Branes and symmetries for $\mathcal{N} = 3$ SCFTs

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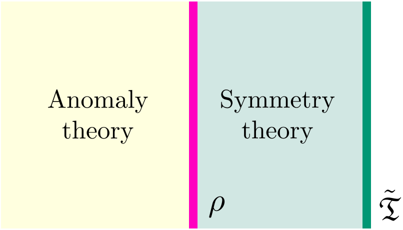
Symmetries and the symmetry theory

As discussed in a number of talks in this conference (see the talks by Ibou, Michele, Max, Ling, Sakura, Saghar, Dewi, Thomas, Muldrow and Xingyang), the study of symmetries, in a modern sense, is the study of a topological subsector of the QFT.

This topological subsector can often be very nicely characterised as a subsector of a topological theory in one dimension higher, which is sometimes called the symmetry theory[1] or SymTFT. [Freed, Teleman '12], [Freed, Teleman '18], [Gaiotto, Kulp '20], [Apruzzi, Bonetti, IGE, Hosseini, Schäfer-Nameki '21], [Burbano, Kulp, Neuser '21], [Apruzzi '22], [Freed, Moore, Teleman '22], [Kaidi, Ohmori, Zheng '22], [. . .]

[1] In analogy with the idea of "anomaly theory", reviewed in [Monnier '19].

Symmetries and representation theory



The symmetry theory is a (typically non-invertible) TFT that admits different types of boundary conditions / interfaces:

- A set of gapless interacting edge modes, $\tilde{\mathfrak{T}}$, which we think of as the local degrees of freedom of the theory whose symmetries we want to understand.
- A gapped interface ρ to some invertible theory (the anomaly).

Different choices of (anomaly, ρ) correspond to different global structures for the theory with local dynamics $\tilde{\mathfrak{T}}$.

Example: $\mathcal{N}=4$ SYM

Consider for instance the case of $\mathcal{N}=4$ with gauge algebra $\mathfrak{su}(N)$. Ignoring the R-symmetry (and its anomaly) for simplicity, the symmetry theory in this case is a higher discrete $\mathbb{Z}_N$ gauge theory, which can be presented in BF form:

$$S_{BF} = 2\pi i N \int_{X^5} B_2 \wedge dC_2 \,,$$

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The equations of motion in this theory are $dB_2 = 0$ and $dC_2 = 0$. The non-trivial operators are

$$U(\Sigma_2) = \exp\left(2\pi i \int_{\Sigma_2} B_2\right) \qquad V(\Theta_2) = \exp\left(2\pi i \int_{\Theta_2} C_2\right)$$

subject to the constraint $U^N = V^N = 1$. Because B_2 is canonically conjugate to NC_2, these operators have commutation relations

$$U(\Sigma_2)V(\Theta_2) = e^{-2\pi i\, \Sigma_2\cdot\Theta_2/N} V(\Theta_2)U(\Sigma_2)\,.$$

Example: $\mathcal{N}=4$ SYM (continued)

We would like to understand the gapped boundary conditions for this discrete theory. These arise from giving from fields Dirichlet boundary conditions on the 4d boundary. Let me call the space where the boundary lives $\mathcal{M}^4$.

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We cannot just set $B_2 = C_2 = 0$ in general, since this would imply $U(\Sigma_2) = V(\Theta_2) = 1$ for all Σ_2 and Θ_2. If $H_2(\mathcal{M}^4; \mathbb{R}) \neq 0$ this leads to an inconsistency with the commutation relation

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The right approach is as usual in quantum mechanics when we have non-commuting operators: choose a maximal subset of commuting operators, and specify a basis of states by giving the eigenvalues for this subset.

Example: $\mathcal{N} = 4$ SYM (continued)

Let's focus on $N = 2$, for concreteness. There are three non-trivial operators (up to signs): U, V and UV, and the following "universal" (independent of $\mathcal{M}^4$) choices of a maximal commuting set L: $\{U\}$, $\{V\}$ and $\{UV\}$.

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Choosing L $= \{U\}$ means that we are choosing Dirichlet boundary conditions for B_2. For example, $B_2 = 0$, but more generally some fixed phases $U(\Sigma_2)$ for each Σ_2:

$$U(\Sigma_2)\,|\Phi_2\rangle = e^{-2\pi i\,\Sigma_2\cdot\Phi_2/N}\,|\Phi_2\rangle$$

with $\Phi_2 \in H_2(\mathcal{M}^4)$. We can think of Φ_2 as the background for the 1-form symmetry in the system. (More on this in a second.)

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The operator $U(\Sigma_2) = \exp(2\pi i \int_{\Sigma_2} B_2)$ then becomes a trivial operator when pushed to the boundary. It can nevertheless *end* on the boundary, since when fixing boundary conditions we don't sum over gauge related configurations.

Example: $\mathcal{N} = 4$ SYM (continued)

We have chosen the maximal commuting set $\mathsf{L} = \{U\}$, such that the U operators become trivial (constant) when pushed to the boundary.

Thinking of the U and V operators as discrete (exponentiated) versions of position and momentum, this implies that we choose an eigenstate $|\Phi_2\rangle$ of position. This implies that it is a superposition of momenta eigenstate:

$$|\Phi_2\rangle = \sum_{\Psi_2} e^{2\pi i\, \Phi_2 \cdot \Psi_2 / N} \, |\Psi_2\rangle \quad \text{with} \quad V(\Theta_2)\, |\Psi_2\rangle = e^{2\pi i\, \Theta_2 \cdot \Psi_2 / N} \, |\Psi_2\rangle \ .$$

Note that this implies $V(\Theta_2)\, |\Phi_2\rangle = |\Phi_2 + \Theta_2\rangle$ as expected. (Momentum shifts position.)

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Note that this implies $V(\Theta_2)\,|\Phi_2\rangle = |\Phi_2+\Theta_2\rangle$ as expected. (Momentum shifts position.) So the path integral with this choice of boundary conditions involves a sum over $|\Psi_2\rangle$, and $V(\Theta_2)$ is a non-trivial (1-form symmetry) operator when pushed to the boundary.

The general case

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Depending on the choice of boundary conditions, some bulk branes will be able to end of the boundary (line operators in the field theory) and some will be able to survive being pushed to the boundary (1-form symmetries).

Goals for today

I will explain how to obtain the 1-form symmetries for $\mathcal{N} = 3$ S-fold SCFTs. [IGE, Regalado '15] [Aharony, Tachikawa '16] These theories have (in most cases) no known Lagrangian, so these are genuine predictions.

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In addition to the 1-form symmetries, I will argue that $\mathcal{N}=3$ theories can have (for suitable choices of global form) non-invertible symmetries.

$\mathcal{N} = 3$ S-folds

Calabi-Yau fourfolds of the form $(\mathbb{C}^3 \times T^2)/\mathbb{Z}_k$ can be classified completely: the orbifold actions preserving susy were classified in [Morrison, Stevens '84], [Anno '03], [Font, López '04]. We focus on the cases preserving at least 12 supercharges.

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In the F-theory limit we have $\mathbb{C}^3/\mathbb{Z}_k$ with a non-trivial $SL(2,\mathbb{Z})$ bundle on top. Adding N D3 branes on top:

- $k=1$ gives IIB string theory $\rightarrow$ 4d $\mathfrak{u}(N)$ $\mathcal{N}=4$ SYM.
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- $k=2$ gives IIB w/ O3 plane $\rightarrow$ 4d $\mathcal{N}=4$ SYM w/ orthogonal or symplectic algebras.
- $k=3,4,6$ give IIB w/ exotic S-fold $\rightarrow$ 4d $\mathcal{N}=3$ SCFTs.

So, at this level, there are two parameters: $k \in \{2,3,4,6\}$ and $N \in \{0,1,\ldots\}$. In fact, there is an extra flux parameter ℓ, which gives two different variants of each S-fold (except for $k=6$). [Aharony, Tachikawa '16]

$\mathcal{N}=3$ S-folds, holographically

If we place N D3 branes on the singular point on $\mathbb{C}^3/\mathbb{Z}_k$, and look to the near horizon geometry, we have (as in Markus' talk yesterday) F-theory on $\mathrm{AdS}_5 \times (S^5 \times T^2)/\mathbb{Z}_k$. (This background was already studied by [Ferrara, Porrati, Zaffaroni '98].)

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- Figure out the symmetry operators in the 5d bulk.
- Find out their commutation relations, so that we can classify the possible boundary conditions (symmetry structures) as states this algebra acts on.

Symmetry theory for $\mathcal{N} = 3$ S-folds

As in the $\mathfrak{su}(N)$ case [Witten '98], we want to think of the discrete 0-form and 1-form symmetry generators as branes, as discussed in [Apruzzi, Bah, Bonetti, Schäfer-Nameki '22], [IGE '22], [Heckman, Hübner, Torres, Zhang '22]. One way to motivate this requirement is that we want the symmetry generators to satisfy multiple constraints:

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A concrete consequence of this guess is that we naturally have gauge theories living on the symmetry generators. These lead to the non-invertible fusion algebra, more on this later.

(Ibou gave in his talk an alternative reasoning that leads to the same conclusion.)

The commutation algebra

Before identifying the symmetry generators, let us review how brane insertions might fail to commute in certain situations. [Gukov, Rangamani, Witten '98], [Moore '04], [Freed, Moore, Segal '06]

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Consider generalised Maxwell

$$S_{\mathsf{gM}} = \int F_{p+1} \wedge \star F_{p+1} \,,$$

with $F_{p+1} = dA_p$. Assume that we quantise this theory on $\mathcal{M}_d \times \mathbb{R}$, with $\mathbb{R}$ the time direction, and that $\operatorname{Tor} H^{d-p}(\mathcal{M}_d) = \operatorname{Tor} H^{p+1}(\mathcal{M}_d) \neq 0$.

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Choose elements a, b of these groups, with Poincaré duals (in $\mathcal{M}_d$) $\mathrm{PD}[a] \in \operatorname{Tor} H_p(\mathcal{M}_d)$ and $\mathrm{PD}[b] \in \operatorname{Tor} H_{d-p-1}$.

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$$U(a) = \exp\left(2\pi i \int_{\mathrm{PD}[a]} A_p\right) \quad ; \quad V(b) = \exp\left(2\pi i \int_{\mathrm{PD}[b]} A^D_{d-p-1}\right)$$

where $F^D_{d-p} = dA^D_{d-p-1} = \star F_{p+1}$ is the magnetic dual field strength.

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We have [Freed, Moore, Segal '06]

$$U(a)V(b) = \exp(2\pi i L(a,b))V(b)U(a)\,,$$

with the linking pairing

$$L\colon\ \mathrm{Tor}\, H^{d-p}(\mathcal{M}_d) \times \mathrm{Tor}\, H^{p+1}(\mathcal{M}_d) \to \mathbb{Q}/\mathbb{Z}$$

defined as follows: choose some integer n and chain C_a such that $n\,\mathrm{PD}[a] = \partial C_a$. Then

$$L(a,b) = \frac{C_a \cdot \mathrm{PD}[b]}{n} \mod 1\,.$$

(This is independent of choices.)

The general story

The S-fold holography case has a number of complications:

- Some of the relevant fields are self-dual ($F_5 = dC_4$). This was already discussed in [Freed, Moore, Segal '06], and there's an appropriate notion of self-linking applicable here:

$$U(a_5)U(b_5) = \exp(2\pi i L(a_5, b_5))U(b_5)U(a_5)\,.$$

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- There are Chern-Simons terms. These do complicate the story, but are essential for the details to come out right. See the paper for details.
- We are in a non-trivial $SL(2,\mathbb{Z})$ background, so for (B_2, C_2) for example we locally have a doublet of fields, but globally a single object. We need a notion of linking/non-commutativity in this case. (See Max's talk and [Heckman, Hübner, Torres, Zhang '22] for a similar setup.)

Cohomology with local coefficients

The pair (B_2, C_2) is a doublet of $SL(2,\mathbb{Z})$, and as we move around our S-fold background $S^5/\mathbb{Z}_k$ the components mix. This is an example of *cohomology with coefficients*:

$$(dB_2, dC_2) \in H^3(S^5/\mathbb{Z}_k; (\mathbb{Z}\oplus\mathbb{Z})_{\rho_k})$$

where ρ_k is the action on the coefficients as we go around the non-trivial one-cycle in $S^5/\mathbb{Z}_k$:

$$\rho_2 = -\mathbb{I} \quad ; \quad \rho_3 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \rho_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \rho_6 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} .$$

In this case, this can also be understood as the components of the C_3 field on the torus fiber of $(\mathbb{C}^3 \times T^2)/\mathbb{Z}_k$.

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$$\rho_2 = -\mathbb{I} \quad ; \quad \rho_3 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \rho_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \rho_6 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} .$$

In this case, this can also be understood as the components of the C_3 field on the torus fiber of $(\mathbb{C}^3 \times T^2)/\mathbb{Z}_k$. Either way, a straightforward computation gives [Aharony, Tachikawa '16]

$$H^3(S^5/\mathbb{Z}_k; (\mathbb{Z}\oplus\mathbb{Z})_{\rho_k}) = \mathsf{C}_k := \operatorname{coker}(\mathbb{I}-\rho_k) = \begin{cases} \mathbb{Z}_2\oplus\mathbb{Z}_2 & \text{for } k=2\,, \\ \mathbb{Z}_3 & \text{for } k=3\,, \\ \mathbb{Z}_2 & \text{for } k=4\,, \\ \mathbb{Z}_1 & \text{for } k=6\,. \end{cases} .$$

Branes

So far I have described the fluxes on the background. The symmetry generators themselves come from wrapping branes on appropriate cycles of the internal space: D3 branes are $SL(2,\mathbb{Z})$ singlets, so they wrap cycles in ordinary (untwisted) homology:

$$H_*(S^5/\mathbb{Z}_k;\mathbb{Z}) = \{\mathbb{Z}, \mathbb{Z}_k, 0, \mathbb{Z}_k, 0, \mathbb{Z}\}\,,$$

leading (for the $\mathbb{Z}_k$ terms) to 3-surfaces and lines in AdS_5 (generators of 0-form and 2-form symmetries when pushed to the boundary).

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leading (for the $\mathbb{Z}_k$ terms) to 3-surfaces and lines in AdS_5 (generators of 0-form and 2-form symmetries when pushed to the boundary). On the other hand, 1-branes and 5-branes wrap elements of

$$H_*(S^5/\mathbb{Z}_k;(\mathbb{Z}\oplus\mathbb{Z})_{\rho_k}) = \{\mathsf{C}_k, 0, \mathsf{C}_k, 0, \mathsf{C}_k, 0\}\,,$$

leading to generators of -1 and 3-form symmetries and 1-form symmetries. Focusing on the latter, we find that the k S-fold has 1-form symmetry (depending on boundary conditions) C_k.

Freed-Witten anomalies

The discussion so far is not quite complete:

- It is possible to introduce 3-form fluxes living on $H^3(S^5/\mathbb{Z}_k; (\mathbb{Z} \oplus \mathbb{Z})_{\rho_k}) = \mathsf{C}_k$. Whenever this happens, some of the symmetry generators are projected out, by a generalisation of the Freed-Witten anomaly.[2]

[2] An open question in our paper concerns the right modification of the right hand side in the $[H_3] = W_3$ Freed-Witten condition in our cases. We assume (and give circumstantial evidence) $W_3 = 0$.

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- Some of the symmetries can have 't Hooft anomalies, which we compute: for $k>2$

$$S_{\text{anomaly}} = 2\pi i \mathsf{q}_k \int A_1 \smile B_2 \smile B_2\,,$$

with $\mathsf{q}_k = -1/k$ for $k=3$ and $k=4$ and $\mathsf{q}_k = 0$ for $k=6$.

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Checks (I)

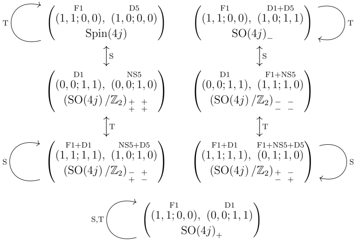

Figure 1: The SL$(2,\mathbb{Z})$ webs shared by both $\mathfrak{so}(8j)$ and $\mathfrak{so}(8j+4)$ theories.

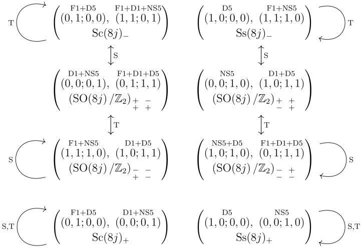

Figure 2: SL$(2,\mathbb{Z})$ webs unique to $\mathfrak{so}(8j)$ theories.

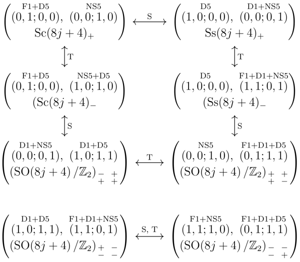

Figure 3: SL$(2,\mathbb{Z})$ webs unique to $\mathfrak{so}(8j+4)$ theories.

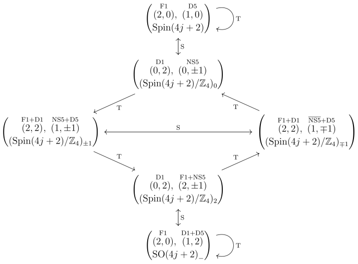

Figure 4: SL$(2,\mathbb{Z})$ webs for $\mathfrak{so}(4j+2)$ theories.

Checks (II)

I have derived the properties of the symmetry theory by directly studying how the operators in the effective theory commute (by seeing how the parent branes commute). The resulting theory can also be described in BF form. If we do this, we reproduce (and extend) the theory proposed by [Bergman, Hirano '22] in the $k = 2$ case.

Checks (III)

In [Zafrir '20], two $\mathcal{N} = 1$ theories are constructed that are conjectured to flow to the same conformal manifold as $\mathcal{N} = 3$ theories. (Similar to Craig's talk, although the details are different.)

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- A $\mathcal{N} = 1$ theory with a $\mathbb{Z}_3$ 1-form symmetry flows to the conformal manifold of the rank 3 $k = 3$ theory with no internal flux. This also agrees with our analysis ($\mathsf{C} := \mathrm{coker}(\mathbb{I} - \rho_3) = \mathbb{Z}_3$).

Non-invertibles

From the holographic dual we find

$$S_{\mathsf{anomaly}} = 2\pi i \mathsf{q}_k \int A_1 \smile B_2 \smile B_2 \,,$$

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Following [Kaidi, Ohmori, Zheng '21], this implies that gauging B_2 will lead to a non-invertible 0-form symmetry generator. The non-invertibility comes from a non-trivial topological theory living on the worldvolume on the symmetry generator. (As in Ibou's talk.)

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In the holographic setting, we expect in general that this theory comes from the degrees of freedom on the brane, reduced on the internal space [Apruzzi, Bah, Bonetti, Schäfer-Nameki '22], [IGE '22], [Heckman, Hübner, Torres, Zhang '22].

Non-invertibles (continued)

In particular, the $k=2$ case was already studied in [IGE '22], and it reproduced perfectly the field theory expectations in [Bhardwaj, Bottini, Schäfer-Nameki, Tiwari '22].

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Amusingly, the mechanism giving rise to the TFT on the symmetry defect is an old friend: non-commutativity of electric and magnetic operators in torsional backgrounds! Non-trivial duality bundles on the D3 symmetry generators, when lifted to M-theory, lead to M5 branes on torsional backgrounds.

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There's the general question of how to perform this kind of derivation systematically in the non-abelian M5 brane stack case (equivalently, non-abelian $\mathcal{N} = 4$ on duality bundle backgrounds). $\Longrightarrow$ Upcoming work with S. Hosseini.

Conclusions

$\mathcal{N} = 3$ theories are fairly mysterious, but using geometric engineering techniques we were able to obtain some rather valuable information about them, including some non-trivial supporting evidence for field theory conjectures.

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The basic picture seems to hold, but there is still much to learn on how to extract the symmetry theory in many cases!