Online appendix to "Leverage dynamics and credit quality"

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Abstract

We present proofs for two extensions of the baseline model. The first extension allows borrowers borrow through long-term debt. The second extension modifies the distribution of collateral growth to be continuous with bounded support.

1. Long-term debt

We extend the baseline model to allow borrowers to borrow through two-period debt in period 1. We find that good borrowers strictly prefer one-period debt to two-period debt (except in the region labeled Proposition 3′ in Figure 5, where they are indifferent between the two maturities). The intuition is that one-period debt creates opportunities for good borrowers to costlessly separate by deleveraging in period 2. Two-period debt rules out the possibility of separation by deleveraging since separation in the terminal period can only happen through default, which is ex-ante costly.

Let $F_{1,3}$ denote two-period debt issued in period 1 that matures in period 3. Instead of equation (8), the expected repayment for a type i borrower is

$$C'_{i,1} = \frac{\Pr(D_{i,3} \geq F_{1,3})F_{1,3} + \Pr(D_{i,3} < F_{1,3})\mathbb{E}[X_3 + V_3 | D_{i,3} < F_{1,3}]}{R^2}.$$

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Instead of equation (12), the value of the non-pledgeable asset conditional on repayment is

$$\begin{aligned}W'_{i,1} =& -(\mathbb{1}_g(i) - \pi_1)(C_{g,1} - C_{b,1}) + \mathbb{1}_g(i)\left(\frac{Y_1}{R} + \frac{\mathbb{E}_1[Y_2]}{R^2}\right) \\ &+ \frac{\Pr(D_{i,3} \geq F_{1,3})\mathbb{E}_1[W_{i,3}|D_{i,3} \geq F_{1,3}]}{R^2} \\ &+ \frac{\Pr(D_{i,3} < F_{1,3})\mathbb{E}_1\left[\widehat{W}_{i,3}|D_{i,3} < F_{1,3}\right]}{R^2}.\end{aligned} \tag{OA.1}$$

Proposition 4. *Good borrowers strictly prefer one-period debt to two-period debt in all regions of the parameter space, except when $(1-p)\overline{x} < 0.5$ and $\frac{\overline{x}}{\underline{x}} \geq \min\{d_g, z\}$. In this region, good borrowers are indifferent between the two maturities.*

Proof. To simplify notation, let lowercase letters denote the corresponding variables divided by X_t. That is, $f_{1,3} = \frac{F_{1,3}}{X_1}$, $d_{i,t} = \frac{D_{i,t}}{X_t}$, $w'_{i,t} = \frac{W'_{i,t}}{X_t}$, and $c'_{i,t} = \frac{C'_{i,t}}{X_t}$.

If $f_0 \leq x_1$, Lemma 4 implies no updating of reputation in period 1 so that $\pi_1 = \pi_0$. Good borrowers choose two-period debt $f_{1,3}$ that maximizes $w'_{g,1}$, and bad borrowers mimic good borrowers.

In period 1, expected repayment for a type i borrower is

$$c'_{i,1} = \begin{cases} \frac{1}{R(R-1)} & \text{if } f_{1,3} > d_{i,3}\overline{x}^2 \\ \frac{(1-p)^2 f_{1,3}}{R^2} + \frac{1-(1-p)^2\overline{x}^2}{R(R-1)} & \text{if } f_{1,3} \in (d_{i,3}\underline{x}\overline{x}, d_{i,3}\overline{x}^2] \\ \frac{(1-p^2)f_{1,3}}{R^2} + \frac{p^2\underline{x}^2}{R(R-1)} & \text{if } f_{1,3} \in (d_{i,3}\underline{x}^2, d_{i,3}\underline{x}\overline{x}] \\ \frac{f_{1,3}}{R^2} & \text{if } f_{1,3} \leq d_{i,3}\underline{x}^2. \end{cases}.$$

If $d_{b,3}\overline{x}^2 < d_{g,3}\underline{x}^2$ or $\frac{\overline{x}}{\underline{x}} < \left(\frac{d_{g,3}}{d_{b,3}}\right)^{0.5}$, the difference in expected repayment is

$$c'_{g,1} - c'_{b,1} = \begin{cases} 0 & \text{if } f_{1,3} > d_{g,3}\overline{x}^2 \\ \frac{(1-p)^2}{R}\left(\frac{f_{1,3}}{R} - \frac{\overline{x}^2}{R-1}\right) & \text{if } f_{1,3} \in (d_{g,3}\underline{x}\overline{x}, d_{g,3}\overline{x}^2] \\ \frac{1}{R}\left(\frac{(1-p^2)f_{1,3}}{R} - \frac{1-p^2\underline{x}^2}{R-1}\right) & \text{if } f_{1,3} \in (d_{g,3}\underline{x}^2, d_{g,3}\underline{x}\overline{x}] \\ \frac{1}{R}\left(\frac{f_{1,3}}{R} - \frac{1}{R-1}\right) & \text{if } f_{1,3} \in (d_{b,3}\overline{x}^2, d_{g,3}\underline{x}^2] \\ \frac{1}{R}\left(\frac{(1-(1-p)^2)f_{1,3}}{R} - \frac{1-(1-p)^2\overline{x}^2}{R-1}\right) & \text{if } f_{1,3} \in (d_{b,3}\underline{x}\overline{x}, d_{b,3}\overline{x}^2] \\ \frac{p^2}{R}\left(\frac{f_{1,3}}{R} - \frac{\underline{x}^2}{R-1}\right) & \text{if } f_{1,3} \in (d_{b,3}\underline{x}^2, d_{b,3}\underline{x}\overline{x}] \\ 0 & \text{if } f_{1,3} \leq d_{b,3}\underline{x}^2. \end{cases}. \tag{OA.2}$$

If $d_{b,3}\underline{x}\overline{x} < d_{g,3}\underline{x}^2 \leq d_{b,3}\overline{x}^2 < d_{g,3}\underline{x}\overline{x}$ or $\frac{\overline{x}}{\underline{x}} \in \left[\left(\frac{d_{g,3}}{d_{b,3}}\right)^{0.5}, \frac{d_{g,3}}{d_{b,3}}\right)$, the difference in expected repayment is

$$c'_{g,1} - c'_{b,1} = \begin{cases} 0 & \text{if } f_{1,3} > d_{g,3}\overline{x}^2 \\ \frac{(1-p)^2}{R}\left(\frac{f_{1,3}}{R} - \frac{\overline{x}^2}{R-1}\right) & \text{if } f_{1,3} \in (d_{g,3}\underline{x}\overline{x}, d_{g,3}\overline{x}^2] \\ \frac{1}{R}\left(\frac{(1-p^2)f_{1,3}}{R} - \frac{1-p^2\underline{x}^2}{R-1}\right) & \text{if } f_{1,3} \in (d_{b,3}\overline{x}^2, d_{g,3}\underline{x}\overline{x}] \\ \frac{2p(1-p)}{R}\left(\frac{f_{1,3}}{R} - \frac{\underline{x}\overline{x}}{R-1}\right) & \text{if } f_{1,3} \in (d_{g,3}\underline{x}^2, d_{b,3}\overline{x}^2] \\ \frac{1}{R}\left(\frac{(1-(1-p)^2)f_{1,3}}{R} - \frac{1-(1-p)^2\overline{x}^2}{R-1}\right) & \text{if } f_{1,3} \in (d_{b,3}\underline{x}\overline{x}, d_{g,3}\underline{x}^2] \\ \frac{p^2}{R}\left(\frac{f_{1,3}}{R} - \frac{\underline{x}^2}{R-1}\right) & \text{if } f_{1,3} \in (d_{b,3}\underline{x}^2, d_{b,3}\underline{x}\overline{x}] \\ 0 & \text{if } f_{1,3} \le d_{b,3}\underline{x}^2. \end{cases} \quad \text{(OA.3)}$$

If $d_{g,3}\underline{x}\overline{x} \le d_{b,3}\overline{x}^2$ and $d_{g,3}\underline{x}^2 \le d_{b,3}\underline{x}\overline{x}$ or $\frac{\overline{x}}{\underline{x}} \ge \frac{d_{g,3}}{d_{b,3}}$, the difference in expected repayment is

$$c'_{g,1} - c'_{b,1} = \begin{cases} 0 & \text{if } f_{1,3} > d_{g,3}\overline{x}^2 \\ \frac{(1-p)^2}{R}\left(\frac{f_{1,3}}{R} - \frac{\overline{x}^2}{R-1}\right) & \text{if } f_{1,3} \in (d_{b,3}\overline{x}^2, d_{g,3}\overline{x}^2] \\ 0 & \text{if } f_{1,3} \in (d_{g,3}\underline{x}\overline{x}, d_{b,3}\overline{x}^2] \\ \frac{2p(1-p)}{R}\left(\frac{f_{1,3}}{R} - \frac{\underline{x}\overline{x}}{R-1}\right) & \text{if } f_{1,3} \in (d_{b,3}\underline{x}\overline{x}, d_{g,3}\underline{x}\overline{x}] \\ 0 & \text{if } f_{1,3} \in (d_{g,3}\underline{x}^2, d_{b,3}\underline{x}\overline{x}] \\ \frac{p^2}{R}\left(\frac{f_{1,3}}{R} - \frac{\underline{x}^2}{R-1}\right) & \text{if } f_{1,3} \in (d_{b,3}\underline{x}^2, d_{g,3}\underline{x}^2] \\ 0 & \text{if } f_{1,3} \le d_{b,3}\underline{x}^2. \end{cases} \quad \text{(OA.4)}$$

If $\frac{\overline{x}}{\underline{x}} < \left(\frac{d_{g,3}}{d_{b,3}}\right)^{0.5}$, equations (3), (OA.1), and (OA.2) imply that

$$w'_{g,1} = \begin{cases} \frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} & \text{if (1) } f_{1,3} > d_{g,3}\overline{x}^2 \\ -(1-\pi_1)\frac{(1-p)^2}{R}\left(\frac{f_{1,3}}{R} - \frac{\overline{x}^2}{R-1}\right) & \\ +\frac{y}{R-1} - (1-\pi_1)\frac{(1-(1-p)^2\overline{x}^2)y}{R^2(R-1)} & \text{if (2) } f_{1,3} \in (d_{g,3}\underline{x}\overline{x}, d_{g,3}\overline{x}^2] \\ -\frac{1-\pi_1}{R}\left(\frac{(1-p^2)f_{1,3}}{R} - \frac{1-p^2\underline{x}^2}{R-1}\right) & \\ +\frac{y}{R-1} - \frac{(1-\pi_1)p^2\underline{x}^2 y}{R^2(R-1)} & \text{if (3) } f_{1,3} \in (d_{g,3}\underline{x}^2, d_{g,3}\underline{x}\overline{x}] \\ -\frac{1-\pi_1}{R}\left(\frac{f_{1,3}}{R} - \frac{1}{R-1}\right) + \frac{y}{R-1} & \text{if (4) } f_{1,3} \in (d_{b,3}\overline{x}^2, d_{g,3}\underline{x}^2] \\ -\frac{1-\pi_1}{R}\left(\frac{(1-(1-p)^2)f_{1,3}}{R} - \frac{1-(1-p)^2\overline{x}^2}{R-1}\right) & \\ +\frac{y}{R-1} - \frac{(1-\pi_1)(1-p)^2\overline{x}^2 y}{R^2(R-1)} & \text{if (5) } f_{1,3} \in (d_{b,3}\underline{x}\overline{x}, d_{b,3}\overline{x}^2] \\ -(1-\pi_1)\frac{p^2}{R}\left(\frac{f_{1,3}}{R} - \frac{\underline{x}^2}{R-1}\right) & \\ +\frac{y}{R-1} - (1-\pi_1)\frac{(1-p^2\underline{x}^2)y}{R^2(R-1)} & \text{if (6) } f_{1,3} \in (d_{b,3}\underline{x}^2, d_{b,3}\underline{x}\overline{x}] \\ \frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} & \text{if (7) } f_{1,3} \le d_{b,3}\underline{x}^2. \end{cases}$$

None of these values are greater than $\frac{y}{R-1}$, which is attained by one-period debt for any amount $f_1 \in (\overline{x}, d_{b,2}\underline{x}]$.

If $\frac{\overline{x}}{\underline{x}} \in \left[\left(\frac{d_{g,3}}{d_{b,3}}\right)^{0.5}, \frac{d_{g,3}}{d_{b,3}}\right)$, equations (3), (OA.1), and (OA.3) imply that

$$w'_{g,1} = \begin{cases} \frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} & \text{if (1) } f_{1,3} > d_{g,3}\overline{x}^2 \\ -(1-\pi_1)\frac{(1-p)^2}{R}\left(\frac{f_{1,3}}{R} - \frac{\overline{x}^2}{R-1}\right) & \\ +\frac{y}{R-1} - (1-\pi_1)\frac{(1-(1-p)^2\overline{x}^2)y}{R^2(R-1)} & \text{if (2) } f_{1,3} \in (d_{g,3}\underline{x}\overline{x}, d_{g,3}\overline{x}^2] \\ -\frac{1-\pi_1}{R}\left(\frac{(1-p^2)f_{1,3}}{R} - \frac{1-p^2\underline{x}^2}{R-1}\right) & \\ +\frac{y}{R-1} - \frac{(1-\pi_1)p^2\underline{x}^2 y}{R^2(R-1)} & \text{if (3) } f_{1,3} \in (d_{b,3}\overline{x}^2, d_{g,3}\underline{x}\overline{x}] \\ -(1-\pi_1)\frac{2p(1-p)}{R}\left(\frac{f_{1,3}}{R} - \frac{\underline{x}\overline{x}}{R-1}\right) & \\ +\frac{y}{R-1} - (1-\pi_1)\frac{(1-2p\underline{x}(1-p)\overline{x})y}{R^2(R-1)} & \text{if (4) } f_{1,3} \in (d_{g,3}\underline{x}^2, d_{b,3}\overline{x}^2] \\ -\frac{1-\pi_1}{R}\left(\frac{(1-(1-p)^2)f_{1,3}}{R} - \frac{1-(1-p)^2\overline{x}^2}{R-1}\right) & \\ +\frac{y}{R-1} - (1-\pi_1)\frac{(1-(1-p)^2\overline{x}^2)y}{R^2(R-1)} & \text{if (5) } f_{1,3} \in (d_{b,3}\underline{x}\overline{x}, d_{g,3}\underline{x}^2] \\ -(1-\pi_1)\frac{p^2}{R}\left(\frac{f_{1,3}}{R} - \frac{\underline{x}^2}{R-1}\right) + \frac{y}{R-1} - \frac{(1-\pi_1)p^2\underline{x}^2 y}{R^2(R-1)} & \text{if (6) } f_{1,3} \in (d_{b,3}\underline{x}^2, d_{b,3}\underline{x}\overline{x}] \\ \frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} & \text{if (7) } f_{1,3} \leq d_{b,3}\underline{x}^2. \end{cases}.$$

None of these values are greater than $\frac{y}{R-1}$, which is attained by one-period debt for any amount $f_1 \in (\overline{x}, d_{b,2}\underline{x}]$.

If $\frac{\overline{x}}{\underline{x}} \geq \frac{d_{g,3}}{d_{b,3}}$, equations (3), (OA.1), and (OA.4) imply that

$$w'_{g,1} = \begin{cases} \frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} & \text{if (1) } f_{1,3} > d_{g,3}\overline{x}^2 \\ -(1-\pi_1)\frac{(1-p)^2}{R}\left(\frac{f_{1,3}}{R} - \frac{\overline{x}^2}{R-1}\right) & \\ +\frac{y}{R-1} - (1-\pi_1)\frac{(1-(1-p)^2\overline{x}^2)y}{R^2(R-1)} & \text{if (2) } f_{1,3} \in (d_{b,3}\overline{x}^2, d_{g,3}\overline{x}^2] \\ \frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} & \text{if (3) } f_{1,3} \in (d_{g,3}\underline{x}\overline{x}, d_{b,3}\overline{x}^2] \\ -(1-\pi_1)\frac{2p(1-p)}{R}\left(\frac{f_{1,3}}{R} - \frac{\underline{x}\overline{x}}{R-1}\right) & \\ +\frac{y}{R-1} - \frac{(1-\pi_1)(1-p)^2\overline{x}^2 y}{R^2(R-1)} & \text{if (4) } f_{1,3} \in (d_{b,3}\underline{x}\overline{x}, d_{g,3}\underline{x}\overline{x}] \\ \frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} & \text{if (5) } f_{1,3} \in (d_{g,3}\underline{x}^2, d_{b,3}\underline{x}\overline{x}] \\ -(1-\pi_1)\frac{p^2}{R}\left(\frac{f_{1,3}}{R} - \frac{\underline{x}^2}{R-1}\right) & \\ +\frac{y}{R-1} - (1-\pi_1)\frac{(1-p^2\underline{x}^2)y}{R^2(R-1)} & \text{if (6) } f_{1,3} \in (d_{b,3}\underline{x}^2, d_{g,3}\underline{x}^2] \\ \frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} & \text{if (7) } f_{1,3} \leq d_{b,3}\underline{x}^2. \end{cases}.$$

Note that $w'_{g,1}$ is decreasing in $f_{1,3}$ in regions (2), (4), and (6). In the other regions, $w'_{g,1}$ is independent of $f_{1,3}$. Let $w'_{g,1}(n)$ denote the maximized value of $w'_{g,1}$ in region (n). $w'_{g,1}(2)$ is greater than $w'_{g,1}(1)$, $w'_{g,1}(3)$, $w'_{g,1}(5)$, and $w'_{g,1}(7)$. $w'_{g,1}(4)$ is greater than $w'_{g,1}(6)$. $w'_{g,1}(2)$ is greater than $w'_{g,1}(4)$ if and only if $(1-p)\overline{x} > 0.5^{0.5}$, That is, $w'_{g,1}$ is maximized for $f_{1,3} = d_{b,3}\underline{x}\overline{x} + \varepsilon_{1,3}$ when $(1-p)\overline{x} \leq 0.5^{0.5}$ and $f_{1,3} = d_{b,3}\overline{x}^2 + \varepsilon_{1,3}$ when $(1-p)\overline{x} > 0.5^{0.5}$, for $\varepsilon_{1,3} > 0$ arbitrarily close to zero.

To determine whether good borrowers prefer one- or two-period debt, we compare the maximum values of $w_{g,1}$ and $w'_{g,1}$ when $\frac{\overline{x}}{\underline{x}} \geq d_b$. For one-period debt,

$$w_{g,1} = \begin{cases} \frac{y}{R-1} - (1-\pi_1)p\left(\frac{\overline{x}}{R} - \frac{\underline{x}}{R-1}\right) & \text{if } \frac{\overline{x}}{\underline{x}} < \min\{d_g, z\} \\ \frac{y}{R-1} - \frac{(1-\pi_1)(1-p)^2\overline{x}^2 y}{R^2(R-1)} & \text{if } \frac{\overline{x}}{\underline{x}} \geq \min\{d_g, z\} \text{ and } (1-p)\overline{x} < 0.5 \\ \frac{y}{R-1} - \frac{(1-\pi_1)(1-p)\overline{x}p\underline{x}y}{R^2(R-1)} & \text{if } \frac{\overline{x}}{\underline{x}} \geq \min\{d_g, z\} \text{ and } (1-p)\overline{x} \geq 0.5 \end{cases},$$

where

$$z = \begin{cases} \frac{R}{R-1} + \frac{((1-p)\overline{x})^2}{1-(1-p)\overline{x}} \frac{y}{R(R-1)} & \text{if } (1-p)\overline{x} < 0.5 \\ \frac{R}{R-1} + \frac{(1-p)\overline{x}}{R} \frac{y}{R-1} & \text{if } (1-p)\overline{x} \geq 0.5 \end{cases}.$$

For two-period debt,

$$w'_{g,1} = \begin{cases} \frac{y}{R-1} - \frac{(1-\pi_1)(1-p)^2\overline{x}^2 y}{R^2(R-1)} & \text{if } (1-p)\overline{x} < 0.5^{0.5} \\ \frac{y}{R-1} - (1-\pi_1)\frac{(1-(1-p)^2\overline{x}^2)y}{R^2(R-1)} & \text{if } (1-p)\overline{x} \geq 0.5^{0.5}. \end{cases}.$$

For $(1-p)\overline{x} < 0.5$ and $\frac{\overline{x}}{\underline{x}} < \min\{d_g, z\}$, good borrowers prefer one-period debt since

$$\begin{aligned} \frac{\overline{x}}{\underline{x}} < z \Longrightarrow p\left(\frac{\overline{x}}{R} - \frac{\underline{x}}{R-1}\right) &< \frac{(1-p)^2\overline{x}^2 y}{R^2(R-1)} \\ \Longrightarrow \frac{y}{R-1} - (1-\pi_1)p\left(\frac{\overline{x}}{R} - \frac{\underline{x}}{R-1}\right) &> \frac{y}{R-1} - \frac{(1-\pi_1)(1-p)^2\overline{x}^2 y}{R^2(R-1)}, \end{aligned}$$

where the second inequality follows from $\frac{1}{p} = \frac{\underline{x}}{1-(1-p)\overline{x}}$. For $(1-p)\overline{x} < 0.5$ and $\frac{\overline{x}}{\underline{x}} \geq \min\{d_g, z\}$, good borrowers are indifferent between one- and two-period debt because $w_{g,1}$ is the same in both cases. For $(1-p)\overline{x} \in [0.5, 0.5^{0.5})$ and $\frac{\overline{x}}{\underline{x}} < \min\{d_g, z\}$, good borrowers prefer one-period debt since

$$\begin{aligned} \frac{\overline{x}}{\underline{x}} < z \Longrightarrow p\left(\frac{\overline{x}}{R} - \frac{\underline{x}}{R-1}\right) &< \frac{(1-p)\overline{x}p\underline{x}y}{R^2(R-1)} < \frac{(1-p)^2\overline{x}^2 y}{R^2(R-1)} \\ \Longrightarrow \frac{y}{R-1} - (1-\pi_1)p\left(\frac{\overline{x}}{R} - \frac{\underline{x}}{R-1}\right) &> \frac{y}{R-1} - \frac{(1-\pi_1)(1-p)^2\overline{x}^2 y}{R^2(R-1)}, \end{aligned}$$

where the second inequality follows from $p\underline{x} < (1-p)\overline{x}$. For $(1-p)\overline{x} \in [0.5, 0.5^{0.5})$ and $\frac{\overline{x}}{\underline{x}} > \min\{d_g, z\}$, good borrowers prefer one-period debt since

$$\begin{aligned} p\underline{x} < (1-p)\overline{x} \Longrightarrow \frac{(1-p)\overline{x}p\underline{x}y}{R^2(R-1)} &< \frac{(1-p)^2\overline{x}^2 y}{R^2(R-1)} \\ \Longrightarrow \frac{y}{R-1} - \frac{(1-\pi_1)(1-p)\overline{x}p\underline{x}y}{R^2(R-1)} &> \frac{y}{R-1} - \frac{(1-\pi_1)(1-p)^2\overline{x}^2 y}{R^2(R-1)}. \end{aligned}$$

For $(1-p)\overline{x} \geq 0.5^{0.5}$ and $\frac{\overline{x}}{\underline{x}} < \min\{d_g, z\}$, good borrowers prefer one-period debt since

$$\begin{aligned}
\frac{\overline{x}}{\underline{x}} < z \Longrightarrow p\left(\frac{\overline{x}}{R} - \frac{\underline{x}}{R-1}\right) < \frac{(1-p)\overline{x}p\underline{x}y}{R^2(R-1)} < \frac{(1-(1-p)^2\overline{x}^2)y}{R^2(R-1)} \\
\Longrightarrow \frac{y}{R-1} - (1-\pi_1)p\left(\frac{\overline{x}}{R} - \frac{\underline{x}}{R-1}\right) > \frac{y}{R-1} - (1-\pi_1)\frac{(1-(1-p)^2\overline{x}^2)y}{R^2(R-1)},
\end{aligned}$$

where the second inequality follows from $1-(1-p)^2\overline{x}^2 = p^2\underline{x}^2 + 2(1-p)\overline{x}p\underline{x} > (1-p)\overline{x}p\underline{x}$. For $(1-p)\overline{x} \geq 0.5^{0.5}$ and $\frac{\overline{x}}{\underline{x}} \geq \min\{d_g, z\}$, good borrowers prefer one-period debt since

$$\begin{aligned}
&(1-p)\overline{x}p\underline{x} < p^2\underline{x}^2 + 2(1-p)\overline{x}p\underline{x} = 1-(1-p)^2\overline{x}^2 \\
&\Longrightarrow \frac{(1-p)\overline{x}p\underline{x}y}{R^2(R-1)} < \frac{(1-(1-p)^2\overline{x}^2)y}{R^2(R-1)} \\
&\Longrightarrow \frac{y}{R-1} - \frac{(1-\pi_1)(1-p)\overline{x}p\underline{x}y}{R^2(R-1)} > \frac{y}{R-1} - (1-\pi_1)\frac{(1-(1-p)^2\overline{x}^2)y}{R^2(R-1)}.
\end{aligned}$$

2. Continuous distribution of collateral growth

We modify the distribution of collateral growth to be continuous with bounded support. Our main results remain the same. When uncertainty is low, good borrowers fully separate by deleveraging, that is borrowing a sufficiently high amount such that subsequent repayment reveals the presence of unobservable income. When uncertainty is higher, good borrowers pay an adverse selection cost through a higher interest rate, and full separation may not be optimal.

We continue to use Assumptions 1 (regarding off-equilibrium beliefs) and 2 (that unobservable income is a constant proportion of observable income). However, we modify Assumption 3 as follows.

Assumption 3″. *The growth rate of observable income* $x_t = \frac{X_t}{X_{t-1}}$ *is drawn from a continuous distribution* $G(x)$ *with mean one and bounded support* $[\underline{x}, \overline{x}]$.

Proposition 1 in the binomial case continues to hold under this alternative distributional assumption.

Proposition 1″ (Costless full separation). *Suppose that uncertainty in collateral value is low. That is,* $\frac{\overline{x}}{\underline{x}} < \frac{R}{R-1}$. *In period 1, all borrowers borrow* $F_1 \in (X_1\overline{x}, RV_1\underline{x}]$ *at the interest rate* $P_1^{-1} = R$. *In period 2, borrower type is fully revealed. Good borrowers repay by rolling over* $F_2 \in [R\max\{0, F_1 - X_2 - Y_2\}, R(F_1 - X_2))$. *Bad borrowers repay by rolling over* $F_2 \in [R(F_1 - X_2), RV_2]$.

Proof. Lemma 11 below implies the equilibrium in period 1. Lemma 5 implies the equilibrium in period 2.

Propositions 2 and 3 in the binomial case are combined into a single proposition under the alternative distributional assumption.

Proposition 2″ (Costly full separation versus partial separation). *Suppose that uncertainty in collateral value is higher. That is, $\frac{\overline{x}}{\underline{x}} \geq \frac{R}{R-1}$. In period 1, all borrowers borrow $F_1 \in [RV_1\underline{x}, X_1\overline{x}]$ at an interest rate $P_1^{-1} > R$.*

With probability $\Pr(F_1 \leq X_2) > 0$, borrower type is not revealed in period 2. All borrowers repay by rolling over $F_2 \in (RV_2\underline{x}, RV_2\overline{x}]$ at an interest rate $P_2^{-1} > R$. Subsequently in period 3, all borrowers repay with probability $\Pr(F_2 \leq D_{b,3}) > 0$, in which case borrower type is not revealed. Otherwise, only bad borrowers default with probability $\Pr(D_{b,3} < F_2 \leq D_{g,3}) > 0$, in which case borrower type is fully revealed. Otherwise, all borrowers default with probability $\Pr(D_{g,3} < F_2) > 0$, in which case borrower type is not revealed.

With probability $\Pr(X_2 < F_1 \leq D_{b,2}) > 0$, borrower type is fully revealed in period 2. Good borrowers repay by rolling over $F_2 \in [R\max\{0, F_1 - X_2 - Y_2\}, R(F_1 - X_2))$. Bad borrowers repay by rolling over $F_2 \in [R(F_1 - X_2), RV_2]$.

With probability $\Pr(D_{b,2} < F_1 \leq D_{g,2}) > 0$, only bad borrowers default, in which case borrower type is fully revealed in period 2.

If $\frac{\overline{x}}{\underline{x}} \geq \frac{R}{R-1} + y$, all borrowers default in period 2 with probability $\Pr(D_{g,2} < F_1) > 0$, in which case borrower type is not revealed.

Proof. Lemma 11 implies the equilibrium in period 1, and Lemma 10 implies the equilibrium in period 2 when there is no separation. Lemmas 4, 5, 6, and 7 imply the equilibrium in periods 2 and 3.

2.1. Lemmas used to prove Propositions 1″ and 2″

To simplify notation, let lowercase letters denote the corresponding variables divided by X_t. That is, $f_t = \frac{F_t}{X_t}$, $d_{i,t} = \frac{D_{i,t}}{X_t}$, $w_{i,t} = \frac{W_{i,t}}{X_t}$, and $c_{i,t} = \frac{C_{i,t}}{X_t}$.

Lemma 10. *If $f_1 \leq x_2$ in period 2, all borrowers borrow $f_2 \in (d_{b,3}\underline{x}, d_{b,3}\overline{x}]$ at an interest rate $P_2^{-1} > R$. The value of non-pledgeable assets for good borrowers satisfies $w_{g,2} < \frac{y}{R-1}$.*

Proof. If $f_1 \leq x_2$, Lemma 4 implies no updating of reputation in period 2 so that $\pi_2 = \pi_1$. Good borrowers choose f_2 that maximizes $w_{g,2}$, and bad borrowers mimic good borrowers.

In period $t \in \{1, 2\}$, expected repayment for a type i borrower is

$$c_{i,t} = \begin{cases} \frac{1}{R-1} & \text{if } f_t > d_{i,t+1}\overline{x} \\ \int_{\frac{f_t}{d_{i,t+1}}}^{\overline{x}} \left(\frac{f_t}{R} - \frac{x}{R-1}\right) dG(x) + \frac{1}{R-1} & \text{if } f_t \in (d_{i,t+1}\underline{x}, d_{i,t+1}\overline{x}] \\ \frac{f_t}{R} & \text{if } f_t \leq d_{i,t+1}\underline{x}. \end{cases}.$$

If $d_{b,t+1}\overline{x} < d_{g,t+1}\underline{x}$, the difference in expected repayment is

$$c_{g,t} - c_{b,t} = \begin{cases} 0 & \text{if } f_t > d_{g,t+1}\overline{x} \\ \int_{\frac{f_t}{d_{g,t+1}}}^{\overline{x}} \left(\frac{f_t}{R} - \frac{x}{R-1}\right) dG(x) & \text{if } f_t \in (d_{g,t+1}\underline{x}, d_{g,t+1}\overline{x}] \\ \frac{f_t}{R} - \frac{1}{R-1} & \text{if } f_t \in (d_{b,t+1}\overline{x}, d_{g,t+1}\underline{x}] \\ \int_{\underline{x}}^{\frac{f_t}{d_{b,t+1}}} \left(\frac{f_t}{R} - \frac{x}{R-1}\right) dG(x) & \text{if } f_t \in (d_{b,t+1}\underline{x}, d_{b,t+1}\overline{x}] \\ 0 & \text{if } f_t \leq d_{b,t+1}\underline{x}. \end{cases}. \quad \text{(OA.5)}$$

If $d_{b,t+1}\overline{x} \geq d_{g,t+1}\underline{x}$, the difference in expected repayment is

$$c_{g,t} - c_{b,t} = \begin{cases} 0 & \text{if } f_t > d_{g,t+1}\overline{x} \\ \int_{\frac{f_t}{d_{g,t+1}}}^{\overline{x}} \left(\frac{f_t}{R} - \frac{x}{R-1}\right) dG(x) & \text{if } f_t \in (d_{b,t+1}\overline{x}, d_{g,t+1}\overline{x}] \\ \int_{\frac{f_t}{d_{g,t+1}}}^{\frac{f_t}{d_{b,t+1}}} \left(\frac{f_t}{R} - \frac{x}{R-1}\right) dG(x) & \text{if } f_t \in (d_{g,t+1}\underline{x}, d_{b,t+1}\overline{x}] \\ \int_{\underline{x}}^{\frac{f_t}{d_{b,t+1}}} \left(\frac{f_t}{R} - \frac{x}{R-1}\right) dG(x) & \text{if } f_t \in (d_{b,t+1}\underline{x}, d_{g,t+1}\underline{x}] \\ 0 & \text{if } f_t \leq d_{b,t+1}\underline{x}. \end{cases} \quad \text{(OA.6)}$$

If $d_{b,3}\overline{x} < d_{g,3}\underline{x}$, equations (3), (12), and (OA.5) imply that

$$w_{g,2} = \begin{cases} \frac{y}{R-1} - \frac{(1-\pi_2)y}{R(R-1)} & \text{if (1) } f_2 > d_{g,3}\overline{x} \\ -(1-\pi_2)\int_{\frac{f_2}{d_{g,3}}}^{\overline{x}} \left(\frac{f_2}{R} - \frac{x}{R-1}\right) dG(x) & \\ +\frac{y}{R-1} - \frac{(1-\pi_2)y}{R(R-1)} \int_{\underline{x}}^{\frac{f_2}{d_{g,3}}} x dG(x) & \text{if (2) } f_2 \in (d_{g,3}\underline{x}, d_{g,3}\overline{x}] \\ -(1-\pi_2)\left(\frac{f_2}{R} - \frac{1}{R-1}\right) + \frac{y}{R-1} & \text{if (3) } f_2 \in (d_{b,3}\overline{x}, d_{g,3}\underline{x}] \\ -(1-\pi_2)\int_{\underline{x}}^{\frac{f_2}{d_{b,3}}} \left(\frac{f_2}{R} - \frac{x}{R-1}\right) dG(x) & \\ +\frac{y}{R-1} - \frac{(1-\pi_2)y}{R(R-1)} \int_{\frac{f_2}{d_{b,3}}}^{\overline{x}} x dG(x) & \text{if (4) } f_2 \in (d_{b,3}\underline{x}, d_{b,3}\overline{x}] \\ \frac{y}{R-1} - \frac{(1-\pi_2)y}{R(R-1)} & \text{if (5) } f_2 \leq d_{b,3}\underline{x}. \end{cases}$$

Note that $w_{g,2}$ is continuous in f_2. Lemma 12 implies that $w_{g,2}$ is strictly decreasing in region (2). Moreover, $w_{g,2}$ is strictly decreasing in region (3). Therefore, $w_{g,2}$ is maximized for some $f_2 \in (d_{b,3}\underline{x}, d_{b,3}\overline{x}]$, attaining a value $w_{g,2} < \frac{y}{R-1}$.

If $d_{b,3}\overline{x} \geq d_{g,3}\underline{x}$, equations (3), (12), and (OA.6) imply that

$$w_{g,2} = \begin{cases} \frac{y}{R-1} - \frac{(1-\pi_2)y}{R(R-1)} & \text{if (1) } f_2 > d_{g,3}\overline{x} \\ -(1-\pi_2)\int_{\frac{f_2}{d_{g,3}}}^{\overline{x}} \left(\frac{f_2}{R} - \frac{x}{R-1}\right) dG(x) & \\ +\frac{y}{R-1} - \frac{(1-\pi_2)y}{R(R-1)} \int_{\underline{x}}^{\frac{f_2}{d_{g,3}}} x dG(x) & \text{if (2) } f_2 \in (d_{b,3}\overline{x}, d_{g,3}\overline{x}] \\ -(1-\pi_2)\int_{\frac{f_2}{d_{g,3}}}^{\frac{f_2}{d_{b,3}}} \left(\frac{f_2}{R} - \frac{x}{R-1}\right) dG(x) & \\ \frac{y}{R-1} - \frac{(1-\pi_2)y}{R(R-1)} \left(\int_{\frac{f_2}{d_{b,3}}}^{\overline{x}} + \int_{\underline{x}}^{\frac{f_2}{d_{g,3}}}\right) x dG(x) & \text{if (3) } f_2 \in (d_{g,3}\underline{x}, d_{b,3}\overline{x}] \\ -(1-\pi_2)\int_{\underline{x}}^{\frac{f_2}{d_{b,3}}} \left(\frac{f_2}{R} - \frac{x}{R-1}\right) dG(x) & \\ +\frac{y}{R-1} - \frac{(1-\pi_2)y}{R(R-1)} \int_{\frac{f_2}{d_{b,3}}}^{\overline{x}} x dG(x) & \text{if (4) } f_2 \in (d_{b,3}\underline{x}, d_{g,3}\underline{x}] \\ \frac{y}{R-1} - \frac{(1-\pi_2)y}{R(R-1)} & \text{if (5) } f_2 \leq d_{b,3}\underline{x}. \end{cases}$$

Note that $w_{g,2}$ is continuous in f_2. Lemma 12 implies that $w_{g,2}$ is strictly decreasing in region (2). Therefore, $w_{g,2}$ is maximized for some $f_2 \in (d_{b,3}\underline{x}, d_{b,3}\overline{x}]$, attaining a value $w_{g,2} < \frac{y}{R-1}$.

Equations (OA.5) and (OA.6) imply that $c_{g,2} - c_{b,2} > 0$ for $f_2 \in (d_{b,3}\underline{x}, d_{b,3}\overline{x}]$, so that $P_2^{-1} > R$.

Lemma 11. *Assume that $f_0 \leq x_1$ in period 1. If $\frac{\overline{x}}{\underline{x}} \in [1, d_{b,2})$, all borrowers borrow $f_1 \in (\overline{x}, d_{b,2}\underline{x}]$ at an interest rate $P_1^{-1} = R$. If $\frac{\overline{x}}{\underline{x}} \geq d_{b,2}$, all borrowers borrow $f_1 \in [d_{b,2}\underline{x}, \overline{x}]$ at an interest rate $P_1^{-1} > R$.*

Proof. If $f_0 \leq x_1$, Lemma 4 implies no updating of reputation in period 1 so that $\pi_1 = \pi_0$. Good borrowers choose f_1 that maximizes $w_{g,1}$, and bad borrowers mimic good borrowers. Equations (OA.5) and (OA.6) imply that $c_{g,1} - c_{b,1} > 0$ for $f_1 \in [d_{b,2}\underline{x}, \overline{x}]$, so that $P_1^{-1} > R$.

In period $t \in \{1, 2\}$, Assumption 2 and Lemma 3 imply that the default boundaries satisfy the inequality

$$\frac{d_{g,t}}{d_{b,t}} \leq \frac{\frac{R}{R-1} + \frac{y}{R^{3-t}(R-1)}}{\frac{R}{R-1}} \leq \frac{R+y}{R} < \frac{R}{R-1} \leq d_{b,2}.$$

In addition, Lemmas 5, 6, and 10 imply that

$$w_{g,2} = \begin{cases} \frac{y}{R-1} & \text{if } f_1 \in (x_2, d_{g,2}x_2] \\ w_{g,2}(3,4) < \frac{y}{R-1} & \text{if } f_1 \leq x_2 \end{cases}, \tag{OA.7}$$

where $w_{g,2}(3, 4)$ is the maximum value that $w_{g,2}$ attains if $f_1 \leq x_2$, as stated in Lemma 10.

If $\frac{\overline{x}}{\underline{x}} < \frac{d_{g,2}}{d_{b,2}}$, equations (3), (12), (OA.5), and (OA.7) imply that

$$w_{g,1} = \begin{cases} \frac{y}{R-1} - \frac{(1-\pi_0)y}{R^2(R-1)} & \text{if (1) } f_1 > d_{g,2}\overline{x} \\ -(1-\pi_1)\int_{\frac{f_1}{d_{g,2}}}^{\overline{x}} \left(\frac{f_1}{R} - \frac{x}{R-1}\right) dG(x) & \\ +\frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} \int_{\underline{x}}^{\frac{f_1}{d_{g,2}}} x dG(x) & \text{if (2) } f_1 \in (d_{g,2}\underline{x}, d_{g,2}\overline{x}] \\ -(1-\pi_1)\left(\frac{f_1}{R} - \frac{1}{R-1}\right) + \frac{y}{R-1} & \text{if (3) } f_1 \in (d_{b,2}\overline{x}, d_{g,2}\underline{x}] \\ -(1-\pi_1)\int_{\underline{x}}^{\frac{f_2}{d_{b,2}}} \left(\frac{f_1}{R} - \frac{x}{R-1}\right) dG(x) + \frac{y}{R-1} & \text{if (4) } f_1 \in (d_{b,2}\underline{x}, d_{b,2}\overline{x}] \\ \frac{y}{R-1} & \text{if (5) } f_1 \in (\overline{x}, d_{b,2}\underline{x}] \\ \frac{y}{R-1} - \frac{1}{R}\left(\frac{y}{R-1} - w_{g,2}(3,4)\right)\int_{f_1}^{\overline{x}} x dG(x) & \text{if (6) } f_1 \in (\underline{x}, \overline{x}] \\ \frac{y}{R-1} - \frac{1}{R}\left(\frac{y}{R-1} - w_{g,2}(3,4)\right) & \text{if (7) } f_1 \leq \underline{x}. \end{cases}$$

Note that $w_{g,1}$ is maximized in region (5) for any $f_1 \in (\overline{x}, d_{b,2}\underline{x}]$.

If $\frac{\overline{x}}{\underline{x}} \in [\frac{d_{g,2}}{d_{b,2}}, d_{b,2})$, equations (3), (12), (OA.6), and (OA.7) imply that

$$w_{g,1} = \begin{cases} \frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} & \text{if (1) } f_1 > d_{g,2}\overline{x} \\ -(1-\pi_1)\int_{\frac{f_1}{d_{g,2}}}^{\overline{x}} \left(\frac{f_1}{R} - \frac{x}{R-1}\right) dG(x) & \\ +\frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} \int_{\underline{x}}^{\frac{f_1}{d_{g,2}}} xdG(x) & \text{if (2) } f_1 \in (d_{b,2}\overline{x}, d_{g,2}\overline{x}] \\ -(1-\pi_1)\int_{\frac{f_2}{d_{g,2}}}^{\frac{f_1}{d_{b,2}}} \left(\frac{f_1}{R} - \frac{x}{R-1}\right) dG(x) & \\ +\frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} \int_{\underline{x}}^{\frac{f_1}{d_{g,2}}} xdG(x) & \text{if (3) } f_1 \in (d_{g,2}\underline{x}, d_{b,2}\overline{x}] \\ -(1-\pi_1)\int_{\underline{x}}^{\frac{f_1}{d_{b,2}}} \left(\frac{f_1}{R} - \frac{x}{R-1}\right) dG(x) + \frac{y}{R-1} & \text{if (4) } f_1 \in (d_{b,2}\underline{x}, d_{g,2}\underline{x}] \\ \frac{y}{R-1} & \text{if (5) } f_1 \in (\overline{x}, d_{b,2}\underline{x}] \\ \frac{y}{R-1} - \frac{1}{R}\left(\frac{y}{R-1} - w_{g,2}(3,4)\right)\int_{f_1}^{\overline{x}} xdG(x) & \text{if (6) } f_1 \in (\underline{x}, \overline{x}] \\ \frac{y}{R-1} - \frac{1}{R}\left(\frac{y}{R-1} - w_{g,2}(3,4)\right) & \text{if (7) } f_1 \leq \underline{x}. \end{cases}$$

Note that $w_{g,1}$ is maximized in region (5) for any $f_1 \in (\overline{x}, d_{b,2}\underline{x}]$.

If $\frac{\overline{x}}{\underline{x}} \in [d_{b,2}, d_{g,2})$, equations (3), (12), (OA.6), and (OA.7) imply that

$$w_{g,1} = \begin{cases} \frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} & \text{if (1) } f_1 > d_{g,2}\overline{x} \\ -(1-\pi_1)\int_{\frac{f_1}{d_{g,2}}}^{\overline{x}} \left(\frac{f_1}{R} - \frac{x}{R-1}\right) dG(x) & \\ +\frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} \int_{\underline{x}}^{\frac{f_1}{d_{g,2}}} xdG(x) & \text{if (2) } f_1 \in (d_{b,2}\overline{x}, d_{g,2}\overline{x}] \\ -(1-\pi_1)\int_{\frac{f_2}{d_{g,2}}}^{\frac{f_1}{d_{b,2}}} \left(\frac{f_1}{R} - \frac{x}{R-1}\right) dG(x) & \\ +\frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} \int_{\underline{x}}^{\frac{f_1}{d_{g,2}}} xdG(x) & \text{if (3) } f_1 \in (d_{g,2}\underline{x}, d_{b,2}\overline{x}] \\ -(1-\pi_1)\int_{\underline{x}}^{\frac{f_1}{d_{b,2}}} \left(\frac{f_1}{R} - \frac{x}{R-1}\right) dG(x) + \frac{y}{R-1} & \text{if (4) } f_1 \in (\overline{x}, d_{g,2}\underline{x}] \\ -(1-\pi_1)\int_{\underline{x}}^{\frac{f_1}{d_{b,2}}} \left(\frac{f_1}{R} - \frac{x}{R-1}\right) dG(x) & \\ \frac{y}{R-1} - \frac{1}{R}\left(\frac{y}{R-1} - w_{g,2}(3,4)\right)\int_{f_1}^{\overline{x}} xdG(x) & \text{if (5) } f_1 \in (d_{b,2}\underline{x}, \overline{x}] \\ \frac{y}{R-1} - \frac{1}{R}\left(\frac{y}{R-1} - w_{g,2}(3,4)\right)\int_{f_1}^{\overline{x}} xdG(x) & \text{if (6) } f_1 \in (\underline{x}, d_{b,2}\underline{x}] \\ \frac{y}{R-1} - \frac{1}{R}\left(\frac{y}{R-1} - w_{g,2}(3,4)\right) & \text{if (7) } f_1 \leq \underline{x}. \end{cases}$$

Note that $w_{g,1}$ is continuous in f_1 and constant in regions (1) and (7). Lemma 12 implies that $w_{g,1}$ is strictly decreasing in region (2), and Lemma 13 implies that $w_{g,1}$ is strictly decreasing in region (3). Moreover, $w_{g,1}$ is strictly decreasing in region (4) and strictly increasing in region (6). Therefore, $w_{g,1}$ is maximized for some $f_1 \in [d_{b,2}\underline{x}, \overline{x}]$, attaining a value $w_{g,1} < \frac{y}{R-1}$.

If $\frac{\overline{x}}{\underline{x}} \geq d_{g,2}$, equations (3), (12), (OA.6), and (OA.7) imply that

$$w_{g,1} = \begin{cases} \frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} & \text{if (1) } f_1 > d_{g,2}\overline{x} \\ -(1-\pi_1)\int_{\frac{f_1}{d_{g,2}}}^{\overline{x}} \left(\frac{f_1}{R} - \frac{x}{R-1}\right) dG(x) & \\ +\frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} \int_{\underline{x}}^{\frac{f_1}{d_{g,2}}} x dG(x) & \text{if (2) } f_1 \in (d_{b,2}\overline{x}, d_{g,2}\overline{x}] \\ -(1-\pi_1)\int_{\frac{f_1}{d_{g,2}}}^{\frac{f_1}{d_{b,2}}} \left(\frac{f_1}{R} - \frac{x}{R-1}\right) dG(x) & \\ +\frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} \int_{\underline{x}}^{\frac{f_1}{d_{g,2}}} x dG(x) & \text{if (3) } f_1 \in (\overline{x}, d_{b,2}\overline{x}] \\ -(1-\pi_1)\int_{\frac{f_1}{d_{g,2}}}^{\frac{f_1}{d_{b,2}}} \left(\frac{f_1}{R} - \frac{x}{R-1}\right) dG(x) & \\ \frac{y}{R-1} - \frac{(1-\pi_1)y}{R^2(R-1)} \int_{\underline{x}}^{\frac{f_1}{d_{g,2}}} x dG(x) & \\ -\frac{1}{R}\left(\frac{y}{R-1} - w_{g,2}(3,4)\right)\int_{f_1}^{\overline{x}} x dG(x) & \text{if (4) } f_1 \in (d_{g,2}\underline{x}, \overline{x}] \\ -(1-\pi_1)\int_{\underline{x}}^{\frac{f_1}{d_{b,2}}} \left(\frac{f_1}{R} - \frac{x}{R-1}\right) dG(x) & \\ \frac{y}{R-1} - \frac{1}{R}\left(\frac{y}{R-1} - w_{g,2}(3,4)\right)\int_{f_1}^{\overline{x}} x dG(x) & \text{if (5) } f_1 \in (d_{b,2}\underline{x}, d_{g,2}\underline{x}] \\ \frac{y}{R-1} - \frac{1}{R}\left(\frac{y}{R-1} - w_{g,2}(3,4)\right)\int_{f_1}^{\overline{x}} x dG(x) & \text{if (6) } f_1 \in (\underline{x}, d_{b,2}\underline{x}] \\ \frac{y}{R-1} - \frac{1}{R}\left(\frac{y}{R-1} - w_{g,2}(3,4)\right) & \text{if (7) } f_1 \leq \underline{x}. \end{cases}.$$

Note that $w_{g,1}$ is continuous in f_1 and constant in regions (1) and (7). Lemma 12 implies that $w_{g,1}$ is strictly decreasing in region (2), and Lemma 13 implies that $w_{g,1}$ is strictly decreasing in region (3). Moreover, $w_{g,1}$ is strictly increasing in region (6). Therefore, $w_{g,1}$ is maximized for some $f_1 \in [d_{b,2}\underline{x}, \overline{x}]$, attaining a value $w_{g,1} < \frac{y}{R-1}$.

Lemma 12. *Let*

$$h(f) = \int_{\frac{f}{d_g}}^{\overline{x}} \left(\frac{f}{R} - \frac{x}{R-1}\right) dG(x) + \frac{y}{R^t(R-1)} \int_{\underline{x}}^{\frac{f}{d_g}} x dG(x),$$

where $d_g = \frac{R}{R-1} + y$ *and* $t \in [0,1]$. *Then* $h(f)$ *is strictly increasing.*

Proof.

$$h'(f) = -\left(\frac{f}{R} - \frac{1}{R-1}\frac{f}{d_g}\right) g\left(\frac{f}{d_g}\right)\frac{1}{d_g} + \int_{\frac{f}{d_g}}^{\overline{x}} \frac{1}{R} dG(x) + \frac{y}{R^t(R-1)}\frac{f}{d_g} g\left(\frac{f}{d_g}\right)\frac{1}{d_g},$$

where $g(\cdot)$ denotes the density function of $G(\cdot)$. This expression can be rewritten as

$$h'(f) = \frac{1}{Rd_g}\left[y\frac{f}{d_g} g\left(\frac{f}{d_g}\right)\left(\frac{1}{R^{t-1}(R-1)} - 1\right) + d_g\left(1 - G\left(\frac{f}{d_g}\right)\right)\right] > 0.$$

Lemma 13. *Let*

$$h(f) = \int_{\frac{f}{d_g}}^{\frac{f}{d_b}} \left(\frac{f}{R} - \frac{x}{R-1} \right) dG(x) + \frac{y}{R^2(R-1)} \int_{\underline{x}}^{\frac{f}{d_g}} x dG(x),$$

where $d_b = \frac{R}{R-1}$ *and* $d_g = \frac{R}{R-1} + y$*. Then* $h(f)$ *is strictly increasing for* $f \in (d_g \underline{x}, d_b \overline{x}]$*.*

Proof.

$$h'(f) = \left(\frac{f}{R} - \frac{1}{R-1} \frac{f}{d_b} \right) g\left(\frac{f}{d_b} \right) \frac{1}{d_b} - \left(\frac{f}{R} - \frac{1}{R-1} \frac{f}{d_g} \right) g\left(\frac{f}{d_g} \right) \frac{1}{d_g} + \int_{\frac{f}{d_g}}^{\frac{f}{d_b}} \frac{1}{R} dG(x) + \frac{y}{R^2(R-1)} \frac{f}{d_g} g\left(\frac{f}{d_g} \right) \frac{1}{d_g},$$

where $g(\cdot)$ denotes the density function of $G(\cdot)$. This expression can be rewritten as

$$h'(f) = \frac{1}{R d_g} \left[y \frac{f}{d_g} g\left(\frac{f}{d_g} \right) \left(\frac{1}{R(R-1)} - 1 \right) + d_g \left(G\left(\frac{f}{d_b} \right) - G\left(\frac{f}{d_g} \right) \right) \right] > 0.$$