$D_{i,t} = X_t + V_t + \min\left\{\mathbb{1}_g(i)Y_t + \max_{F_t}\ P_tF_t - V_t, W_{i,t} - \widehat{W}_{i,t}\right\}, \tag{13}$$

where

$$\begin{aligned} W_{i,t} - \widehat{W}_{i,t} =& -(\mathbb{1}_g(i) - \pi_t)(C_{g,t} - C_{b,t}) + \frac{\mathbb{E}_t[\widehat{\pi}_{t+1}Y_{t+1}] - \widehat{\pi}_tY_t}{R^{3-t}(R-1)} \\ &+ \frac{\Pr(D_{i,t+1} \geq F_t)\mathbb{E}_t\left[W_{i,t+1} - \widehat{W}_{i,t+1}|D_{i,t+1} \geq F_t\right]}{R}. \end{aligned} \tag{14}$$

Proof. See Appendix A.

In case of repayment in period $t \in \{1, 2\}$, the borrower chooses F_t to maximize his net worth (11). However, all components of net worth are predetermined, except for the value of the non-pledgeable asset. Therefore, the borrower's maximization problem simplifies to

$$\max_{F_t}\ W_{i,t} \text{ subject to } X_t + \mathbb{1}_g(i)Y_t + P_tF_t \geq F_{t-1}.$$

As we discussed in Section 2, the assumption that the borrower cares directly about reputation is a reduced-form representation of the fact that borrowing capacity in the terminal period would naturally be increasing in reputation.

3.2. Benchmark with perfect information

Private information about whether or not the borrower has the non-pledgeable asset is the only friction in our model. The benchmark with perfect information is a special case of our model where reputation is $\pi_{t-1} \in \{0, 1\}$. In this special case, we recover the standard result that debt (or leverage) is indeterminate.

Lemma 2 (Modigliani and Miller (1958)). *If $F_{t-1} \leq X_t$ and $\pi_{t-1} \in \{0, 1\}$, borrowers are indifferent between any loan amount such that $P_tF_t \leq V_t$. The equilibrium interest rate is $P_1^{-1} = R$.*

Proof. See Appendix A.

3.3. Signaling through deleveraging or default

In the presence of asymmetric information, good borrowers have an incentive to signal through repayment and the new loan amount conditional on repayment. Bad borrowers have an incentive to mimic good borrowers' actions in order to delay (or if possible avoid) information revelation.[6] The incentive of bad borrowers to mimic good borrowers comes from two sources. First, bad borrowers pay interest that is lower than under perfect information, given that they are more likely to default in the future. This source is captured by the first term, $\pi_t(C_{g,t} - C_{b,t}) \geq 0$, in equation (12). Second, there is a higher terminal value of the non-pledgeable asset if bad borrowers can altogether avoid information revelation. This source is captured by the last two terms in equation (12).

The following lemma formally establishes that bad borrowers are more likely to default than good borrowers.

Lemma 3. *The default boundary for good borrowers is higher than that for bad borrowers:*

$$X_t + V_t \leq D_{b,t} \leq D_{g,t} \leq X_t + Y_t + V_t. \tag{15}$$

In the event of full separation in period t, the first and third inequalities are equalities, and the second inequality is strict.

Proof. See Appendix A.

Based on Lemma 3, we define four regions for the face value of maturing debt relative to the realized collateral value and the default boundaries, as illustrated in Figure 1. Lemmas 4 to 7 that follow correspond to the four regions. For each region, we state the optimal strategy of good borrowers and whether there is separation in equilibrium.

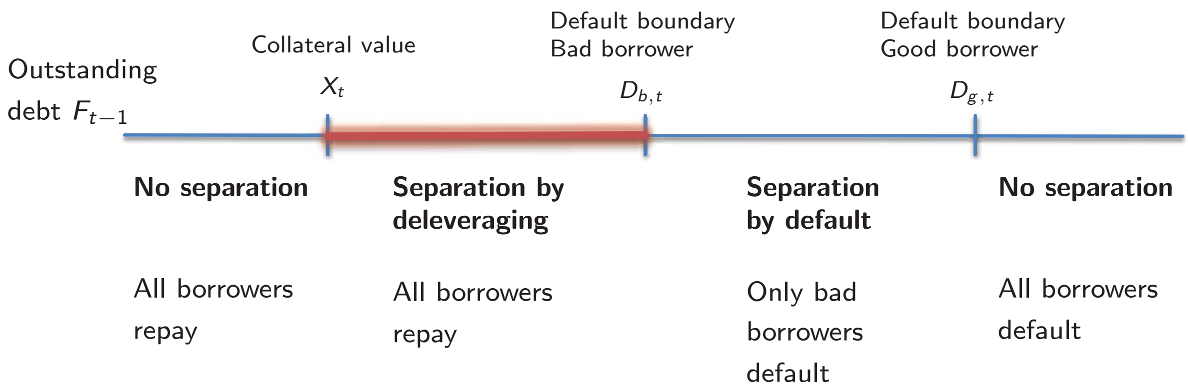

Figure 1: Signaling regions

[6]To simplify the statement of our results, we follow the convention that bad borrowers mimic good borrowers in the knife-edge case of indifference.

Lemma 4 (No separation). *Suppose that* $F_{t-1} \leq X_t$ *in period* $t \in \{1,2\}$ *or* $F_2 \leq D_{b,3}$ *in period 3. All borrowers repay, so borrower type is not revealed.*

Proof. If $F_{t-1} \leq X_t$, both types of borrowers can repay without rolling over debt.

Lemma 5 (Separation by deleveraging). *Suppose that* $F_{t-1} \in (X_t, D_{b,t}]$ *in period* $t \in \{1,2\}$. *Good borrowers repay by rolling over* $F_t \in [R\max\{0, F_{t-1} - X_t - Y_t\}, R(F_{t-1} - X_t))$. *Bad borrowers repay by rolling over* $F_t \in [R(F_{t-1} - X_t), RV_t]$. *Thus, borrower type is fully revealed.*

Proof. In this region, it is optimal for all borrowers to repay. Good borrowers can repay by rolling over at least $P_tF_t \geq \max\{0, F_{t-1} - X_t - Y_t\}$. Bad borrowers can repay by rolling over at least $P_tF_t \geq F_{t-1} - X_t$. Therefore, good borrowers separate by rolling over at most $P_tF_t < F_{t-1} - X_t$. Lemma 2 implies that the equilibrium interest rate is $P_t^{-1} = R$.

Lemma 6 (Separation by default). *Suppose that* $F_{t-1} \in (D_{b,t}, D_{g,t}]$ *in any period* t. *Only bad borrowers default, so borrower type is fully revealed. In period* $t \in \{1,2\}$, *good borrowers repay by rolling over* $F_t \in [R\max\{0, F_{t-1} - X_t - Y_t\}, RV_t]$.

Proof. In this region, bad borrowers are forced to default. In period $t \in \{1,2\}$, good borrowers can repay by rolling over at least $P_tF_t \geq \max\{0, F_{t-1} - X_t - Y_t\}$. Lemma 2 implies that the equilibrium interest rate is $P_t^{-1} = R$.

Lemma 7 (No separation). *Suppose that* $F_{t-1} > D_{g,t}$ *in any period* t. *All borrowers default, so borrower type is not revealed.*

Lemmas 5 and 6 establish that there are two ways in which the borrower type is fully revealed in period 2. First consider a low amount of maturing debt $F_1 \in (X_2, D_{b,2}]$, shown as the red shaded region in Figure 1. In this region, good borrowers can roll over less debt by repaying with unobservable income. Bad borrowers, who do not have unobservable income, must roll over more debt in order to repay. Therefore, rolling over a lower amount than the maturing debt minus observable income signals that the borrower is good because only borrowers with unobservable income can follow such a strategy.

Next consider a higher amount of maturing debt $F_1 > D_{b,2}$ in Figure 1. In this region, good borrowers can repay with unobservable income, while bad borrowers are forced to default. Therefore, repayment signals that the borrower is good.

Lemmas 5 and 6 describe the optimal strategy conditional on the face value of maturing debt and the realized collateral value in period 2. In Section 4, we will work backwards to solve for the optimal choice of debt in period 1. Before we go into the formal analysis, we discuss the intuition for the tradeoff that good borrowers face in choosing the optimal loan amount in period 1. Good borrowers have a choice of borrowing a lower amount to prepare for signaling by *deleveraging* or a higher amount to prepare for signaling by *forcing default* in period 2. If possible, good borrowers prefer deleveraging because equation (12) implies that the value of the non-pledgeable asset is $W_{g,1} = \frac{Y_1}{R-1}$ under deleveraging and

$$W_{g,1} = -(1-\pi_t)(C_{g,1} - C_{b,1}) + \frac{Y_1}{R-1} \tag{16}$$

under forcing default. Forcing default is costly because good borrowers must pay higher interest in period 1 due to adverse selection, captured by the first term in equation (16).

Recall that asymmetric information is the only friction in our model. We do not have deadweight costs of default, which is the key friction that allows good borrowers to signal through higher debt in the static model (Ross, 1977). In a dynamic setting, deleveraging is a superior way of signaling, which is ruled out by construction in the static model.

In the absence of uncertainty in collateral value, X_2 is known when borrowers choose F_1 in period 1. In that case, good borrowers can choose $F_1 \in (X_2, D_{b,2}]$ to always separate by deleveraging in period 2. When there is uncertainty in collateral value, however, good borrowers may not be able to ensure that $F_1 \in (X_2, D_{b,2}]$ in all states. A sufficiently high loan amount that ensures full separation through deleveraging when collateral value rises could cause bad borrowers to default when collateral value falls. Thus, good borrowers face a tradeoff between the benefit of separation and a higher interest cost that arises from the possibility of default. We analyze how this tradeoff depends on uncertainty in collateral value in the next section.

4. Characterization of the equilibrium

The tradeoff between the benefit of separation and a higher interest cost depends on parametric assumptions about uncertainty in collateral value. We make such assumptions and fully characterize how the equilibrium depends on uncertainty in collateral value.

4.1. Parametric assumptions

Our first assumption is that unobservable income is a constant proportion of observable income. Moreover, unobservable income is less than the collateral value so that signaling plays a non-trivial role in the model.

Assumption 2. *Unobservable income is a constant proportion y of observable income. Moreover, unobservable income is less than the collateral value:*

$$y = \frac{Y_t}{X_t} < \frac{X_t + V_t}{X_t} = \frac{R}{R-1}.$$

Our second assumption is that observable income follows a binomial version of the geometric random walk.

Assumption 3. *The growth rate of observable income is distributed as*

$$x_t = \frac{X_t}{X_{t-1}} = \begin{cases} \overline{x} & \text{with probability } 1-p \\ \underline{x} & \text{with probability } p \end{cases},$$

where $\overline{x} \geq \underline{x}$ and $(1-p)\overline{x} + p\underline{x} = 1$.

There are only two free parameters between $\overline{x}$, $\underline{x}$, and p because of the normalization that the mean growth rate of observable income is one. In characterizing the equilibrium, it is convenient to divide the parameter space into regions along $\frac{\overline{x}}{\underline{x}}$ and $(1-p)\overline{x}$. $\frac{\overline{x}}{\underline{x}}$ captures uncertainty in collateral value, and $(1-p)\overline{x}$ captures asymmetry in the distribution of collateral value. In this section, we present the results for the case $(1-p)\overline{x} \geq 0.5$, where collateral value is unlikely to fall and thus has negative skew. We present the results for the complementary case $(1-p)\overline{x} < 0.5$ in Appendix C.

As we discuss in Section 5, our main results are robust to an alternative distributional assumption that the growth rate is continuous and bounded between $\underline{x}$ and $\overline{x}$. Thus, the binomial assumption is not important per se, but the fact that the distribution has bounded support is important for the set of signaling strategies that we consider. To be precise about the interpretation of $\frac{\overline{x}}{\underline{x}}$, the relevant notion of uncertainty is the maximum range of possible values for the growth rate.

4.2. Low uncertainty in collateral value

As we discussed in Section 3, deleveraging is a costless and optimal strategy for separation in the absence of uncertainty in collateral value. By continuity, the equilibrium should remain full separation through deleveraging as long as uncertainty in collateral value is low.

Proposition 1 (Costless full separation). *Suppose that uncertainty in collateral value is low. That is, $\frac{\overline{x}}{\underline{x}} < \frac{R}{R-1}$. In period 1, all borrowers borrow $F_1 \in (X_1\overline{x}, RV_1\underline{x}]$ at the interest rate $P_1^{-1} = R$. In period 2, borrower type is fully revealed. Good borrowers repay by rolling* *over* $F_2 \in [R\max\{0, F_1 - X_2 - Y_2\}, R(F_1 - X_2))$*. Bad borrowers repay by rolling over* $F_2 \in [R(F_1 - X_2), RV_2]$*.*

Proof. Lemma 9 in Appendix B implies the equilibrium in period 1. Lemma 5 implies the equilibrium in period 2.

In period 2, good borrowers can fully separate by deleveraging if the face value of maturing debt F_1 is greater than observable income X_2. In that case, good borrowers repay from their observable income, part of their unobservable income, and the remainder from rolling over debt. Bad borrowers, who do not have unobservable income, must roll over more debt in order to repay. This implies that in period 1, good borrowers must borrow at least $F_1 > X_1\overline{x} \geq X_2$ to ensure separation in period 2, even if the realized collateral value is high. Thus, $X_1\overline{x}$ is the lower bound on F_1 for costless full separation. This is illustrated as the red shaded region in the upper line in Figure 2.

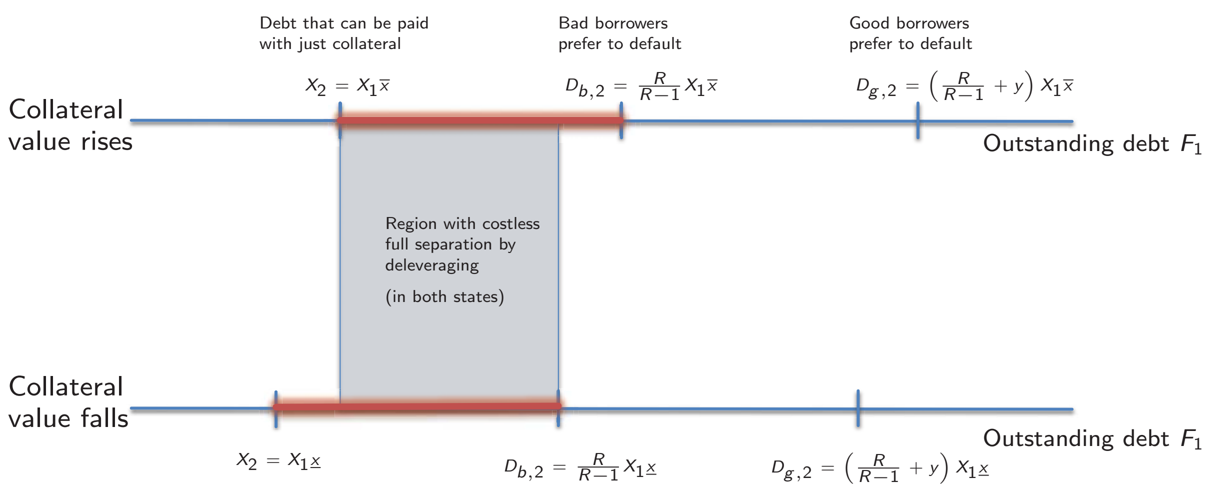

Figure 2: Signaling regions for low uncertainty in collateral value

Bad borrowers can repay in period 2 as long as the face value of maturing debt F_1 is less than the collateral value $X_2 + V_2$. Therefore, good borrowers do not want to borrow more than $F_1 \leq RV_1\underline{x} \leq X_2 + V_2$ to ensure that bad borrowers do not default even if the collateral value falls. As we discussed in Section 3, good borrowers prefer not to force default because they bear a higher interest cost due to adverse selection in period 1. Thus, $RV_1\underline{x}$ is the upper bound on F_1 for costless full separation. This is illustrated as the red shaded region in the lower line in Figure 2.

The range of equilibrium debt in period 1, which is the gray shaded region of overlap in Figure 2, shrinks as uncertainty in collateral value rises. This is because the lower bound on debt must rise so that bad borrowers cannot repay with observable income alone, even if the collateral value rises in period 2. At the same time, the upper bound on debt must fall to prevent bad borrowers from defaulting and surrendering collateral, even if the collateral value falls in period 2. The range of equilibrium debt shrinks until it becomes a point at which $F_1 = X_1\overline{x} = RV_1\underline{x}$, which is equivalent to $\frac{\overline{x}}{\underline{x}} = \frac{R}{R-1}$. At this point, good borrowers face a tradeoff. On the one hand, a higher loan amount in period 1 would force bad borrowers to default if collateral value falls in period 2. On the other hand, a lower loan amount in period 1 would allow even bad borrowers to repay without rolling over debt if collateral value rises in period 2.

4.3. *Higher uncertainty in collateral value*

When uncertainty in collateral value is higher such that $\frac{\overline{x}}{\underline{x}} \geq \frac{R}{R-1}$, good borrowers cannot costlessly separate by deleveraging in all states. A sufficiently high loan amount that allows good borrowers to separate by deleveraging when collateral value rises causes bad borrowers to default when collateral value falls. A lower loan amount allows good borrowers to separate by deleveraging when collateral value falls, but it does not allow good borrowers to separate when collateral value rises.

When uncertainty in collateral value is intermediate, the equilibrium turns out to be full separation through deleveraging in good states and default in bad states. The optimal loan amount in period 1 is $F_1 > X_1\overline{x}$, illustrated as the red shaded region in the upper line in Figure 3. If the collateral value rises in period 2, good borrowers fully separate by rolling over less debt than bad borrowers. If the collateral value falls in period 2, good borrowers fully separate by repaying, while bad borrowers default. Thus, full separation is costly in the sense that bad borrowers default in bad states.

Proposition 2 (Costly full separation). *Suppose that uncertainty in collateral value is intermediate. That is,* $\frac{R}{R-1} \leq \frac{\overline{x}}{\underline{x}} < \min\{d_{g,2}, z\}$, *where*

$$d_{g,2} = \frac{R}{R-1} + y,$$
$$z = \frac{R}{R-1} + \frac{(1-p)\overline{x}y}{R(R-1)}.$$

In period 1, all borrowers borrow $F_1 > X_1\overline{x}$ *at an interest rate* $P_1^{-1} > R$ *that satisfies*

$$P_1F_1 = \frac{(1-(1-\pi_0)p)F_1}{R} + \frac{(1-\pi_0)pX_1\underline{x}}{R-1}.$$

If the collateral value rises in period 2 (i.e., $x_2 = \overline{x}$*), borrower type is fully revealed. Good*

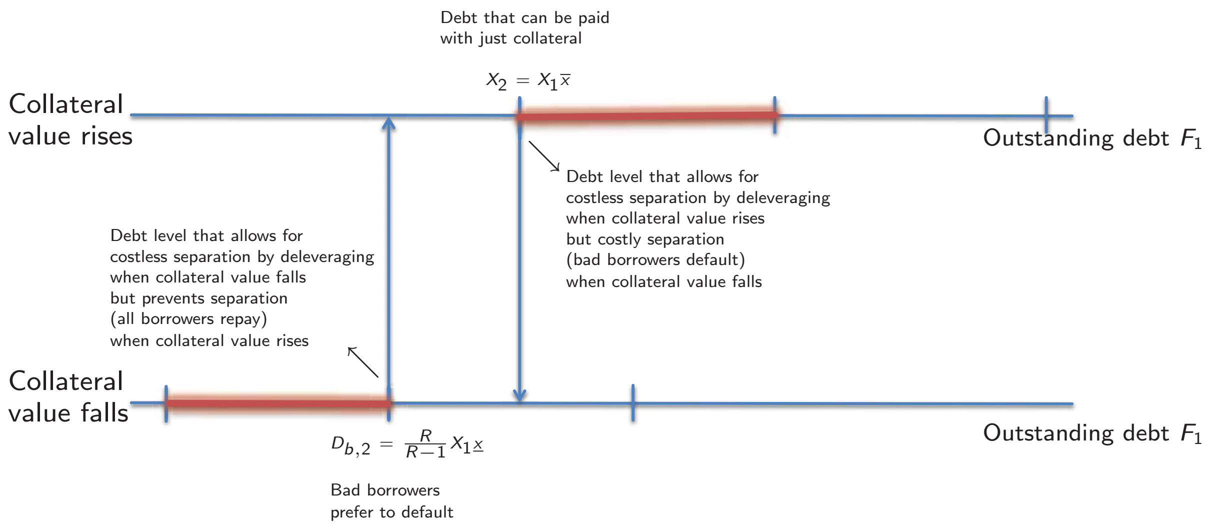

Figure 3: Signaling regions for intermediate uncertainty in collateral value

borrowers repay by rolling over $F_2 = 0$, and bad borrowers repay by rolling over $F_2 \in (0, RV_2]$. If the collateral value falls in period 2 (i.e., $x_2 = \underline{x}$), only bad borrowers default, so borrower type is fully revealed. Good borrowers repay by rolling over $F_2 \in [R\max\{0, F_1 - X_2 - Y_2\}, RV_2]$.

Proof. Lemma 9 in Appendix B implies the equilibrium in period 1. Lemma 5 implies the equilibrium if the collateral value rises in period 2. Lemma 6 implies the equilibrium if the collateral value falls in period 2.

When uncertainty in collateral value is high, the equilibrium turns out to be partial separation. The optimal loan amount in period 1 is $F_1 \in (X_1\underline{x}, RV_1\underline{x}]$, which is illustrated as the red shaded region in the lower line in Figure 4. If the collateral value rises in period 2, all borrowers roll over the same amount, so borrower type is not revealed. If the collateral value falls in period 2, good borrowers fully separate by rolling over less debt than bad borrowers.

Proposition 3 (Partial separation). *Suppose that uncertainty in collateral value is high. That is, $\frac{\overline{x}}{\underline{x}} \geq \min\{d_{g,2}, z\}$. In period 1, all borrowers borrow $F_1 \in (X_1\underline{x}, RV_1\underline{x}]$ at the interest rate $P_1^{-1} = R$.*

If the collateral value rises in period 2 (i.e., $x_2 = \overline{x}$), borrower type is not revealed. All borrowers repay by rolling over $F_2 > RV_2\overline{x}$ at an interest rate $P_2^{-1} > R$ that satisfies

$$P_2F_2 = V_2 + \pi_0(1-p)\left(\frac{F_2}{R} - V_2\overline{x}\right).$$

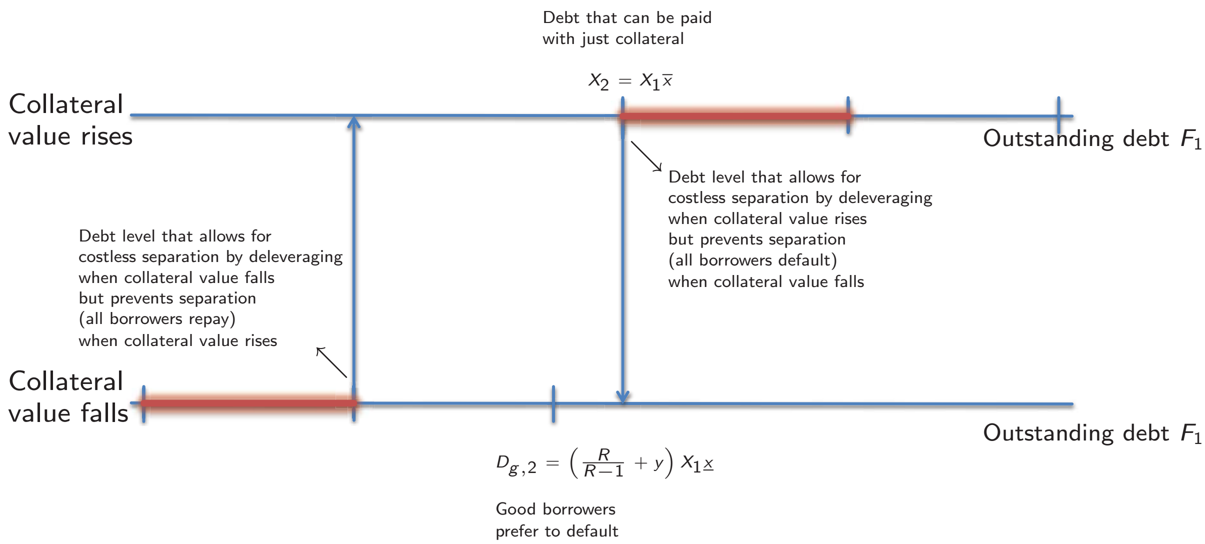

Figure 4: Signaling regions for high uncertainty in collateral value

Subsequently, if the collateral value rises in period 3 (i.e., $x_3 = \overline{x}$), only bad borrowers default, so borrower type is fully revealed. If the collateral value falls instead (i.e., $x_3 = \underline{x}$), all borrowers default, so borrower type is not revealed.

If the collateral value falls in period 2 (i.e., $x_2 = \underline{x}$), borrower type is fully revealed. Good borrowers repay by rolling over $F_2 \in [R\max\{0, F_1 - X_2 - Y_2\}, R(F_1 - X_2))$. Bad borrowers repay by rolling over $F_2 \in [R(F_1 - X_2), RV_2]$.

Proof. Lemma 9 in Appendix B implies the equilibrium in period 1. Lemma 8 implies the equilibrium if the collateral value rises in period 2. Lemma 6 implies the equilibrium if the collateral value subsequently rises in period 3, and Lemma 7 implies the equilibrium if the collateral value falls instead. Lemma 5 implies the equilibrium if the collateral value falls in period 2.

We now discuss the intuition for Propositions 2 and 3 by sketching out the essential elements of the formal proofs in Appendix B. As we discussed above, full separation through deleveraging alone is no longer possible when $\frac{\overline{x}}{\underline{x}} \geq \frac{R}{R-1}$. Good borrowers then face a tradeoff between *costly full separation* and *partial separation.*

Costly full separation occurs if good borrowers borrow $F_1 > X_1\overline{x}$ as in Proposition 2. The value of the non-pledgeable asset under this strategy is

$$W_{g,1} = -(1-\pi_0)p\left(\frac{F_1}{R} - \frac{X_1\underline{x}}{R-1}\right) + \frac{Y_1}{R-1}.$$

The first term is the interest cost in period 1, which arises from bad borrowers defaulting if the collateral value falls in period 2 with probability p. The second term is the benefit of full separation. To minimize the interest cost, good borrowers optimally choose F_1 that is arbitrarily close to $X_1\overline{x}$ so that

$$W_{g,1} = -\frac{(1-\pi_0)X_1 p\underline{x}}{R}\left(\frac{\overline{x}}{\underline{x}} - \frac{R}{R-1}\right) + \frac{Y_1}{R-1}. \tag{17}$$

Fixing $(1-p)\overline{x}$ (or equivalently fixing $p\underline{x} = 1-(1-p)\overline{x}$), the interest cost increases with uncertainty $\frac{\overline{x}}{\underline{x}}$.

Partial separation occurs if good borrowers borrow $F_1 \leq RV_1\underline{x}$ as in Proposition 3. Under this strategy, good borrowers avoid the higher interest cost at the sacrifice of not being able to fully separate in all future states. The value of the non-pledgeable asset under this strategy is

$$W_{g,1} = \frac{Y_1}{R-1} - \frac{(1-\pi_0)(1-p)\overline{x}p\underline{x}Y_1}{R^2(R-1)}. \tag{18}$$

The first term is the benefit of full separation. The second term accounts for the fact that good borrowers cannot separate if the collateral value rises in period 2 with probability $1-p$, then falls in period 3 with probability p. Fixing $(1-p)\overline{x}$, equation (18) is constant in uncertainty $\frac{\overline{x}}{\underline{x}}$.

Comparing equations (17) and (18), good borrowers prefer costly full separation to partial separation when

$$-\frac{(1-\pi_0)X_1 p\underline{x}}{R}\left(\frac{\overline{x}}{\underline{x}} - \frac{R}{R-1}\right) > -\frac{(1-\pi_0)(1-p)\overline{x}p\underline{x}Y_1}{R^2(R-1)},$$

which is equivalent to

$$\frac{\overline{x}}{\underline{x}} < \frac{R}{R-1} + \frac{(1-p)\overline{x}y}{R(R-1)} = z.$$

That is, good borrowers prefer costly full separation when uncertainty in collateral value is sufficiently low. Because the interest cost increases with uncertainty, the preferred strategy switches to partial separation at the point z. Even if costly full separation is preferred, it may not be feasible if the face value of debt must be so high that even good borrowers would want to default. Therefore, the boundary between Propositions 2 and 3 is the minimum of z (i.e., the point at which partial separation becomes preferred) and $d_{g,2}$ (i.e., the point at which full separation becomes infeasible).

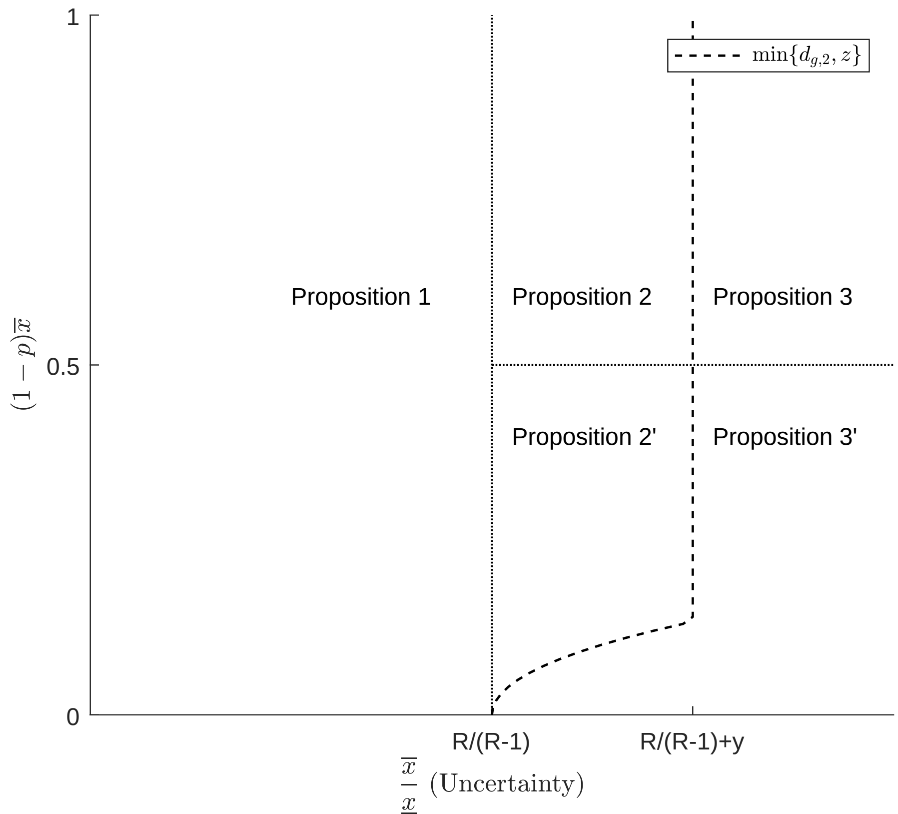

Figure 5: Regions of the parameter space. The proposition that describes the equilibrium depends on uncertainty in collateral value ($\frac{\overline{x}}{\underline{x}}$) and asymmetry in the distribution of collateral value ($(1-p)\overline{x}$).

In this section, we have presented the results for the case $(1-p)\overline{x} > 0.5$. In Appendix C, we present the results for the complementary case $(1-p)\overline{x} < 0.5$ as summarized by Figure 5. The condition for costless full separation remains the same as Proposition 1. The results for Proposition 2 and 3 also remain essentially the same, except for two small differences. First, the threshold z for uncertainty in collateral value at which partial separation becomes preferred to full separation takes a different expression. Second, the optimal loan amount when the collateral value falls in period 2 is lower than in Proposition 3. Therefore, if the collateral value rises in period 3, all borrowers repay, so borrower type is not revealed. If the collateral value falls instead, only bad borrowers default, so borrower type is fully revealed.

5. Extensions

We have made simplifying assumptions to highlight how leverage dynamics could signal credit quality and how the equilibrium depends on uncertainty in collateral value. We now discuss three extensions to show the robustness of our main results to various modeling assumptions. First, we discuss why good borrowers prefer short-term debt to long-term debt in our model. Second, we discuss how the possibility of hidden savings only affects the boundary between Propositions 2 and 3 and not the fundamental structure of the equilibrium. Finally, we relax the binomial distribution for collateral growth and consider a continuous distribution with bounded support instead.

5.1. Long-term debt

In our baseline model, we have assumed that borrowers can only borrow through one-period debt. We consider an extension where borrowers can also borrow through two-period debt. In Ordoñez et al. (2019), we prove that good borrowers strictly prefer one-period debt to two-period debt (except in the region labeled Proposition 3′ in Figure 5, where they are indifferent between the two maturities). The intuition is that one-period debt creates opportunities for good borrowers to costlessly separate by deleveraging in period 2. Two-period debt rules out the possibility of separation by deleveraging since separation in the terminal period can only happen through default, which is ex-ante costly. The only situation in which borrowers are indifferent is the combination of parameters (i.e., high collateral uncertainty and positive skew) such that there is no separation when the collateral value rises in both periods. In this case, two-period debt can achieve the same equilibrium as one-period debt.

5.2. Hidden savings

In our baseline model, we have assumed that income is perishable and cannot be saved. In principle, this assumption restricts good borrowers from separating by using hidden savings to repay debt. If good borrowers could save their unobservable income in period 1, could they separate more effectively in period 2? The possibility of hidden savings simply shifts the boundary between Propositions 2 and 3 in Figure 5 to the right and otherwise leaves our results intact.

The region "separation by deleveraging" in Figure 1 is defined by the collateral value X_t and the default boundary for bad borrowers $D_{b,t}$, neither of which depends on unobservable income. The intuition is that good borrowers need only an infinitesimal amount of unobservable income to separate by rolling over an infinitesimally smaller amount than bad borrowers. Since separation through deleveraging does not require a large amount of unobservable income, good borrowers do not need hidden savings.

The region "separation by default" in Figure 1 is defined by the default boundaries for the two types of borrowers, $D_{b,t}$ and $D_{g,t}$. To see how hidden savings affect this region, we assume that unobservable income that is saved earns a gross return x_t (i.e., the same return as collateral). Inequality (15) in Lemma 3 that determines the region in which there is separation by default becomes

$$X_t + V_t \leq D_{b,t} \leq D_{g,t} \leq X_t + \sum_{s=1}^{t} x_t^{t-s} Y_s + V_t.$$

The default boundary for the good borrower is bounded by a larger amount than in the baseline case because of hidden savings. The corresponding expressions in Proposition 2 become

$$\begin{aligned} d_{g,2} = & \frac{R}{R-1} + 2y, \\ z = & \frac{R}{R-1} + \frac{(1-p)\overline{x}2y}{R(R-1)}. \end{aligned}$$

Since the boundary between Propositions 2 and 3 is $\min\{d_{g,2}, z\}$, hidden savings simply shifts this boundary to the right.

In summary, the possibility of hidden savings does not alter our conclusions about costless separation by deleveraging. However, the possibility of hidden savings expands the parameter region under which good borrowers choose to separate by costly default, by reducing the likelihood of default for a given loan amount.

Hidden savings could play a more important role if good borrowers could signal through a durable-good purchase or capital investment in the terminal period. By saving their unobservable income until the terminal period, good borrowers could self-finance a higher share of the durable-good purchase than bad borrowers. This signaling strategy is essentially Bester (1985), where good borrowers signal through higher collateral. It requires that lenders know the realized returns on hidden savings and the total cost of the durable good. Otherwise, the loan amount may not be a reliable signal of hidden savings because of high returns on hidden savings or a low cost of the durable good.

In this extension, deleveraging is a better signaling strategy as long as collateral uncertainty is sufficiently low. There is no reason for good borrowers to wait until the terminal period if they can fully separate by deleveraging earlier. When collateral uncertainty is higher, hidden savings could enrich the set of signaling strategies in the terminal period. To model this interaction, we must introduce a durable-good purchase in the terminal period, which we leave for future work.

5.3. *Continuous distribution of collateral growth*

In our baseline model, we have assumed that collateral growth follows a binomial distribution. In Ordoñez et al. (2019), we consider an alternative distributional assumption that the growth rate is continuous and bounded between $\underline{x}$ and $\overline{x}$. This extension delivers two insights.

First, costless full separation is feasible as long as $\frac{\overline{x}}{\underline{x}} \leq \frac{R}{R-1}$, exactly as in Proposition 1 for the binomial case. Thus, this result appears robust to the shape of the underlying distribution. The intuition is that a loan amount that allows good borrowers to separate through deleveraging when collateral growth is the maximum possible does not induce default if collateral growth is the minimum possible. In other words, the extent to which debt dynamics can sustain costless full separation only depends on the range between the highest and lowest possible growth rates, not on the underlying distribution. If the distribution were unbounded, costless full separation would not be feasible.

Second, the equilibrium entails default in some states when $\frac{\overline{x}}{\underline{x}} > \frac{R}{R-1}$. In this case, the point at which good borrowers prefer to switch from costly full separation to partial separation (i.e., the boundary between Propositions 2 and 3 in Figure 5) does depend on the underlying distribution. Intuitively, the distribution of collateral growth determines how the relative probabilities of no separation versus bad borrowers defaulting change with the loan amount. As we characterize in Ordoñez et al. (2019), the equilibrium is essentially a hybrid of Propositions 2 and 3 for the binomial case, as illustrated in Figure 6.

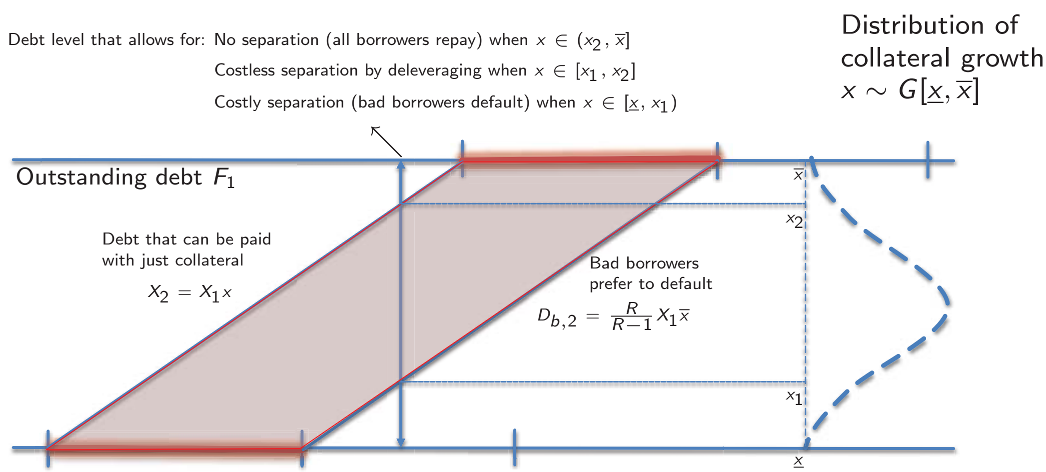

Figure 6: Signaling regions for a continuous distribution of collateral growth

6. Conclusion

We have developed a dynamic model of credit markets with asymmetric information to allow for a richer set of signaling strategies through the path of debt and repayment, which were ruled out by assumption in a previous literature dominated by static models. In particular, we have shown the importance of deleveraging strategies in which good borrowers borrow a sufficiently high amount such that subsequent repayment reveals the presence of unobservable income. The precision of deleveraging as a signal depends on the degree of uncertainty in collateral value. When uncertainty is high, information revelation could entail default of bad borrowers or no information revelation in equilibrium.

If we interpret uncertainty of collateral value as a property of an asset class, our model shows that assets with more certain values allow borrowers to signal through the path of debt and repayment in a dynamic setting. Interestingly, Dang et al. (2015) reach a similar conclusion that optimal collateral in credit markets is debt instead of equity because of its low information sensitivity.

If we interpret uncertainty in collateral value as systematic uncertainty that is common across borrowers, our model provides an alternative view of leverage cycles in the macroeconomy. The previous literature solidified a view that a wave of leveraging followed by deleveraging or default is a negative consequence of financial frictions or credit constraints (Kiyotaki and Moore, 1997; Geanakoplos, 2009; Fostel and Geanakoplos, 2008). In our model, leverage is not an outcome of constraints but rather determined by the optimal choice of borrowers trying to resolve asymmetric information. Deleveraging is a signaling mechanism that reveals credit quality, sometimes through default of bad borrowers in equilibrium. Such information revelation is more likely when collateral value falls, such as in periods of falling home or stock prices. Our model thus highlights a potentially positive side of deleveraging that complements existing theories of leverage cycles.

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Appendix A. Proofs of Lemmas 1 to 3

Proof of Lemma 1. We show that equations (2) and (11) are equivalent by induction. Suppose that equation (11) holds for period $t+1$. Then the continuation value is

$$\begin{aligned}\frac{\mathbb{E}_t[J_{i,t+1}]}{R} =& \frac{\Pr(D_{i,t+1} \geq F_t)\mathbb{E}_t[X_{t+1} + \mathbb{1}_g(i)Y_{t+1} + V_{t+1} + W_{i,t+1} - F_t | D_{i,t+1} \geq F_t]}{R} \\ &+ \frac{\Pr(D_{i,t+1} < F_t)\mathbb{E}_t\left[\mathbb{1}_g(i)Y_{t+1} + \widehat{W}_{i,t+1} | D_{i,t+1} < F_t\right]}{R} \\ =& V_t - C_{i,t} + \frac{\Pr(D_{i,t+1} \geq F_t)\mathbb{E}_t[\mathbb{1}_g(i)Y_{t+1} + W_{i,t+1} | D_{i,t+1} \geq F_t]}{R} \\ &+ \frac{\Pr(D_{i,t+1} < F_t)\mathbb{E}_t\left[\mathbb{1}_g(i)Y_{t+1} + \widehat{W}_{i,t+1} | D_{i,t+1} < F_t\right]}{R}.\end{aligned} \tag{A.1}$$

Substituting equations (7) and (A.1) into equation (2), equation (11) holds for period t. Equations (10) and (11) imply equation (13).

Proof of Lemma 2. When there is no further updating of reputation, the value of the non-pledgeable asset is $W_{i,t} = \widehat{W}_{i,t}$. That is, the borrower's objective function is independent of F_t. Therefore, the borrower is indifferent between any loan amount such that repayment is feasible (i.e., $P_t F_t \geq F_{t-1} - X_t - \mathbb{1}_g(i)Y_t$). Moreover, the maximum possible loan amount is

$$\begin{aligned}P_t F_t =& \frac{\Pr(X_{t+1} + V_{t+1} \geq F_t)F_t + \Pr(X_{t+1} + V_{t+1} < F_t)\mathbb{E}_t[X_{t+1} + V_{t+1} | X_{t+1} + V_{t+1} < F_t]}{R} \\ \leq& \frac{\Pr(X_{t+1} + V_{t+1} \geq F_t)\mathbb{E}_t[X_{t+1} + V_{t+1} | X_{t+1} + V_{t+1} \geq F_t]}{R} \\ &+ \frac{\Pr(X_{t+1} + V_{t+1} < F_t)\mathbb{E}_t[X_{t+1} + V_{t+1} | X_{t+1} + V_{t+1} < F_t]}{R} = V_t,\end{aligned}$$

with equality when $F_t = \mathbb{E}_t[X_{t+1} + V_{t+1} | X_{t+1} + V_{t+1} \geq F_t]$. That is, debt is fully collateralized and riskless.

Proof of Lemma 3. We first show that inequality (15) holds in period 3. By equation (9), it suffices to show that $0 \leq \pi_3 - \widehat{\pi}_3 \leq 1$, which would imply that $0 \leq W_3 - \widehat{W}_3 \leq \frac{Y_3}{R-1}$. If $F_2 \leq \min\{D_{b,3}, D_{g,3}\}$, all borrowers repay. Therefore, equation (6) and Assumption 1 imply that $\pi_3 = \pi_2$ and $\widehat{\pi}_3 = 0$. If $F_2 > \max\{D_{b,3}, D_{g,3}\}$, all borrowers default. Therefore, equation (5) and Assumption 1 imply that $\pi_3 = 1$ and $\widehat{\pi}_3 = \pi_2$.

If $F_2 \in (\min\{D_{b,3}, D_{g,3}\}, \max\{D_{b,3}, D_{g,3}\}]$, we show that $D_{b,3} \leq D_{g,3}$ by contradiction. Suppose that $D_{b,3} > D_{g,3}$. Equations (5) and (6) imply that $\pi_3 = 0$ and $\widehat{\pi}_3 = 1$. Equation (9) then implies that $D_{b,3} = D_{g,3} = X_3 + V_3 - \frac{Y_3}{R-1}$, which contradicts $D_{b,3} > D_{g,3}$. Therefore,

$D_{b,3} \leq D_{g,3}$. In the event of full separation in period 3, $\pi_3 = 1$, $\widehat{\pi}_3 = 0$, and $W_3 - \widehat{W}_3 = \frac{Y_3}{R-1}$. Therefore, the default boundary (9) simplifies to $D_{i,3} = X_3 + V_3 + \mathbb{1}_g(i)Y_3$.

We now show that inequality (15) holds in period $t \in \{1, 2\}$. By equation (13), it suffices to show that

$$\max_{F_t} \ P_t F_t \geq V_t \tag{A.2}$$

and

$$0 \leq W_{b,t} - \widehat{W}_{b,t} \leq W_{g,t} - \widehat{W}_{g,t} \leq \frac{Y_t}{R^{3-t}(R-1)}. \tag{A.3}$$

Suppose that inequalities (15) and (A.3) hold in period $t+1$. The proof is by induction.

Equation (8) implies that

$$C_{g,t} - C_{b,t} = \Pr(F_t \in (D_{b,t+1}, D_{g,t+1}])\mathbb{E}_t\left[\frac{F_t}{R} - \frac{X_{t+1}}{R-1}|F_t \in (D_{b,t+1}, D_{g,t+1}]\right] \geq 0, \tag{A.4}$$

where the inequality follows from

$$\frac{F_t}{R} - \frac{X_{t+1}}{R-1} \geq 0 \Longleftrightarrow F_t \geq \frac{RX_{t+1}}{R-1} = X_{t+1} + V_{t+1}$$

and the induction hypothesis. Inequality (A.2) then follows from $\max_{F_t} \ C_{b,t} = V_t$.

We now prove the first part of inequality (A.3). Inequality (A.4) implies that the first term in equation (14) is weakly positive for bad borrowers. The third term in equation (14) is weakly positive by the induction hypothesis. The numerator in the second term of equation (14) can be rewritten as

$$\mathbb{E}_t[\widehat{\pi}_{t+1}Y_{t+1}] - \widehat{\pi}_t Y_t = \begin{cases} \pi_t \Pr(D_{g,t+1} < F_t)\mathbb{E}_t[Y_{t+1}|D_{g,t+1} < F_t] & \text{if } F_{t-1} \leq \min\{D_{b,t}, D_{g,t}\} \\ \Pr(D_{g,t+1} < F_t)\mathbb{E}_t[Y_{t+1}|D_{g,t+1} < F_t] & \text{if } F_{t-1} \in (D_{b,t}, D_{g,t}] \\ -Y_t & \text{if } F_{t-1} \in (D_{g,t}, D_{b,t}] \\ 0 & \text{if } F_{t-1} > \max\{D_{b,t}, D_{g,t}\} \end{cases}, \tag{A.5}$$

which is weakly positive unless $F_{t-1} \in (D_{g,t}, D_{b,t}]$. In this case, we show that $D_{b,t} \leq D_{g,t}$ by contradiction. Suppose that $D_{b,t} > D_{g,t}$. Equations (4) and (5) imply that $\pi_t = 0$ and $\widehat{\pi}_t = 1$. Moreover, Lemma 2 implies that $D_{b,t+1} = D_{g,t+1}$ and $W_{i,t+1} = \widehat{W}_{i,t+1}$. Equations (13) and (A.2) then imply that $D_{b,t} = D_{g,t} = X_t + V_t - \frac{Y_t}{R^{3-t}(R-1)}$, which contradicts $D_{b,t} > D_{g,t}$.

Therefore, $D_{b,t} \leq D_{g,t}$.

We now prove the third part of inequality (A.3). Inequality (A.4) implies that the first term in equation (14) is weakly negative for good borrowers. If $F_{t-1} \leq D_{b,t}$, the sum of the second and third terms of equation (14) is less than or equal to

$$\frac{\pi_t \Pr(D_{g,t+1} < F_t)\mathbb{E}_t[Y_{t+1}|D_{g,t+1} < F_t]}{R^{3-t}(R-1)} + \frac{\Pr(D_{g,t+1} \geq F_t)\mathbb{E}_t[Y_{t+1}|D_{g,t+1} \geq F_t]}{R^{3-t}(R-1)}$$
$$< \frac{Y_t}{R^{3-t}(R-1)}$$

by equation (A.5) and the induction hypothesis. If $F_{t-1} \in (D_{b,t}, D_{g,t}]$, the second term of equation (14) is less than or equal to $\frac{Y_t}{R^{3-t}(R-1)}$ by equation (A.5), and the third term is zero by Lemma 2.

We now prove the second part of inequality (A.3). Equations (14) and (A.4) imply that

$$W_{g,t} - \widehat{W}_{g,t} - \left(W_{b,t} - \widehat{W}_{b,t}\right) =$$
$$\Pr(F_t \in (D_{b,t+1}, D_{g,t+1}])\mathbb{E}_t\left[\frac{Y_{t+1}}{R^{3-t}(R-1)} - \frac{F_t}{R} + \frac{X_{t+1}}{R-1}|F_t \in (D_{b,t+1}, D_{g,t+1}]\right]$$
$$+ \frac{\Pr(D_{b,t+1} \geq F_t)\mathbb{E}_t\left[W_{g,t+1} - \widehat{W}_{g,t+1} - \left(W_{b,t+1} - \widehat{W}_{b,t+1}\right)|D_{b,t+1} \geq F_t\right]}{R}.$$

The first term is positive if

$$\frac{Y_{t+1}}{R^{3-t}(R-1)} - \frac{F_t}{R} + \frac{X_{t+1}}{R-1} \geq 0 \Longleftrightarrow F_t \leq X_{t+1} + V_{t+1} + \frac{Y_{t+1}}{R^{2-t}(R-1)},$$

which holds by the induction hypothesis. The second term is also positive by the induction hypothesis.

In the event of full separation in period t, $\max_{F_t}\ P_t F_t = \max_{F_t}\ C_{i,t} = V_t$ and $W_{i,t} - \widehat{W}_{i,t} = \frac{\mathbb{1}_g(i)Y_t}{R^{3-t}(R-1)}$. Therefore, the default boundary (13) simplifies to $D_{i,t} = X_t + V_t + \mathbb{1}_g(i)Y_t$.

Appendix B. Lemmas used to prove Propositions 1 to 3

To simplify notation, let lowercase letters denote the corresponding variables divided by X_t. That is, $f_t = \frac{F_t}{X_t}$, $d_{i,t} = \frac{D_{i,t}}{X_t}$, $w_{i,t} = \frac{W_{i,t}}{X_t}$, and $c_{i,t} = \frac{C_{i,t}}{X_t}$.

Lemma 8. *If $f_1 \leq x_2$ in period 2, all borrowers borrow $f_2 = d_{b,3}\overline{x} + \varepsilon_2$ for an arbitrarily* *small $\varepsilon_2 > 0$ at an interest rate $P_2^{-1} < R$ that satisfies*

$$P_2 f_2 = \begin{cases} \frac{1}{R-1} + \pi_1 \left(\frac{\overline{x}-1}{R-1} + \frac{\varepsilon_2}{R} \right) & \textit{if } \frac{\overline{x}}{\underline{x}} < \frac{d_{g,3}}{d_{b,3}} \\ \frac{1}{R-1} + \frac{\pi_1 (1-p) \varepsilon_2}{R} & \textit{if } \frac{\overline{x}}{\underline{x}} \geq \frac{d_{g,3}}{d_{b,3}} \end{cases}. \tag{B.1}$$

The value of non-pledgeable assets for good borrowers is

$$w_{g,2} = \begin{cases} -(1-\pi_1) \left(\frac{\overline{x}-1}{R-1} + \frac{\varepsilon_2}{R} \right) + \frac{y}{R-1} & \textit{if } \frac{\overline{x}}{\underline{x}} < \frac{d_{g,3}}{d_{b,3}} \\ -\frac{(1-\pi_1)(1-p)\varepsilon_2}{R} + \frac{y}{R-1} - \frac{(1-\pi_1) p \underline{x} y}{R(R-1)} & \textit{if } \frac{\overline{x}}{\underline{x}} \geq \frac{d_{g,3}}{d_{b,3}} \end{cases}. \tag{B.2}$$

Proof. If $f_1 \leq x_2$, Lemma 4 implies no updating of reputation in period 2 so that $\pi_2 = \pi_1$. Good borrowers choose f_2 that maximizes $w_{g,2}$, and bad borrowers mimic good borrowers.

In period $t \in \{1, 2\}$, expected repayment for a type i borrower is

$$c_{i,t} = \begin{cases} \frac{1}{R-1} & \text{if } f_t > d_{i,t+1}\overline{x} \\ \frac{(1-p) f_t}{R} + \frac{p\underline{x}}{R-1} & \text{if } f_t \in (d_{i,t+1}\underline{x}, d_{i,t+1}\overline{x}] \\ \frac{f_t}{R} & \text{if } f_t \leq d_{i,t+1}\underline{x}. \end{cases}.$$

If $d_{b,t+1}\overline{x} < d_{g,t+1}\underline{x}$, the difference in expected repayment is

$$c_{g,t} - c_{b,t} = \begin{cases} 0 & \text{if } f_t > d_{g,t+1}\overline{x} \\ (1-p)\left(\frac{f_t}{R} - \frac{\overline{x}}{R-1} \right) & \text{if } f_t \in (d_{g,t+1}\underline{x}, d_{g,t+1}\overline{x}] \\ \frac{f_t}{R} - \frac{1}{R-1} & \text{if } f_t \in (d_{b,t+1}\overline{x}, d_{g,t+1}\underline{x}] \\ p\left(\frac{f_t}{R} - \frac{\underline{x}}{R-1} \right) & \text{if } f_t \in (d_{b,t+1}\underline{x}, d_{b,t+1}\overline{x}] \\ 0 & \text{if } f_t \leq d_{b,t+1}\underline{x}. \end{cases}. \tag{B.3}$$

If $d_{b,t+1}\overline{x} \geq d_{g,t+1}\underline{x}$, the difference in expected repayment is

$$c_{g,t} - c_{b,t} = \begin{cases} 0 & \text{if } f_t > d_{g,t+1}\overline{x} \\ (1-p)\left(\frac{f_t}{R} - \frac{\overline{x}}{R-1} \right) & \text{if } f_t \in (d_{b,t+1}\overline{x}, d_{g,t+1}\overline{x}] \\ 0 & \text{if } f_t \in (d_{g,t+1}\underline{x}, d_{b,t+1}\overline{x}] \\ p\left(\frac{f_t}{R} - \frac{\underline{x}}{R-1} \right) & \text{if } f_t \in (d_{b,t+1}\underline{x}, d_{g,t+1}\underline{x}] \\ 0 & \text{if } f_t \leq d_{b,t+1}\underline{x}. \end{cases}. \tag{B.4}$$

If $d_{b,3}\overline{x} < d_{g,3}\underline{x}$, equations (3), (12) and (B.3) imply that

$$w_{g,2} = \begin{cases} \frac{y}{R-1} - \frac{(1-\pi_2)y}{R(R-1)} & \text{if (1) } f_2 > d_{g,3}\overline{x} \\ -(1-\pi_2)(1-p)\left(\frac{f_2}{R} - \frac{\overline{x}}{R-1}\right) + \frac{y}{R-1} - \frac{(1-\pi_2)p\underline{x}y}{R(R-1)} & \text{if (2) } f_2 \in (d_{g,3}\underline{x}, d_{g,3}\overline{x}] \\ -(1-\pi_2)\left(\frac{f_2}{R} - \frac{1}{R-1}\right) + \frac{y}{R-1} & \text{if (3) } f_2 \in (d_{b,3}\overline{x}, d_{g,3}\underline{x}] \\ -(1-\pi_2)p\left(\frac{f_2}{R} - \frac{\underline{x}}{R-1}\right) + \frac{y}{R-1} - \frac{(1-\pi_2)(1-p)\overline{x}y}{R(R-1)} & \text{if (4) } f_2 \in (d_{b,3}\underline{x}, d_{b,3}\overline{x}] \\ \frac{y}{R-1} - \frac{(1-\pi_2)y}{R(R-1)} & \text{if (5) } f_2 \le d_{b,3}\underline{x}. \end{cases}$$

Note that $w_{g,2}$ is decreasing in f_2 in regions (2), (3) and (4). In the other regions, $w_{g,2}$ is independent of f_2. Let $w_{g,2}(n)$ denote the maximized value of $w_{g,2}$ in region (n). $w_{g,2}(3)$ is greater than $w_{g,2}(2)$. $w_{g,2}(4)$ is greater than $w_{g,2}(1)$ and $w_{g,2}(5)$. $w_{g,2}(3)$ is greater than $w_{g,2}(4)$ when $(1-p)\overline{x} \ge 0.5$. Therefore, $w_{g,2}$ is maximized in region (3) when $f_2 = d_{b,3}\overline{x} + \varepsilon_2$.

If $d_{b,3}\overline{x} \ge d_{g,3}\underline{x}$, equations (3), (12) and (B.4) imply that

$$w_{g,2} = \begin{cases} \frac{y}{R-1} - \frac{(1-\pi_2)y}{R(R-1)} & \text{if (1) } f_2 > d_{g,3}\overline{x} \\ -(1-\pi_2)(1-p)\left(\frac{f_2}{R} - \frac{\overline{x}}{R-1}\right) + \frac{y}{R-1} - \frac{(1-\pi_2)p\underline{x}y}{R(R-1)} & \text{if (2) } f_2 \in (d_{b,3}\overline{x}, d_{g,3}\overline{x}] \\ \frac{y}{R-1} - \frac{(1-\pi_2)y}{R(R-1)} & \text{if (3) } f_2 \in (d_{g,3}\underline{x}, d_{b,3}\overline{x}] \\ -(1-\pi_2)p\left(\frac{f_2}{R} - \frac{\underline{x}}{R-1}\right) + \frac{y}{R-1} - \frac{(1-\pi_2)(1-p)\overline{x}y}{R(R-1)} & \text{if (4) } f_2 \in (d_{b,3}\underline{x}, d_{g,3}\underline{x}] \\ \frac{y}{R-1} - \frac{(1-\pi_2)y}{R(R-1)} & \text{if (5) } f_2 \le d_{b,3}\underline{x}. \end{cases}$$

Note that $w_{g,2}$ is decreasing in f_2 in regions (2) and (4). In the other regions, $w_{g,2}$ is independent of f_2. $w_{g,2}(4)$ is greater than $w_{g,2}(1)$, $w_{g,2}(3)$, and $w_{g,2}(5)$. $w_{g,2}(2)$ is greater than $w_{g,2}(4)$ when $(1-p)\overline{x} \ge 0.5$. Therefore, $w_{g,2}$ is maximized in region (2) when $f_2 = d_{b,3}\overline{x} + \varepsilon_2$.

We obtain equation (B.1) by substituting the value of f_2 that maximizes $w_{g,2}$, $\pi_2 = \pi_1$, as well as equations (B.3) and (B.4) into equation (7).

Lemma 9. *If $f_0 \le x_1$ in period 1, all borrowers borrow*

$$f_1 \in \begin{cases} (\overline{x}, d_{b,2}\underline{x}] & \textit{if } \frac{\overline{x}}{\underline{x}} \in [1, d_{b,2}) \\ \overline{x} + \varepsilon_1 & \textit{if } \frac{\overline{x}}{\underline{x}} \in [d_{b,2}, \min\{d_{g,2}, z\}) \\ (\underline{x}, d_{b,2}\underline{x}] & \textit{if } \frac{\overline{x}}{\underline{x}} \ge \min\{d_{g,2}, z\} \end{cases},$$

for an arbitrarily small $\varepsilon_1 > 0$ *at an interest rate* $P_1^{-1} < R$ *that satisfies*

$$P_1 f_1 = \begin{cases} \frac{f_1}{R} & \textit{if } \frac{\overline{x}}{\underline{x}} \in [1, d_{b,2}) \\ \frac{(1-(1-\pi_0)p)f_1}{R} + \frac{(1-\pi_0)p\underline{x}}{R-1} & \textit{if } \frac{\overline{x}}{\underline{x}} \in [d_{b,2}, \min\{d_{g,2}, z\}) \\ \frac{f_1}{R} & \textit{if } \frac{\overline{x}}{\underline{x}} \geq \min\{d_{g,2}, z\}, \end{cases} \tag{B.5}$$

Proof. If $f_0 \leq x_1$, Lemma 4 implies no updating of reputation in period 1 so that $\pi_1 = \pi_0$. Good borrowers choose f_1 that maximizes $w_{g,1}$, and bad borrowers mimic good borrowers. We obtain equation (B.5) by substituting $\pi_1 = \pi_0$ as well as equations (B.3) and (B.4) into equation (7).

In period $t \in \{1, 2\}$, Lemma 3 and Assumption 2 imply that the default boundaries satisfy the inequality

$$\frac{d_{g,t}}{d_{b,t}} \leq \frac{\frac{R}{R-1} + \frac{y}{R^{3-t}(R-1)}}{\frac{R}{R-1}} \leq \frac{R+y}{R} < \frac{R}{R-1} \leq d_{b,2}.$$

In addition, Lemmas 5, 6, and 8 imply that

$$w_{g,2} = \begin{cases} \frac{y}{R-1} & \text{if } f_1 \in (x_2, d_{g,2}x_2] \\ w_{g,2}(2,3) < \frac{y}{R-1} & \text{if } f_1 \leq x_2 \end{cases}, \tag{B.6}$$

where $w_{g,2}(2,3)$ denotes equation (B.2).

If $\frac{\overline{x}}{\underline{x}} < \frac{d_{g,2}}{d_{b,2}}$, equations (3), (12), (B.3), and (B.6) imply that

$$w_{g,1} = \begin{cases} \frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} & \text{if (1) } f_1 > d_{g,2}\overline{x} \\ -(1-\pi_1)(1-p)\left(\frac{f_1}{R} - \frac{\overline{x}}{R-1}\right) + \frac{y}{R-1} - \frac{(1-\pi_1)p\underline{x}y}{R^2(R-1)} & \text{if (2) } f_1 \in (d_{g,2}\underline{x}, d_{g,2}\overline{x}] \\ -(1-\pi_1)\left(\frac{f_1}{R} - \frac{1}{R-1}\right) + \frac{y}{R-1} & \text{if (3) } f_1 \in (d_{b,2}\overline{x}, d_{g,2}\underline{x}] \\ -(1-\pi_1)p\left(\frac{f_1}{R} - \frac{\underline{x}}{R-1}\right) + \frac{y}{R-1} & \text{if (4) } f_1 \in (d_{b,2}\underline{x}, d_{b,2}\overline{x}] \\ \frac{y}{R-1} & \text{if (5) } f_1 \in (\overline{x}, d_{b,2}\underline{x}] \\ \frac{y}{R-1} - \frac{(1-p)\overline{x}}{R}\left(\frac{y}{R-1} - w_{g,2}(2,3)\right) & \text{if (6) } f_1 \in (\underline{x}, \overline{x}] \\ \frac{y}{R-1} - \frac{1}{R}\left(\frac{y}{R-1} - w_{g,2}(2,3)\right) & \text{if (7) } f_1 \leq \underline{x}. \end{cases} \tag{B.7}$$

Note that $w_{g,1}$ is maximized in region (5) for any $f_1 \in (\overline{x}, d_{b,2}\underline{x}]$.

If $\frac{\overline{x}}{\underline{x}} \in [\frac{d_{g,2}}{d_{b,2}}, d_{b,2})$, equations (3), (12), (B.4), and (B.6) imply that

$$w_{g,1} = \begin{cases} \frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} & \text{if (1) } f_1 > d_{g,2}\overline{x} \\ -(1-\pi_1)(1-p)\left(\frac{f_1}{R} - \frac{\overline{x}}{R-1}\right) + \frac{y}{R-1} - \frac{(1-\pi_1)p\underline{x}y}{R^2(R-1)} & \text{if (2) } f_1 \in (d_{b,2}\overline{x}, d_{g,2}\overline{x}] \\ \frac{y}{R-1} - \frac{(1-\pi_1)p\underline{x}y}{R^2(R-1)} & \text{if (3) } f_1 \in (d_{g,2}\underline{x}, d_{b,2}\overline{x}] \\ -(1-\pi_1)p\left(\frac{f_1}{R} - \frac{\underline{x}}{R-1}\right) + \frac{y}{R-1} & \text{if (4) } f_1 \in (d_{b,2}\underline{x}, d_{g,2}\underline{x}] \\ \frac{y}{R-1} & \text{if (5) } f_1 \in (\overline{x}, d_{b,2}\underline{x}] \\ \frac{y}{R-1} - \frac{(1-p)\overline{x}}{R}\left(\frac{y}{R-1} - w_{g,2}(2,3)\right) & \text{if (6) } f_1 \in (\underline{x}, \overline{x}] \\ \frac{y}{R-1} - \frac{1}{R}\left(\frac{y}{R-1} - w_{g,2}(2,3)\right) & \text{if (7) } f_1 \leq \underline{x}. \end{cases} \quad \text{(B.8)}$$

Note that $w_{g,1}$ is maximized in region (5) for any $f_1 \in (\overline{x}, d_{b,2}\underline{x}]$.

If $\frac{\overline{x}}{\underline{x}} \in [d_{b,2}, d_{g,2})$, equations (3), (12), (B.4), and (B.6) imply that

$$w_{g,1} = \begin{cases} \frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} & \text{if (1) } f_1 > d_{g,2}\overline{x} \\ -(1-\pi_1)(1-p)\left(\frac{f_1}{R} - \frac{\overline{x}}{R-1}\right) + \frac{y}{R-1} - \frac{(1-\pi_1)p\underline{x}y}{R^2(R-1)} & \text{if (2) } f_1 \in (d_{b,2}\overline{x}, d_{g,2}\overline{x}] \\ \frac{y}{R-1} - \frac{(1-\pi_1)p\underline{x}y}{R^2(R-1)} & \text{if (3) } f_1 \in (d_{g,2}\underline{x}, d_{b,2}\overline{x}] \\ -(1-\pi_1)p\left(\frac{f_1}{R} - \frac{\underline{x}}{R-1}\right) + \frac{y}{R-1} & \text{if (4) } f_1 \in (\overline{x}, d_{g,2}\underline{x}] \\ -(1-\pi_1)p\left(\frac{f_1}{R} - \frac{\underline{x}}{R-1}\right) & \\ -\frac{(1-\pi_1)(1-p)^2\overline{x}\varepsilon_2}{R^2} + \frac{y}{R-1} - \frac{(1-\pi_1)(1-p)\overline{x}p\underline{x}y}{R^2(R-1)} & \text{if (5) } f_1 \in (d_{b,2}\underline{x}, \overline{x}] \\ -\frac{(1-\pi_1)(1-p)^2\overline{x}\varepsilon_2}{R^2} + \frac{y}{R-1} - \frac{(1-\pi_1)(1-p)\overline{x}p\underline{x}y}{R^2(R-1)} & \text{if (6) } f_1 \in (\underline{x}, d_{b,2}\underline{x}] \\ -\frac{(1-\pi_1)(1-p)\varepsilon_2}{R^2} + \frac{y}{R-1} - \frac{(1-\pi_1)p\underline{x}y}{R^2(R-1)} & \text{if (7) } f_1 \leq \underline{x}. \end{cases}.$$

Note that $w_{g,1}$ is decreasing in f_1 in regions (2), (4) and (5). In the other regions, $w_{g,1}$ is independent of f_1. Let $w_{g,1}(n)$ denote the maximized value of $w_{g,1}$ in region (n). $w_{g,1}(3)$ is greater than $w_{g,1}(1)$ and $w_{g,1}(2)$. $w_{g,1}(6)$ is greater than $w_{g,1}(3)$, $w_{g,1}(5)$, and $w_{g,1}(7)$. Moreover, $w_{g,1}(4)$ is greater than $w_{g,1}(6)$ if and only if $\frac{\overline{x}}{\underline{x}} < z$. Therefore, $w_{g,1}$ is maximized in region (4) for $f_1 = \overline{x} + \varepsilon_1$ if $\frac{\overline{x}}{\underline{x}} < z$. Otherwise, $w_{g,1}$ is maximized in region (6) for any $f_1 \in (\underline{x}, d_{b,2}\underline{x}]$.

If $\frac{\overline{x}}{\underline{x}} \geq d_{g,2}$, equations (3), (12), (B.4), and (B.6) imply that

$$w_{g,1} = \begin{cases} \frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} & \text{if (1) } f_1 > d_{g,2}\overline{x} \\ -(1-\pi_1)(1-p)\left(\frac{f_1}{R} - \frac{\overline{x}}{R-1}\right) + \frac{y}{R-1} - \frac{(1-\pi_1)p\underline{x}y}{R^2(R-1)} & \text{if (2) } f_1 \in (d_{b,2}\overline{x}, d_{g,2}\overline{x}] \\ \frac{y}{R-1} - \frac{(1-\pi_1)p\underline{x}y}{R^2(R-1)} & \text{if (3) } f_1 \in (\overline{x}, d_{b,2}\overline{x}] \\ -\frac{(1-\pi_1)(1-p)^2\overline{x}\varepsilon_2}{R^2} + \frac{y}{R-1} - \frac{(1-\pi_1)(1+(1-p)\overline{x})p\underline{x}y}{R^2(R-1)} & \text{if (4) } f_1 \in (d_{g,2}\underline{x}, \overline{x}] \\ -(1-\pi_1)p\left(\frac{f_1}{R} - \frac{\underline{x}}{R-1}\right) & \\ -\frac{(1-\pi_1)(1-p)^2\overline{x}\varepsilon_2}{R^2} + \frac{y}{R-1} - \frac{(1-\pi_1)(1-p)\overline{x}p\underline{x}y}{R^2(R-1)} & \text{if (5) } f_1 \in (d_{b,2}\underline{x}, d_{g,2}\underline{x}] \\ -\frac{(1-\pi_1)(1-p)^2\overline{x}\varepsilon_2}{R^2} + \frac{y}{R-1} - \frac{(1-\pi_1)(1-p)\overline{x}p\underline{x}y}{R^2(R-1)} & \text{if (6) } f_1 \in (\underline{x}, d_{b,2}\underline{x}] \\ -\frac{(1-\pi_1)(1-p)\varepsilon_2}{R^2} + \frac{y}{R-1} - \frac{(1-\pi_1)p\underline{x}y}{R^2(R-1)} & \text{if (7) } f_1 \leq \underline{x}. \end{cases}$$

Note that $w_{g,1}$ is decreasing in f_1 in regions (2) and (5). In the other regions, $w_{g,1}$ is independent of f_1. $w_{g,1}(3)$ is greater than $w_{g,1}(1)$ and $w_{g,1}(2)$. $w_{g,1}(6)$ is greater than $w_{g,1}(3)$, $w_{g,1}(4)$, $w_{g,1}(5)$, and $w_{g,1}(7)$. Therefore, $w_{g,1}$ is maximized in region (6) for any $f_1 \in (\underline{x}, d_{b,2}\underline{x}]$.

Appendix C. Characterization of the equilibrium for $(1-p)\overline{x} < 0.5$

We first present Lemmas 8′ and 9′ for the case $(1-p)\overline{x} < 0.5$, which correspond to Lemmas 8 and 9 for the case $(1-p)\overline{x} \geq 0.5$. We then present Propositions 2′ and 3′ for the case $(1-p)\overline{x} < 0.5$, which correspond to Propositions 2 and 3 for the case $(1-p)\overline{x} \geq 0.5$.

Lemma 8′. *Suppose that $(1-p)\overline{x} < 0.5$ and $\frac{\overline{x}}{\underline{x}} \geq \frac{d_{g,3}}{d_{b,3}}$. If $f_1 \leq x_2$ in period 2, all borrowers borrow $f_2 = d_{b,3}\underline{x} + \varepsilon_2$ for an arbitrarily small $\varepsilon_2 > 0$ at an interest rate $P_2^{-1} < R$ that satisfies*

$$P_2 f_2 = \frac{(1-(1-\pi_1)p)f_2}{R} + \frac{(1-\pi_1)p\underline{x}}{R-1}.$$

The value of non-pledgeable assets for good borrowers is

$$w_{g,2} = -\frac{(1-\pi_1)p\varepsilon_2}{R} + \frac{y}{R-1} - \frac{(1-\pi_1)(1-p)\overline{x}y}{R(R-1)}. \tag{C.1}$$

Proof. The proof essentially follows that for Lemma 8. The only difference is that $w_{g,2}(4)$ is greater than $w_{g,2}(2)$ when $(1-p)\overline{x} < 0.5$. Therefore, $w_{g,2}$ is maximized in region (4) when $f_2 = d_{b,3}\underline{x} + \varepsilon_2$.

Lemma 9′. *Suppose that* $(1-p)\overline{x} < 0.5$. *If* $f_0 \le x_1$ *in period 1, all borrowers borrow*

$$f_1 \in \begin{cases} (\overline{x}, d_{b,2}\underline{x}] & \text{if } \frac{\overline{x}}{\underline{x}} \in [1, d_{b,2}) \\ \overline{x} + \varepsilon_1 & \text{if } \frac{\overline{x}}{\underline{x}} \in [d_{b,2}, \min\{d_{g,2}, z'\}) \\ (\underline{x}, d_{b,2}\underline{x}] & \text{if } \frac{\overline{x}}{\underline{x}} \ge \min\{d_{g,2}, z'\} \end{cases}$$

for an arbitrarily small $\varepsilon_1 > 0$ *at an interest rate* $P_1^{-1} < R$ *that satisfies*

$$P_1 f_1 = \begin{cases} \frac{f_1}{R} & \text{if } \frac{\overline{x}}{\underline{x}} \in [1, d_{b,2}) \\ \frac{(1-(1-\pi_0)p)f_1}{R} + \frac{(1-\pi_0)p\underline{x}}{R-1} & \text{if } \frac{\overline{x}}{\underline{x}} \in [d_{b,2}, \min\{d_{g,2}, z'\}) \\ \frac{f_1}{R} & \text{if } \frac{\overline{x}}{\underline{x}} \ge \min\{d_{g,2}, z'\} \end{cases}.$$

Proof. The proof essentially follows that for Lemma 9. Lemmas 5, 6, and 8′ imply that

$$w_{g,2} = \begin{cases} \frac{y}{R-1} & \text{if } f_1 \in (x_2, d_{g,2}x_2] \\ w_{g,2}(4) < \frac{y}{R-1} & \text{if } f_1 \le x_2 \end{cases}, \tag{C.2}$$

where $w_{g,2}(4)$ denotes equation (C.1).

If $\frac{\overline{x}}{\underline{x}} < \frac{d_{g,2}}{d_{b,2}}$, $w_{g,1}$ is given by equation (B.7) with $w_{g,2}(2,3)$ replaced by $w_{g,2}(4)$. Similarly, if $\frac{\overline{x}}{\underline{x}} \in [\frac{d_{g,2}}{d_{b,2}}, d_{b,2})$, $w_{g,1}$ is given by equation (B.8) with $w_{g,2}(2,3)$ replaced by $w_{g,2}(4)$. In both cases, $w_{g,1}$ is maximized in region (5) for any $f_1 \in (\overline{x}, d_{b,2}\underline{x}]$.

If $\frac{\overline{x}}{\underline{x}} \in [d_{b,2}, d_{g,2})$, equations (3), (12), (B.4), and (C.2) imply that

$$w_{g,1} = \begin{cases} \frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} & \text{if (1) } f_1 > d_{g,2}\overline{x} \\ -(1-\pi_1)(1-p)\left(\frac{f_1}{R} - \frac{\overline{x}}{R-1}\right) + \frac{y}{R-1} - \frac{(1-\pi_1)p\underline{x}y}{R^2(R-1)} & \text{if (2) } f_1 \in (d_{b,2}\overline{x}, d_{g,2}\overline{x}] \\ \frac{y}{R-1} - \frac{(1-\pi_1)p\underline{x}y}{R^2(R-1)} & \text{if (3) } f_1 \in (d_{g,2}\underline{x}, d_{b,2}\overline{x}] \\ -(1-\pi_1)p\left(\frac{f_1}{R} - \frac{\underline{x}}{R-1}\right) + \frac{y}{R-1} & \text{if (4) } f_1 \in (\overline{x}, d_{g,2}\underline{x}] \\ -(1-\pi_1)p\left(\frac{f_1}{R} - \frac{\underline{x}}{R-1}\right) & \\ -\frac{(1-\pi_1)p(1-p)\overline{x}\varepsilon_2}{R^2} + \frac{y}{R-1} - \frac{(1-\pi_1)(1-p)^2\overline{x}^2 y}{R^2(R-1)} & \text{if (5) } f_1 \in (d_{b,2}\underline{x}, \overline{x}] \\ -\frac{(1-\pi_1)p(1-p)\overline{x}\varepsilon_2}{R^2} + \frac{y}{R-1} - \frac{(1-\pi_1)(1-p)^2\overline{x}^2 y}{R^2(R-1)} & \text{if (6) } f_1 \in (\underline{x}, d_{b,2}\underline{x}] \\ -\frac{(1-\pi_1)p\varepsilon_2}{R^2} + \frac{y}{R-1} - \frac{(1-\pi_1)(1-p)\overline{x}y}{R^2(R-1)} & \text{if (7) } f_1 \le \underline{x}. \end{cases}$$

Note that $w_{g,1}$ is decreasing in f_1 in regions (2), (4) and (5). In the other regions, $w_{g,1}$ is independent of f_1. Let $w_{g,1}(n)$ denote the maximized value of $w_{g,1}$ in region (n). $w_{g,1}(3)$ is greater than $w_{g,1}(1)$ and $w_{g,1}(2)$. $w_{g,1}(6)$ is greater than $w_{g,1}(3)$, $w_{g,1}(5)$, and $w_{g,1}(7)$. $w_{g,1}(4)$

is greater than $w_{g,1}(6)$ if and only $\frac{\overline{x}}{\underline{x}} < z'$, where z' is given by equation (C.3). Therefore, $w_{g,1}$ is maximized in region (4) for $f_1 = \overline{x} + \varepsilon_1$ if $\frac{\overline{x}}{\underline{x}} < z'$. Otherwise, $w_{g,1}$ is maximized in region (6) for any $f_1 \in (\underline{x}, d_{b,2}\underline{x}]$.

If $\frac{\overline{x}}{\underline{x}} \geq d_{g,2}$, equations (3), (12), (B.4), and (C.2) imply that

$$w_{g,1} = \begin{cases} \frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} & \text{if (1) } f_1 > d_{g,2}\overline{x} \\ -(1-\pi_1)(1-p)\left(\frac{f_1}{R} - \frac{\overline{x}}{R-1}\right) + \frac{y}{R-1} - \frac{(1-\pi_1)p\underline{x}y}{R^2(R-1)} & \text{if (2) } f_1 \in (d_{b,2}\overline{x}, d_{g,2}\overline{x}] \\ \frac{y}{R-1} - \frac{(1-\pi_1)p\underline{x}y}{R^2(R-1)} & \text{if (3) } f_1 \in (\overline{x}, d_{b,2}\overline{x}] \\ -\frac{(1-\pi_1)p(1-p)\overline{x}\varepsilon_2}{R^2} + \frac{y}{R-1} - \frac{(1-\pi_1)(p\underline{x}+(1-p)^2\overline{x}^2)y}{R^2(R-1)} & \text{if (4) } f_1 \in (d_{g,2}\underline{x}, \overline{x}] \\ -(1-\pi_1)p\left(\frac{f_1}{R} - \frac{\underline{x}}{R-1}\right) & \\ -\frac{(1-\pi_1)p(1-p)\overline{x}\varepsilon_2}{R^2} + \frac{y}{R-1} - \frac{(1-\pi_1)(1-p)^2\overline{x}^2y}{R^2(R-1)} & \text{if (5) } f_1 \in (d_{b,2}\underline{x}, d_{g,2}\underline{x}] \\ -\frac{(1-\pi_1)p(1-p)\overline{x}\varepsilon_2}{R^2} + \frac{y}{R-1} - \frac{(1-\pi_1)(1-p)^2\overline{x}^2y}{R^2(R-1)} & \text{if (6) } f_1 \in (\underline{x}, d_{b,2}\underline{x}] \\ -\frac{(1-\pi_1)p\varepsilon_2}{R^2} + \frac{y}{R-1} - \frac{(1-\pi_1)(1-p)\overline{x}y}{R^2(R-1)} & \text{if (7) } f_1 \leq \underline{x} \end{cases}.$$

Note that $w_{g,1}$ is decreasing in f_1 in regions (2) and (5). In the other regions, $w_{g,1}$ is independent of f_1. $w_{g,1}(3)$ is greater than $w_{g,1}(1)$ and $w_{g,1}(2)$. $w_{g,1}(6)$ is greater than $w_{g,1}(3)$, $w_{g,1}(4)$, $w_{g,1}(5)$, and $w_{g,1}(7)$. Therefore, $w_{g,1}$ is maximized in region (6) for any $f_1 \in (\underline{x}, d_{b,2}\underline{x}]$.

Proposition 2′. *Suppose that $(1-p)\overline{x} < 0.5$ and uncertainty in collateral value is intermediate. That is, $\frac{R}{R-1} \leq \frac{\overline{x}}{\underline{x}} < \min\{d_{g,2}, z'\}$, where*

$$\begin{aligned} d_{g,2} &= \frac{R}{R-1} + y, \\ z' &= \frac{R}{R-1} + \frac{((1-p)\overline{x})^2 y}{R(R-1)(1-(1-p)\overline{x})}. \end{aligned} \tag{C.3}$$

In period 1, all borrowers borrow $F_1 > X_1\overline{x}$ at an interest rate $P_1^{-1} > R$ that satisfies

$$P_1 F_1 = \frac{(1-(1-\pi_0)p)F_1}{R} + \frac{(1-\pi_0)pX_1\underline{x}}{R-1}.$$

If the collateral value rises in period 2 (i.e., $x_2 = \overline{x}$), borrower type is fully revealed. Good borrowers repay by rolling over $F_2 = 0$. Bad borrowers repay by rolling over $F_2 \in (0, RV_2]$. If the collateral value falls in period 2 (i.e., $x_2 = \underline{x}$), only bad borrowers default, so borrower type is fully revealed. Good borrowers repay by rolling over $F_2 \in [R\max\{0, F_1 - X_2 - Y_2\}, RV_2]$.

Proof. Lemma 9′ implies the equilibrium in period 1. Lemma 5 implies the equilibrium if the collateral value rises in period 2. Lemma 6 implies the equilibrium if the collateral value falls in period 2.

Proposition 3′. *Suppose that* $(1-p)\overline{x} < 0.5$ *and uncertainty in collateral value is high. That is,* $\frac{\overline{x}}{\underline{x}} \geq \min\{d_{g,2}, z'\}$. *In period 1, all borrowers borrow* $F_1 \in (X_1\underline{x}, RV_1\underline{x}]$ *at the interest rate* $P_1^{-1} = R$.

If the collateral value rises in period 2 (i.e., $x_2 = \overline{x}$*), borrower type is not revealed. All borrowers repay by rolling over* $F_2 > RV_2\underline{x}$ *at an interest rate* $P_2^{-1} > R$ *that satisfies*

$$P_2F_2 = \frac{(1-(1-\pi_0)p)F_2}{R} + \frac{(1-\pi_0)pX_2\underline{x}}{R-1}.$$

Subsequently, if the collateral value rises in period 3 (i.e., $x_3 = \overline{x}$*), all borrowers repay, so borrower type is not revealed. If the collateral value falls instead (i.e.,* $x_3 = \underline{x}$*), only bad borrowers default, so borrower type is fully revealed.*

If the collateral value falls in period 2 (i.e., $x_2 = \underline{x}$*), borrower type is fully revealed. Good borrowers repay by rolling over* $F_2 \in [R\max\{0, F_1 - X_2 - Y_2\}, R(F_1 - X_2))$. *Bad borrowers repay by rolling over* $F_2 \in [R(F_1 - X_2), RV_2]$.

Proof. Lemma 9′ implies the equilibrium in period 1. Lemma 8′ below implies the equilibrium if the collateral value rises in period 2. Lemma 4 implies the equilibrium if the collateral value subsequently rises in period 3, and Lemma 6 implies the equilibrium if the collateral value falls instead. Lemma 5 implies the equilibrium if the collateral value falls in period 2.