Notes on Nash Equilibrium †

ECON 201B - Game Theory

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1 Strategic Form Games

A game in strategic (normal) form is:
- A finite set of players $i = 1, ..., N$
- For each player i, a non-empty set of actions S_i (pure strategies)
- For each player i, a utility or payoff (as a function of strategy profiles) $u_i(s) : S \equiv \times_{i=1}^{N} S_i$ (usually interpreted as von Neumann-Morgenstern utility)

A useful notation is
$s_{-i} \in S_{-i} \equiv \times_{j \neq i} S_j$, the "player i's opponents" strategies profile
$s = (s_i, s_{-i}) \in S$
$u_i(s) = u_i(s_i, s_{-i}) = u_i(s_i | s_{-i})$

As can be seen u_i depends on actions of ALL players and can be extended to mixed strategies and mixed strategies profiles, defined as $\sigma_i \in \Sigma_i \equiv P(S_i)$, such that
$\sigma_{-i} \in \Sigma_{-i} \equiv \times_{j \neq i} \Sigma_j$
$\sigma = (\sigma_i, \sigma_{-i}) \in \Sigma$
$u_i(\sigma) = u_i(\sigma_i, \sigma_{-i}) = u_i(\sigma_i | \sigma_{-i})$

the expected utility or payoff can be expressed as $u_i(\sigma) = \sum_{s \in S} \left[u_i(s) \prod_{j=1}^{N} \sigma_j(s_j) \right]$ (if S is finite, this is a polynomial in σ (hence continuous), affine in σ_i)

[0] † These notes were prepared as a back up material for TA session. If you have any questions or comments, or notice any errors or typos, please drop me a line at guilord@ucla.edu

2 Dominated Strategies

Definition 1 *A pure strategy s_i is **strictly dominated** for player i if there exists $\sigma_i' \in \Sigma_i$ such that*

$$u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i}) \text{ \textbf{\textit{for all}} } s_{-i} \in S_{-i}$$

Definition 2 *A pure strategy s_i is **weakly dominated** for player i if there exists $\sigma_i' \in \Sigma_i$ such that*

$$u_i(\sigma_i', s_{-i}) \geq u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}$$

and strict for at least one $s_{-i} \in S_{-i}$

In words, a strictly dominated strategy is just an strategy that would not be used NO MATTER how opponents play.

When applying a iterated elimination of dominated strategies it's necessary to check domination of all strategies after each round of elimination. Typically it is the case that a strategy not dominated in the original game is dominated after the elimination of some of the opponents' strategies.

3 Nash Equilibrium

3.1 Definition

A Nash Equilibrium (NE) is a profile of strategies such that each player's strategy is an optimal response to the other players' strategies.

Definition 3 *A mixed-strategy profile σ is a Nash Equilibrium if, for each i and for all $\sigma_i' \neq \sigma_i$*

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

A pure-strategy Nash Equilibrium is a pure-strategy profile that satisfies the same conditions.

A Nash Equilibrium is strict if each player has a unique best response to his rivals' strategies. That is, s is a strict equilibrium if, for each i and for all $s_i' \neq s_i$, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$. As denoted by FT, by definition, a strict equilibrium is necessarily a pure-strategy equilibrium.

3.2 Computation (The recipe)

How to compute Nash Equilibria?

a) Construct the Normal form of the game (where players, actions and payoffs are the components as defined at the beginning).

b) Find the best responses that each player have against the opponents' pure strategies

For each player take each possible combination of opponents' pure-strategies and choose the own strategy that maximizes payoffs in each case.

c) Proceed iteratively to eliminate all strictly dominated pure strategies

Eliminate strategies that are never best response, no matter what the others are doing, (i.e. strategies that will never be used). Since this follows a iterative elimination it's necessary to check all the strategies after each round of elimination.

d) Check if there is a Nash equilibrium in pure strategies

A NE in pure strategies is the combination of strategies where all the best responses in pure strategies concide. In this case there is no incentives for anybody to deviate.

e) Find all mixed strategy equilibria by listing all possible combinations for the support of mixed strategies for each player and checking the necessary and sufficient conditions for a Nash equilibrium in mixed strategies.

3.3 Examples

3.3.1 Example 1: Matching Pennies (ROW chooses row, COL chooses column)[1]

	L	R
U	$1, -1$	$-1, 1$
D	$-1, 1$	$1, -1$

Following the steps of the recipe above,

a) Given by the chart below

b) Best responses in bold for each possible play of the opponent.

	L	R
U	$\mathbf{1}, -1$	$-1, \mathbf{1}$
D	$-1, \mathbf{1}$	$\mathbf{1}, -1$

[1] This is a zero sum game, (zero sum games are characterized by the fact that $\sum_{i=1}^{N} u_i(s) = 0$, for all $s \in S$). In fact, as denoted by FT, the key is that the sum of the utilities is a constant and setting the constant equal to 0 is just a normalization.

A best response against the opponents' pure strategies is the best reaction to EACH of the opponents' pure strategies profile.

For example, in this game ROW only has one opponent (COL), who has two possible pure strategies, L and R. ROW should think what is the best to do against each possible action potentially played by COL.

If COL plans to play L, the best ROW can do is to play U (since playing U delivers a payoff of 1 instead of -1 that would be result from playing D),

If COL plans to play R, then ROW prefers to play D instead of U.

Now, to obtain the best responses for COL, we should proceed in the same way considering ALL and EACH possible ROW's play.

If ROW plans to play U, then COL prefers to play R intead of L.

If ROW plans to play D, then COL prefers to play L intead of R.

c) Elimination of strictly dominated pure strategies

In this case there is no strictly dominated pure strategies since there is no action that never would be used.

d) Pure strategies NE

As can be seen there is no combination of pure strategies in which the best responses coincide. Take for example the case (U, L). If the players think this is the situation that will arise, ROW will be happy staying at that position but COL will prefer to play R instead of L. In order (U, L) to be a NE, both players should be happy at that particular position. The same analysis applies to the four pure strategies combinations that belong to the pure strategies profile. This is not the case in any of the four possibilities though.

e) Mixed strategies NE

In this case we have to allow for the possibility of randomization between strategies.

Define $\sigma_i(s)$ the probability of player i to play the strategy s. For example $\sigma_R(U)$ is the probability ROW play U

		$\sigma_C(L)$	$\sigma_C(R)$
		L	R
$\sigma_R(U)$	U	$1, -1$	$-1, 1$
$\sigma_R(D)$	D	$-1, 1$	$1, -1$

Naturally, $\sigma_R(U) + \sigma_R(D) = 1$ and $\sigma_C(L) + \sigma_C(R) = 1$

The key question is how each player react to the randomization of the other player.

Naturally, ROW will play U whenever the expected payoff from playing U is greater than the expected payoff from playing D (i.e. $u_i(U) > u_i(D)$)

$$1.\sigma_C(L) + (-1)\sigma_C(R) > (-1)\sigma_C(L) + 1.\sigma_C(R)$$

which paired with the fact that $\sigma_C(R) = 1 - \sigma_C(L)$, allows us to conclude that ROW will play U (or which is the same $\sigma_R(U) = 1$) whenever $\sigma_C(L) > \frac{1}{2}$

Summarizing, ROW's best response to COL's randomization is
$\sigma_R(U) = 1$ if $\sigma_C(L) > \frac{1}{2}$
$\sigma_R(U) = 0$ if $\sigma_C(L) < \frac{1}{2}$
$\sigma_R(U) \in [0, 1]$ if $\sigma_C(L) = \frac{1}{2}$

In a symmetry way we can obtain COL's best response to ROW's randomization
$\sigma_C(L) = 1$ if $\sigma_R(U) < \frac{1}{2}$
$\sigma_C(L) = 0$ if $\sigma_R(U) > \frac{1}{2}$
$\sigma_C(L) \in [0, 1]$ if $\sigma_R(U) = \frac{1}{2}$

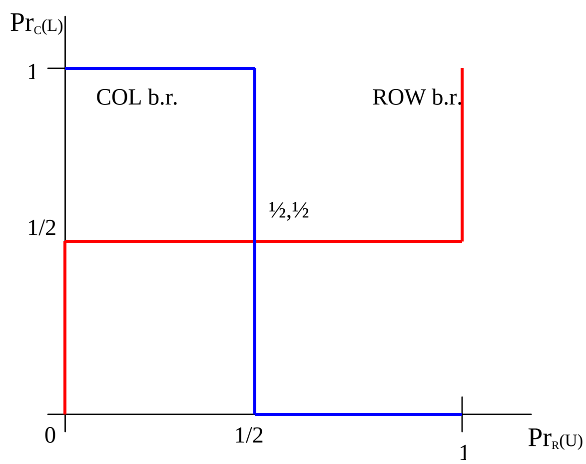
The unique mixed strategy in this case is $\sigma_R(U) = \sigma_C(L) = \frac{1}{2}$.

Why? Take any other possible randomization. Say COL randomize with $\sigma_C(L) = \frac{2}{3}$. In this case, from the best response above, ROW best action is to play U ($\sigma_R(U) = 1$). But if this is the case, COL best action is to play L for sure ($\sigma_C(L) = 1$ since $1 > \frac{1}{2}$) instead of randomizing.

Now take any possible probability different than $\frac{1}{2}$ for any player and the same argument is valid. The only consistent case is the case in which $\sigma_R(U) = \sigma_C(L) = \frac{1}{2}$, the ONLY possible equilibrium.

This consistency check is a key to understand the idea of a NE. If I'm planning to do something you will react in some way (specified by your best response to my actions). That situation is an equilibrium only if my best response to your reaction is exactly the same strategy I originally planned to play. In this neither of us would like to deviate from those plays.

A useful exercise is to graph the best responses (also called reaction functions or more precisely reaction correspondence) to see how the equilibrium arise when the best responses coincide.



(In the graph $\Pr_R(U) = \sigma_R(U)$)

As can be seen the only point in which the two best responses coincide (and no player has an ncentive to deviate) is the mixed Nash Equilibrium $\{\frac{1}{2}U + \frac{1}{2}D, \frac{1}{2}L + \frac{1}{2}R\}$.

3.3.2 Example 2: Battle of the Sexes (ROW chooses row, COL chooses column)

The previous game was characterized by a unique Nash equilibrium. The following game (belonging to a type called Battle of the sexes) show many Nash equilibrium. In order to make "refined predictions" about which will be the outcome of the game and the strategies followed, we will apply a refinement called trembling hand perfection (THP)

a)

	L	R
U	$2,1$	$0,0$
D	$0,0$	$1,5$

b) Best responses

	L	R
U	$\mathbf{2,1}$	$0,0$
D	$0,0$	$\mathbf{1,5}$

c) Elimination of strictly dominated pure strategies

In this case there is no strictly dominated pure strategies since there is no action that would never be used.

d) Pure strategies NE

In the cases (U, L) and (D, R) best responses coincide, which means we have two pure strategies Nash Equilibrium

e) Mixed strategies NE

In this case we have to allow for the possibility of randomization between strategies.

Define $\sigma_i(s)$ the probability of player i to play the strategy s. For example $\sigma_R(U)$ is the probability ROW play U

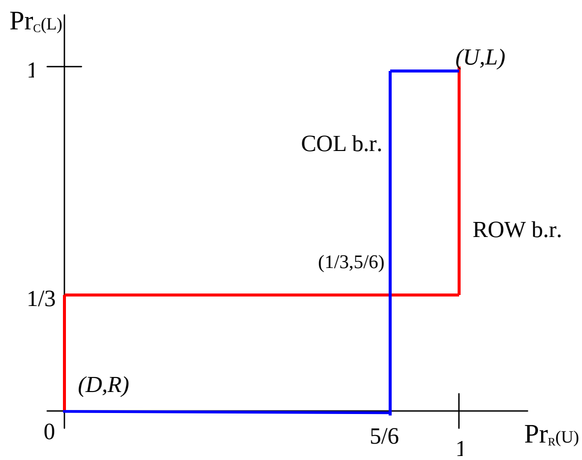
		$\sigma_C(L)$	$\sigma_C(R)$
		L	R
$\sigma_R(U)$	U	$2,1$	$0,0$
$\sigma_R(D)$	D	$0,0$	$1,5$

Naturally, $\sigma_R(U) + \sigma_R(D) = 1$ and $\sigma_C(L) + \sigma_C(R) = 1$

ROW's best reaction to a randomization by COL is
$\sigma_R(U) = 1$ if $\sigma_C(L) > \frac{1}{3}$
$\sigma_R(U) = 0$ if $\sigma_C(L) < \frac{1}{3}$
$\sigma_R(U) \in [0, 1]$ if $\sigma_C(L) = \frac{1}{3}$

COL's best reaction to a randomization by ROW
$\sigma_C(L) = 1$ if $\sigma_R(U) > \frac{5}{6}$
$\sigma_C(L) = 0$ if $\sigma_R(U) < \frac{5}{6}$
$\sigma_C(L) \in [0, 1]$ if $\sigma_R(U) = \frac{5}{6}$

The unique mixed strategy in this case is the one in which the players randomize $\sigma_R(U) = \frac{5}{6}$ and $\sigma_C(L) = \frac{1}{3}$.This can be expressed as $\{\frac{5}{6}U + \frac{1}{6}D, \frac{1}{3}L + \frac{2}{3}R\}$



Note that the Best responses collapses all the equilibrium both pure and mixed stratgies ones.

Since we have three equilibria, it's important to ask which ones are more likely. To do this, there exist some refinements that try to impose harder conditions to the set of equilibriums to see which ones are able to pass them and prove robustness in the prediction. In this case we will analyze

Trembling Hand Perfection (THP)

In a NE each player's equilibrium strategy is a best response to the other player's strategies. A THP equilibrium additionally imposes the NE should be robust to small perturbations (or mistakes) in the strategies played by other players. Hence a THP equilibrium is robust to an arbitrarily small and strictly positive probability that other players play all others pure strategies, such that each player's equilibrium strategy is still a best response to the other players' perturbed strategies.

Definition 4 *Strategy profile σ is a trembling hand perfect (THP) equilibrium if there exists a sequence of totally mixed strategy profiles $\sigma^n \to \sigma$ such that, for all i,*

$$u_i(\sigma_i, \sigma_{-i}^n) \geq u_i(s_i, \sigma_{-i}^n) \text{ for all } s_i \in S_i$$

An important NOTE:

In order a NE be THP it's necessary just to find ONE sequence that fulfills the definition (that's why a completely mixed strategy NE is always THP).

For a NE not to be THP it should not exist ANY sequence such that the definition is fulfilled.

Basically THP allows us to eliminate weakly equilibrium such as pure strategies NE based on weakly dominated strategies.

Consider the case before. There are two strict NE (strict because when defining the best responses, the choices corresponding to the NE were strictly preferred to other possible strategies) and one completely mixed NE (completely because all pure strategies are used with a strictly positive probability).

These two cases are THP equilibrium, hence the THP refinement do not help us to eliminate any of the three equilibria obtained in this case.

3.3.3 Example 3: Equilibrium with weak dominated strategies

a)

	L	R
U	$4,3$	$5,3$
D	$6,2$	$2,1$

b) Best responses

	L	R
U	$4,\mathbf{3}$	$\mathbf{5},\mathbf{3}$
D	$\mathbf{6},\mathbf{2}$	$2,1$

c) Elimination of strictly dominated pure strategies

There is no strictly dominated pure strategies since there is no action that would never be used.

d) Pure strategies NE

In the cases (D, L) and (U, R) best responses coincide, which means we have two pure strategies Nash Equilibrium

e) Mixed strategies NE

In this case we have to allow for the possibility of randomization between strategies.

Define $\sigma_i(s)$ the probability of player i to play the strategy s. For example $\sigma_R(U)$ is the probability ROW play U

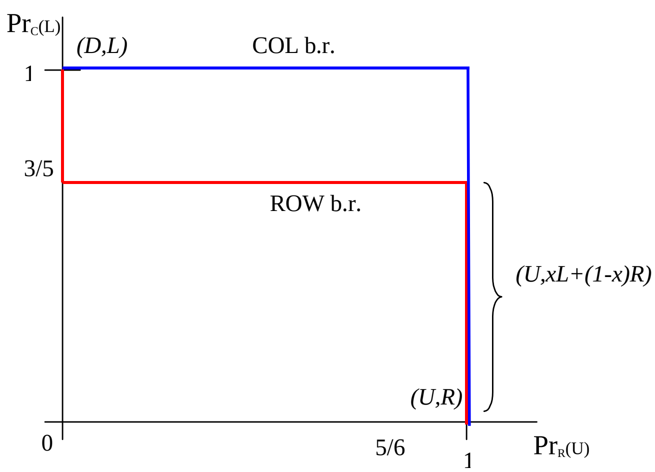
		$\sigma_C(L)$	$\sigma_C(R)$
		L	R
$\sigma_R(U)$	U	$4, 3$	$5, 3$
$\sigma_R(D)$	D	$6, 2$	$2, 1$

and, $\sigma_R(U) + \sigma_R(D) = 1$ and $\sigma_C(L) + \sigma_C(R) = 1$

ROW's best reaction to a randomization by COL is
$\sigma_R(U) = 1$ if $\sigma_C(L) < \frac{3}{5}$
$\sigma_R(U) = 0$ if $\sigma_C(L) > \frac{3}{5}$
$\sigma_R(U) \in [0, 1]$ if $\sigma_C(L) = \frac{3}{5}$

COL's best reaction to a randomization by ROW
$\sigma_C(L) = 1$ if $\sigma_R(U) < 1$
$\sigma_C(L) = 0$ if $\sigma_R(U) > 1$
$\sigma_C(L) \in [0, 1]$ if $\sigma_R(U) = 1$

In this particular case there many mixed strategy NE that can be expressed as $\{U; xL + (1 - x)R\}$ where $x \leq \frac{3}{5}$



Is (D, L) **THP?**

Take a $\sigma_R^n \to (0,1)$ (For example $(\frac{1}{n}, 1-\frac{1}{n})$ as $n \to \infty$).This is a sequence that approaches to the case in which ROW plays D for sure and then, for sure, does not play U.

Given this sequence, as $n \to \infty$, the question is: Will COL still prefer to play L?

To answer this, we need to compare,

$u_C(L, \sigma_R^n) = 3\sigma_R^n(U) + 2\sigma_R^n(D) = 3\frac{1}{n} + 2(1-\frac{1}{n})$

and

$u_C(R, \sigma_R^n) = 3\sigma_R^n(U) + \sigma_R^n(D) = 3\frac{1}{n} + 1 - \frac{1}{n}$

As can be seen, $u_C(L, \sigma_R^n) > u_C(R, \sigma_R^n)$ as $n \to \infty$, hence COL prefers to play L even when ROW can tremble.

Now we need to check what happens with ROW. If COL tremble, will ROW still prefer to play D?

Take a $\sigma_C^n \to (1,0)$.To answer this, we need to compare,

$u_R(D, \sigma_C^n) = 6\sigma_C^n(L) + 2\sigma_C^n(R) = 6(1-\frac{1}{n}) + 2\frac{1}{n}$

and

$u_R(U, \sigma_C^n) = 4\sigma_C^n(L) + 5\sigma_C^n(R) = 4(1-\frac{1}{n}) + 5\frac{1}{n}$

As can be seen, $u_R(D, \sigma_C^n) > u_R(U, \sigma_C^n)$ as $n \to \infty$, hence ROW prefers to play D even when COL can tremble. Hence (D, L) **is THP** since we've found ONE specific sequence that fulfills the definition.

Is (U, R) **THP?**

Take a $\sigma_R^n \to (1,0)$ in general that approaches to the case in which ROW plays U for sure and, for sure, does not play D. Will COL still prefer to play R for any sequence?

To answer this, we need to compare,

$u_C(L, \sigma_R^n) = 3\sigma_R^n(U) + 2\sigma_R^n(D)$

and

$u_C(R, \sigma_R^n) = 3\sigma_R^n(U) + \sigma_R^n(D)$

As can be seen, $u_C(L, \sigma_R^n) > u_C(R, \sigma_R^n)$ no matter what is $\sigma_R^n(D)$ and the sequence related, hence COL prefers to play L when ROW can tremble. Hence (U, R) **is not THP** since it's not possible to find ANY sequence that fulfills the definition.

Is $(U, xL + (1-x)R)$ **THP?**

Take a $\sigma_R^n \to (1,0)$ in general that approaches that approaches to the case in which ROW plays U for sure and, for sure, does not play D.

Will COL still prefer to randomize for any sequence?

To answer this, we need to compare,

$u_C(xL + (1-x)R, \sigma_R^n) = x[3\sigma_R^n(U) + 2\sigma_R^n(D)] + (1-x)[3\sigma_R^n(U) + \sigma_R^n(D)]$

and

$u_C(L, \sigma_R^n) = 3\sigma_R^n(U) + 2\sigma_R^n(D)$

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As can be seen, $u_C(L, \sigma_R^n) > u_C(xL+(1-x)R, \sigma_R^n)$ no matter what is $\sigma_R^n(D)$ and the sequence related, hence COL prefers to play L when ROW can tremble than randomize. Hence $(U, xL + (1-x)R)$ **is not THP** since it's not possible to find ANY sequence that fulfills the definition.

3.3.4 Example 4: More than 2 players

Consider the following 3 player game: ROW chooses U or D, COL chooses L or R, MAT chooses A or B. In each box, first entry is payoff for ROW, second is for COL and third is for MAT.

A

	L	R
U	$4, 2, 6$	$0, 3, 5$
D	$2, 2, 2$	$4, 1, 4$

B

	L	R
U	$5, 8, 3$	$9, 6, 5$
D	$4, 8, 1$	$3, 3, 3$

First note that MAT strictly prefer to play A to B except in the case in which ROW plays U and COL plans to play R. Since the preference is weak, in fact the only possible equilibrium in which strategy II can be used by MAT is $(U, R, xA + (1-x)B)$. In order for this to be an equilibrium ROW must prefer to play U to D and COL should prefer to play R to L.

For ROW, $0x + 9(1-x) \geq 4x + 3(1-x) \Longrightarrow \boxed{x \leq \frac{3}{5}}$

For COL, $3x + 6(1-x) \geq 2x + 8(1-x) \Longrightarrow \boxed{x \geq \frac{2}{3}}$

But, since $\frac{3}{5} \leq \frac{2}{3}$, there is no equilibrium in which B is played. Hence B can be eliminated and we can restrict our attention to matrix A. AS can be easily seen there is no pure-strategy NE. To obtain the mixed-strategy NE, the following conditions should hold.

To make ROW indifferent between playing U and D, $4\sigma_C(L) + 0\sigma_C(R) = 2\sigma_C(L) + 4\sigma_C(R)$. Hence $\sigma_C(L) = \frac{2}{3}$

To make COL indifferent between playing L and R, $2\sigma_R(U) + 2\sigma_R(D) = 3\sigma_R(U) + \sigma_R(D)$. Hence $\sigma_R(U) = \frac{1}{2}$

Hence, the only equilibrium in this game is a mixed-strategy Nash Equilibrium is $(\frac{1}{2}U + \frac{1}{2}D, \frac{2}{3}L + \frac{1}{3}R, A)$

3.3.5 Example 5: More than 2 actions

Take the following game from Myerson,

	L	M	R
U	$\mathbf{7}, 2$	$2, \mathbf{7}$	$3, 6$
D	$2, \mathbf{7}$	$\mathbf{7}, 2$	$\mathbf{4}, 5$

As can be seen there is no dominated strategy and there is no pure-strategy Nash Equilibrium either.

Then, we can look for mixed-strategy Nash Equilibrium, but we need to be carefull since we need to check all the possible combinations given player COL has 3 possible actions.

First of all, ROW should randomize between U and D and COL must randomize on at least two options. We will consider the following possible cases:

i) COL randomizes using the three actions L, M and R (i.e. $\sigma_C(L) > 0$, $\sigma_C(M) > 0$ and $\sigma_C(R) > 0$)

To make ROW indifferent between U and D.

$$7\sigma_C(L) + 2\sigma_C(M) + 3\sigma_C(R) = 2\sigma_C(L) + 7\sigma_C(M) + 4\sigma_C(R) \tag{1}$$

To make COL indifferent among L, M and R.

$$2\sigma_R(U) + 7\sigma_R(D) = 7\sigma_R(U) + 2\sigma_R(D) = 6\sigma_R(U) + 5\sigma_R(D) \tag{2}$$

and the obvious probability restrictions

$$\sigma_R(U) + \sigma_R(D) = 1 \quad \text{and} \quad \sigma_C(L) + \sigma_C(M) + \sigma_C(R) = 1 \tag{3}$$

Basically we have 5 equations and 5 unknowns (the five probabilities)

From the first equality on (2), $\sigma_R(U) = \sigma_R(D) = \frac{1}{2}$

But given this, the second equality in (2) does not hold because $7\frac{1}{2} + 2\frac{1}{2} \neq 6\frac{1}{2} + 5\frac{1}{2}$

Which means **there is no possible equilibrium in which COL randomizes among the three actions.**

ii) COL randomizes using M and R (i.e. $\sigma_C(L) = 0$)

To make ROW indifferent between U and D.

$$2\sigma_C(M) + 3\sigma_C(R) = 7\sigma_C(M) + 4\sigma_C(R) \tag{4}$$

To make COL indifferent between M and R.

$$7\sigma_R(U) + 2\sigma_R(D) = 6\sigma_R(U) + 5\sigma_R(D) \tag{5}$$

and the obvious probability restrictions

$$\sigma_R(U) + \sigma_R(D) = 1 \quad \text{and} \quad \sigma_C(M) + \sigma_C(R) = 1 \tag{6}$$

Basically we have 4 equations and 4 unknowns (the four probabilities)

From (5) and the first restriction on (6), $\sigma_R(U) = \frac{1}{4}$ and $\sigma_R(D) = \frac{3}{4}$

From (4) and the second restriction on (6), $\sigma_C(M) = -\frac{1}{4}$ and $\sigma_C(R) = \frac{5}{4}$

which is naturally impossible because the probabilities should be greater or equal than zero. Hence **there is no equilibrium in which COL randomizes between M nd R.**

iii) COL randomizes using L and M (i.e. $\sigma_C(R) = 0$)

To make ROW indifferent between U and D.

$$7\sigma_C(L) + 2\sigma_C(M) = 2\sigma_C(L) + 7\sigma_C(M) \tag{7}$$

To make COL indifferent between M and R.

$$2\sigma_R(U) + 7\sigma_R(D) = 7\sigma_R(U) + 2\sigma_R(D) \tag{8}$$

and the obvious probability restrictions

$$\sigma_R(U) + \sigma_R(D) = 1 \quad \text{and} \quad \sigma_C(L) + \sigma_C(M) = 1 \tag{9}$$

We also have 4 equations and 4 unknowns (the four probabilities)

From (8) and the first restriction on (9), $\sigma_R(U) = \frac{1}{2}$ and $\sigma_R(D) = \frac{1}{2}$

From (7) and the second restriction on (9), $\sigma_C(L) = \frac{1}{2}$ and $\sigma_C(M) = \frac{1}{2}$

It's easy to check that the expected payoff of this case would be 4.5 for ROW and 4.5 for COL.

Even when this case seems to fulfill all the conditions to be an equilibrium we need to check if COL would not prefer to play R instead of randomizing between L and M given the proposed randomization of ROW by half and half. To do this we need to obtain the expected payoff for COL of playing R given the proposed strategy for ROW.

Hence $u_C(R, \frac{1}{2}U + \frac{1}{2}D) = 6\frac{1}{2} + 5\frac{1}{2} = 5.5$, which means this is not an equilibrium because COL would prefer to play R, in which case ROW would prefer to play D instead of randomizing. Hence **there is no equilibrium in which COL randomizes between L nd M.**

iv) COL randomizes using L and R (i.e. $\sigma_C(M) = 0$)

To make ROW indifferent between U and D.

$$7\sigma_C(L) + 3\sigma_C(R) = 2\sigma_C(L) + 4\sigma_C(R) \tag{10}$$

To make COL indifferent between M and R.

$$2\sigma_R(U) + 7\sigma_R(D) = 6\sigma_R(U) + 5\sigma_R(D) \tag{11}$$

and the obvious probability restrictions

$$\sigma_R(U) + \sigma_R(D) = 1 \quad \text{and} \quad \sigma_C(L) + \sigma_C(R) = 1 \tag{12}$$

We also have 4 equations and 4 unknowns (the four probabilities)

From (11) and the first restriction on (12), $\sigma_R(U) = \frac{1}{3}$ and $\sigma_R(D) = \frac{2}{3}$

From (10) and the second restriction on (12), $\sigma_C(L) = \frac{1}{6}$ and $\sigma_C(R) = \frac{5}{6}$

It's easy to check that the expected payoff of this case would be $\frac{11}{3}$ for ROW and $\frac{16}{3}$ for COL.

Again we need to check whether or not it's preferred for COL to play M instead of randomize. To do this we need to obtain the expected payoff for COL of playing M given the proposed strategy for ROW.

Hence $u_C(M, \frac{1}{3}U + \frac{2}{3}D) = 7\frac{1}{3} + 2\frac{2}{3} = \frac{11}{3} < \frac{16}{3}$, which means this is an equilibrium because COL would prefer to play in the proposed profile than to play M. Hence **the unique Nash Equilibrium on this game would be** $(\frac{1}{3}U + \frac{2}{3}D, \frac{1}{6}L + \frac{5}{6}R)$