Notes on Signaling Games†

ECON 201B - Game Theory

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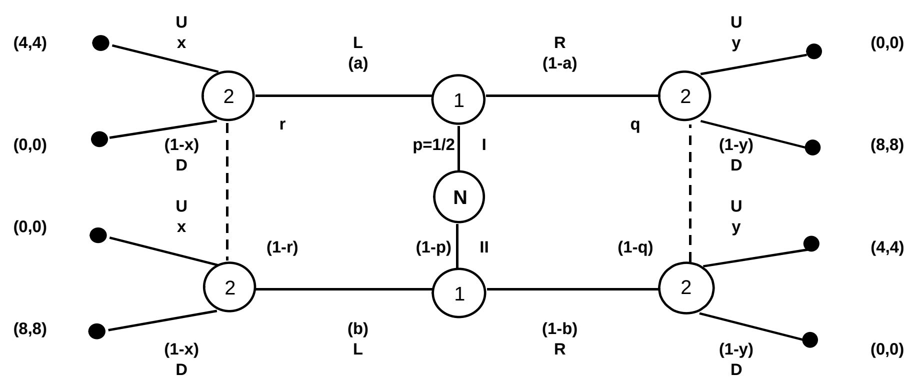
1 Understanding a Signaling Game

Consider the following game.



This is a typical signaling game that follows the classical "beer and quiche" structure. In these notes I will discuss the elements of this particular type of games, equilibrium concepts to use and how to solve for pooling, separating, hybrid and completely mixed sequential equilibria.

1.1 Elements of the game

The game starts with a "decision" by Nature, which determines whether player 1 is of type I or II with a probability $p = \Pr(I)$ (in this example we assume $p = \frac{1}{2}$). Player 1, after learning his type, must decide whether to play L or R. Since player 1 knows his type, the action will be decided conditioning on it.

[0]† These notes were prepared as a back up material for TA session. If you have any questions or comments, or notice any errors or typos, please drop me a line at guilord@ucla.edu

After player 1 moves, player 2 is able to see the action taken by player 1 but not the type of player 1. Hence, conditional on the action observed she has to decide whether to play U or D.

Hence, it's possible to define strategies for each player (π_i) as

- For player 1, a mapping from types to actions

  $\pi_1 : I \longrightarrow aL + (1-a)R$ and $II \longrightarrow bL + (1-b)R$

- For player 2, a mapping from player 1's actions to her own actions

  $\pi_2 : L \longrightarrow xU + (1-x)D$ and $R \longrightarrow yU + (1-y)D$

In this game there is no subgame since we cannot find any single node where a game completely separated from the rest of the tree starts. This is why we cannot use the concept of Subgame perfection in this type of games. However it's necessary to identify subforms, which are trees separated from the whole game that starts from an information set instead than from a single node. In this case, there are two subforms, the one that start after player 2 sees player 1 played L and the other after observing R. In general we will represent information sets for each player i by $h_i \in H_i$

But, important elements are missing here: **Beliefs ($\mathbf{b}_i$)**.

Player 1 should have some beliefs about how player 2 will eventually play. We will call these beliefs "*expectations about opponents' strategies*" (μ_i). In the example these are $x = \Pr(U|L)$ and $y = \Pr(U|R)$. Basically these are player 1's beliefs about how player 2 will react to the own action

Player 2 should have beliefs about the type of player 1 given the action she observes. We will call these beliefs "*assesments*" (α_i). In the example these are $r = \Pr(I|L)$ and $q = \Pr(I|R)$. Basically these are player 2's beliefs about who is player 1 conditional on the actions she saw.

These assesments cannot be just anything. We require these are consistent in the sense they can be constructed from the expected play of opponents in a reasonable way. Specifically we requiere assesments to be constructed following Bayes rule. In the example, let's define $a = \Pr(L|I)$ and $b = \Pr(L|II)$

Hence by Bayes rule definition

$$r = \Pr(I|L) = \frac{\Pr(L|I)\Pr(I)}{\Pr(L|I)\Pr(I) + \Pr(L|II)\Pr(II)} = \frac{ap}{ap + b(1-p)} \qquad (1)$$

and

$$q = \Pr(I|R) = \frac{\Pr(R|I)\Pr(I)}{\Pr(R|I)\Pr(I) + \Pr(R|II)\Pr(II)} = \frac{(1-a)p}{(1-a)p + (1-b)(1-p)} \tag{2}$$

The only problem we may encounter is that Bayes rule does not work at information sets that are not reached with positive probability. How can we define assesments about player 1 being of type I given he plays L if it's the case that both I and II play R for sure? To solve this we use the notion of consistency.

Definition 1 *A belief $b_i = (\alpha_i, \mu_i)$ is consistent if $\alpha_i = \lim \alpha_i^n$, being α_i^n the assesment that was constructed by using Bayes rule on a strictly positive sequence $\mu_i^n \longrightarrow \mu_i$*

Now we understood the informational structure of the game and defined strategies and beliefs, we can discuss expected payoffs in order to make decisions. The expected payoff will be obtained according to the distribution over terminal nodes induced by a behavior strategy π and beliefs $b_i = (\alpha_i, \mu_i)$ for each player.

Hence $u_i(\pi_i|b_i) = \sum_z u_i(z|\pi_i, \mu_i)\Pr(z|\alpha_i, \pi_i, \mu_i)$

We can also define expected payoffs in each information set h_i

$u_i^{h_i}(\pi_i|b_i) = \sum_z u_i(z|\pi_i^{h_i}, \mu_i^{h_i})\Pr(z|\alpha_i^{h_i}, \pi_i^{h_i}, \mu_i^{h_i})$

1.2 Sequential Equilibrium

Definition 2 *A behavior strategy π_i is sequentially rational for player i if there does not exist a profitable deviation $\pi_i' \in \Pi_i$, given beliefs b_i for player i at any of i's information sets, i.e*

$$u_i^{h_i}(\pi_i|b_i) \geq u_i^{h_i}(\pi_i'|b_i), \text{ for all } \pi_i' \in \Pi_i \text{ and for all } h_i \in H_i$$

Definition 3 *A sequential equilibrium is a behavior strategy profile π and an assesment α_i for every player i such that (α_i, π_{-i}) is consistent and π_i is sequentially rational for every player i*

2 Finding Equilibria in a Signaling Game

We will solve for pure strategy equilibria (both separating and pooling) and hybrid equilibrium for the game presented above.

2.1 Separating and Pooling Equilibria

	1's Plan	2's assesments	2's b.r.	Optimal for 1?
		Separating Equilibria		
$A)$	$I \longrightarrow L; II \longrightarrow R$	$r = 1; q = 0$	$L \longrightarrow U; R \longrightarrow U$	YES
$B)$	$I \longrightarrow R; II \longrightarrow L$	$r = 0; q = 1$	$L \longrightarrow D; R \longrightarrow D$	YES
		Pooling Equilibria		
$C)$	$I \longrightarrow L; II \longrightarrow L$	$r = \frac{1}{2}; q = ?$		
		Case $q < \frac{1}{3}$	$L \longrightarrow D; R \longrightarrow U$	YES
		Case $q > \frac{1}{3}$	$L \longrightarrow D; R \longrightarrow D$	NO $(I \longrightarrow R)$
$D)$	$I \longrightarrow R; II \longrightarrow R$	$r = ?; q = \frac{1}{2}$		
		Case $r < \frac{2}{3}$	$L \longrightarrow D; R \longrightarrow D$	NO $(II \longrightarrow L)$
		Case $r > \frac{2}{3}$	$L \longrightarrow U; R \longrightarrow D$	YES

Hence, there are two separating equilibria:

- Type I plays L, type II plays R and player 2 always plays U. Assesments are $r = 1$ and $q = 0$
- Type I plays R, type II plays L and player 2 always plays D. Assesments are $r = 0$ and $q = 1$

and there are two pooling equilibria:

- Both types play L, player 2 plays D when observes L and U when observes R. Assesments should be $r = \frac{1}{2}$ and $q < \frac{1}{3}$
- Both types play R, player 2 plays U when observes L and D when observes R. Assesments should be $r > \frac{2}{3}$ and $q = \frac{1}{2}$

Now we have to solve for hybrid and completely mixed equilibria. (At this point we will work with the cutoff points we sat aside from the chart above, $r = \frac{2}{3}$ and $q = \frac{1}{3}$)

2.2 Hybrid equilibria

These equilibria are defined as those in which one of the types play something for sure while the other randomizes. I'll analyze just one of the four possible cases here.

Let's say the action plan for player 1 is the following:

$$I \longrightarrow L \text{ and } II \longrightarrow bL + (1-b)R$$

Step 1: *Determine how player 2 will play in the information set that is reached only by one of the types.*

In this example the information set followed by R is reached only by type II who randomizes, never by type I. Hence $q = 0$ and player 2 will play $R \longrightarrow U$ (or which is the same $y = 1$).

Step 2: *Determine the conditions under which one type randomizes.*

In this example, Type II is indifferent ($b \in [0, 1]$) if $8(1 - x) = 4y$. From Step 1, this means $x = \frac{1}{2}$

Step 3: *Determine the conditions for player 2 to randomize in the information set reached by both types.*

In this example, after observing the action L, player 2 should be indifferent between choosing U or D, obtaining an equation in terms of r (i.e. the probability L is being played by a type I). Player 2 is indifferent between playing U and D when observes L ($x \in [0, 1]$) if $4r = 8(1 - r)$ or $r = \frac{2}{3}$

Step 4: *Determine player 1's randomization that is consistent with player 2's assesments (following a Bayes rule).*

In this example, since type II is the only one randomizing and $a = 1$, from equation (1)

$$r = \frac{2}{3} = \frac{ap}{ap + b(1-p)} = \frac{1}{1+b}$$

which implies that type II's randomization given by $b = \frac{1}{2}$ is consistent with the assesments that make player 2 to randomize in the information set followed by L.

Step 5: *Finally, check the player not randomizing is playing optimally.*

In this example, type I will play L since he gains 2, while if he deviates will end up with 0.

Hence, this is a hybrid sequential equilibrium:

- Player 1: $I \longrightarrow L$ and $II \longrightarrow \frac{1}{2}L + \frac{1}{2}R$

  Player 2: $L \longrightarrow \frac{1}{2}U + \frac{1}{2}D$ and $R \longrightarrow U$

  Beliefs: $r = \Pr(I|L) = \frac{2}{3}$ and $q = \Pr(I|R) = 0$

Prove there is another hybrid equilibrium in which

- Player 1: $I \longrightarrow \frac{1}{2}L + \frac{1}{2}R$ and $II \longrightarrow R$

  Player 2: $L \longrightarrow U$ and $R \longrightarrow \frac{1}{2}U + \frac{1}{2}D$

  Beliefs: $r = \Pr(I|L) = 1$ and $q = \Pr(I|R) = \frac{1}{3}$

and also prove that the cases where $I \longrightarrow R$ and $II \longrightarrow \frac{1}{2}L + \frac{1}{2}R$ and $I \longrightarrow \frac{1}{2}L + \frac{1}{2}R$ and $II \longrightarrow L$ cannot be sustained as an equilibrium.

2.3 Completely mixed equilibria

These equilibria are defined as those in which both types randomize.

Step 1: *Determine the conditions for each type to randomize between choices.*

In this example, for type I we have to equalize the payoffs from playing L with the payoffs from playing R. In this way we'll end up with one equation in terms of x and y. Doing the same for type II it's possible to obtain a similar equation. At the end we'll have two equations and two unknowns (x and y), which are player 2's randomization such that both types randomize.

Type I is indifferent ($a \in [0, 1]$) if $4x = 8(1 - y)$

Type II is indifferent ($b \in [0, 1]$) if $8(1 - x) = 4y$

Solving this system of two equations and two unknowns we obtain that $x = y = \frac{2}{3}$

Step 2: *Determine the conditions for player 2 to randomize between choices after observing any action.*

In this example, if player 2 observes player 1 played L we have to equalize the expected payoffs from playing U and D in that information set, obtaining an equation in terms of r (i.e. the probability L is being played by a type I). Similarly, q can be obtained from the equalization of the expected payoffs when player 2 sees R.

Player 2 is indifferent between playing U and D when observes L ($x \in [0,1]$) if $4r = 8(1-r)$ or $r = \frac{2}{3}$

Player 2 is indifferent between playing U and D when observes R ($y \in [0,1]$) if $4(1-q) = 8q$ or $q = \frac{1}{3}$

Step 3: *Determine player 1's randomization that is consistent with player 2's assesments (following a Bayes rule).*

In this example, from equation (1)

$$r = \frac{2}{3} = \frac{ap}{ap + b(1-p)} = \frac{a}{a+b}$$

and from equation (2)

$$q = \frac{1}{3} = \frac{(1-a)p}{(1-a)p + (1-b)(1-p)} = \frac{1-a}{2-a-b}$$

This is a system of two equations and two unknowns (a and b). Solving it, we get, $a = \frac{2}{3}$ and $b = \frac{1}{3}$, which is the randomization by player 1 that are consistent with the assesments that make player 2 to also randomize.

Hence, the completely mixed sequential equilibrium is:

- Player 1: $I \longrightarrow \frac{2}{3}L + \frac{1}{3}R$ and $II \longrightarrow \frac{1}{3}L + \frac{2}{3}R$

  Player 2: $L \longrightarrow \frac{2}{3}U + \frac{1}{3}D$ and $R \longrightarrow \frac{2}{3}U + \frac{1}{3}D$

  Beliefs: $r = \Pr(I|L) = \frac{2}{3}$ and $q = \Pr(I|R) = \frac{1}{3}$