Notes on Common Knowledge †

ECON 201B - Game Theory

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Game Theory is explicitly interested in understanding the strategic behavior of players. If you're participating in a game against another player, you will be interested not only in knowing your own structure of payoffs and to have information about the environment, but also in **knowing** what the other player **knows** about that environment. In these notes we will focus on how individuals may use their information about others' information in order to have a better assesment of the environment.

1 Notation and definitions

In these notes we will be dealing mostly with individuals who know that some "event" is true but do not really know which "event" is true. In this sense they need to update probabilities as new information is revealed. Individuals can obtain information both from own experimentation and from pieces of information inferred from other players.

Let me first introduce some notation that will be useful to formalize the main idea of common knowledge.

The world consists of a set of **states** $\omega \in \Omega$. An **event** E is a set of those states (i.e., a subset of Ω). One way to define the extent of the individual's knowledge is to introduce an information function.

Definition 1 *An* ***information function*** $I(\omega)$ *is a function that associates with every state* $\omega \in \Omega$ *a nonempty subset* $I(\omega) \in \Omega$

This means that if the true state is ω, the individual only knows that the true state is in $I(\omega)$. This is important because it restricts the assignment of probabilities about the true state ω

Definition 2 *A* ***knowledge function*** $K_i(E)$ *for an individual* i *is a set of all states in which the individual knows* E *happened, such that,*

$$K_i(E) = \{\omega \in \Omega : I_i(\omega) \subseteq E\}$$

[0]† These notes were prepared as a back up material for TA session. If you have any questions or comments, or notice any errors or typos, please drop me a line at guilord@ucla.edu

If $I_i(\omega) \subseteq E$, an individual i who observes ω, will know that a state of the event E has occurred.

Definition 3 *An event E is **"mutual knowledge"** in a state ω if and only if that state ω is an element of the knowledge function $K_i(E)$ for all individuals i.*

Hence, if a given state ω occurs such that all individuals know the event E must have happened, that event is "mutual knowledge" for all players. However this is not the same as "common knowledge", which is a much more involving definition.

Quoting Aumann (1976), *"Two people, 1 and 2, are said to have **common knowledge** of an event E if both know it (mutual knowledge), 1 knows that 2 knows it, 2 knows that 1 knows it, 1 knows that 2 knows that 1 knows it, and so on"*

Formally

Definition 4 *An event E is **"common knowledge"** between two individuals if both know E, both know that both know E, etc ad infinitum. The common knowledge operator $K(E)$ is then defined by*

$$K(E) = K_1(E) \cap K_2(E) \cap K_1(K_2(E)) \cap K_2(K_1(E)) \cap K_1(K_2(K_1(E))) \cap .....$$

Let I (without subscript) be the meet (finest common coarsening) of the partitions (information functions) I_1 and I_2 and denote $I(\omega)$ the cell of I that contains ω, then

$$K(E) = \{\omega \in \Omega : I(\omega) \subseteq E\}$$

2 Agreeing to Disagree

Aumann's 3 pages paper not only provided the previous formal definition of common knowledge but also used it to prove that *"if two people have the same priors, and their posteriors for an event E are common knowledge, then the posteriors must be equal. This is so even though they may base their posteriors on quite different information. In brief, **people with the same priors cannot agree to disagree**"*

2.1 An example

Let me provide here an example to show the intuition behind this conclusion.

Assume a murder has been committed and there are 5 suspects: Alex, Bev, Cynthia, Dorothy and Elmer. While A and E are guys, B, C and D are girls.

It's also known that A and B have blood type $O+$, C and D have blood type $O-$ and E has blood type AB.

A stain of blood has been found in the crime scene. There are two detectives, each one having access to different blood tests. Detective 1 can only perform a test that determines the blood type while detective 2 can only perform a test that determines the gender.

They have exactly the same prior about the probability the suspects have committed the crime, given by

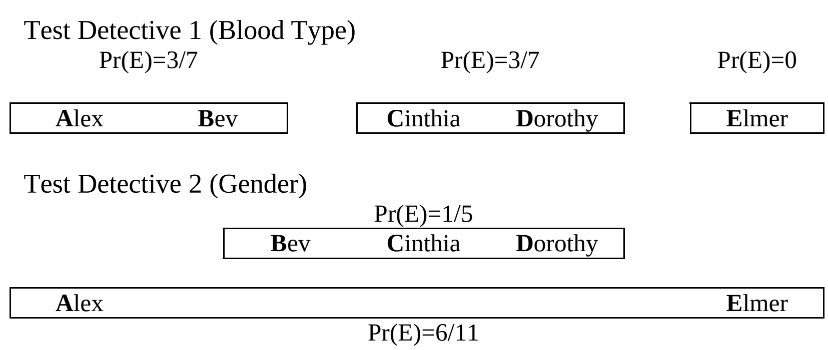
Suspect	**A**lex	**B**ev	**C**inthia	**D**orothy	**E**lmer
Pr(culprit)	6/26	8/26	3/26	4/26	5/26

Assume Alex and Cynthia are a couple and they ran away. The detectives want to know which is the probability one of them has committed the crime. Hence the event we want to focus on is $E = \{A, C\}$. Assume the detectives cannot share the results of their respective test but they can mention to the other the updated probabilities assigned to the couple have committed the crime. Aumann showed it is not possible for the detectives to have different posteriors under common knwledge, even though they have access to different pieces of information. In other words, after the share of information they cannot agree to disagree.

The probability the couple committed the crime in each one of the possible results of the blood tests can be obtained by Bayes' Rule in the following way. Suppose the result of the test that determines the blood type is that the murderer has a type $O+$. Hence the probability the couple committed the crime (event E) given that result (blood type $O+$) is

$$\Pr(E|O+) = \frac{\Pr(E \cap O+)}{\Pr(O+)} = \frac{\Pr(A)}{\Pr(\{A, B\})} = \frac{6/26}{6/26 + 8/26} = \frac{3}{7}$$

Following this procedure we can write the probabilities of the event in each one of the possible test results.



Now, **assume the real murderer was Dorothy**. Hence, Detective 1 will determine the murderer has a blood type $O-$ and Detective 2 will determine the murderer is a woman. Hence Detective 1 will inform he believes the probability the couple committed the crime is $3/7$ while Detective 2 will inform he believes the couple committed the crime with a probability $1/5$.

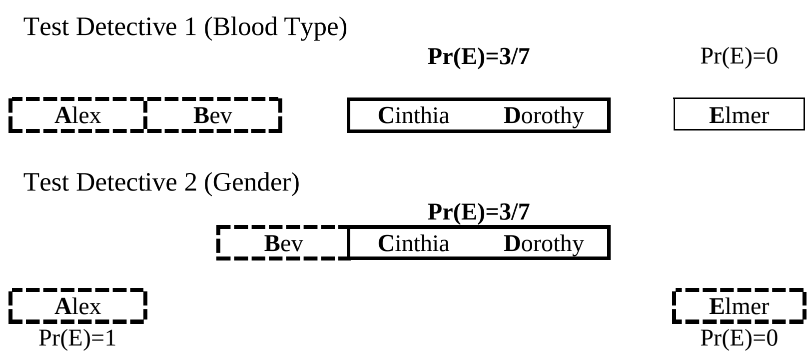
The question is, can they agree in having that disagreement about posteriors? **The answer is NO**.

Detective 2 knows that if Detective 1 assigns a probability $3/7$ to the event, then it's not possible for Elmer to have been the murderer (since in that case Detective 1 would have obtained a result of blood type AB and would have assigned a probability 0 to the couple committing the crime because nor Alex nor Bev have a blood type AB). However Detective 2 cannot infer from this result whether the blood test was $O+$ or $O-$.

In the same way, Detective 1 knows that if Detective 2 assigns a probability $1/5$ to the event, then it's not possible for Alex to have been the murderer (since in that case Detective 2 would have determined the murderer was a man and would have assigned a probability $6/11$ to the couple committing the crime). However this does not affect Detective 1's posterior of $3/7$. It would have affected that posterior only in case the blood type would have been $O+$ because he would have erased Alex and would have determined that Bev was the murderer for sure.

After this iteration, assume they meet again and see their posteriors did not change. However this conveys a lot of new information.

Detective 2 knows that Detective 1 would have modified his posterior in case of a $O+$ test result. Hence, Detective 2 will infer the test of Detective 1 should have been $O-$. If this is the case, Bev is a woman that could not have been involved in the crime. Hence, the new posterior for Detective 2 will be $3/7$, exactly as Detective 1.



When one detective announced his subjective estimate, the other detective found that announcement informative and revised his own estimate accordingly. At the end of the process of exchanges, the announcement by one detective of his estimate did not make the other detective change his estimate.

3 Extensions

The previous considerations about common knowledge and the result by Aumann has been extended in several directions. Some of the most important extensions have been "Don't bet on it" by Sebenius and Geanakoplos (1983) and "No trade" theorems by Milgrom and Stokey (1983).

3.1 "Don't bet on it"

Aumann's result can be extended from the probability of an event to the expectation of a random variable. The importance of this extension is that it told us that it cannot be common knowledge between two risk - neutral individuals they both expect to profit from a bet (or two risk-averse individuals they both are willing to participate in the bet). If it's common knowledge that they both expect to gain from the bet, then it's common knowledge that for one the expectation of the bet is positive and for the other is negative, which is a contradiction.

3.2 "No trade"

Also along these lines Milgrom (1981) and Milgrom and Stokey (1983) proved a result that is typically considered as establishing the impossibility of speculative trade. Assume that the two traders agree on an ex ante efficient allocation of goods. After the traders get new information, there is no transaction with the property that it is common knowledge that both traders are willing to carry it out.