Lecture Notes

Macroeconomics - ECON 510a, Fall 2010, Yale University

Growth

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1 Introduction

Growth is an important topic in macroeconomics, which represents a large fraction of the literature and seeks to explain some facts in the long-term behavior of economies, complementing the also large literature about business cycle. These two topics are so relevant because macroeconomics is mostly worried about the first and second moments of macroeconomic activity (this is, development and cycles).

In what follows we will motivate the empirical regularities that the literature on growth seeks to explain. Then, after describing our economy, we will discuss exogenous growth models, in which an exogenous change in the production technology generates income growth as a theoretical result. The third section introduces technological change as a decision variable, and hence the growth rate becomes endogenously determined. Finally, we will discuss how, even when these models are successful in explaining growth in the last century, they are not useful in explaining the long stagnation before the Industrial Revolution. In this sense, our framework can also accommodates the Malthusian points that in economies based on agriculture the population growth is an increasing function of income per capita.

1.1 Kaldor's Stylized Facts

1. The growth rate of output g_y is constant over time.
2. The capital-labor ratio K/L grows at a constant rate.
3. The capital-income ratio K/Y is constant.
4. Capital and labor shares of income are constant.
5. Real rates of return are constant.
6. Growth rates persistently vary across countries.

1.2 Other Important Facts

1. Labor productivity Y/L is very dispersed across countries.
2. The distribution of Y/L does not seem to spread out (although the variance has increased somewhat).
3. Countries with low incomes in 1960 did not show on average higher subsequent growth (this phenomenon is sometimes referred to as "no absolute convergence").
4. There is "conditional convergence": Within groups classified by 1960 human capital measures (such as schooling), 1960 savings rates, and other indicators, a higher initial income Y_0 (in 1960) was positively correlated with a lower growth rate g_y: This is studied by performing the "growth regression":

   $$g_{y,i}^{1960-1990} = \alpha + \beta y_{0,i} + \gamma edu_{0,i} + \epsilon_i, \qquad i = 1, ..., n$$

   Then controlling for the initial level of education, the growth rate was negatively correlated with initial income for the period 1960-1990: $\beta < 0$. Whereas if the regression is performed without controlling for the level of education, the result for the period is $\beta = 0$, i.e. no absolute convergence, as mentioned above.
5. Foreign trade volume seems to correlate positively with growth.
6. Demographic growth (fertility) is negatively correlated with income.

7. Growth in factor inputs (capital, labor) is not enough to explain output growth. The idea of an ”explanation” of growth is due to Solow, who envisaged the method of ”growth accounting”. Based on a neoclassical production function $Y = AF(K; L)$; the variable A captures the idea of technological change. If goods production is performed using a constant-returns-to-scale technology, operated under perfect competition, then (by an application of the Euler Theorem) it is possible to estimate how much out of total production growth is due to each production factor, and how much to the technological factor A. The empirical studies have shown that the contribution of A (the Solow residual) to output growth is very significant.

8. Workers tend to migrate into high-income countries.

2 The Economy

The economy has a constant population composed by identical households whose utility function is

$$U_0 = \sum_{t=0}^{\infty} \beta^t u(C_t)$$

where we will assume

$$u(C) = \frac{C^{1-\sigma}}{1-\sigma}$$

for $\sigma \in [0, \infty)$ where $\sigma = 1$ is taken to be the logarithmic utility function.

We will assume the production function is Cobb-Douglas with constant return of scale

$$Y_t = AK_t^{\theta}(X_t N_t)^{1-\theta}$$

Naturally this production function is an example of a function with positive but diminishing marginal products and Inada conditions. Finally, the law of motion of capital is

$$K_{t+1} = (1-\delta)K_t + I_t$$

Given these conditions it is straightforward to check the First and Second Welfare Theorems. We will start analyzing the planner's solution.

2.1 No growth

Trivially we can write all variables per effective labor, denoting $c_t = \frac{C_t}{X_t N_t}$, $y_t = \frac{Y_t}{X_t N_t}$, $i_t = \frac{I_t}{X_t N_t}$ and $k_t = \frac{K_t}{X_t N_t}$. Then, households' problem is

$$max \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

subject to

$$\begin{aligned} y_t &= Ak_t^\theta \\ k_{t+1} &= (1-\delta)k_t + i_t \\ y_t &= c_t + i_t \end{aligned}$$

We can write this in dynamic programming form

$$v(k) = max_{k'}\{u(Ak^\theta - k' + (1-\delta)k) + \beta v(k')\}$$

The first order condition is (with $v_k = \frac{\partial v(k)}{\partial k}$)

$$-u_c + \beta v_{k'} = 0$$

The envelope condition is

$$v_k = u_c \left(\theta A k^{\theta-1} + 1 - \delta\right)$$

From these two conditions we get the Euler equation that determines the evolution of consumption

$$u_c = \beta u_{c'} \left(\theta A k'^{\theta-1} + 1 - \delta\right)$$

and evaluating at steady state

$$\beta \left(\theta A k^{*,\theta-1} + 1 - \delta\right) = 1$$

The steady state with a constant effective labor input must therefore be a constant consumption level and capital-labor ratio k^* obtained from the equation above. Naturally this is a benchmark economy without growth.

2.2 Exogenous growth

As in Solow's (1956) classical work, the simplest way to ensure consumption steady state growth (or a balance growth path) is to assume an exogenous growth of population (N_t) or of labor-augmenting technological change (X_t) at a constant rate. This is, assume

$$\begin{aligned} X_t &= (1+\gamma)^t \qquad for\ X_0\ given \\ N_t &= (1+n)^t \qquad for\ N_0\ given \end{aligned}$$

Now, we can redefine (normalize $N_0 = X_0 = 1$, multiply and divide by $N_t X_t$) households' problems as maximizing

$$max \sum_{t=0}^{\infty} \bar{\beta}^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

where $\bar{\beta} \equiv \beta[(1+\gamma)(1+n)]^{1-\sigma}$, subject to

$$\begin{aligned} y_t &= Ak_t^\theta \\ (1+\gamma)(1+n)k_{t+1} &= (1-\delta)k_t + i_t \\ y_t &= c_t + i_t \end{aligned}$$

We can write this in dynamic programming form

$$v(k) = max_{k'}\{u(Ak^\theta - (1+\gamma)(1+n)k' + (1-\delta)k) + \bar{\beta}v(k')\}$$

The first order condition is (with $v_k = \frac{\partial v(k)}{\partial k}$)

$$-u_c(1+\gamma)(1+n) + \bar{\beta}v_{k'} = 0$$

The envelope condition is

$$v_k = u_c\left(\theta Ak^{\theta-1} + 1 - \delta\right)$$

From these two conditions we get the Euler equation that determines the evolution of consumption

$$u_c(1+\gamma)(1+n) = \bar{\beta} u_{c'} \left(\theta A k'^{\theta-1} + 1 - \delta\right)$$

and evaluating at steady state we get

$$\bar{\beta} \left(\theta A k^{*,\theta-1} + 1 - \delta\right) = (1+\gamma)(1+n)$$

or, which is the same

$$\beta \left(\theta A k^{*,\theta-1} + 1 - \delta\right) = [(1+\gamma)(1+n)]^{\sigma}$$

As can be seen in this case, $k = \frac{K_t}{X_t N_t}$ is constant. Since $X_t N_t$ grow at a rate $(1+\gamma)(1+n)$, K_t and also Y_t grow at the same rate. Naturally, this means that production and consumption per capita grows at an exogenous rate $(1+\gamma)$.

3 Endogenous growth

The question is whether a government can do something to promote growth. Naturally this is not the case under exogenous growth, which is also efficient. We will review two different ways to endogeneize growth: By human capital investment and by research and monopolistic competition. In both cases the competitive equilibrium is not efficient and falls short in providing growth in an economy. For this reason we will also characterize the competitive equilibrium.

3.1 Human capital

In this case, all factors of production are reproducible, which means the economy can invest not only on capital but also on human capital. In this case we interpret X_t as human capital, which increases the effective hours of labor in an economy composed by a population N_t. Now we will have two types of capital, possibly with different depreciation rates δ_K and δ_X.

First, let's define variables per capita (this is, dividing them by the constant population N), such that

$$\hat{Y}_t = A\hat{K}_t^{\theta} X_t^{1-\theta}$$

We can redefine the planner's problems as maximizing

$$max \sum_{t=0}^{\infty} \beta^t \frac{\hat{c}_t^{1-\sigma}}{1-\sigma}$$

where $\hat{C}_t = C_t/N$, subject to

$$\begin{aligned}
\hat{Y}_t &= A\hat{K}_t^{\theta} X_t^{1-\theta} \\
\hat{K}_{t+1} &= (1-\delta)\hat{K}_t + \hat{I}_t^K \\
X_{t+1} &= (1-\delta)X_t + \hat{I}_t^X \\
\hat{Y}_t &= \hat{C}_t + \hat{I}_t^K + \hat{I}_t^X
\end{aligned}$$

We can write this in dynamic programming form

$$v(\hat{K}, X) = max_{\hat{K}', X'}\{u\left(A\hat{K}_t^{\theta} X_t^{1-\theta} - \hat{K}' + (1-\delta_K)\hat{K} - X' + (1-\delta_X)X\right) + \beta v(\hat{K}', X')\}$$

The first order condition is (with $v_k = \frac{\partial v(\hat{K}, X)}{\partial \hat{K}}$ and $v_x = \frac{\partial v(\hat{K}, X)}{\partial X}$)

$$\begin{aligned}
\beta v_{x'} &= u_c \\
\beta v_{k'} &= u_c
\end{aligned}$$

The envelope condition is

$$\begin{aligned}
v_x &= u_c\left((1-\theta)\frac{\hat{Y}}{X} + 1 - \delta_X\right) \\
v_k &= u_c\left(\theta\frac{\hat{Y}}{\hat{K}} + 1 - \delta_K\right)
\end{aligned}$$

From these last conditions we get the Euler equation that determines the evolution of consumption in terms of capital and human capital

$$\begin{aligned} u_c &= \beta u_{c'}\left((1-\theta)\frac{\hat{Y}'}{X'}+1-\delta_X\right) \\ u_c &= \beta u_{c'}\left(\theta\frac{\hat{Y}'}{\hat{K}'}+1-\delta_K\right) \end{aligned}$$

Taking the difference between these two expressions

$$(1-\theta)\frac{\hat{Y}'}{X'}-\theta\frac{\hat{Y}'}{\hat{K}'}=\delta_X-\delta_K$$

Since $k=\frac{\hat{K}}{X}$ and $y=\frac{\hat{Y}}{X}$

$$\begin{aligned} \theta\frac{\hat{Y}}{\hat{K}} &= \theta A k^{\theta-1} \\ (1-\theta)\frac{\hat{Y}}{X} &= (1-\theta)Ak^{\theta}=y-\theta Ak^{\theta-1}k \end{aligned}$$

Then

$$y-(1+k)\theta Ak^{\theta-1}=\delta_X-\delta_K$$

or

$$\theta Ak^{\theta-1}=\frac{y-(\delta_X-\delta_K)}{1+k}$$

From the Euler equation for capital

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma}=\beta\left[\frac{y}{1+k}+1-\frac{(\delta_X+k\delta_K)}{1+k}\right]$$

In steady state

$$\beta\left[\frac{Ak^{*,\theta}}{1+k^*}+1-\frac{(\delta_X+k^*\delta_K)}{1+k^*}\right]=1$$

In this model X_t is growing endogenously at a rate determined by the investment in human capital. That comes together with the need of capital K_t to grow also at that rate in order to maintain the rate $k=\frac{K}{XN}$ to be constant at k^*.

In this case, a competitive equilibrium replicates the optimal solution. To see this recall the households that endeavor both in investment of capital and human capital internalizes the effects of both types of capital in the production of the consumption good. In this sense, there are no externalities and no role for the government to intervene in improving growth. Naturally, there are many stories about why human capital generates positive externalities, in which case the government has a natural role. You will discuss one of these cases in the problem set.

3.2 Research and Monopolistic Competition

The main formulation to cope with the demand of differentiated products comes from Dixit and Stiglitz (1977), and it was extended to differentiated inputs in the production process by Ethier (1982).

In this economy, the source of growth is not the factor X that augments the labor productivity, human capital or effective hours (in fact we will assume $X_t = 1$), but the variety of inputs a worker uses. Capital in this economy will be related to the variety of inputs. Total production is

$$Y_t = L^{1-\theta} \int_0^{A_t} Z_t(i)^\theta di$$

Recall that, since inputs have a decreasing returns to scale ($\theta < 1$), it is maximized having as many varieties as possible. This is, in fact, a version of the importance of production "specialization" to the welfare of nations that Adam Smith discussed more than two centuries ago.

We will assume inputs fully depreciate in one period and one unit of input can be generated from one unit of production. It is also possible to endeavor in research that increases the variety of inputs at a constant marginal cost κ. Hence

$$I_t = \int_0^{A_{t+1}} Z_{t+1}(i)^\theta di + (A_{t+1} - A_t)\kappa$$

We will analyze a symmetric allocation where the quantity of an intermediate input is the same across all existing types. Then $Z_t(i) = Z_t$ for $i \in [0, A_t]$, and the resource constraint can then be written as

$$C_t + A_{t+1}Z_{t+1} + (A_{t+1} - A_t)\kappa = L^{1-\theta} A_t Z_t^\theta$$

As you can see here, the source of growth will come not from a labor augmenting technology, as the human capital case, but from a total factor productivity that increases the technology of all the factors. If $A_t = A$ in all periods and $\delta = 1$, the result replicates our steady state without growth.

In this case the results from a competitive equilibrium and a planner will differ because the production function is increasing returns of scale in the three factors (labor, inputs and range of inputs).

In the case of a competitive equilibrium, we assume perfect competition in the final good sector. Since it exhibits constant return to scale on labor and inputs (to final good producers the range of inputs is given), each input will be paid its marginal product. Then

$$\begin{aligned} w_t &= (1-\theta)L^{-\theta}\int_0^{A_t} Z_t(i)^{\theta}di \\ p_t(i) &= \theta L^{1-\theta}Z_t(i)^{\theta-1} \end{aligned}$$

where $p_t(i)$ is the price of intermediate input i at time t in terms of the final good.

Denote $1 + R_m$ the steady state interest rate along the balanced growth path that we are seeking. This is exactly the marginal cost of producing inputs since it is the opportunity cost of giving up consumption in period t to generate a productive input in period $t+1$. Then the profits of input i at period t is

$$\pi_t(i) = [p_t(i) - (1+R_m)]Z_t(i)$$

Input producers maximize these profits. Then

$$\max_{Z_t(i)} \pi_t(i) = [\theta L^{1-\theta}Z_t(i)^{\theta-1} - (1+R_m)]Z_t(i)$$

Then

$$p_t(i) = \frac{1+R_m}{\theta}$$

Digression: *In general*

$$p_t(i) = \frac{MC_t(i)}{1+\varepsilon_t(i)}$$

where $MC_t(i)$ is the marginal cost of production of input i and $\varepsilon_t(i)$ is the price elas-

ticity of input i

$$\varepsilon_t(i) = \left[\frac{\partial p_t(i)}{\partial Z_t(i)} \frac{Z_t(i)}{p_t(i)}\right] < 0$$

In our case $\frac{\partial p_t(i)}{\partial Z_t(i)} = (\theta - 1)\frac{p_t(i)}{Z_t(i)}$, hence $\varepsilon_t(i) = \theta$

From this equation we can obtain $Z_t(i)$ as a function of parameters.

$$Z_t(i) = \left(\frac{\theta^2}{1 + R_m}\right)^{\frac{1}{1-\theta}} L \equiv Z_m$$

plugging this back into the profits

$$\pi_t(i) = (1 - \theta)\theta^{\frac{1}{1-\theta}} \left(\frac{\theta}{1 + R_m}\right)^{\frac{\theta}{1-\theta}} L \equiv \Omega_m(R_m)$$

In an equilibrium with free entry, the cost κ of inventing new inputs must be equal to the discounted stream of future profits associated with being the sole supplier of that input for the rest of that input's life.

$$\sum_{t=1}^{\infty}(1 + R_m)^{-t}\Omega_m(R_m) = \frac{\Omega_m(R_m)}{R_m}$$

Hence, the free entry condition delivers

$$R_m\kappa = \Omega_m(R_m)$$

which determines R_m in equilibrium and characterizes the balance growth path,

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta(1 + R_m)$$

In steady state there is growth as soon as $\beta(1 + R_m) \geq 1$. Substituting Z_m into the resource constraint and dividing by A_t, where $c_t = \frac{C_t}{A_t}$ and $a_t = \frac{A_{t+1}}{A_t}$

$$c_t + a_t Z_m + (a_t - 1)\kappa = L^{1-\theta} Z_m^{\theta}$$

When a_t is constant, implies c_t should be constant. In other words, the range of inputs must grow at the same rate as consumption.

This model predicts larger countries (with larger L) grow at faster rates. Can you see why?

Now we solved for the competitive equilibrium, we can see there is a difference to what a planner would like to do. This difference arises from increasing returns to scale introduced by the research and invention of new inputs.

To analyze the planner problem, let's star defining $1 + R_s$ the social rate of interest along the balance growth path. For a given R_s and a range of inputs A_t, the planner would choose the quantities of inputs that maximize

$$L^{1-\theta}\int_0^{A_t} Z_t(i)^\theta di - (1+R_s)\int_0^{A_t} Z_t(i)^\theta di$$

from first order conditions with respect to $Z_t(i)$

$$Z_t(i) = \left(\frac{\theta}{1+R_s}\right)^{\frac{1}{1-\theta}} L \equiv Z_s$$

Hence, the quantity of an input is the same across all inputs and constant over time.

The planner wants to maximize utility subject to resource constraint with quantity of intermediate inputs Z_s, controlling A_{t+1}, We can write this in dynamic programming form

$$v(A) = \max_{A'}\{u(AL^{1-\theta}Z_s^\theta - A'Z_s - (A'-A)\kappa) + \beta v(A')\}$$

The first order condition is (with $v_A = \frac{\partial v(A)}{\partial A}$)

$$\beta v_{A'} = u_C(Z_s + \kappa)$$

The envelope condition is

$$v_A = u_C\left(L^{1-\theta}Z_s^\theta + \kappa\right)$$

From these two conditions we get the Euler equation that determines the evolution of consumption

$$u_C(Z_s+\kappa) = \beta u_{C'}\left(L^{1-\theta}Z_s^\theta + \kappa\right)$$

and evaluating at steady state

$$\left(\frac{C_{t+1}}{C_t}\right)^\sigma = \beta\frac{L^{1-\theta}Z_s^\theta + \kappa}{Z_s+\kappa} = \beta(1+R_s)$$

Substituting for the solution of Z_s

$$R_s\kappa = (1-\theta)\left(\frac{\theta}{1+R_s}\right)^{\frac{\theta}{1-\theta}} L \equiv \Omega_s(R_s)$$

Comparing Ω_m with Ω_s for a given R, we can show the later is larger, which means $R_s > R_m$ (recall $R_s = \frac{\Omega_s}{\kappa}$ and $R_m = \frac{\Omega_m}{\kappa}$). This is because,

$$\Omega_m(R) = \theta^{1/(1-\theta)}\Omega_s(R)$$

Since $R_s > R_m$, the optimal growth rate exceed the competitive equilibrium outcome. Similarly, we can show the competitive equilibrium supply of intermediate inputs fall short of the socially optimal one. This is because

$$Z_m = \left(\theta\frac{1+R_s}{1+R_m}\right)^{1/(1-\theta)} Z_s$$

Hence $Z_m < Z_s$ if $\theta\frac{1+R_s}{1+R_m} < 1$. All this means that the competitive equilibrium is characterized by a smaller supply of each intermediate input and a lower growth rate than would be socially optimal. The reason for inefficiencies and the potential positive role of the planner is that suppliers of intermediate inputs do not internalize the full contribution of their inventions and so their monopolistic pricing results in less than socially efficient quantities of inputs.

4 Putting some numbers

Let's consider again the exogenous growth model, characterized by a balance growth path

$$\beta\left(\theta A k^{*,\theta-1} + 1 - \delta\right) = [(1+\gamma)(1+n)]^{\sigma}$$

We need to choose values for the following parameters

$$\delta,\ \gamma.\ n,\ \theta,\ \sigma,\ \beta$$

1. What is depreciation δ? It is the loss of capital due to use, machines wear out. How do we measure depreciation? King and Rebelo, for example, use 10% but

this is probably too high. To see this note that on average, depreciation is about 10% of total output (this is $\delta k = 0.1y$). Since the capital-output ratio (defining capital as the capital in the business sector) is about $k/y = 1.5$, then

$$\delta = 6.67\%$$

2. Growth rate of technology γ can be measured as the long-run average growth rate of total factor productivity. This is about 1.3 percent, then

$$\gamma = 0.013$$

3. Growth rate of population n

$$n = 0.014$$

4. The capital's share of income θ can be measured by separating capital and labor income. First, consider national income, which is the sum of compensation of employees, rental income of persons, net interests, corporate profits and proprietors income. Of these, compensation of employees is unambiguous labor income, rental income of persons, net interests and corporate profits are unambiguous capital income, while proprietors income is ambiguous.

   So what we do is to take national income, subtract out proprietors income and add in depreciation, also a unambiguous capital income. We measure θ as the ratio of capital income to the sum of capital and labor income. The result is

$$\theta = 1/3$$

5. To obtain the preferences parameters, σ and β, we will use data on real return to physical investment. Considering $R = 0.065$ and $R = MPK + 1 - \delta$. From the Euler equation of this model

$$\beta 1.065 = [(1+\gamma)(1+n)]^{\sigma}$$

   Typically, values for σ range from 1 (log) to about 4. King and Rebelo choose a benchmark of 1,

$$\sigma = 1$$

which delivers

$$\beta = 0.9645$$

By using this calibration, King and Rebelo examines the transition path assuing that the initial capital stock is such that output at date 0 is 40 percent of steady state output. In their Figure 4, they show transitions are very fast, with high I/Y and very high interest rates. What drives these results? One factor, a high marginal product of capital, which results from lots of curvature in the marginal product of capital schedule

$$MPK = \theta A k^{\theta-1}$$

As θ becomes small then the percent change in MPK tends to be linear in k. This is because

$$d\ln(MPK) = (\theta - 1)d\ln(k)$$

The high initial MPK have the following implication: It raises interest rates. High interest rates induce households to increase savings. High savings means the capital stock grows quickly, which results in high I/Y, rapid growth of output and a fast transition.

How might we change the neoclassical model so that the transition is slower and both the I/Y ratio and the interest rates are not so high? In what follows we will consider a number of modifications.

4.1 Modifications to the Benchmark Neoclassical Model

For the neoclassical model to be consistent with slow transitions, we need a modification that slows down the rate of investment without generating counterfactual predictions for other variables (such as the value of stock market interest rates, relative income shares, etc).

1. **Increase parameter θ**

   A higher value of θ yields predictions hat are closer to the data. The main reason is that a higher θ reduces the curvature in the marginal product of capital schedule. This prevents the interest rate from rising so much, which prevents savings from rising so much, which slows down the transition. King and Rebelo argues that the drawback of this modification is that it leads to way too much investment in the steady state. Without a framework for understanding what this extra capital is, this modification is not entirely successful.

2. **Increase parameter σ**

   A high σ slows down transition because it governs intertemporal elasticity of substitution (IES).

   $$IES = \frac{\partial c'/c}{\partial R}\frac{R}{c'/c} = \frac{\partial \ln(c'/c)}{\partial \ln R} = \frac{1}{\sigma}$$

   This is obtained from taking logarithms in the Euler condition $\sigma \ln(c'/c) = \ln R$, where $R = MPK + 1 - \delta$.

   High σ slows down consumption growth, which implies that it reduces savings, which further implies the transition is slower. However, this also keeps interest rates high for a long time. This is because high rates of savings quickly drives down the MPK. However, with high σ savings is reduced, so the economy experiences high interest rates for a protracted period.

   This modification is not successful because it predicts counterfactually high real interest rates for a number of periods during the transition.

3. **Change preferences to a Stone-Geary specification**

   Stone-Geary utility functions impose a subsistence level of consumption $\bar{c}_t$

   $$u(c) = \frac{(c_t - \bar{c}_t)^{1-\sigma}}{1-\sigma}$$

   As total consumption approaches $\bar{c}_t$ marginal utility approaches infinity. This specification is one example of non-homothetic preferences.

   This also slows down transition. By having such high marginal utility early in the transition, savings is lower. But this has roughly the same effect as raising σ. It increases the number of periods with high interest rates, by lowering the savings rate in the early part of the transition.

4. **Different elasticity of substitution between capital and labor**

Consider the constant elasticity of substitution function, such that

$$Y = (\alpha L^\theta + (1-\alpha)K^\theta)^{1/\theta}, \qquad \theta \in [1, -\infty]$$

The elasticity of substitution is $\frac{1}{1-\theta}$. The marginal product of capital is

$$MPK = (1-\alpha)\left(\frac{K}{Y}\right)^{\theta-1}$$

Taking logarithms and differentiating

$$d\ln(MPK) = (\theta-1)[d\ln(K) - d\ln(Y)]$$

Taking a Taylor series expansion of Y around its steady state values, assuming labor is constant, we get

$$\ln(Y) \approx \ln(Y^*) + MPK(\ln(K) - \ln(K^*))$$

or

$$d\ln(Y) \approx MPK d\ln(K)$$

This implies

$$d\ln(MPK) = (\theta-1)d\ln(K)\left[1-(1-\alpha)\left(\frac{K}{Y}\right)^{\theta-1}\right]$$

Note that, as θ approaches 1 (perfect substitutes), the marginal product of capital is relatively unaffected by deviations from steady state.

Since this is the key object which governs the speed of the transition nd the level of interest rates, it is possible to slow down the transition by choosing θ near 1. But what other implications does this have? Given the MPK, let's look at the capital's share of income

$$\frac{MPK * K}{Y} = (1-\alpha)\left(\frac{K}{Y}\right)^{\theta}$$

For $\theta = 0$, the income share is constant and equal to $(1-\alpha)$. Assuming the capital's share rise for θ greater than 0. This modification is at odds with available evidence, since in many countries the production technology seems well approximated by a Cobb-Douglas technology.

5. **Adjustment Costs**

   This modification drops the assumption that one unit of output can be transformed into one unit of capital. Instead suppose that it takes more than one unit of output to produce one unit of the new investment, and that the marginal rate of transformation depends on the level of investment and the existing capital stock. Consider the following model.

$$k_{t+1} = z_t + (1-\delta)k_t$$

$$y_t = c_t + z_t(1 + h(z_t, k_t))$$

   where h, the adjustment cost, is continuously differentiable, convex. homogenous of degree 0 and $h(\delta) = 0$ and $dh(\delta) = 0$. These last two assumption just guarantee that the steady state of the model does not depend on adjustment costs. The assumption of homogeneity of degree 0 is made so that marginal rates are equal to average changes.

   Assume the adjustment cost function is quadratic

$$h\left(\frac{z_t}{k_t}\right) = \frac{\phi}{2}\left(\frac{z_t}{k_t}\right)^2$$

   where ϕ is a cost function parameter.

   The resource constraint is

$$F(k_t) \geq c_t + z_t(1 + h(z_t/k_t))$$

   and the investment constraint is

$$z_t \geq k_{t+1} - (1-\delta)k_t$$

   The lagrangian is

$$\mathcal{L} = \max \sum \beta^t \left\{u(c_t) + \lambda_t \left(F(k_t) - c_t - z_t(1 + h(z_t/k_t))\right) + \mu_t(z_t + (1-\delta)k_t - k_{t+1})\right\}$$

First Order Conditions are

$$\begin{aligned}
\{c_t\} &: \quad \beta^t U_{c_t} = \beta^t \lambda_t \\
\{z_t\} &: \quad \beta^t \lambda_t (1 + (z_t/k_t) dh + h) = \beta^t \mu_t \\
\{k_{t+1}\} &: \quad \beta^t \mu_t = \beta^{t+1} \left[\lambda_{t+1} \left[F_{k_{t+1}} - dh \left(\frac{z_{t+1}}{k_{t+1}} \right)^2 \right] + \mu_{t+1}(1-\delta) \right]
\end{aligned}$$

The adjustment cost literature focuses on the Tobin's q, this is the price of existing capital relative to the cost of producing new capital. The main questions are, How much output should one pay for one unit of capital already in place? or, in other words, What is the value of the stock market, which prices capital already in place?

The answer to these questions is based on arbitrage. No arbitrage requires that the price of existing capital be equal to the marginal cost of producing new capital (since capital in this model is homogenous). Hence, to compute the price of existing capital, it is necessary to compute the marginal cost of investment. Recall in the standard model Tobin's q is always 1, since it is possible to transform one unit of capital with one unit of output always. In the model with adjustment costs, the Tobin's q is the marginal cost of investment (from differentiating the investment cost function).

$$MC = 1 + h + \left(\frac{z}{k} \right) dh$$

hence, an individual is indifferent between undertaking a new investment with marginal cost $1 + h + \left(\frac{z}{k} \right) dh$ or paying $1 + h + \left(\frac{z}{k} \right) dh$ for a piece of capital already in place.

Adjustment costs can slow down the transition if the marginal cost of investment is sufficiently high to discourage rapid changes in investment. King and Rebelo find that a Tobin's q of around 4 would be needed (this means it takes about 4 units of output to produce one unit of capital). This is much higher than any data observed in the US.

5 Malthus: Growth Before the Industrial Revolution

The models discussed so far do an excellent job of accounting for the growth experiences of industrialized and developing countries over the past 50 to 100 years. From a historical perspective, however, this recent period is rather special. The two most important facts about economic growth in the present are that living standards tend to increase over time in most countries of the world, and that differences in living standards across countries are huge. Both these facts (sustained economic growth and large income differences) emerged only after the Industrial Revolution began to take hold in Britain a little over 200 years ago. From early human history until about 1800, there was no sustained growth in income per capita and living standards in any country of the world. Historically, stagnation, not growth in income per capita was the typical situation. Likewise, income differences across countries used to be small. The current world income distribution emerged only after some countries experienced an industrial takeoff that lifted income per capita in these countries to stratospheric levels relative to prior experience, while other countries continued to stagnate at a low level of income.

Given these observations, it seems natural to regard the relative timing of the industrial takeoff in different countries as a key determinant of modern income distribution across countries. However, if we want to understand why some countries escaped stagnation and others did not, we first need to know why there was stagnation in the first place. The explanation suggested by one of the earliest writers on the subject, British economist Thomas Malthus in his Essay on the Principle of Population of 1798, is widely accepted to the present day. The Malthusian model relies on two key ingredients: an agricultural production function that uses the fixed factor land, and an income-population feedback where the population growth rate is an increasing function of income per capita.

Consider an aggregate production function of the form

$$Y_t = (A_t X)^{\alpha} N_t^{1-\alpha}$$

where Y_t denotes output in period t, A_t is productivity, in this case X is a fixed amount of land and N_t is the size of the population. Dividing by N_t on both sides, we can see that income per capita $y_t = Y_t/N_t$ is given by

$$y_t = \left(\frac{A_t X}{N_t}\right)^{\alpha}$$

The equation implies that income per capita is an increasing function of productivity, but a decreasing function of population: when the size of the population increases, there is less land for each person to work with, which lowers income per capita.

To develop a theory of stagnation based on this production function, we need to specify how productivity and population evolve over time. As in the Solow model, we will assume that productivity A_t grows at the constant rate g, so that:

$$A_{t+1} = (1+g)A_t$$

Population growth, in turn, is assumed to be an increasing function of income per capita y_t,

$$\frac{N_{t+1} - N_t}{N_t} = f(y_t)$$

where $f'(y_t) > 0$. A number of different justifications can be given for this relationship. One possibility is that children enter the utility function of parents as normal goods. A rise in income would then increase the demand for children, leading to higher population growth. Alternatively, the mechanism could also work through mortality. If higher income leads to better nutrition and, as a consequence, lower mortality rates, a positive relationship between income per capita and population growth follows. As an empirical matter, the assumption of a positive relationship appears to fit the experience of most pre-industrial economies rather well.

For concreteness, let's now specify population growth more precisely in a framework in which parents optimally choose their fertility, and children are a normal good. In other words, when income goes up, more children are consumed by the parents. I assume that parents have children for their enjoyment only, that is, I abstract from issues like child labor. Consider a utility function over consumption c_t and number of children n_t of the form

$$u(c_t, n_t) = \ln(c_t) + \ln(n_t)$$

The parent has income y_t, and the cost of raising a child in terms of goods is p. The budget constraint is then

$$c_t + pn_t = y_t$$

By substituting for consumption, we can write the utility maximization problem as

$$\max_{n_t}\{\ln(y_t - pn_t) + \ln(n_t)\}$$

The first order condition with respect to n_t is

$$-\frac{p}{y_t - pn_t} + \frac{1}{n_t} = 0$$

then

$$n_t = \frac{y_t}{2p}$$

Thus, the higher the parents' income, the more children are going to be produced. If we assume that people live only for one period, the number of children per adult n_t determines population in the next period N_{t+1}

$$N_{t+1} = n_t N_t = \frac{y_t}{2p} N_t$$

That is, the population tomorrow equals population today times number of children per person. Now that income per capita y_t as well as the laws of motion for productivity and population are specified, we can derive the long-run behavior of our economy. Plugging income per capita into the law of motion of population yields

$$N_{t+1} = \frac{1}{2p}(A_t X)^{\alpha} N_t^{1-\alpha}$$

The qualitative properties of this law of motion for population are similar to those of the law of motion for capital in the growth model observed above. Most importantly, the law of motion has the convergence property in the sense that population growth is higher when population is low.

$$\frac{N_{t+1}}{N_t} = \frac{1}{2p}\left(\frac{A_t X}{N_t}\right)^{\alpha}$$

That is, if A_t were fixed and constant over time, the population would converge to a fixed level, regardless of the initial condition. If A_t is growing, i.e., $g > 0$, we can redefine variables such that the modified law of motion reaches a steady state. Define

$$m_t = \frac{N_t}{A_t X}$$

The term $A_t X$ is referred to as effective land units. The variable m_t is a measure of population density, but rather than simply taking the ratio of population and land (this is, people per square mile), land is weighted by its productive capacity. Rewriting the law of motion by substituting $N_t = A_t X m_t$ and $N_{t+1} = A_{t+1} X m_{t+1}$ gives

$$A_{t+1} X m_{t+1} = \frac{1}{2p}(A_t X)^{\alpha}(A_t X m_t)^{1-\alpha}$$

Since we have $A_{t+1} = (1+g)A_t$, this can be written as

$$(1+g)A_t X m_{t+1} = \frac{1}{2p} A_t X m_t^{1-\alpha}$$

Dividing by $A_t X$ and $(1+g)$ on both sides gives

$$m_{t+1} = \frac{1}{(1+g)2p} m_t^{1-\alpha}$$

This law of motion depends on m_t only, and since we have $1-\alpha < 1$, m_t converges to a steady state from any initial condition. The steady state ratio $\bar{m}$ of population to effective land units has to satisfy. In steady state

$$\bar{m} = \left[\frac{1}{(1+g)2p}\right]^{\frac{1}{\alpha}}$$

This implies that in steady state, population and productivity have to grow at the same rate. This is, fertility rate in steady state is $\bar{n} = 1+g$.

So far, we have a developed a theory of population size as a function of productivity. Specifically, we found that population growth depends negatively on initial population. This implies that population growth should be expected to rise after a sudden decrease in population, for example because of an epidemic or a war. This prediction is in line with historical observations, for example, the fast recovery of European population after the Black Death. The theory also implies that population growth and therefore density will be higher in countries or areas where productivity growth

g is high. Again, this prediction is confirmed by evidence. Throughout history, population density is closely related to the technological knowledge of a society. For example, in the last 2000 years first China and later Western Europe were the leaders in technological progress, and both these areas ended up with a very high population density.

What are the implications of the model for living standards? In the model, income per person y_t is given by

$$y_t = m_t^{-\alpha}$$

Hence, income per capita is a decreasing function of population per effective land unit. This is not surprising: Given that land is fixed, if the size of the population increases, every person has less land to work with, and therefore produces less output. The dependence of income per capita on m_t also leads to a somewhat depressing conclusion: given that we established that m_t is constant in the long run, y_t will be constant as well. In other words, the model predicts stagnation in living standards, despite sustained productivity growth at rate g. The reason for this surprising result is that productivity growth is fully offset by population growth. For any doubling of productivity the size of the population also doubles. As a result, the amount of land that each person can work with is cut in half; even though land is twice as productive, income per capita then remains constant.

We can compute the level of income per capita in the steady state using our expression for $\bar{m}$

$$\bar{y} = \bar{m}^{-\alpha} = (1+g)2p$$

Then

$$\bar{n} = \frac{\bar{y}}{2p} = 1+g$$

Of course, this much we knew already: population growth has to be equal to productivity growth in the steady state, because otherwise m_t would not be constant. In fact, income per capita y_t adjusts to exactly the level that implements population growth at the right rate. This explains why income per capita depends both on the growth rate of productivity and on the cost of having children. If productivity growth is high, more population growth can be sustained in the long-run equilibrium. If population growth is initially lower than the long-run level, income per capita will rise until population growth just offsets productivity growth. Likewise, if the cost of having children is high, income per capita must be high as well to induce people to have many children.

Consider, for example, the adjustment after a sudden increase in the cost of children. Other things equal, people will initially reduce their fertility rate, which lowers population growth. Low population growth, in turn, lowers population density, so that the next generations have more land to work with per person. This increases income per capita above the previous level. Through this process, income per capita will continue to rise until population growth once again offsets productivity growth. As our formula shows, the required increase in income per capita is exactly proportional to the increase in the cost of children.

The Malthusian model shows that it is easy to produce stagnation in living standards in an economic model. Only two assumptions are needed. First, there have to be decreasing aggregate returns to the size of the population. This feature arises naturally if there is an important factor of production that is fixed in supply. Historically, most of production was generated in agriculture, which relies on the fixed factor land. Thus, the first assumption is entirely realistic for all countries of the world, at least until the time of the Industrial Revolution. Second, there has to be a positive relationship between income per capita and population growth. This relationship arises either if people consciously choose their number of children and regard children as normal good, or if mortality rates are endogenous and respond strongly to the level of income per capita. As an empirical matter, the assumption used to be satisfied in essentially all countries until about 200 years ago. In other words, more productive countries with higher income per capita used to experience faster population growth, which of course tended to offset high living standards by crowing more people onto the fixed amount of land. Even within countries, rich people used to have more children than poor people, especially when survival rates are taken into account.

We therefore see that both key assumptions of the Malthus model are likely to be satisfied in most countries from Stone Age until the Industrial Revolution. And indeed, the predictions of the Malthus model line up almost perfectly with evidence from the period; in terms of empirical success, the Malthus model is the biggest success stories in the social sciences. The available evidence strongly confirms the prediction that income per capita stagnated over time, and that income differences across countries or regions were small. Likewise, differences in the productivity of agriculture were reflected in population densities (a fact that Adam Smith comments on in the Wealth of Nations), rather than in income per capita.

One might argue that some developing countries that still rely mostly on agriculture continue to be well-described by the Malthus model to the present day. We now know, however, that a large set of countries ultimately managed to leave the Malthusian stagnation trap behind. These countries no longer rely on agriculture as their main mode of production. Moreover, the Malthusian relationship between income per capita and population growth is also severed: Nowadays the richest countries tend to have low rates of population growth, and within rich countries it is usually the poorest people that have the most children. The first country to start the transition from Malthusian Stagnation to modern growth was Britain with the start of the Industrial Revolution, soon followed by other European nations and the former European colonies in the Americas. More countries, many of them in Asia, underwent a similar transition in the post-war period of the twentieth century. One way to interpret the current large income differences across countries is to view them as resulting from differences in the timing of the takeoff from Malthusian stagnation: the rich countries managed to achieve this almost 200 years ago, todays middle-countries escaped 30 to 60 years ago, and todays poorest countries may still be caught up in the Malthusian mechanism. To understand income differences, we therefore need to understand how countries manage to escape from the Malthusian trap.

6 From Malthus to Solow

If we want to understand why some countries are rich today and others are poor, we have to find out how countries can escape from the Malthusian trap. The Malthus model is based on two key ingredients. First, the fixed factor land plays a key role in production, which implies that average income declines with the size of the population. Second, population growth is an increasing function of income per capita. We can engineer an escape from the Malthusian trap by either altering the production technology, or by modifying the income-population relationship. We will consider each possibility in turn.

6.1 Accelerated Technological Process

One scenario in which the income-population link breaks down is if technological progress is so fast that population growth can no longer keep up. In the Malthus model in the previous section, population growth was determined by the equation $n_t = y_t/2p$. Implicitly, this equation is based on the unrealistic assumption that population growth can be arbitrarily large. In reality, there are limits to how many children each woman can have. If n^{max} is the highest fertility rate that is biologically feasible, the equation should be modified to state:

$$n_t = \min\left\{\frac{y_t}{2p}, n^{max}\right\}$$

With the modified equation, if productivity growth is large enough to outrun population growth (i.e., $1 + g > n^{max}$), growth in income per capita will follow. In this case even at maximum population growth the detrimental effect of increasing population density does not suffice to negate productivity improvements. If $(1 + g) > n^{max}$,

$$\begin{aligned}
\log(y_{t+1}) - \log(y_t) &= \alpha[\log(A_{t+1}) - \log(A_t) - (\log(N_{t+1}) - \log(N_t))] \\
&= \alpha[\log(1 + g) - \log(n^{max})] \\
&\approx \alpha(1 + g - n^{max}) > 0
\end{aligned}$$

If accelerated technological progress was indeed the correct explanation for growth takeoffs in industrializing countries, we would expect the rise of living standards to be accompanied by extremely fast population growth. To some extent, this is true for the early phase of the Industrial Revolution, when Britain indeed had high population growth. Later on, however, population growth declined substantially despite fast income growth. Therefore, we will next focus on explanations for the takeoff where population growth ceases well below the biological maximum.

6.2 Fertility Control Policies

Perhaps the easiest way to engineer an economic takeoff in the Malthus model is by introducing a government policy that simply forbids parents to have more than a specified number of children. Consider, for example, a policy forcing every parent to have exactly one child, $n = 1$. This policy would immediately stabilize population at a constant level, thus population growth is zero. The growth rate of income per capita is then given by:

$$\begin{aligned}\log(y_{t+1}) - \log(y_t) &= \alpha[\log(A_{t+1}) - \log(A_t) - (\log(N_{t+1}) - \log(N_t))] \\ &= \alpha[\log(1+g) - \log(1)] \\ &\approx \alpha g > 0\end{aligned}$$

While this policy works well in theory, a more difficult question is how well the policy would work in practice. Choosing how many children to have is an important private decision for parents, and there might be considerable resistance against a government mandate of low fertility. Moreover, many parents might attempt to circumvent the policy. Historically, at least, fertility control policies did not play a major role in the economic takeoff of most countries. The only major country that has placed strong emphasis on policies of this kind is China with the one child policy. However, this policy has only been in place for about 30 years now, and enforcement varies across regions.

Despite the absence of official fertility control policies, in many countries population growth ultimately did decline. Evidently, this decline in population growth must reflect that parents no longer find having lots of children to be optimal. We therefore next turn to reasons why modern-day parents appear to prefer small families to large ones.

6.3 Demographic Transition

Historically, the Malthusian relationship between income and population growth did indeed break down in every single country that experienced a growth takeoff. In a pattern known as the demographic transition, the high fertility and mortality rates of the pre-industrial era gave way to a new regime in which fertility, mortality, and population growth are low. In modern data, the relationship between income per capita and population growth is negative (both in a cross section of countries and in the time series for most rich countries), which is the opposite of what the Malthusian model assumes.

For the most part, this change in fertility behavior does not seem to be due to government policies. Rather, the decline in fertility is usually interpreted as a substitution of child quantity (a large number of children) by child quality (fewer children in which parents invest in terms of education or human capital). As an example of a model capturing this tradeoff, consider the decision problem of a parent with preferences,

$$u(c, n, h) = (1 - \beta)\log(c) + \beta[\log(n) + \gamma \log(h)]$$

where h are children's human capital, $\beta > 0$ and $0 < \gamma < 1$. The parent has to spend a fraction ϕ of his time to raise each child, and can choose to spend an additional per-child fraction e on educating the children. The total child-bearing time is $(\phi + e)n$ and the budget constraint for the parent becomes

$$c \leq (1 - (\phi + e)n)w$$

where w is the wage of the parent. A child's human capital do depend on the education time e.

$$h = e^{\theta}$$

where θ is just the elasticity of education technology.

Of course, in the pre-industrial era most children received little or no education, as most people worked in unskilled professions that did not require formal education. The basic idea is therefore that the expanded decision problem on child quality and child quantity replaced the more basic decision problem of the previous section only after human capital became an important factor of production.

Let us now determine how the return on education affects optimal fertility and education choices in our model. Plugging the constraints into the utility function, the decision problem is just

$$\max_{n,e}(1 - \beta)\log((1 - (\phi + e)n)w) + \beta[\log(n) + \gamma \log(h)]$$

From first order conditions

$$n = \frac{\beta}{\phi + e}$$

According to this result, the fertility rate n is decreasing in the level of education e. Intuitively, investing a lot in the children's education makes each child more expensive. Parents who choose a high e therefore prefer to economize on the size of the family, and choose a low n.

Since

$$e = \frac{\gamma\theta\phi}{1-\gamma\theta}$$

we can solve n only as a function of parameters,

$$n = \frac{\beta(1-\gamma\theta)}{\phi}$$

Given these results, an escape from the Malthusian trap is possible if a rising return to human capital induces parents to invest more in their childrens education. Such a change will lower the fertility rate, so that productivity growth is no longer offset by increased population density.

Whereas the initial phase of the Industrial Revolution was accompanied by fast population growth, the evidence suggests that the story of this section was relevant for the second phase of the Industrial Revolution. From the middle of the nineteenth century, education and schooling levels starting to rise rapidly in all industrializing countries. Soon thereafter, fertility rates started to drop substantially. In Britain, for example, the average fertility rate (i.e., the number of children per woman) fell from 5 in 1830 to only 2 in 1930. Thus, the endogenous switch from child quantity to child quality may have been one of the forces that contributed to fast economic growth from the middle of the nineteenth century.

6.4 Structural Change

Apart from endogenous population growth, the Malthusian model also relies on the presence of the fixed factor land to generate stagnation. A second potential trigger for a growth takeoff is therefore structural change that decreases the role of land. In pre-industrial economies, agriculture was the main mode of production. In contrast, in modern industrial economies the share of agriculture in output is small, and consequently land is less important. We can engineer a theory of the industrial takeoff by adding a second, industrial production technology to the Malthus model of the previous section.

Consider a variant of the Malthus model where in addition to the agricultural production function, an industrial, constant-returns technology is also available:

$$Y_t^I = A_t^I N_t^I$$

where superindices I refer to industrial sector. Productivity grows at a constant rate

$$A_{t+1}^I = (1 + g^I) A_t^I$$

The main characteristic of this new production technology is that it does not rely on the fixed factor land. As a consequence, population growth does not have any detrimental effect on the productivity of this sector; the same output per capita can be produced at any population size. One might think that once this technology is available, it will always be optimal to use it right away, in order to escape from the Malthusian trap. Under closer inspection, however, this does not turn out to be the case.

Consider a version of the Malthusian model where both production technologies are always available. People will choose to use whatever production technology yields higher output per capita; if y_t^A is output per capita in the agricultural technology and y_t^I is output per capita in industry, the industrial technology will be used if $y_t^I > y_t^A$, or, which is the same

$$A_t^I > \left(\frac{A_t X}{N_t} \right)^{\alpha}$$

Let us now assume that initially industrial productivity A_t^I is very low, so that only the agricultural sector is used. This is exactly the case that was analyzed in the previous section. Why is it that only the agricultural sector is used in the early phase of development? As it turns out, the presence of the fixed factor land, which is a curse when the population size is large, turns out to be a blessing when the population size is small. Even if the agricultural technology is very unproductive, workers can achieve a sizable income per capita as long as the population density is small.

When the Malthusian economy reaches the steady state, Given that productivity growth in industry is positive ($g^I > 0$), at some point A_t^I has to reach this thresholdthe takeoff from Malthusian stagnation is inevitable. Once the threshold is reached, income per capita will increase every period at the rate g^I regardless of population size, because the industrial technology exhibits constant returns with respect to the size of the labor force. Viewed through the lens of this model, what initially appears as a structural break in economic history is merely the outcome of an optimal sectoral allocation decision in an otherwise stable economic environment.

The structural-change model predicts that the takeoff in income per capita should coincide with a structural transformation whereby agriculture is replaced by industry and, later on, services as the main sector of production. The data strongly confirms this prediction. As the term Industrial Revolution indicates, the economic transformation at the beginning of the nineteenth century was mostly driven by the expansion of the new industrial sector.