

Notes on Bayesian Games†

ECON 201B - Game Theory

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1 Bayesian games

So far we have been assuming that everything in the game was common knowledge for everybody playing. But in fact players may have private information about their own payoffs, about their type or preferences, etc. The way to modelling this situation of asymmetric or incomplete information is by recurring to an idea generated by Harsanyi (1967). The key is to introduce a move by the Nature, which transforms the uncertainty by converting an incomplete information problem into an imperfect information problem.

The idea is the Nature moves determining players' types, a concept that embodies all the relevant private information about them (such as payoffs, preferences, beliefs about other players, etc)

Definition 1 *A Bayesian Game is a game in normal form with incomplete information that consists of:*

- 1) *Players $i \in \{1, 2, \dots, I\}$*
- 2) *Finite action set for each player $a_i \in A_i$*
- 3) *Finite type set for each player $\theta_i \in \Theta_i$*
- 4) *A probability distribution over types $p(\theta)$ (common prior beliefs about the players' types)*
- 5) *Utilities $u_i : A_1 \times A_2 \times \dots \times A_I \times \Theta_1 \times \Theta_2 \times \dots \times \Theta_I \rightarrow \mathbb{R}$*

It is important to discuss a little bit each part of the definition.

Players' types contain all relevant information about certain player's private characteristics. The type θ_i is only observed by player i , who uses this information both to make decisions and to update his beliefs about the likelihood of opponents' types (using the conditional probability $p(\theta_{-i}|\theta_i)$)

Combining actions and types for each player it's possible to construct the strategies. Strategies will be given by a mapping from the type space to the

⁰† These notes were prepared as a back up material for TA session. If you have any questions or comments, or notice any errors or typos, please drop me a line at guilord@ucla.edu

action space, $s_i : \Theta_i \rightarrow A_i$, with elements $s_i(\theta_i)$. In words a strategy may assign different actions to different types.

Finally, utilities are calculated by each player by taking expectations over types using his or her own conditional beliefs about opponents' types. Hence, if player i uses the pure strategy s_i , other players use the strategies s_{-i} and player i 's type is θ_i , the expected utility can be written as

$$Eu_i(s_i|s_{-i}, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} u_i(s_i, s_{-i}(\theta_{-i}), \theta_i, \theta_{-i})p(\theta_{-i}|\theta_i)$$

2 Bayesian Nash Equilibrium (BNE)

A Bayesian Nash Equilibrium is basically the same concept than a Nash Equilibrium with the addition that players need to take expectations over opponents' types. Hence

Definition 2 *A Bayesian Nash Equilibrium (BNE) is a Nash Equilibrium of a Bayesian Game, i.e.*

$$Eu_i(s_i|s_{-i}, \theta_i) \geq Eu_i(s'_i|s_{-i}, \theta_i)$$

for all $s'_i(\theta_i) \in S_i$ and for all types θ_i occurring with positive probability

Theorem 3 *Every finite Bayesian Game has a Bayesian Nash Equilibrium*

3 Computing BNE

3.1 Example 1

Consider the following Bayesian game:

- 1) Nature decides whether the payoffs are as in Matrix I or Matrix II, with equal probabilities
- 2) ROW is informed of the choice of Nature, COL is not
- 3) ROW chooses U or D , COL chooses L or R (choices are made simultaneously)
- 4) Payoffs are as in the Matrix chosen by Nature

For each of these games, find all the Bayesian Nash equilibria. Write each equilibrium in mixed behavioral strategies.

Matrix I		
	L	R
U	1, 1	0, 0
D	0, 0	0, 0

Matrix II		
	L	R
U	0, 0	0, 0
D	0, 0	2, 2

3.1.1 Pure strategy BNE

We will first collapse the incomplete information problem as a static extended game with all the possible strategies (call it $\hat{\Gamma}$). It can be shown, following Harsanyi, that the Nash Equilibrium in $\hat{\Gamma}$ is the same equilibrium of the imperfect game presented. The idea is to collapse a game such that all the ways the game can follow is considered in the extended game $\hat{\Gamma}$.

The first step, as always is to determine the strategies for each player.

COL has only two strategies (L and R) because he does not know in which matrix the game is played.

ROW knows in which Matrix the game occurs, and the strategies are UU (play U in case of being in Matrix I and U in case he is in Matrix II), UD , DU and DD .

Knowing the probability (a half) the Nature locate the game in each matrix (which is needed to obtain expected payoffs), the new extended game $\hat{\Gamma}$ can be written as:

	L	R
UU	$\frac{1}{2}, \frac{1}{2}$	0, 0
UD	$\frac{1}{2}, \frac{1}{2}$	$\underline{1}, \underline{1}$
DU	0, 0	0, 0
DD	0, 0	$\underline{1}, \underline{1}$

Recall DU is a dominated strategy for ROW. After eliminating that possibility, the game has three pure Nash Equilibrium $\{(UU, L); (UD, R); (DD, R)\}$ (shown by squares in the Matrix above)

3.1.2 Mixed strategy BNE

In order to obtain the mixed strategies we will make another kind of analysis and try to replicate the three pure BNE obtained before.

Assume the probabilities of playing each action are as shown in the matrices below (y is the probability COL plays L (not conditional on the matrix since this is an information COL does not have), x is the probability ROW plays U if the game is in Matrix I and z is the probability ROW plays U if the game is in Matrix II).

Matrix I

		y	$(1 - y)$
		L	R
x	U	1, 1	0, 0
$(1 - x)$	D	0, 0	0, 0

Matrix II

		y	$(1 - y)$
		L	R
z	U	0, 0	0, 0
$(1 - z)$	D	0, 0	2, 2

Players' best responses

- ROW's best response in Matrix I:

ROW would play U instead of D ($x = 1$) if $1y + 0(1 - y) > 0 \implies y > 0$

which can be summarized as:

- i) $y > 0 \implies x = 1$
- ii) $y = 0 \implies x \in [0, 1]$

- ROW's best response in Matrix II:

ROW would play D instead of U ($z = 0$) if $0 < 2(1 - y) \implies y < 1$

which can be summarized as:

- iii) $y < 1 \implies z = 0$
- iv) $y = 1 \implies z \in [0, 1]$

- COL's best response

(recall COL does not know in which Matrix he or she is playing, assigning a probability of $\frac{1}{2}$ to be in one or the other).

COL would play L instead of R ($y = 1$) if

$$\begin{aligned} \frac{1}{2}[1x + 0(1 - x)] + \frac{1}{2}[0z + 0(1 - z)] &> \frac{1}{2}[0x + 0(1 - x)] + \frac{1}{2}[0z + 2(1 - z)] \\ \implies \frac{x}{2} > 1 - z &\implies x > 2(1 - z) \end{aligned}$$

which can be summarized as:

- v) $x = 2(1 - z) \implies y \in [0, 1]$
- vi) $x > 2(1 - z) \implies y = 1$
- vii) $x < 2(1 - z) \implies y = 0$

Now, we can check all the possibilities in order to find the Nash Equilibria, i.e. those strategies consistent for all players. Let's start by checking COL's strategies since there are less combinations.

Mixed Equilibria

Possible case 1

$y = 0$, then from ii) $x \in [0, 1]$ and from iii) $z = 0$.

Now we need to check this is a Equilibrium from COL's point of view. From vii) we can see that when $z = 0$, then $x < 2$ which always hold and that $y = 0$.

This Nash Equilibrium supports two of the three pure BNE found before: (DD, R) , which is the same as $y = 0$, $x = 0$ and $z = 0$ and (UD, R) which is the same as $y = 0$, $x = 1$ and $z = 0$

So, we have Nash Equilibria of the form $y = 0$, $x \in [0, 1]$ and $z = 0$.

There are many BNE in which Column plays R and Row plays $xU + (1 - x)D$, where $x \in [0, 1]$ if Matrix I occurs and D if Matrix II occurs.

Possible case 2

$y = 1$, then from iv) $z \in [0, 1]$ and from i) $x = 1$.

From vi) we can see that when $x = 1$, then it should be the case that $z \geq \frac{1}{2}$ in order to be true that $y = 1$. Hence, these BNE are restricted to

$y = 1, z \in (\frac{1}{2}, 1]$ and $x = 1$.

This BNE supports the third pure Nash Equilibrium found before: (UU, L) , which is the same as $y = 1, x = 1$ and $z = 1$

There are many BNE in which Column plays L and Row plays U if Matrix I occurs and $zU + (1 - z)D$, where $z \in [\frac{1}{2}, 1]$ if Matrix II occurs

Possible case 3

$y \in (0, 1)$, then from i) $x = 1$ and from iii) $z = 0$.

From v) we can see that in order y belongs to the interval between 0 and 1 it should be the case that $x = 2(1 - z)$. But it is impossible this equality to hold when both $z = 0$ and $x = 1$.

Hence, the case when $y \in (0, 1)$ is not a BNE.

3.2 Example 2

Consider the following Bayesian game:

- 1) Nature decides whether the payoffs are as in Matrix I or Matrix II, with equal probabilities
- 2) ROW is informed of the choice of Nature, COL is not
- 3) ROW chooses U or D , COL chooses L or R (choices are made simultaneously)
- 4) Payoffs are as in the Matrix chosen by Nature

For each of these games, find all the Bayesian Nash equilibria. Write each equilibrium in mixed behavioral strategies.

Matrix I		
	L	R
U	1, 1	0, 2
D	0, 2	1, 1

Matrix II		
	L	R
U	2, 2	0, 1
D	4, 4	2, 3

3.2.1 Pure strategy BNE

Again we will collapse the incomplete information problem as a static extended game with all possible strategies (call it also $\hat{\Gamma}$) and we will look for the Nash Equilibrium in $\hat{\Gamma}$

As before COL has only two strategies (L and R) and ROW's strategies are UU (play U in Matrix I and U in Matrix II), UD , DU and DD .

Hence

	L	R
UU	$\frac{3}{2}, \frac{3}{2}$	$0, \frac{3}{2}$
UD	$\frac{5}{2}, \frac{5}{2}$	$1, \frac{5}{2}$
DU	$1, 2$	$\frac{1}{2}, 1$
DD	$2, 3$	$\frac{3}{2}, 2$

First it is important to realize we can eliminate two dominated strategies for ROW, UU and DU . Hence, ROW will never play U if he knows he's in Matrix II. The game has just one pure Bayesian Nash Equilibrium given by (UD, L)

3.2.2 Mixed strategy BNE

Matrix I

	y	$(1-y)$	
	L	R	
x	U	$1, 1$	$0, 2$
$(1-x)$	D	$0, 2$	$1, 1$

Matrix II

	y	$(1-y)$	
	L	R	
z	U	$2, 2$	$0, 1$
$(1-z)$	D	$4, 4$	$2, 3$

Players' best responses

- ROW's best response in Matrix I:

ROW would play U instead of D ($x = 1$) if $1y + 0(1-y) > 0y + 1(1-y) \implies y > 1 - y$

which can be summarized as:

- i) $y > \frac{1}{2} \implies x = 1$
- ii) $y < \frac{1}{2} \implies x = 0$
- iii) $y = \frac{1}{2} \implies x \in [0, 1]$

- ROW's best response in Matrix II:

iv) When ROW observes Matrix II she will always play D , since U is a strictly dominated strategy. In other words $z = 0$ always.

- COL's best response

(recall he does not know in which Matrix he's playing, assigning a probability $\frac{1}{2}$ to be in one or the other)

COL would play L instead of R ($y = 1$) if $\frac{1}{2}[1x + 2(1-x)] + \frac{1}{2} \cdot 4 > \frac{1}{2}[2x + 1(1-x)] + \frac{1}{2} \cdot 3$

$$6 - x > 4 + x \implies x < 1$$

which can be summarized as:

$$\text{v) } \quad x = 1 \implies y \in [0, 1]$$

$$\text{vi) } \quad x < 1 \implies y = 1$$

Mixed Equilibria

Possible case 1

$$y = \frac{1}{2}, \text{ then from iii) } x \in [0, 1] \text{ and from iv) } z = 0.$$

We need to check if this is a Equilibrium from COL's point of view. From vi) we can see that, when $x < 1$ then $y = 1$, which is not the case here. Hence the only possibility is, by v), $x = 1$, in which case $y \in [0, 1]$ (and naturally contains the value $\frac{1}{2}$).

$$\text{Hence we have just one BNE } y = \frac{1}{2}, x = 1 \text{ and } z = 0.$$

There is one BNE in which Column plays $\frac{1}{2}\mathbf{L} + \frac{1}{2}\mathbf{R}$ and Row plays \mathbf{U} if Matrix I occurs and \mathbf{D} if Matrix II occurs.

Possible case 2

$$y > \frac{1}{2}, \text{ then from i) } x = 1 \text{ and from iv) } z = 0.$$

From v) when $x = 1$, then $y \in [0, 1]$

$$\text{So, these Nash Equilibria are } y \in (\frac{1}{2}, 1], x = 1 \text{ and } z = 0.$$

which supports the unique pure Nash Equilibrium found before, i.e. (UD, L) , or which is the same $y = 1, x = 1$ and $z = 0$

There are many BNE in which Column plays $y\mathbf{U} + (1 - y)\mathbf{D}$, where $y \in (\frac{1}{2}, 1]$ and Row plays \mathbf{U} if Matrix I occurs and \mathbf{D} if Matrix II occurs. Recall this is an extension of the BNE obtained before.

Possible case 3

$$y < \frac{1}{2}, \text{ then from ii) } x = 0 \text{ and from iv) } z = 0.$$

From vi), if $x = 0$, then y should be 1, which is clearly impossible.

So, the case when $y \in [0, \frac{1}{2})$ is not a BNE.

3.3 Example 3 (for you to practice)

Now consider this game (based on a Bill Zame's exercise)

ROW and COL play a game in which ROW knows his type but COL does not. With probability $\frac{3}{5}$ ROW is a person that plays considering Matrix I and with probability $\frac{2}{5}$ the game is played in Matrix II.

Matrix I		
	L	R
U	2, 4	0, 0
D	0, 0	4, -10

Matrix II		
	L	R
U	4, -10	0, 0
D	0, 0	4, 4

It's clear that if both ROW and COL knew everything (specifically in which Matrix the game occurs), in each case there would be a pure strategy Nash Equilibrium (namely (U, L) in Matrix I and (D, R) in Matrix II.

Let me claim that once we introduce incomplete information, there is no BNE in which COL plays a pure strategy. Can you show this?

Answer: There is a unique BNE where COL plays $\frac{2}{3}L + \frac{1}{3}R$, ROW plays $\frac{5}{9}U + \frac{4}{9}D$ if the true game is played in Matrix I and U if the true game is in Matrix II.