Lecture Notes

Macroeconomics - ECON 510a, Fall 2010, Yale University

Limited Commitment.

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1 Introduction

So far, we have focused on a complete market benchmark and studied its success and failure to match the data in several dimensions (business cycles, asset pricing and long run growth) and to understand optimal taxation. A key assumption in that context was perfect enforcement of contracts (in particular, perfect enforcement of state contingent claims). A key result in that context is perfect smoothing of consumption: agents can trade contingent claims that are perfectly enforceable to maintain their consumption constant over time (perfect insurance against income shocks).

However, in national, regional and household consumption data, the null hypothesis of complete insurance is routinely rejected. Tests of complete insurance involve running the following regression of consumption growth on income growth.

$$\Delta \log \hat{c}_{t+1} = \alpha_0 + \alpha_1 \Delta \log \hat{y}_{t+1} + \epsilon_{t+1}$$

where $\hat{c}_{t+1}$ is the household's consumption share and $\hat{y}_{t+1}$ is the household's income share. Under the null hypothesis of complete insurance $\alpha_1 = 0$. This null is rejected in data. Household consumption shares respond to idiosyncratic income shocks, even though our benchmark with complete markets predicts it should not, because households can trade all of the idiosyncratic risk away. As we've seen, the benchmark model's asset pricing implications are also off quite a bit. It seems like there are other sources of aggregate risk, besides aggregate consumption growth risk, the the complete markets/perfect insurance benchmark misses.

Consumption and asset pricing data reveal that households are unable to trade away all of their idiosyncratic risk. In these notes, we examine whether the evidence from asset prices and consumption data is consistent with a modified version of the benchmark model where agents are not fully able to commit to their contracts: If an agent promised to pay lots of consumption in good states of the world, he may consider reneging on that promise when the good state indeed arrives. The break down of complete insurance crucially depends on this, since lack of commitment introduces a friction to transfer units of consumption from good states to bad states.

In what follows, we consider the following modification of the benchmark complete markets model: Agents can still trade a complete set of contingent claims, but they can walk away from their debts. The financial interpretation we will give to walking away from a contract is that of committing default. We will assume that if households default they are excluded from trading forever, hence we derive endogenous restrictions on trading when there is an inability to commit.

2 General Environment

We start with the same general environment that we have been dealing with so far, maintaining throughout the assumption of separability of preferences.

A planner wants to solve the following problem

$$\max_{c^j(s^t);n^j(s^t)} \sum_{j=1}^{J} \mu^j \sum_{s^t} \beta^t \pi(s^t)[u^j(c^j(s^t)) - v^j(n^j(s^t))]$$

subject to

$$\sum_{j=1}^{J} \mu^j c^j(s^t) \leq A(s^t)F(K(s^{t-1}), N(s^t)) - K(s^t) + (1-\delta)K(s^{t-1}) \qquad, \forall s^t,$$

$$N(s^t) = \sum_{j=1}^{J} \mu^j \theta^j(s^t) n^j(s^t) \qquad , \forall s^t$$

and a limited commitment constraint

$$\sum_{s^\tau | s^t} \beta^{\tau - t} \frac{\pi(s^\tau)}{\pi(s^t)} [u^j(c^j(s^t)) - v^j(n^j(s^t))] \geq D^j(s^t) \qquad , \forall j, s^t$$

This last constraint states that the continuation value of staying in society (under the contract) must be at least as large as the value of walking away. In this setup, the value of walking away for each of the j classes, is given **exogenously**. We will endogeneize this constraint later. The constraint alters consumption and labor dynamics.

Constructing the lagrangean

$$\begin{aligned}
\mathcal{L} \;\; = \;\; & \sum_{j=1}^{J} \mu^j \sum_{s^t} \beta^t \pi(s^t) [u^j(c^j(s^t)) - v^j(n^j(s^t))] \\
& + \sum_{s^t} \lambda(s^t) \left\{ A(s^t) F(K(s^{t-1}), N(s^t)) - K(s^t) + (1 - \delta) K(s^{t-1}) - \sum_{j=1}^{J} \mu^j c^j(s^t) \right\} \\
& + \sum_{j=1}^{J} \sum_{s^t} \psi^j(s^t) \left\{ \sum_{s^\tau} \beta^\tau \pi(s^\tau) [u^j(c^j(s^t)) - v^j(n^j(s^t))] - \beta^t \pi(s^t) D^j(s^t) \right\}
\end{aligned}$$

we can recover the first order conditions. For capital, the FOC is the same as before. Consumption and labor will look different though. Before we do this, we need to introduce the concept of a **cumulative multiplier**. Define Ψ to be such that

$$\Psi^j(s^t) \equiv \Psi^j(s^{t-1}) + \psi^j(s^t) \qquad \forall t \geq 1 \quad and \quad \Psi^j(s^0) = \psi^j(s^0)$$

The idea is that, whenever the agent strictly prefers to stay in the contract, the original multiplier takes on a value of zero. Whenever the agent is indifferent between staying and walking away, the multiplier becomes positive. Thus, the cumulative multiplier goes up (is non-decreasing) over time. When the limited commitment constraint is binding, the multiplier gets pumped up, when it is not binding, the cumulative multiplier stays the same.

Next, we merge the last constraint with the objective function using the cumulative multiplier to get our modified lagrangean

$$\begin{aligned}
\mathcal{L} &= \sum_{j=1}^{J}\sum_{s^t}\{\mu^j + \Psi^j(s^t)\}\beta^t\pi(s^t)[u^j(c^j(s^t)) - v^j(n^j(s^t))] \\
&+ \sum_{s^t}\lambda(s^t)\left\{A(s^t)F(K(s^{t-1}), N(s^t)) - K(s^t) + (1-\delta)K(s^{t-1}) - \sum_{j=1}^{J}\mu^j c^j(s^t)\right\} \\
&- \sum_{s^t}\{\Psi^j(s^t) - \Psi^j(s^{t-1})\}\beta^t\pi(s^t)\sum_{j=1}^{J}D^j(s^t)
\end{aligned}$$

Notice that the planner cannot control the last term, he weights utilities stochastically according to $D^j(s^t)$. The new FOC conditions for labor and consumption are,

$$\begin{aligned}
\{n^j(s^t)\} &: \quad \{\mu^j + \Psi^j(s^t)\}\beta^t\pi(s^t)v_n^j(n^j(s^t)) = \mu^j\lambda(s^t)A(s^t)F_N(K(s^{t-1}), N(s^t))\theta^j(s^t) \\
\{c^j(s^t)\} &: \quad \{\mu^j + \Psi^j(s^t)\}\beta^t\pi(s^t)u_c^j(c^j(s^t)) = \mu^j\lambda(s^t)
\end{aligned}$$

Notice that whenever $\Psi^j(s^t) = 0$ this is the equation we have for the standard model.

Regarding the decentralization in this economy, as the state dependent multiplier $\lambda(s^t)$ raises, so does the relative value of current consumption. Lower interest rates generate incentives to run down the savings. This in turn implies that the overall share of output will shrink. In such context, whenever the limited commitment constraint for an agent is binding, her consumption allocation gets pumped up. In this specification the $D^j(s^t)$ are such that some risk sharing is still optimal.

2.1 Some important observations

First, given the state dependent multipliers $\lambda(s^t)$, the **first order conditions for capital remain unchanged**. Second, the **labor-consumption margin remains unchanged**. Third, the distortion generated by the participation constraints implies that the utility weight of class j agents will increase whenever the constraint is binding. Fourth, in an open economy, taking the sequence of multipliers $\{\lambda(s^t)\}$ as given, capital decisions remain unchanged, but consumption paths are changed following the **"Backloading Principle"** by which consumption goes up over time for those whose constraint is binding. The logic is that consumption after several periods out (say T) becomes more helpful to relax T constraints: by promising to raise their consumption profile, those whose constraint is binding are going to be willing to stick to the contract. Remember that consumption today only relaxes today's constraints whereas consumption T periods afterwards helps to relax the T constraints, from today to period T. This is a principle commonly observed in labor contracts, where agents tend to get very well paid after their productivity peak. In such contracts, employees agree to take not very high wages today after experiencing a positive productivity shock instead of walking away from the contract if they are promised to get an increasing stream of payments through time. Here we also observe that compensation is backloaded so as to relax more effectively the budget constraints.

Finally, in a closed economy it is not possible to backload everyone since, by assumption, there are no external sources of financing. The implication is that those who get a positive shock that causes their constraint to bind get pumped up whereas all the others actually get pumped down. Savings incentives are weakened through low individual rationality constraints. This transfer of resources from the "pumped down" to the "pumped up" can be implemented through savings incentives. For the decentralization of this economy, the multipliers $\lambda(s^t)$ are to be taken as state prices. The introduction of limited commitment will lower interest rates, which will feed back into capital accumulation.

3 Simplifying Assumptions

In this section we will introduce some simplifying assumptions to analyze the evolution of consumption to shocks without making our lives too hard.

Our assumptions are

1. We will study an endowment economy, where endowment of a household j in state s^t is $y_t^j(s^t)$.
2. Assume a CRRA utility function

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

3. In an endowment economy, we can easily compute the utility derived from being in autarky, which is

$$D^j(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} \beta^{\tau-t} \pi(s^\tau | s^t) u(y^j_\tau(s^\tau))$$

Also in this case, the first order condition for consumption for household j in node s^t is

$$\{\mu^j + \Psi^j(s^t)\} \beta^t \pi(s^t) u^j_c(c^j(s^t)) = \mu^j \lambda(s^t)$$

Defining

$$\xi^j(s^t) = \frac{\mu^j + \Psi^j(s^t)}{\mu^j}$$

we can write the ratio of first order conditions for two households i and j is

$$\frac{u^i_c(c^j(s^t))}{u^j_c(c^j(s^t))} = \frac{\xi^j(s^t)}{\xi^i(s^t)}$$

The complementarity slackness conditions are

$$\lambda(s^t) \left\{ \sum_{j=1}^{J} \mu^j [y^j(s^t) - c^j(s^t)] \right\} = 0 \qquad \forall s^t$$

and

$$\psi^j(s^t) \left\{ \sum_{s^\tau} \beta^\tau \pi(s^\tau) u^j(c^j(s^t)) - \beta^t \pi(s^t) D^j(s^t) \right\} = 0 \qquad \forall j \texttt{ and } s^t$$

Using explicitly the utility function, conjecture a risk-sharing rule.

$$c^i_t(s^t) = \frac{\xi^i_t(s^t)^{1/\sigma}}{\sum_{j=1}^{J} \xi^i_t(s^t)^{1/\sigma}} \sum_{j=1}^{J} y^j_t(s^t)$$

It is easy to verify that the risk sharing rule satisfies first order conditions and market clearing. History dependence and time-varying weights are signatures of enforcement and private information problems.

Result: *Typically, a household's participation constraint binds when it switches to a high income state, if the endowment is persistent. In those states of the world, the value of autarchy is high and agents are tempted to default. When agents switch to* *these states, their multiplier increases, raising their consumption shares, depending on how severely other agents are constrained (captured by the denominator of the risk sharing rule).*

Remark: *Amnesia property: A household's consumption share decreases as long as it does not switch to a state with a binding constraint, but when it does, its consumption share increases to some cutoff level that does not depend on the history* (s^t) *if the endowment process is first-order Markov.*

Proof The first part follows from the risk sharing rule and the fact that $\{\xi_t^i(s^t)\}$ is a non-decreasing process for all i. The second part follows from the complementary slackness condition, which says that, when constraint binds

$$\left[\sum_{s^\tau|s^t} \beta^{\tau-t}\pi(s^\tau|s^t)u^j(c^j(s^t)) - D^j(s^t)\right] = 0$$

Now, if y is first order Markov, then $D^j(s^t)$ only depends on s_t. This implies $\{c_\tau^i(s^\tau)\}_{\tau=1}^t$ cannot depend on s^t, only on s_t. Q.E.D.

3.1 Shadow Prices

Recall the first order condition for household's i consumption

$$\xi_t^j(s^t)\beta^t\pi(s^t)u_c^j(c^j(s^t)) = \lambda(s^t)$$

This implies that the planner values resources in state s^t (in units of s^{t-1} consumption) as follows

$$\beta\pi(s^t|s^{t-1})\frac{\xi_t^j(s^t)u_c^j(c^j(s^t))}{\xi_{t-1}^j(s^{t-1})u_c^j(c^j(s^{t-1}))} = \frac{\lambda(s^t)}{\lambda(s^{t-1})}$$

This equation holds for any i. Using the risk sharing rule

$$c_t^i(s^t) = \frac{\xi_t^i(s^t)^{1/\sigma}}{\sum_{j=1}^J \xi_t^i(s^t)^{1/\sigma}} \sum_{j=1}^J y_t^j(s^t)$$

For any household i

$$q_{t-1}^{t}(s^{t}) = \frac{\lambda(s^{t})}{\lambda(s^{t-1})} = \beta\pi(s^{t}|s^{t-1}) \left(\frac{\sum_{j} y_{t}^{j}(s^{t+1})}{\sum_{j} y_{t-1}^{j}(s^{t})}\right)^{-\sigma} \left(\frac{\sum_{j} \xi_{t}^{j}(s^{t+1})^{1/\sigma}}{\sum_{j} \xi_{t-1}^{j}(s^{t})^{1/\sigma}}\right)^{\sigma}$$

The shadow price of aggregate consumption increases in those states of the world in which lots of agents are severely constrained, because in that case $\left(\frac{\sum_{j} \xi_{t}^{j}(s^{t+1})^{1/\sigma}}{\sum_{j} \xi_{t-1}^{j}(s^{t})^{1/\sigma}}\right)$ would be large (i.e., much larger than one). Note that if no one is constrained, the last part simply drops out.

3.2 Simple Recursive Characterization

Actually, solving the saddle point problem (recall in the problem we want to maximize c and minimize ψ and λ) is computationally challenging. Instead we can try to use what we know about constrained efficient allocations to ease the burden of computation. We use consumption weights as state variables instead of cumulative multipliers because we want to use stationary state variables.

The consumption share of household i in period t, node s^t is defined as

$$\omega_{t}^{i} = \frac{\xi_{t}^{i}(s^{t})^{1/\sigma}}{\sum_{j} \xi_{t}^{j}(s^{t})^{1/\sigma}}$$

When the household i does NOT switch to a state with a binding constraint, its consumption share next period is

$$\omega_{t+1}^{i} = \omega_{t}^{i} \frac{\sum_{j} \xi_{t}^{j}(s^{t})^{1/\sigma}}{\sum_{j} \xi_{t+1}^{j}(s^{t+1})^{1/\sigma}}$$

When the household i does switch to a state with a binding constraint, its consumption share next period is

$$\omega_{t+1}^{i} = \omega_{t}^{i} \frac{\sum_{j} \xi_{t}^{j}(s^{t})^{1/\sigma}}{\sum_{j} \xi_{t+1}^{j}(s^{t+1})^{1/\sigma}} \frac{\xi_{t+1}^{i}(s^{t+1})^{1/\sigma}}{\xi_{t}^{i}(s^{t})^{1/\sigma}}$$

but this consumption share cannot depend directly on this individual household's history s^t, because of the amnesia property: the value of the outside option only depends on the current shock s.

Define an $(I-1)\times 1$ vector of consumption shares $\overrightarrow{\omega}$ where the i^{th} element is such that

$$c_{t-1}^i = \omega_{t-1}^i \sum_j y_{t-1}^i, \qquad for\ i = 2, ..., I$$

The consumption share of the first household is just the residual $1 - \sum_{i\neq 1} \omega_{t-1}^i$. A natural choice for the states variables is $(\overrightarrow{\omega}, s)$. In the simplest case of two agents, we would keep track only of (ω_1, s).

3.2.1 Cutoff Rule for Consumption

At the start of next period, we compare the agent's consumption share in the previous period ω_{t-1}^i to the cutoff value for the current state of the world: $\underline{\omega}^i(\overrightarrow{\omega}, s)$. If the weight exceeds the cutoff weight, the consumption weight ω^i is left unchanged and the agent's actual consumption $\omega^{i'}$ is

$$\omega^{i'} = \omega^i$$

If the consumption weight ω^i is smaller than the cutoff weight, the agent's actual consumption is

$$\omega^{i'} = \underline{\omega}^i(\overrightarrow{\omega}, s)$$

The cutoff rule is determined such that the constraint binds exactly

$$U(c^i(s^t)) = D^j(s^t)$$

when the consumption weight equals the cutoff weight $\omega_t^i = \underline{\omega}_t^i$. At the end of each period we store the vector $\overrightarrow{\omega}$ which contains the consumption shares

$$\frac{\omega^{i'}}{\sum_j \omega^{j'}}$$

These consumption weights are exactly like $\xi^{1/\sigma}$, but they are normalized to make sure they sum to one at the end of each period.

Summary: *The cutoff rule for the consumption weights is given by:*

$$\begin{aligned} \omega^{i'} &= \omega^i & if \quad \omega^i > \underline{\omega}^i(\overrightarrow{\omega}, s) \\ \omega^{i'} &= \underline{\omega}^i(\overrightarrow{\omega}, s) & if \quad \omega^i \leq \underline{\omega}^i(\overrightarrow{\omega}, s) \end{aligned}$$

while actual consumption is given by

$$c^i = \frac{\omega^{i'}}{\sum_j \omega^{j'}} \sum_j y^j$$

for each agent i. At the end of each period we store $\omega_t^i = \frac{\omega^{i'}}{\sum_j \omega^{j'}}$ in a vector $\overrightarrow{\omega}$ as our state variable.

Proof The optimality of this cutoff rule follows immediately from the first order conditions in the saddle point problem of the planner. Q.E.D.

3.3 A Two-Agent Example

We consider the simplest example with two agents, two states $\{L, H\}$ and no aggregate uncertainty. In state L agent 1 draws a low endowment, in state H agent 1 draws a high endowment. The endowment sum to one in each state. How do we solve for the constrained efficient allocations? First, solve for the value of autarchy

$$\begin{aligned} D^1(L) &= u(L) + \beta \sum_{y'} \pi(y'|L) D^1(y') \\ D^1(H) &= u(H) + \beta \sum_{y'} \pi(y'|H) D^1(y') \end{aligned}$$

This implies (using vector notation) that the value of autarchy in each state can be recovered from the following equation

$$D^1(y) = u(y)(I - \beta \Pi)^{-1}$$

For the second household, we solve the following equation

$$D^2(y) = u(1 - y)(I - \beta \Pi)^{-1}$$

Remark: *Necessary and sufficient condition for perfect risk-sharing*

$$U(1/2) \geq D^1(L) \qquad \textit{and} \qquad U(1/2) \geq D^1(H)$$

Remark: *Note that, as* $\beta \to 0$*, this inequality cannot be satisfied because* $u(1/2) < u(H)$

$$u(y) + \beta \sum_{y'} \pi(y'|y) D^1(y') = u(1/2) + \beta \sum_{y'} \pi(y'|y) U(1/2)$$

Similarly, as the persistence increases (this is, $\pi(H|H) \to 1$*),* $D^1(H) \to u(H)/(1-\beta) > u(1/2)/(1-\beta)$*. If the labor income process is too persistent or if agents are too impatient, perfect risk sharing is not feasible.*

3.3.1 Consumption Dynamics

The only state variable is the consumption share of the first agent. Next, we solve for the cutoff values,

$$\begin{aligned} U_1(\underline{\omega}_1(y), y) &= u(\underline{\omega}_1) + \beta \sum_{y'} \pi(y'|y) U_1(\omega') = D^1(y) \\ U_2(\overline{\omega}_1(y), y) &= u(1 - \overline{\omega}_1) + \beta \sum_{y'} \pi(y'|y) U_2(\omega') = D^2(y) \end{aligned}$$

where ω' in the next period is found by applying the following rule, the analogue of the more general rule we described earlier.

$$\begin{aligned} if \quad \underline{\omega}_1(y) &< \omega_1 < \overline{\omega}_1(y), \quad \omega_1' = \omega_1 \\ if \quad \underline{\omega}_1(y) &> \omega_1, \qquad\qquad \omega_1' = \underline{\omega}_1 \\ if \quad \overline{\omega}_1(y) &< \omega_1, \qquad\qquad \omega_1' = \overline{\omega}_1 \end{aligned}$$

When is perfect risk sharing feasible? Perfect risk sharing is feasible when the intersection of the two intervals $(\underline{\omega}_1(y), \overline{\omega}_1(y))$ is non-empty. If it is non-empty, it contains a consumption share of a half.

3.4 A Continuum of Agents

As we discussed, when there are many agents I in the economy, the solution to the previous problem is not straightforward since we have to keep track of $I-1$ consumption weights $(\overrightarrow{\omega}, s)$, which will depend on everybody's history of shocks! Then, a much larger number of claims would have to be traded to complete the market. This turns to be a quite complicated problem, even for good computers.

A way to deal with this problem is to assume a continuum of agents. Under this assumption prices only depend on the individual specific histories coming from idiosyncratic shocks, and in the case shocks follow a first order Markovian process, just the current shock. First, redefine the consumption weight as

$$\omega_t^i(s^t) = \frac{(\xi_t^i(s^t))^{1/\sigma}}{h_t}$$

where $h_t = \int \sum_{s^t} \pi(s^t|s^0)(\xi_t^j(s^t))^{1/\sigma} d\Phi$ and Φ is the distribution of multipliers in the economy.

We want to solve for the policy function $\phi(\omega, y)$ that maps current consumption weights and the current shock into a new consumption weight. As before, this policy function has the cutoff rule property.

The procedure works as follows. First, conjecture and aggregate weight shock $\hat{g}_t$. Next, compute the cutoff values $\underline{\omega}(y)$ for all $y \in Y$ at which the constraint binds exactly.

At the start of next period, we compare the agent's consumption share in the previous period ω_{t-1} to the cutoff value for the current state of the world $\underline{\omega}(y)$.

- If the weight exceeds the cutoff weight, the consumption weight ω_{t-1} is left unchanged and the agent's actual consumption is $\omega'(\omega, y) = \omega_{t-1}$
- If the consumption weight ω_{t-1} is smaller than the cutoff weight, the agent's actual consumption is $\omega'(\omega, y) = \underline{\omega}(y)$

The cutoff is determined at the point where the constraint exactly binds. In the state space we have the consumption weight and the current idiosyncratic shock (ω, y).

Define continuation values $v(\omega, y)$

$$v(\omega, y) = u(\frac{\omega}{\hat{g}_k}) + \beta \sum_{y'} \pi(y'|y) v(\omega', y')$$

Choose $\underline{\omega}(y)$ such that $v(\underline{\omega}(y), y) = D(y)$

Now, the actual implied weight growth is $\hat{g}_{k+1} = \int \sum_{y'} \pi(y'|y)\omega'(\omega, y')d\Phi$. At the end of each period we store the distribution of ω which contains the actual consumption shares $\omega'/\hat{g}_{k+1}$. These rescaled consumption weights are stored at the end of the period.

The following algorithm shows how to compute the constrained efficient allocation in a model with limited commitment and continuum of agents, where we approximate the large number of agents by N and we denote as T the number of simulations.

1. Start by conjecturing $\hat{g}_0 = 1$.
2. Compute the cutoff values $\underline{\omega}(y)$ for all $y \in Y$.
3. Compute the implied aggregate weight shock $\hat{g}_1$ by applying the cutoff policy rules for N agents over T periods. We simulate for T periods, so the distribution has time to settle at its long-run value. Don't forget to rescale the weights at the end of each period. Compute the implied g at the end of this simulation for period T by averaging over the weights for all these agents,

$$\hat{g}_1 = \frac{1}{N} \sum_{i=1}^{N} \omega_T^i$$

4. Repeat the same procedure, but now with $\hat{g}_1$ as the initial guess for the weight shock.
5. Continue repeating the same steps. You should obtain a monotonically increasing sequence $\{\hat{g}_k\}_{k=0}^{\infty}$ that converges to g^*.
6. Once you have computed g^* and the cutoff rules $\{\underline{\omega}(y)\}$ you have computed all you need to know about the constrained efficient allocations.

Exercise: Take $\pi(H|H) = \pi(L|L) = 0.75$, $y(L) = 0.35$, $y(H) = 0.65$, $\beta = 0.65$ and $\sigma = 2$, follows the previous algorithm using $N = 5,000$ and $T = 1,000$ and solve for g^* and the cutoff rules $\{\underline{\omega}(L), \underline{\omega}(H)\}$.

4 Kocherlakota (RESTud, 1996)

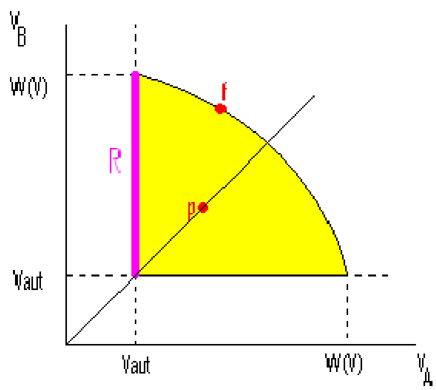
Here we study a model with two-sided limited commitment. This is an insurance scheme or contract between two agents, where any of the agents can decide to default and walk away from the contract. Here is the environment,

- There are two agents in the economy, A and B
- $t = 0, 1, ..., \infty$
- Each period the profile of endowments is $\{y_t^A, y_t^B = 1 - y_t^A\}$.
- For simplicity denote $y_t^A = y_t$, where $y_t = \{y_1, y_2, ..., y_S\}$ with respective probabilities $\pi_1, \pi_1, ..., \pi_S$, corresponding to the S states within the set of states $\mathcal{S} = \{1, 2, ..., S\}$.
- We assume that the distribution is symmetric around $1/2$ over time.
- Preferences of each agent are $\sum_{t=0}^{\infty} \beta^t u(c_t)$, where $u(\cdot)$ is increasing, concave and bounded.
- Each period there is a transfer from player A to player B.
- There is no outside lender, not there is aggregate uncertainty.
- Folk Theorem: As $\beta \to 1$ the first best can be approximated in the sense that we can implement allocations that are arbitrarily close to efficient ones, that are those for which we can eliminate consumption fluctuations over time.

The stationarity of the game allows us to break it between current actions and continuation values that belong to a well defined set that has to be self sustained. Hence we can construct the set of sustainable and efficient allocations. We impose **voluntary transfers**, then the actions prescribed in equilibrium are in the actors' best interests.

In Figure 1 I show a function (frontier) that maps one's continuation value to the other's continuation value. V_A and V_B represent the values for players A and B. V_{AUT} represents the worst punishment that one can inflict on the other; that is, make the other get his value under autarky. The point p can be sustained as a lottery between autarky and a point in the frontier. Player A is indifferent between any point in the region R. he is indifferent between being the only one punished or both of them being punished. We want to sustain a point like f, in the frontier. Supposing we start out from that point, what happens if someone deviates? How can we sustain the best possible outcome? Well, both agents can threaten to move to autarky, (which is the worst possible outcome), if one of the players deviates.

Figure 1: Pareto Frontier



Let's see the problem from a date zero perspective. When constructing the frontier, let $W(V)$ represent the value to player B as a function of V, the value to player A. Player B tries to maximize

$$W(V) = \max b(\cdot) \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) u(1 - y(s^t) + b(s^t))$$

subject to player A getting at least a value V (**promise keeping constraint**)

$$V \leq \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) u(y(s^t) - b(s^t))$$

whenever A has to pay to B, he will be happy to do so (**incentive compatibility constraint**)

$$\sum_{\tau=0}^{\infty}\sum_{s^{t+\tau}}\beta^{\tau}\frac{\pi(s^{t+\tau})}{\pi(s^t)}u(y(s^{t+\tau})-b(s^{t+\tau}))\geq u(y(s^{t+\tau}))+\beta V_{AUT}\qquad,\forall s^t$$

where

$$V_{AUT}=\sum_{\tau=0}^{\infty}\sum_{s^{t+\tau}}\beta^{\tau}\frac{\pi(s^{t+\tau})}{\pi(s^t)}u(y(s^{t+\tau}))=\frac{1}{1-\beta}\sum_{s\in\mathcal{S}}\pi_s u(y(s))$$

and player B receives as least as much utility by sticking to the contract as by walking away (**participation constraint**)

$$\sum_{\tau=0}^{\infty}\sum_{s^{t+\tau}}\beta^{\tau}\frac{\pi(s^{t+\tau})}{\pi(s^t)}u(1-y(s^{t+\tau})+b(s^{t+\tau}))\geq u(1-y(s^{t+\tau}))+\beta V_{AUT}\qquad,\forall s^t$$

Note: *Recall that in fact we have* $V^A_{AUT}=\frac{1}{1-\beta}\sum_{s\in\mathcal{S}}\pi_s u(y(s))$ *and* $V^B_{AUT}=\frac{1}{1-\beta}\sum_{s\in\mathcal{S}}\pi_s u(1-y(s))$*. Since we assumed endowments are asymmetric around* $1/2$*, and then* $V^A_{AUT}=V^B_{AUT}=V_{AUT}=$*. This assumption simplifies exposition but it is not critical otherwise.*

Now, let's express this problem in recursive form, denoting w_s as the continuation value for player A

$$W(V)=\max_{b_s,w_s}\left\{\sum_{s\in\mathcal{S}}\pi_s(u(1-y_s+b_s))+\beta W(w_s)\right\}$$

subject to promise keeping

$$V\leq\sum_{s\in\mathcal{S}}\pi_s(u(y_s-b_s))+\beta w_s$$

incentive compatibility for player A

$$u(y_s-b_s)+\beta w_s\geq u(y_s)+\beta V_{AUT}\qquad,\forall s\in\mathcal{S}$$

and participation constraint for player B

$$u(1-y_s+b_s)+\beta W(w_s)\geq u(1-y_s)+\beta V_{AUT}\qquad,\forall s\in\mathcal{S}$$

Now we can se up the Lagrangean,

$$\begin{aligned}\mathcal{L} &= \sum_{s\in\mathcal{S}} \pi_s(u(1-y_s+b_s)) + \beta W(w_s) + \mu\left(\sum_{s\in\mathcal{S}} \pi_s(u(y_s-b_s)) + \beta w_s - V\right) \\ &+ \sum_{s\in\mathcal{S}} \psi_s\left(u(y_s-b_s) + \beta w_s - u(y_s) - \beta V_{AUT}\right) \\ &+ \sum_{s\in\mathcal{S}} \chi_s\left(u(1-y_s+b_s) + \beta W(w_s) - u(1-y_s) - \beta V_{AUT}\right)\end{aligned}$$

First order conditions are

$$\{b_s\}: \quad \pi_s u'(1-y_s+b_s) - \mu\pi_s u'(y_s-b_s) - \psi_s u'(y_s-b_s) + \chi_s u'(1-y_s+b_s) = 0$$

We can observe that a higher transfer makes agent B happier, tightens the enforcement constraint for player A, and relaxes the enforcement constraint for player B. For w_s we have.

$$\{w_s\}: \quad \beta\pi_s W'(w_s) - \mu\beta\pi_s u'(y_s-b_s) - \psi_s\beta + \chi_s\beta W'(w_s) = 0$$

that is, promising a higher value to player A makes player B worst off.

Rearranging and combining these FOCs,

$$\frac{u'(1-y_s+b_s)}{u'(y_s-b_s)} = \frac{\mu\pi_s+\psi_s}{\pi_s+\chi_s} = -W'(w_s)$$

in which we can see that the marginal rate of substitution between current consumption and the continuation values gets equated for both players. Finally, we have an envelope condition

$$W'(V) = -\mu$$

To proceed with the analysis, notice that there are four cases to be studied

1. $\psi_s > 0$ and $\chi_s > 0$:

   In this case both players A and B' constraints are binding. This is impossible. The recipient party has no incentives to walk away from the contract, so it is possible to compensate him less.

2. $\psi_s = 0$ and $\chi_s = 0$:

   In this case neither constraint binds. This implies $w_s = V$, so there is no change in continuation values for player A and $\mu = \frac{u'(1-c_s)}{u'(c_s)} = -W'(V)$, so the current split does not depend on the state s.

3. $\psi_s > 0$ and $\chi_s = 0$:

   In this case player A is binding, generating a jump in A's continuation value and then in his consumption. Since $-W'(w_s) > -W'(V)$, then $w_s > V$, so we have a point in the right region of the frontier of Figure 1. In this case, we through away the promise that we have given to player A and have to pump his consumption with this equation

   $$\frac{u'(1-c_s)}{u'(c_s)} = -W'(w_s)$$

   From the incentive constraint for player A

   $$u(c_s) + \beta w_s = u(y_s) + \beta V_{AUT}$$

   and then we have that $\{c_s, w_s\}$ only depend on y_s and not on V.

4. $\psi_s = 0$ and $\chi_s > 0$:

   In this case player B is binding. In this case, $-W'(w_s) < -W'(V)$, then $w_s < V$, so that we have to be in a point on the left region of the frontier and

   $$\frac{u'(1-c_s)}{u'(c_s)} = -W'(w_s)$$

   From the participation constraint for player B

   $$u(1-c_s) + \beta W(w_s) = u(1-y_s) + \beta V_{AUT}$$

   and then we have that $\{c_s, w_s\}$ again only depending on y_s and not on V.