Symmetries and self-dual fields in string theory

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Strings and Geometry 2023
University of Pennsylvania

based on

[2112.02092 with F. Apruzzi, and F. Bonetti, and I. García Etxebarria, and S. Schäfer-Nameki '21]
[Work to appear with I. García Etxebarria]

Motivation

As discussed in the talks by I. Bah, M. Del Zotto, M. Hübner, I. García Etxebarria, M. Liu, L. Lin, S. Schäfer-Nameki and F. Apruzzi, D. Gould, E. Leung, T. Waddleton, M. Etheredge, E. Torres, X. Yu and H. Zhang, in order to specify a QFT we need to specify its spectrum of local and higher dimensional operators. There are QFTs with the same local structure but different extended operators and higher symmetries (global structure).

[Aharony, Seiberg, Tachikawa '13], [Gaiotto, Kapustin, Seiberg, Willett '14]

The choices of global structures can be understood in terms of gapped boundary conditions in a bulk BF theory.

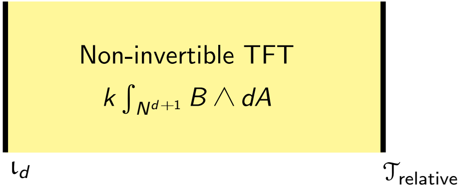
[Witten '98], [Kapustin and Seiberg '14]

In this talk, we discuss how to find the BF theory by torsional reduction from String or M-theory by constructing actions for self-dual fields.

BF theory: choices of global symmetries

[Witten '98], [Kapustin and Seiberg '14]

The relative d-dimensional theory can be viewed as a boundary theory of the $(d+1)$-dimensional Chern-Simons theory



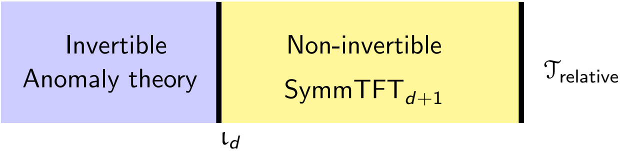
To specify the boundary theory we must specify boundary conditions on ι_d for the gauge fields A and B.

The gauge field gives rise to operators in $\mathcal{T}_{\text{relative}}$ that generate the higher-form symmetry. There can also be open operators ending on the boundary, and they are charged under the higher symmetry.

Symmetry topological field theory

[Cvetič, Dierigl, Lin and Zhang '20], [Bah, Bonetti, Minasian, Nardoni '21], [Gukov, Hsin, Pei '21], [Apruzzi, Bonetti, García Etxebarria, SH, Schäfer-Nameki '21], [Freed, Moore, Teleman '22]

If $\mathcal{T}$ has anomalies



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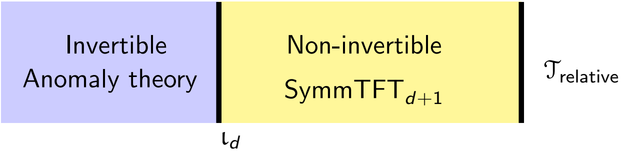
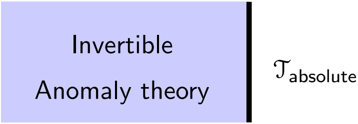


Fixing the boundary condition ι_d results in an absolute theory $\mathcal{T}_{\text{absolute}}$ and the *anomaly theory* $\mathcal{A}_{d+1}$

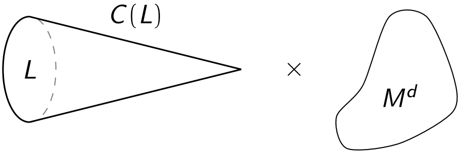


In principle, we want the SymTFT to encode more general categorical symmetries and their anomalies. See talks by I. Bah, M. Del Zotto, M. Hübner, I. García Etxebarria, M. Liu, L. Lin, S. Schäfer-Nameki and F. Apruzzi, ...

Geometric engineering

[Katz, Klemm '96], [Katz, Klemm, Vafa '97]

String theory or M-theory on $C(L) \times M_d$



to find an SCFT $\mathcal{T}_d$ on M^d.

C(L): A singular non-compact Calabi-Yau given as a cone over L

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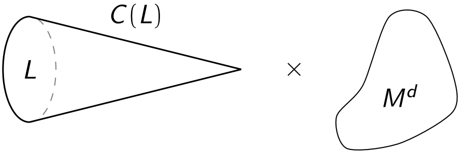

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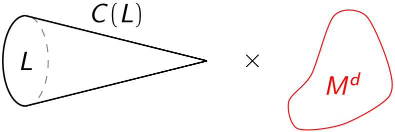
These data only give the local data of the SCFT $\mathcal{T}_{\text{relative}}$ and the data defining a specific SCFT $\mathcal{T}_{\text{absolute}}$ are encoded in a choice of boundary condition at infinity for the supergravity fields.

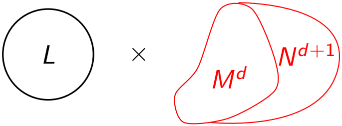
Geometric engineering

[Cvetič, Dierigl, Lin and Zhang '20], [Bah, Bonetti, Minasian, Nardoni '21], [Gukov, Hsin, Pei '21], [Apruzzi, Bonetti, García Etxebarria, SH, Schäfer-Nameki '21]

String theory on X $\xrightarrow{\text{Reduce on } C(L)}$ SCFT $\mathcal{T}_d$ on M^d

$X =$ 

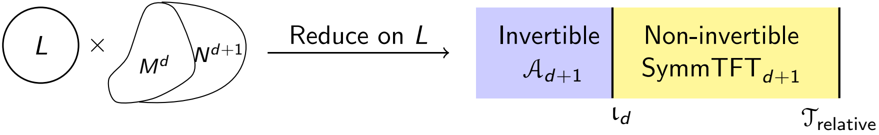
String theory on Y $\xrightarrow{\text{Reduce on } L}$ SymTFT for $\mathcal{T}_d$ on N^{d+1}

$Y =$ 

SymTFT from M-theory

[Cvetič, Dierigl, Lin and Zhang '20], [Bah, Bonetti, Minasian, Nardoni '21], [Apruzzi, Bonetti, García Etxebarria, SH, Schäfer-Nameki '21]

Reducing M-theory over the base L of the cone $C(L)$



$$\int_{L\times N^{d+1}} \frac{1}{6} C_3\wedge G_4\wedge G_4 - C_3\wedge I_8 \xrightarrow{\text{Reduce on } L} \text{Anomalies} \subset \text{SymTFT}$$

$$? \xrightarrow{\text{Reduce on } L} \text{BF theory} \subset \text{SymTFT}$$

Flux non-commutativity

[Diaconescu, Freed, Moore '03], [Moore '04], [Freed, Moore, Segal, '06], [Freed, Moore, Segal, '06], [García Etxebarria, Heidenreich, Regalado '19]

The supergravity fluxes do not commute in the presence of torsion on spaces with boundary

$$\phi_{F_4}\phi_{F_7} = e^{2\pi i L}\phi_{F_7}\phi_{F_4}$$

This implies the non-commutativity of symmetry operators arising from these fluxes.

Thus, it is natural to think BF theory comes from an action such as $\int F_4 \wedge *F_4$ given that $*F_4 = F_7$.

However, the reduction of the latter needs to be done with care.

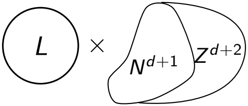
BF theory from M-theory or string theory

[Witten '97], [Freed '00], [Belov, Moore '06], [García Etxebarria, SH, to appear]

Construct a 12-dimensional Chern-Simons CS theory which realises the self-dual field (F_4, F_7) in M-theory as boundary modes. Or 11d CS for self-dual F_5 in IIB.

Extend N^{d+1} to Z^{d+2}, with $\partial Z^{d+2} = N^{d+1}$.

Define Chern-Simons on



$\xrightarrow{\text{Reduce on } L \text{ and Stoke's theorem}}$ BF theory on $N^{d+1} \subset$ SymTFT

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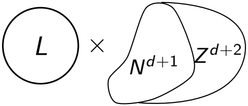

$$\xrightarrow{\text{Reduce on } L \text{ and Stoke's theorem}} \quad \text{BF theory on } N^{d+1} \subset \text{SymTFT}$$

To proceed, we need to define Chern-Simons theory for topologically non-trivial fields. It is the reduction of the torsional sector that results in a non-trivial SymTFT.

3d Chern-Simons theory

[Elitzur, Moore, Schwimmer, Seiberg '89], [Witten '89]

$U(1)$ CS theory at level k

$$CS[A] = k \int_{M^3} A \wedge dA \,.$$

Gauge invariance $\implies$ k, the level, is an integer.

3d spin Chern-Simons theory

[Belov, Moore '05]

On manifolds with spin structure the second Wu class $\nu_2 = 0$, so

$$\int_{M^3=\partial N^4} A \wedge dA = \int_{N^4} dA \wedge dA$$
$$= \int_{N^4} \delta A \cup \delta A = \int_{N^4} \delta A \cup \nu_2 = 0 \quad \text{mod } 2\,,$$

by mapping the differential form A to a cochain.

$\implies$ the level can be half-integer

$$CS[A] = \frac{k}{2} \int_{M^3} A \wedge dA\,.$$

Th factor of half is needed to count the right number of degrees of freedom of the self-dual boundary mode.

Topologically non-trivial fields

[Cheeger and Simons '85], [Hopkins and Singer '02]

A $(p-1)$-form gauge field with

- curvature: $F \in \Omega^p_{\text{closed}}(M)$
- characteristic class: $[N] \in H^p(M;\mathbb{Z})$
- connection: $A \in C^{p-1}(M;\mathbb{R})$

is represented by

$$\breve{A} = (N, A, F)$$

where $dF = \delta N = 0$ and $\delta A = F - N$.

For a topologically trivial field $N = 0$ and $F = \delta A$.

Spin CS for topologically non-trivial fields

[Cheeger and Simons '85], [Hopkins and Singer '02]

For $\breve{A} = (N_A, A, F_A)$ and $\breve{B} = (N_B, B, F_B)$ the product is

$$\breve{A} \cdot \breve{B} = (N_A \cup N_B, N_A \cup B + A \cup F_B + H(F_A, F_B), F_A \wedge F_B)$$

where $H(\, , \,)$ is a homotopy from forms $\Omega^*(M)$ to chains $C^*(M; \mathbb{R})$ satisfying

$$dH(w, w') + H(dw, w') + (-1)^{\deg w} H(w, dw') = w \wedge w' - w \cup w' \, .$$

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Chern-Simons for a 1-form gauge field $\breve{A} = (N, A, F)$

$$\begin{aligned} CS[\breve{A}] &= \frac{k}{2} \int_{M^3} \breve{A} \cdot \breve{A} \\ &= \frac{k}{2} \int_{M^3} \big(N \cup A + A \cup F + H(F, F)\big) . \end{aligned}$$

Chern-Simons for a $(2p-1)$-form field $\breve{A}$

Can we write

$$CS[\breve{A}] = \frac{k}{2}\int_{M^{4p-1}} \breve{A}\cdot\breve{A} \quad ?$$

for $p > 1$?

Assume $M^{4p-1} = \partial N^{4p}$ and that A is flat extending to N^{4p}, then

$$\int_{M^{4p-1}} \breve{A}\cdot\breve{A} = \int_{N^{4p}} N \cup N$$

But the cup product may not be even on an arbitrary $4p$-dimensional manifold!

Wu Chern-Simons theory

[Hopkins and Singer '02], [Monnier '16]

For ν a Wu class of degree $2p$ on a $4p$-fold, we have

$$N \cup N = N \cup \nu \mod 2\,.$$

Thus, we can define the Chern-Simons as

$$\frac{k}{2}\int_{N^{4p}} N \cup (N - \nu)\,.$$

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More generally, without the need to extend to N, we can write

$$CS[\breve{A}] = \frac{k}{2}\int_{M^{4p-1}} \breve{A}\cdot(\breve{A} - \breve{\lambda})$$

where $\breve{\lambda} = (N_\lambda, \lambda, F_\lambda)$ st $N_\lambda = \nu \mod 2$.

Wu structure

The spin structure in 3d CS gets generalized to a Wu structure in $4p-1$-dim CS.

A spin structure is a trivialization of the degree 2 Wu class $\nu_2 = \omega_2$, which coincides with the second Stiefel-Whitney class.

A degree $(2p)$ Wu structure is a trivialization of the degree $(2p)$ Wu class, a certain polynomial in the Stiefel-Whitney classes.

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E.g. For 11d CS on an orientable manifold $\nu_6 = \omega_2\omega_4 + \omega_3^2$.

In the presence of a Wu structure, the Chern-Simons is

$$CS[\breve{A}] = \frac{k}{2}\int_{M^{4p-1}} \breve{A}\cdot\breve{A}\,.$$

Boundary modes

[Witten '96], [Maldacena, Moore, and Seiberg '01], [Hsieh, Tachikawa, Yonekura '20]

How can we realise a self-dual field as the boundary mode of the Chern-Simons theory?

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How can we realise a self-dual field as the boundary mode of the Chern-Simons theory? Consider 2d chiral scalar on X^2.

Set $k = 1$ and add a kinetic term to the CS

$$\frac{S[\breve{A}]}{2\pi} = \frac{1}{2e^2}\int_{M^3} F \wedge \star F - \frac{i}{2}\int_{M^3} \breve{A} \cdot \breve{A}$$

The equation of motion

$$\frac{1}{e^2} d \star F - iF = 0\,.$$

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The equation of motion

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Choose boundary condition $\quad [\breve{A}]\,|_{\partial M^3} = 0\,,$

Take $M^3 = (-\epsilon, 0] \times X^2$ and $F = de^{m\tau} \wedge F_a$ near the boundary.

EoM gives $\quad \star F_a = iF_a\,, \quad d\star F_a = 0\,.$

Type IIB

Self-dual 4-form RR field $\check{C} = (N_5, C_4, F_5)$, boundary mode of Chern-Simons

$$CS[\check{G}] = \frac{1}{2}\int_{N^{11}} \check{G}_6 \cdot \check{G}_6 \,,$$

$\check{G}_6 = (N_G, G_5, F_G)$.

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IIB on $\mathbb{C}^2/\mathbb{Z}_n \times M^6$: 6d (2,0) SCFT of type $\mathfrak{su}(n)$.

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$N^{11} = S^3/\mathbb{Z}_n \times Z^8$, with $\partial Z^8 = N^7$

$$CS[C_3] = \frac{n}{2}\int_{Z^8} C_4 \wedge C_4 = \frac{n}{2}\int_{N^7} C_3 \wedge dC_3 \,,$$

where C_3 is the trivialisation of C_4 assuming $H^4(N^7) = 0$.

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where C_3 is the trivialisation of C_4 assuming $H^4(N^7) = 0$.

Do the boundary conditions of $CS[C_3]$ result in absolute 6d $(2,0)$ theories? Not in general. [Gukov, Hsin, Pei '21]

M-theory

Self-dual pair $\breve{F} = (\breve{F}_4, \breve{F}_7)$, boundary mode of Chern-Simons

$$CS[\breve{F}] = \int_{N^{12}} \breve{G}_5 \cdot \breve{G}_8 \,.$$

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Self-dual pair $\check{F} = (\check{F}_4, \check{F}_7)$, boundary mode of Chern-Simons

$$CS[\check{F}] = \int_{N^{12}} \check{G}_5 \cdot \check{G}_8 \, .$$

M-theory on $\mathbb{C}^2/\mathbb{Z}_n \times M^7$: 7d $N = 1$ SQFT of type $\mathfrak{su}(n)$ on M^7 .

M-theory

Self-dual pair $\check{F} = (\check{F}_4, \check{F}_7)$, boundary mode of Chern-Simons

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M-theory on $\mathbb{C}^2/\mathbb{Z}_n \times M^7$: 7d $N = 1$ SQFT of type $\mathfrak{su}(n)$ on M^7 .

$N^{12} = S^3/\mathbb{Z}_n \times Z^9$, with $\partial Z^9 = N^8$

$$CS[C_3] = n \int_{Z^8} C_3 \wedge C_6 = n \int_{N^7} C_2 \wedge dC_5 \,,$$

where C_2 and C_5 are the trivialisation of C_3 and C_6, respectively, assuming $H^{3,6}(N^7) = 0$.

Conclusion

BF action from the torsional reduction of Chern-Simons action for self-dual fields in string theory.

Outlook:

Find the full SymTFT encoding categorical symmetries and their anomalies from string theory. See talks by Ibrahima Bah, Michele Del Zotto, Max Hübner, Iñaki García Etxebarria, Muyang Liu, Ling Lin, Sakura Schäfer-Nameki and Fabio Apruzzi, Dewi Gould, Enoch Leung, Thomas Waddleton, Muldrow Etheredge, Ethan Torres, Xingyang Yu and Hao Zhang. [Del Zotto, Liu, Oehlmann '22], [Apruzzi, Bah, Bonetti and Schäfer-Nameki '22], [García Etxebarria '22], [Heckman, Hübner, Torres and Zhang '22], [Heckman, Hubner, Torres, Yu, Zhang '22], [Etheredge, García Etxebarria, Heidenreich, Rauch '23]

Study the reduction of defects realised by cobordism conjecture on manifolds with boundaries and their role in SymTFT. See talks by Jake McNamara, Miguel Montero and Markus Dierigl.

Non-invertible and relative theories

[Freed, Teleman '12], [Freed '14]

Similarly the Hilbert space of the Chern-Simons theory for A a chiral $p+1$-form field

$$\frac{n}{2}\int_M A \wedge dA$$

behaves roughly like $n^{\frac{1}{2}\dim H^{p+1}(M)}$. Such a Hilbert space is not invertible, so CS is called a non-invertible theory.

The partition function of the boundary theory $\mathcal{T}$ on ∂N is in 1-1 correspondence with the elements of the Hilbert space. Thus the partition function of $\mathcal{T}$ is not a function but a vector and so it is only well-defined relative to the bulk CS theory. In this case, $\mathcal{T}$ is called a relative theory.

Boundary conditions and Lagrangian subgroups

[Diaconescu, Freed, Moore '03], [Moore '04], [Freed, Moore, Segal, '06], [Freed, Moore, Segal, '06], [García Etxebarria, Heidenreich, Regalado '19]

Geometrically, the expectation values for the supergravity fields cannot be fixed at the special asymptotic slice $L \times M$ due to their non-commutativity. Thus the Hilbert space on the slice $L \times M$ has a grading in terms of $H^m_{\mathrm{Tor}}(L)$. The partition function is in 1-1 correspondence with elements of the Hilbert space so it is a vector.

Upon reduction on $C(L) \times M$, the partition function of M is also a vector. To fix it, must pick a maximal isotropic subgroup $\Lambda \in H^m_{\mathrm{Tor}}(L)$ with respect to the bilinear pairing

$$H^n(M; H^m_{\mathrm{Tor}}(L)) \times H^{d-n}(M; H^m_{\mathrm{Tor}}(L)) \to \mathbb{R}/\mathbb{Z}$$

This gives a choice of global symmetry.

Wu structure

For an oriented manifold M^n with a tangent bundle TM, there is a (homotopy class of) classifying map from M into $BSO(n)$. The second Stiefel-Whitney class w_2 can be realised as a homotopy class of maps from $BSO(n)$ into $K(Z_2, 1)$. The associated homotopy fiber is $BSpin(n)$, and a spin structure on M is a lift of the classifying map of TM from $BSO(n)$ to $BSpin(n)$.

For ν_p the degree p Wu class, we define $BSO[\nu_p](n)$ to be the homotopy fiber of the map from $BSO(n)$ into $K(Z_2, p)$ defined by ν_p. A Wu structure on M is a lift of the classifying map of TM from $BSO(n)$ to $BSO[\nu_p](n)$.

Boundary modes

[Witten '96], [Maldacena, Moore, and Seiberg '01], [Hsieh, Tachikawa, Yonekura '20]

For $\breve{A} = [I, h, F]$ and $\bar{J} = [-\epsilon, 0]$, have boundary condition

$$[\breve{A}] \,|_{\partial M^3} = 0\,.$$

Define $\eta \in H^1(\bar{J}, \partial\bar{J}; \mathbb{Z})$, $\mu \in H^1(\bar{J}, \partial\bar{J}; \mathbb{R})$ and

$$[\breve{A}] \,|_{X^2 \times J} = \Big(\eta \cup I(\breve{a}), \mu \cup h(\breve{a}),\ de^{m\tau} \wedge R(\breve{a})\Big)\,.$$

Then, $\breve{a} \in \breve{H}^1(X)$ is a self-dual field living on the boundary.

Boundary conditions

[Gaiotto, Kapustin, Seiberg, Willett '14], [Kapustin and Seiberg '14], [Gukov, Hsin, Pei '21]

For 6d (2,0) theory of type $\mathfrak{su}(n)$, take $n = k^2$.

Invertible Anomaly theory	$\frac{k^2}{2} \int_{N^7} C_3 \wedge dC_3$ $\subset$ SymmTFT$_7$	Relative 6d (2,0)

ι_6

Fix the boundary condition $C_3 \mid_\partial = H_3$ where H_3 is a $\mathbb{Z}_n$-valued field. This is imposed by

$$2\pi S_\partial = \frac{k}{2} \int_\partial (C_3 - H_3) dY \subset S_{\iota_d}$$

$\frac{k^2}{2} \int_{N^7} H_3 \wedge dH_3$ $\subset$ Anomaly theory	Absolute 6d (2,0)

Boundary conditions

[Gaiotto, Kapustin, Seiberg, Willett '14], [Kapustin and Seiberg '14], [Gukov, Hsin, Pei '21]

Instead, if $n \neq k^2$

$$2\pi\, S_{CS} = CS[C_3] = \frac{n}{2} \int_{N^7} C_3 \wedge dC_3 \,,$$

and

$$2\pi\, S_\partial = \frac{m}{2} \int_\partial (C_3 - H_3) dY \,.$$

such that $m \mid n$.

Requiring the boundary variation to vanish with respect to C_3 and Y, implies C_3 must be $\mathbb{Z}_{n/m}$ and $\mathbb{Z}_m$ valued, respectively. This is only compatible with $n = m^2$.