

# Notes on Extensive Form Games †

## ECON 201B - Game Theory

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### 1 Extensive Form Games

So far we have restricted attention to normal form games, where a player's strategy is just a choice of a single uncontingent action, exactly as in games of simultaneous moves.

The extensive form of a game conveys more information than the strategic form since it shows a particular sequence of moves timing. This is why every game presented in extensive form can be expressed in a strategic form and analyzed with the methods seen in previous notes.

Formally, the extensive form of a game contains the following information:

- 1) The set of players
- 2) The order of moves (who moves when, represented in a "game tree")
- 3) Players' payoffs as a function of the moves that were made.
- 4) What the players' choices are when they move
- 5) What each player knows when he makes his choices
- 6) Probability distributions over any exogenous events (moves by "Nature")

Since the extensive structure conveys more information it allows for an equilibrium concept that refines Nash Equilibrium, which is called Subgame Perfect Equilibrium (SGPE), which basically consists on the elimination of non-credible threats.

### 2 Subgame Perfect Equilibrium (SGPE)

To discuss subgame perfection let me first discuss the idea of a subgame. A subgame is every subset of the game tree that looks like a game itself. In words, it consists on an initial single node and all the nodes that follows it. Naturally the original game is a subgame in itself (the biggest subgame in the

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<sup>0</sup>† These notes were prepared as a back up material for TA session. If you have any questions or comments, or notice any errors or typos, please drop me a line at [guilord@ucla.edu](mailto:guilord@ucla.edu)

game). Subgames that are not the original game sometimes are called "proper subgames".

The main idea and importance of a subgame is that ALL previous actions and history are known by ALL players at the start of the subgame.

**Definition 1** *A subgame perfect equilibrium is a profile of strategies that are a Nash Equilibrium in every subgame (including the game as a whole).*

Because any game is a subgame in itself, a subgame perfect equilibrium profile is necessarily a Nash Equilibrium. If the only subgame in the game is the game as a whole, the sets of Nash and subgame perfect equilibria coincide. If there is any "proper" subgame, some Nash Equilibria may fail to be subgame perfect.

It's easy to see that subgame perfection coincides with backward induction in finite games of perfect information. (However you have to remind that subgame perfection is more general than backward induction). In fact.

**Theorem 2** *(Zermelo 1913; Khun 1953) A finite extensive form game of perfect information has a pure-strategy Nash Equilibrium.*

### 3 Examples

Check examples in Rolf's notes as well.

#### 3.1 Wage Bargaining 1

An important aspect of the bargaining between a Union and a Firm is that the Firm interacts with the market. A very simplified model might go like this:

- 1) The Union makes a take-it-or-leave-it wage offer  $w$ ;
- 2) if the Firm rejects the offer, both the Union and Firm make 0
- 3) if the Firm accepts the offer, the Firm then chooses a production plan to maximize profits given the wage  $w$ .

Suppose i) the objective of the Union is to maximize total income of all workers, ii) the objective of the Firm is to maximize profits, iii) the production plan of the Firm is  $Q = L$ ; iv) the market demand function is  $P = 1 - Q$ .

a) Sketching the game tree

The game states the following:

- 1) The Union makes a take-it-or-leave-it wage offer (its objective is to maximize total income  $I = wL(w)$ ).
- 2) The firm accept or reject the offer (its objective is to maximize profits  $\pi = \pi(w)$ ).

Strategies:

Union:  $w \in [0, \infty)$

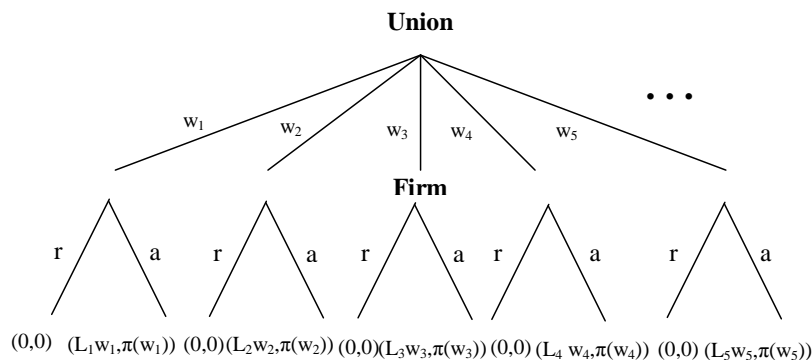
Firm:  $w \rightarrow \{a, r\}$

The payoffs (profits) for the Firm ( $\pi(w)$ ) are determined for each wage offer by a profit maximization for which the Firm decides the labor to hire ( $L(w)$ )

The payoffs (workers' income) for the Union ( $I(w) = L(w)w$ ) depends on the wage offered considering the hiring decision rule of the Firm  $L(w)$ .

Since  $L(w)$  rule follows just a maximization problem by the firm for each  $w$ , it only helps us to get the payoffs in each node.

The game tree and the respective payoffs are sketched in the following figure ( $w_1 < w_2 < \dots < w_n < \dots$ ) (Naturally there is a continuum of offers. In the tree I'm only representing five of the infinite of them).



**b)** Finding the subgame perfect equilibrium (SGPE) (Recall this will be unique since this a perfect information game).

In order to find the SGPE, first we need to compute the firm reaction to an arbitrary wage level  $w$  set by the Union (we need to solve by backwards-induction).

First, in order to get the payoffs the Firm will consider to determine its best response we need to maximize profits given a level of  $w$  choosing the optimal level of employment  $L^*$  the firm will choose once a wage is set.

$$\max \pi = PQ - wL = (1 - Q)Q - wL$$

$$s.t. Q = L$$

The problem can be formulated as:

$$\max \pi = (1 - L)L - wL$$

$$\begin{aligned}\frac{\partial \pi}{\partial L} &= 1 - L - L - w = 0 \\ L^* &= Q^* = \frac{1-w}{2} \\ \pi^* &= \left(\frac{1-w}{2}\right)^2\end{aligned}$$

Therefore, the firm will operate as long as  $w \leq 1$  (since at  $w = 1$  profits are zero). Otherwise, the firm will not produce and will reject the wage offer.

Now, we turn to the problem of the Union whose objective is to maximize total workers' income. The Union should anticipate that the firm's reaction to the wage offer will be to choose the employment level  $L^*(w)$  and the best response will be to accept the offer when  $w \leq 1$ . The maximization problem of the Union is then the following one,

$$\begin{aligned}\max I &= wL^*(w) = w \left(\frac{1-w}{2}\right) \\ \frac{\partial I}{\partial w} &= \left(\frac{1-w}{2}\right) - \frac{w}{2} = 0 \\ w^* &= \frac{1}{2}\end{aligned}$$

The unique subgame perfect equilibrium (SGPE) is:

Union:  $w = \frac{1}{2}$

Firm:

$w < 1 \rightarrow a$

$w = 1 \rightarrow xa + (1-x)r$  with  $x \in [0, 1]$

$w > 1 \rightarrow r$

The outcome of this SGPE is:

Union:  $I^*(w = \frac{1}{2}) = \frac{1}{8}$  (since  $L^* = \frac{1}{4}$ )

Firm:  $\pi^*(w = \frac{1}{2}) = \frac{1}{16}$

### 3.2 Wage Bargaining 2

The story goes the same as above. However now suppose i) the objective of the Union is to maximize total income of all workers, ii) the objective of the Firm is to maximize profits, iii) the production plan of the Firm is  $Q = (LM)^{1/2}$ ; iv) the market demand function is  $P = 1 - Q$ , v) machines cost  $P_M$  per unit.

Find the unique subgame perfect equilibrium (SGPE)

As can be seen the strategies and game tree do not change. The difference arises only on the objective function of the firm that will determine when the firm accepts or rejects the offer.

In order to find the SGPE, first we need to compute the firm reaction to an arbitrary wage level  $w$  set by the Union given an exogenous  $P_M$  (we need to solve also using backwards-induction). In this case, the firm must maximize profits given a level of  $w$  and  $P_M$  choosing the optimal level of employment  $L^*$  and machines  $M^*$ .

$$\max \pi = PQ - wL - P_M M = (1 - Q)Q - wL - P_M M$$

$$s.t. Q = (LM)^{1/2}$$

The problem can be formulated as:

$$\max \pi = (LM)^{1/2} - LM - wL - P_M M$$

$$\frac{\partial \pi}{\partial L} = \frac{1}{2} \left( \frac{M}{L} \right)^{1/2} - M - w = 0$$

$$\frac{\partial \pi}{\partial M} = \frac{1}{2} \left( \frac{L}{M} \right)^{1/2} - L - P_M = 0$$

which gives the following ratio and optimal levels:

$$\frac{M^*}{L^*} = \frac{w}{P_M}$$

$$L^* = \frac{1}{2} \left( \frac{P_M}{w} \right)^{1/2} - P_M$$

$$M^* = \frac{1}{2} \left( \frac{w}{P_M} \right)^{1/2} - w$$

As can be seen,  $L^* \geq 0$  and  $M^* \geq 0$  (hence  $Q^* \geq 0$ ) as long as  $w \leq \frac{1}{4P_M}$ . Otherwise, the firm will not operate and will reject the wage offer. The optimal production and maximum profits are the following:

$$Q^* = \left\{ \left[ \frac{1}{2} \left( \frac{P_M}{w} \right)^{1/2} - P_M \right] \left[ \frac{1}{2} \left( \frac{w}{P_M} \right)^{1/2} - w \right] \right\}^{1/2} = \frac{1}{2} - (wP_M)^{1/2}$$

$$\pi^* = \left( \frac{1}{2} - (wP_M)^{1/2} \right)^2$$

Now, we turn to the problem of the Union whose objective is to maximize total income of the workers. The Union should anticipate that the firm's reaction to the wage offer will be to reject all offers where  $w > \frac{1}{4P_M}$  (and also that the Firm will choose an employment level  $L^*(w, P_M)$ ). The maximization problem of the Union is the following:

$$\max I = wL^*(w, P_M) = w \left( \frac{1}{2} \left( \frac{P_M}{w} \right)^{1/2} - P_M \right) = \frac{1}{2} (wP_M)^{1/2} - wP_M$$

$$\frac{\partial I}{\partial w} = \frac{1}{4} \left( \frac{P_M}{w} \right)^{1/2} - P_M = 0$$

$$w^* = \frac{1}{16P_M}$$

The unique subgame perfect equilibrium (SGPE) is:

Union:  $w = \frac{1}{16P_M}$

Firm:

$$w < \frac{1}{4P_M} \longrightarrow a$$

$$w = \frac{1}{4P_M} \longrightarrow xa + (1-x)r \text{ with } x \in [0, 1]$$

$$w > \frac{1}{4P_M} \longrightarrow r$$

The outcome of this SGPE is:

Union:  $I^*(w = \frac{1}{16P_M}, P_M) = \frac{1}{16}$  (since  $L^* = P_M$ )

Firm:  $\pi^*(w = \frac{1}{16P_M}, P_M) = \frac{1}{16}$

Notice that in this case the production function  $Q = (LM)^{1/2}$  has elasticity of substitution equal to 1. Hence, given the possibility for the firm to substitute  $L$  by  $M$ , the Union is in disadvantage in relation to the result obtained in the first example. Recall the outcome for the Union is worst while the outcome for the Firm in equilibrium is the same irrespective of the machines' price  $P_M$ .

### 3.3 Sequential Bargaining

Consider the following game:

- Game of three stages where 2 players are trying to share a dollar.
- The discount rate per period is  $0 < \delta < 1$
- **At period 1**

- Player 1 offers to take a share  $s_1$  of the dollar, leaving  $1 - s_1$  to player 2
- Player 2 either accepts (and the game ends) or rejects (and the game continues another period).
- **At period 2**
- Player 2 offers that player 1 takes a share  $s_2$ , leaving  $1 - s_2$  for player 2.
- Player 1 either accepts (and the game ends) or rejects (and the game continues another period).
- **At period 3**
- Player 1 receives a share  $s$  and player 2 receives  $1 - s$  (being  $s$  exogenous and  $0 < s < 1$ ).

### Finding all SGPE

We have to solve this problem using backward induction.

**Period 3:** If this period is reached the payoffs are  $(s, 1 - s)$ . Expressed from period 1's stand point (i.e. discounted twice) the payoff will be  $(\delta^2 s, \delta^2(1 - s))$  or which is the same  $(\delta^2 s, \delta^2 - \delta^2 s)$

**Period 2:** Player 2 proposes and player 1 either accepts or rejects. By backward induction we should start analyzing what player 1 (the last one in playing at this period) will do.

If player 1 rejects, he gets  $\delta s$  (what he gets in the last period discounted because he or she has to wait until period 3 to get the money). Hence:

- Player 1 accepts if  $s_2 \geq \delta s$
- Player 1 rejects if  $s_2 < \delta s$

Now, considering this best response by player 1, if player 2 offers  $s_2 < \delta s$ , he or she will be rejected and the payoff will be  $\delta(1 - s)$ . If player 2 offers  $s_2 = \delta s$ , then the offer will be accepted and the payoff will be  $1 - \delta s$ . (Naturally, if player 2 offers  $s_2 > \delta s$  the offer would be accepted but the payoff for player 2 would be smaller than  $1 - \delta s$ , so why he would do that?)

Considering these payoffs, player 2 will prefer to offer  $s_2 = \delta s$  and get  $1 - \delta s > \delta(1 - s)$ . Hence

- Player 2 offers  $s_2 = \delta s$

Given this equilibrium, period 3 will not be reached and the payoffs will be  $(\delta s, 1 - \delta s)$ . Expressed from period 1's stand point (i.e. discounted once) the payoff will be  $(\delta^2 s, \delta(1 - \delta s))$  or which is the same  $(\delta^2 s, \delta - \delta^2 s)$

**Period 1:** Player 1 proposes and player 2 either accepts or rejects. By backward induction we should start analyzing what player 2 (the last one in playing at this period) will do.

If player 2 rejects, he gets  $\delta(1 - \delta s)$  (what he gets in period 2 discounted because he or she has to wait until period 2 to get the money). Hence:

- Player 2 accepts if  $1 - s_1 \geq \delta(1 - \delta s)$
- Player 2 rejects if  $1 - s_1 < \delta(1 - \delta s)$

Now, considering this best response by player 2, if player 1 offers  $1 - s_1 < \delta(1 - \delta s)$ , he or she will be rejected and the payoff will be  $\delta^2 s$ . If player 1

offers  $1 - s_1 = \delta(1 - \delta s)$ , then the offer will be accepted and the payoff will be  $1 - \delta(1 - \delta s)$ . (Naturally, if player 1 offers  $1 - s_1 > \delta(1 - \delta s)$  the offer would be accepted but the payoff for player 1 would be smaller than  $1 - \delta(1 - \delta s)$ , so why he would do that?)

Considering these payoffs, player 1 will prefer to offer  $s_1 = 1 - \delta(1 - \delta s)$  and get  $1 - \delta(1 - \delta s) > \delta^2 s$ . Hence

- Player 1 offers  $s_1 = 1 - \delta(1 - \delta s)$

Given this equilibrium, period 2 will not be reached and the payoffs will be  $(1 - \delta(1 - \delta s), \delta(1 - \delta s))$  or which is the same  $(1 - \delta + \delta^2 s, \delta - \delta^2 s)$

### 3.4 Prison Bargaining

Three prisoners are arguing over the last bottle of beer. The warden, tired of the noise, imposes the following bargaining scheme. Prisoner 1 must propose a division of beer. If Prisoner 2 and 3 agree, the proposed division is implemented and otherwise the warden pours out half the beer (or the warden drinks that half of the beer, whatever you prefer) and send Prisoner 1 back to his cell and Prisoner 2 must propose a division of the remaining beer. If Prisoner 3 agrees the proposal is implemented and otherwise warden pours out half the remaining beer and gives the rest to Prisoner 3 who drinks it.

Find the unique SGPE in pure strategies assuming prisoners always agree when they are indifferent and there is no discounting ( $\delta = 1$ ).

Solving backwards.

1) If the amount of beer remaining is  $y$  and the proposal of Prisoner 2 is  $(y - z, z)$  then Prisoner 3 accepts if  $z \geq \frac{y}{2}$  and reject otherwise. Given this best response by Prisoner 3, Prisoner 2 will make the lowest offer Prisoner 3 will accept (since otherwise Prisoner 2 would just receive a ride to his cell), this is  $z = \frac{y}{2}$

2) If Prisoner 2 or 3 reject the offer by Prisoner 1, half the beer will be eliminated and (considering the strategies discussed above) both Prisoners 2 and 3 would get  $\frac{1}{4}$  each. Hence Prisoner 2 and 3 will accept if the share assigned is greater or equal than  $\frac{1}{4}$  and reject otherwise.

3) Prisoner 1 will propose the worst division that Prisoner 2 and 3 will accept, so Prisoner 1 will propose  $\frac{1}{2}$  for himself and  $\frac{1}{4}$  to each of the other prisoners.