Back to Heterotic ALE Instantonic Little String Theories

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Motivation & Outline

- **Based on**:
  1. 2209.10551 (Part I), 2212.05311(Part II), 2303.xxxxx(Part III) with Del Zotto, Oehlmann
  2. Work in progress with Braun, Del Zotto, Oehlmann

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- **So far**:
  - The Little String Theories (LSTs) associated to the heterotic $Spin(32)/\mathbb{Z}_2$ ALE instantons are Lagrangian and **well-known** [[Blum, Intriligator'97...]
  - On the contrary, most of LSTs for Heterotic $E_8 \times E_8$ ALE instantons are **unknown**, with the **exceptions** studied in [Aspinwall, Morrison'97]

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- **Goal**:
  - Determine novel LSTs by 6d conformal matter methods [Del Zotto, Heckman, Tomasiello, Vafa'14]
  - Study 2-group structure and T-dual partner of them

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- All LSTs have a higher-one form symmetry which forms a 2-group structure $\rightarrow$ relevant data provide a stringent constraint for T-dualities

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- **From field theoretical perspective**:
  1. Employ 6d conformal matter method to explore Heterotic $E_8 \times E_8$ ALE instantons

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  2. Confirm T-duality with the known $Spin(32)/\mathbb{Z}_2$ ALE instantons LSTs

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- **From geometric construction**:
  1. Realise 6d (1,0) LSTs via geometric engineering in F-theory
     [Bhardwaj, Del Zotto, Heckman, Morrison, Rudelius, Vafa'15]
  2. T-dualities are realised as inequivalent elliptic fibrations of the same geometry:
     - Employ **toric** method to construct elliptic fibered Calabi-Yau (CY) threefolds **[Huang, Taylor'19]**
     - Build LSTs from **non-geometric** Heterotic backgrounds on extremal K3 surfaces
       [Braun, Kimura, Watari'13, Font Mayrhofer'17, Font, Garcia-Etxebarria, Lust, Massai Mayrhofer'17]

Results

- **Part I**:
  1. Completely determine all Heterotic $E_8 \times E_8$ ALE instantonic LSTs
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- **Part II**:
  1. Use geometric engineering in F-theory to prove T-dualities
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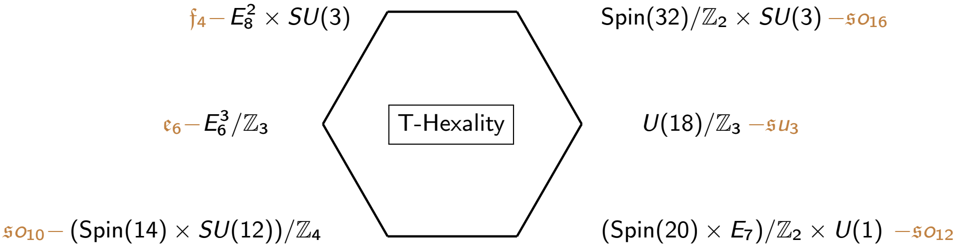

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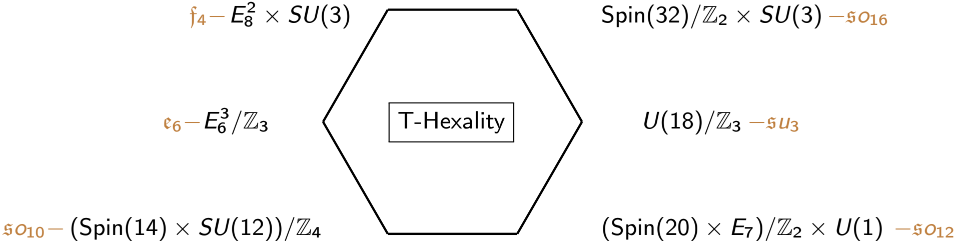

- **Part III** (cf. Michele's Talk):
  Investigate the geometric engineering limit of heterotic strings in ALE space

What is 6d LST?

- LSTs are **intermediate** between local and gravitational theories
  $\rightarrow$ they are related by decompactification:

$$\text{SUGRA} \rightarrow \text{LST} \rightarrow \text{SCFT}$$

  $\rightarrow$ capture some features of SUGRAs and SCFTs

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  2. **T-dualities** arising from a circle reduction of the 6d string compactification
  3. **Decoupled from gravity** and contains interesting **global symmetries**

Notations and Conventions for 6d LST quiver

- Consider a 6d LSTs of rank n_T, with a global zero-form symmetry: $\mathfrak{f}^{(0)} = \prod_{a=1}^{n_f} \mathfrak{f}_a$
- The Generalized quiver $[G_{f_1}] \overset{\mathfrak{g}_1}{n_1} \underset{[N_R=j]}{\overset{\mathfrak{g}_2}{n_2}} \dots \underset{[G_{f_{n_a}}]}{\overset{\mathfrak{g}_k}{n_k}} \overset{\mathfrak{g}_{n_T+1}}{n_{n_T+1}} [G_{f_{n_f}}]$ is encoded by two sets of data:

$$\begin{pmatrix} \eta^{IJ} & \eta^{IA} \\ \eta^{AI} & 0 \end{pmatrix} \qquad \begin{matrix} I, J = 1, \dots, n_T + 1 \\ A = 1, \dots, n_f \end{matrix}, \quad \mathfrak{g} = (\mathfrak{g}_1, \dots, \mathfrak{g}_{n_T+1}, \mathfrak{f}_1, \dots, \mathfrak{f}_{n_f})$$

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- η^{IJ} must be be **non-negative** for LSTs and contains a **unique null eigenvector**:

$$\eta^{IJ} N_J = 0 \,, \qquad \gcd(N_1, ..., N_{n_T+1}) = 1 \qquad N_I > 0 \,.$$

- The charge N_I features the background 2-form tensor field: $B^{(2)}_{\text{LST}} = \sum_{I=1}^{n_T+1} N_I b_I^{(2)}$
  $\leftrightarrow$ relates to a **1-form symmetry** $U(1)^{(1)}_{\textbf{LST}}$

T-duality and 2-groups

- Green-Schwartz mechanism reveals the **mixing** of $U(1)^{(1)}_{\mathrm{LST}}$ and 0-form symmetry:

$$\rightarrow \left(\mathscr{P}^{(0)} \times SU(2)^{(0)}_R \times \prod_a F^{(0)}_a \right) \times_{\widehat{\kappa}_{\mathscr{P}}, \widehat{\kappa}_R, \widehat{\kappa}_{F_a}} U(1)^{(1)}_{LST}$$

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- Two T-dual LSTs must have **matching** (very constraining):

1. $\widehat{\kappa}_{\mathscr{P}}$, $\widehat{\kappa}_R$ and $\widehat{\kappa}_{F_a}$, which control the mixing of the symmetries

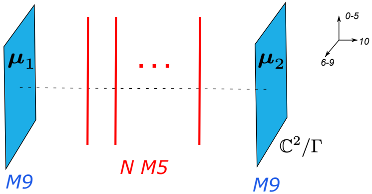
$$\widehat{\kappa}_F = -\sum_{I=1}^{n_T+1} N_I \eta^{IA} \qquad \widehat{\kappa}_R = \sum_{I=1}^{n_T+1} N_I h^{\vee}_{\mathfrak{g}_I} \qquad \widehat{\kappa}_{\mathscr{P}} = -\sum_{I=1}^{n_T+1} N_I (\eta^{II} - 2) \,,$$

2. Coulomb branch dimension and amounts of Wilson line parameters

$$\mathbf{Dim(CB)} = T + \mathrm{rk}(G) \,, \qquad \mathbf{Dim(WL)} = \mathrm{rk}(G_F) \,.$$

Generic $E_8 \times E_8$ heterotic instantons (cf. Michele's talk)

- In M-theory, the exceptional LSTs arise from a stack of N M5 branes [Hořava, Witten'95]:



- The presence of ALE singuarity $\mathbb{C}^2/\Gamma_{\mathfrak{g}}$ makes instantonic configuration **fractional**
- The resulting theory depends on a choice of a flat connection encoded in:

$$\mu_a \colon \pi_1(S^3/\Gamma_{\mathfrak{g}}) \simeq \Gamma_{\mathfrak{g}} \to E_8\,, \quad \text{for } \mu_a \simeq id\,, \text{ see [Aspinwall, Morrison'97]}$$

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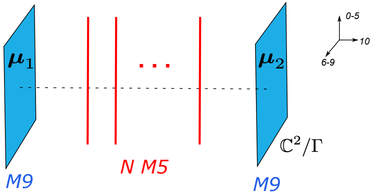

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- $\to$ Now consider **all other** possible choices
- The fractional instantons can be determined by **F-theory** [Del Zotto, Heckman, Tomasiello, Vafa'14]

Generic $E_8 \times E_8$ heterotic instantonic LSTs and T-dualities

- The resulting theories are described as a generalized quiver:

$$\mathcal{K}_N(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2; \mathfrak{g}) = \mathcal{T}(\boldsymbol{\mu}_1, \mathfrak{g}) \overset{\mathfrak{g}}{\text{———}} \mathcal{T}_{N-2}(\mathfrak{g}, \mathfrak{g}) \overset{\mathfrak{g}}{\text{———}} \mathcal{T}(\boldsymbol{\mu}_2, \mathfrak{g})$$

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- $\mathcal{T}(\boldsymbol{\mu}_a, \mathfrak{g})$: 6d orbi-instanton theory associated to a single M9-M5 system
  [Heckman, Morrison, Rudelius, Vafa'15, Mekareeya, Ohmori, Tachikawa, Zafrir'17, Frey, Rudelius'18]

- $\mathcal{T}_{N-2}(\mathfrak{g}, \mathfrak{g})$: 6d conformal matter theory associated to $N-2$ M5 branes
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- $\overset{\mathfrak{g}}{\text{———}}$: fusion of the common factors $\mathfrak{g}$ of the global symmetry of SCFTs
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- Given the matching criteria, we are able to **chart the T-dualities**:

$$\text{novel } \mathcal{K}_N(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2; \mathfrak{g}) \sim \text{known } \widetilde{\mathcal{K}}_{\widetilde{N}}(\boldsymbol{\lambda}; \widetilde{\mathfrak{g}}) \text{ (Heterotic } Spin(32)/\mathbb{Z}_2 \text{ instatons)}$$

Example - Heterotic Instantons on Exceptional Singularities

- Obtain new identified models of the form:

$$\mathcal{K}_N(E_r, E_s; \mathfrak{e}_q) = \mathcal{T}(E_r, \mathfrak{e}_\mathfrak{q}) \overset{\mathfrak{e}_\mathfrak{q}}{\text{———}} \mathcal{T}_{N-2}(\mathfrak{e}_\mathfrak{q}, \mathfrak{e}_\mathfrak{q}) \overset{\mathfrak{e}_\mathfrak{q}}{\text{———}} \mathcal{T}(E_s, \mathfrak{e}_\mathfrak{q}) \;\; , s, r, q \in \{6, 7, 8\}$$

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- Take $q = 6$, the corresponding $Spin(32)/\mathbb{Z}_2$ Heterotic instantonic theory is:

$$\underset{[\mathfrak{so}_{2w_1}]}{\overset{\mathfrak{sp}_{v_1}}{1}} \ \underset{[\mathfrak{sp}_{w_2}]}{\overset{\mathfrak{so}_{2v_2}}{4}} \ \underset{[\mathfrak{so}_{2w_3}]}{\overset{\mathfrak{sp}_{v_3}}{1}} \ \underset{[\mathfrak{u}_{w_4}]}{\overset{\mathfrak{su}_{v_4}}{2}} \ \underset{[\mathfrak{u}_{w_4}]}{\overset{\mathfrak{su}_{v_5}}{2}} \ ,$$

with relevant matching criteria (v_i and w_i are functions of N and determined by $\boldsymbol{\lambda}$):

$$\widehat{\kappa}_R = 2 + v_1 + 3v_3 + 2v_2 + 2v_4 + v_5, \quad \text{Dim(CB)} = 2 + v_1 + v_2 + v_3 + v_4 + v_5$$

Review: LST from F-theory

- An elliptic fibered CY threefold X $\leftrightarrow$ A 6d LST in F-theory
  X may admit multiple inequivalent elliptic fibrations:

$$\begin{array}{ccc} T^2 \rightarrow & X \\ & \downarrow \pi \\ & B_2 \end{array} , \quad \begin{array}{ccc} T^2 \rightarrow & X \\ & \downarrow \widetilde{\pi} \\ & \widetilde{B_2} \end{array}$$

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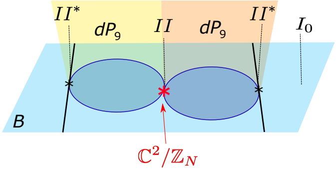
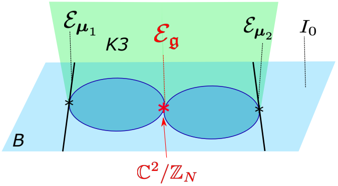
- Obtain the same 5d theory (with inequivalent 6d uplifts) after circle reduction
- Geometrically realise T-duality between these two 6d theories

Geometrizing Heterotic ALE instantons

- Hořava-Witten setup: Heterotic instantons $\leftrightarrow$ N M5 branes parallel to two M9s
- In IIB, N M5 branes emerge as a $\mathbb{C}^2/\mathbb{Z}_N$ singularity
  $\leftrightarrow$ The locus of two -1 curves intersect in the degeneration limit of $K3 \to X$
  (cf. latter slides for the description of K3 fibration)





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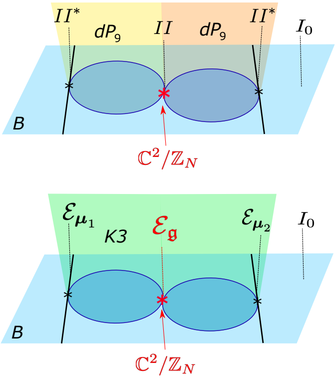

- The bottom renders a singular threefold, which requires a resolution

Toric Preminalry

- Construct non-compact CY threefolds X via 4D polytopes Δ [Batyrev, Borisov'94]
  1. Obtain toric variety $\mathcal{A}$ from a Fan associated to a 4d polytope Δ
  2. Get X as the anti-canonical hypersurface in $\mathcal{A}$.
  3. The elliptic fibration on X is inherited from a fibration structure in $\mathcal{A}$

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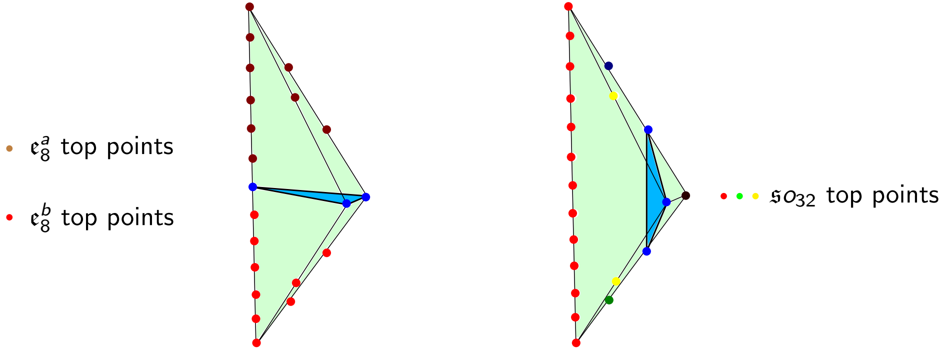
- Note that X also admits the structure of an elliptic K3 fibered over $\mathbb{C}$
  $\leftrightarrow$ Toric ambient space $\mathcal{A}$ can be described as reflexive 3D polytope fibered over $\mathbb{C}$

Toric Visualization

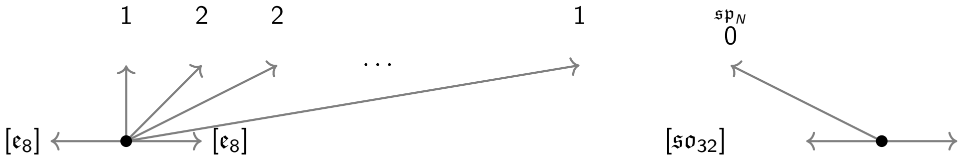
- Use polytope method to obtain the fibration structure **[Anderson, Gao, Gray, Lee'16, Huang Taylor'18]**



- Two depictions of the same 3D polytope that leads to distinct elliptic fibration.
  - Left: a 2D sub-polytope (blud shadow) that identifies two $\mathfrak{e}_8$ top fibers
  - Right: a 2D sub-polytope (blud shadow) of F_{13} type with an $\mathfrak{so}_{32}$ top

Toric Realization of LSTs - Warmup

- Embed into a toric fourfold ambient space with generic LST base B_2
- Left: decorate additional rays, yield N small E_8 instantons on the $E_8 \times E_8$ fibration



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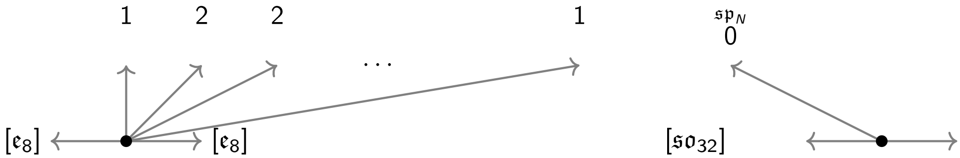

- Right: T-dual $Spin(32)/\mathbb{Z}_2$ theory with $\mathfrak{sp}_N$ gauge group [Aspinwall, Morrison'97]
- Check for T-dual pairs by the **matched** data:

$$\mathrm{Dim(CB)} = N + 1\ , \qquad \widehat{\kappa}_{\mathscr{R}} = h^\vee_{g} = N + 1\ , \qquad \widehat{\kappa}_{\mathscr{P}} = 2$$

More example (novel): The $[\mathfrak{e}_8] - \mathfrak{e}_6^M - [\mathfrak{e}_7]$ LST

- Gauging the $E_8 \times E_7$ flavor factors with $M\times$ $\mathfrak{e}_6$'s, we obtain

$$[\mathfrak{e}_8] \ \underbrace{\overset{\mathfrak{e}_6}{1} \ \overset{\mathfrak{e}_6}{2} \ldots \overset{\mathfrak{e}_6}{2} \ \overset{\mathfrak{e}_6}{1}}_{\times M} \ [\mathfrak{e}_7]$$

- Include conformal matter factors $\mathcal{T}(\mathfrak{e}_6, \mathfrak{e}_6)$, $\mathcal{T}(\mathfrak{e}_8, \mathfrak{e}_6)$ and $\mathcal{T}(\mathfrak{e}_6, \mathfrak{e}_7)$

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- The T-dual fibration has F_{13} fibre type with a $\mathbb{Z}_2$ MW group. The dual chain is :

$$[\mathfrak{so}_{28}] \; \overset{\mathfrak{sp}_{M+7}}{1} \; \overset{\mathfrak{so}_{4M+16}}{4} \; \overset{\mathfrak{sp}_{3N+1}}{1} \; \overset{\mathfrak{su}_{4N+2}}{2} \; \overset{\mathfrak{su}_{2N+2}}{2} \; [\mathfrak{su}_2]$$

More example (novel): The $[\mathfrak{e}_8] - \mathfrak{e}_6^M - [\mathfrak{e}_7]$ LST

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- The 2-groups data and CB dimension can be matched and are given below as

	$[\mathfrak{e}_8] - \mathfrak{e}_6^M - [\mathfrak{e}_7]$	T-dual
Dim(CB)	$22 + 12M$	
$\widehat{\kappa}_{\mathscr{R}}$	$24M + 34$	

Discrete Holonomy Theories

- Explore the geometric models with a non-trivial global structure [Aspinwall, Morrison'98...]
The fibration admits non-trivial section structure to generate an Abelian group:

$$MW(X) = \mathbb{Z}^r \times MW(X)_{Tor}$$

- $MW(X)_{Tor}$ gives rise to non-simply connected symmetries by acting as a diagonal quotient on the (sub-)centers $Z(G_F)$ and $Z(G)$ of flavor and gauge group:

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- They have a beautiful toric realization of the different fibre type F_i respectively

$\mathbb{Z}_2$ Discrete Holonomy LSTs

- We start by the flavor breaking given below:

$$\mathbb{Z}_2 : \mathfrak{e}_8 \to \{\mathfrak{e}_7 \times \mathfrak{su}_2, \mathfrak{so}_{16}\}$$

- Consider a breaking to $\mathfrak{e}_7 \times \mathfrak{su}_2$ and $\mathfrak{e}_7^M$ gaugings:

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- This choice admits two more inequivalent toric fibrations. The first one is given as

$$[\mathfrak{so}_8]\; \overset{\mathfrak{sp}_{M-1}}{1}\; \overset{\mathfrak{so}_{4M+4}}{4}\; \overset{\mathfrak{sp}_{3M-3}}{1}\; \underset{}{\overset{\overset{\overset{\mathfrak{sp}_{2M-4}}{1^*}}{\mathfrak{so}_{8M^*}}}{4^*}}\; \overset{\mathfrak{sp}_{3M-1}}{1}\; \overset{\mathfrak{so}_{4M+12}}{4}\; \overset{\mathfrak{sp}_{M+5}}{1}\; [\mathfrak{so}_{24}]$$

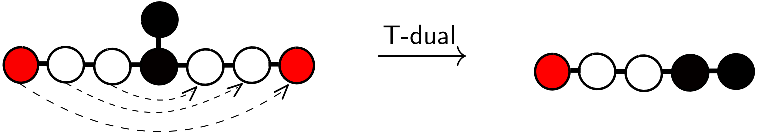
This chain has the full $\mathfrak{e}_7^{(1)}$ topology! $\leftrightarrow$ **Fiber-Base-Duality**

$\mathbb{Z}_2$ Discrete Holonomy LSTs

- The third fibration has the quiver given as

$$[\mathfrak{u}_{16}] \overset{\mathfrak{su}_{2M+10}}{2} \overset{\mathfrak{su}_{4M+4}}{2} \overset{\mathfrak{su}_{6M-2}}{2} \overset{\mathfrak{sp}_{4M-4}}{1} \overset{\mathfrak{so}_{4M+4}}{4^*}$$

- The chain has the topolgy of the twisted affine folded type of the $\mathfrak{e}_7^{(1)} \to \mathfrak{e}_6^{(2)}$ base



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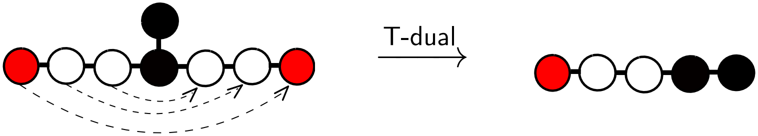

- The 2-groups and CB dimension are matched and given below as

$$\text{Dim(CB)} = 18M + 11\,, \quad \widehat{\kappa}_{\mathscr{R}} = 48M + 2$$

Heterotic LSTs and K3 geometry - Warmup in 8d

- In 8d, the F-theory dual of the Heterotic string is obtained from an elliptic K3 with the stable **degeneration** limit: **[Friedman, Morgan, Witten'97, Donagi'97]**

$$\begin{array}{ccc} T^2 \longrightarrow K3 & & T^2 \longrightarrow dP_9 \vee_{T^2_H} dP_9 \\ \qquad\quad \downarrow f & \rightarrow & \qquad\qquad \downarrow f \\ \qquad\quad \mathbb{P}^1 & & \qquad\qquad \mathbb{P}^1 \end{array}$$

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$$\Lambda_{K3} = U^{\oplus 3} \oplus E_8^{\oplus 2} \cong \Pi_{3,19}$$

- In general, $NS(K3) \subset \Lambda_{K3}$ is the most crucial sublattice **[Braun, Kimura, Watari'13]**:
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- Allow elliptic fibration in K3 $\leftarrow$ embedding hyperbolic lattice U into NS(K3):

$$NS(K3) \cong U \oplus W_{\text{frame}}$$

Heterotic LSTs and K3 fibration - in 6d

- Fiber the elliptic K3s over a non-compact direction with nested fibration structure

$$\begin{array}{ccc} K3 \to & X & \\ & \downarrow & \\ & \mathbb{C} & \end{array} \quad \text{with} \quad \begin{array}{ccc} T^2 \to & K3 & \\ & \downarrow & \\ & \mathbb{P}^1 & \end{array}$$

- Add non-trivial compact curves to X $\to$ **K3 degeneration** $\to$ 6d LST quiver
  - LST is localised at the origin of $\mathbb{C} \leftrightarrow$ additional compact curves at the origin
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  - In particular, focus on an **extremal** K3 surface in the current work
  - Identify distinct T-dual LSTs to **inequivalent elliptic fibrations** of the same K3

K3 degeneration - Kulikov model

- Specific degenerated K3s are described in terms of Kulikov models – three types [Kulikov'77, Braun, Watari'16]
- Smooth degenerated K3s around a neighborhood of the origin of the base $\mathbb{C}$

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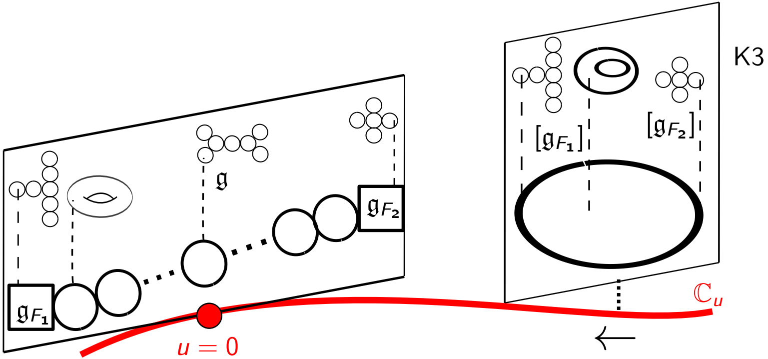
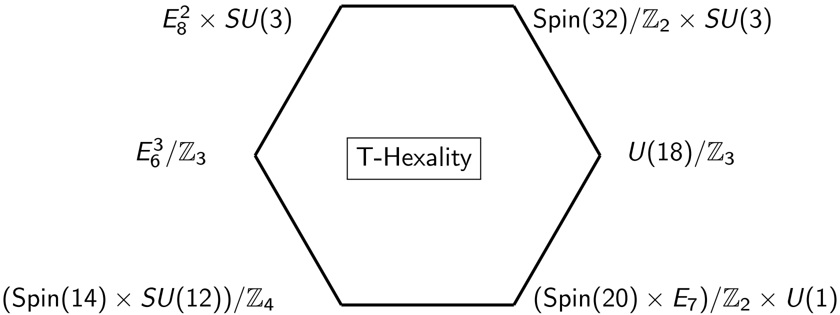
- Smooth degenerated K3s around a neighborhood of the origin of the base $\mathbb{C}$
- Generically, K3 degenerates in a "Kulikov-like" way with intricate fibre structure: [Lee, Weigand'21...]



Example: T-Hexality in non-geometric heterotic string backgrounds

- Six different flavor groups obtained from different elliptic fibration of the same K3
- Those yield the universal flavor groups of six different 6d LSTs



A family of LSTs from T-Hexality

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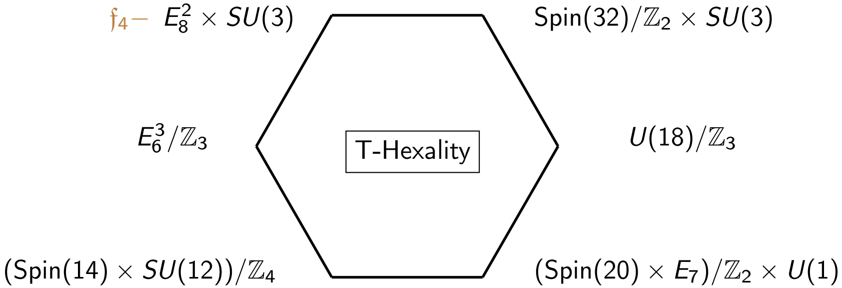

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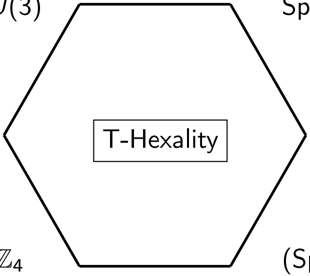
$\mathfrak{f}_4 - \ E_8^2 \times SU(3)$

$\text{Spin}(32)/\mathbb{Z}_2 \times SU(3) \ -\mathfrak{so}_{16}$

$\mathfrak{e}_6 - \ E_6^3/\mathbb{Z}_3$



$U(18)/\mathbb{Z}_3 \ -\mathfrak{su}_3$

$\mathfrak{so}_{10} - \ (\text{Spin}(14) \times SU(12))/\mathbb{Z}_4$

$(\text{Spin}(20) \times E_7)/\mathbb{Z}_2 \times U(1) \ -\mathfrak{so}_{12}$

- In total, we propose different LSTs with the following T-dual data

$$\text{Dim(CB)} = 21\,, \quad \widehat{\kappa}_{\mathscr{R}} = 30$$

Summary and Outlook

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  1. Construct a plethora of novel T-dual LSTs probing various types (ABCDEFG) of singularity T-dualities are verified by Coulomb and tensor branch data matching

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- **Outlook:**
  1. Turn towards non-heterotic LSTs given by systems without M9 branes and incorporate the possibility of twisted compactifications
  2. Relate heterotic LSTs to the classification of degenerate K3 surfaces and study unexplored reducible K3 fibrations occur in the geometry of LSTs

Thank you!

The limit of geometric engineering

- Consider the $N = 0$ (no M5 brane) case in $\mathcal{K}_N(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2; \mathfrak{g})$, one can obtain:

$$[\mathfrak{e}_8] \overset{\mathfrak{g}}{0} [\mathfrak{e}_8] \,,$$

- Fractionalization of the pure heterotic string by fusing the rank zero orbi-instanton theory