Categorical Symmetries

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- Works in progress with Lakshya Bhardwaj

Generalized Symmetry "Revolution"

Generalized Global Symmetries

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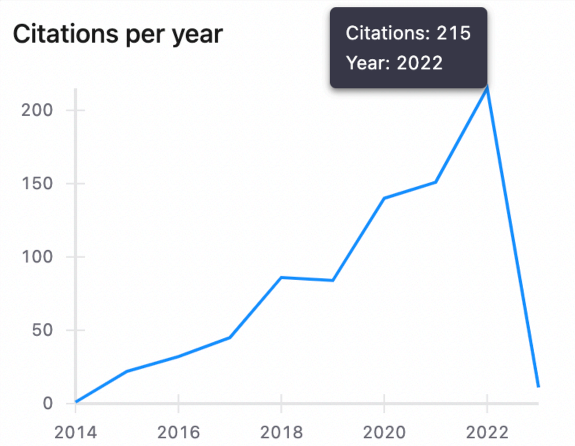


Abstract: (Springer)
A q-form global symmetry is a global symmetry for which the charged operators are of space-time dimension q, e.g. Wilson lines, surface defects, etc., and the charged excitations have q spatial dimensions, e.g. strings, membranes, etc. Many of the properties of ordinary global symmetries (q = 0) apply here. They lead to Ward identities and hence to selection rules on amplitudes. Such global symmetries can be coupled to classical background fields and they can be gauged by summing over these classical fields. These generalized global symmetries can be spontaneously broken (either completely or to a sub-group). They can also have 't Hooft anomalies, which prevent us from gauging them, but lead to 't Hooft anomaly matching conditions. Such anomalies can also lead to anomaly inflow on various defects and exotic Symmetry Protected Topological phases. Our analysis of these symmetries gives a new unified perspective of many known phenomena and uncovers new results.

Note: 49 pages plus appendices. v2: references added

Generalized Symmetry ”Revolution”

We’ve already heard many excellent talks by colleagues and friends on generalized symmetries and the connection to geometry/strings/holography.

1. Ibou Bah
2. McNamara
3. Del Zotto
4. Montero
5. Dierigl
6. Hubner
7. Garcia Extebarria
8. Lin
9. today: Saghar Hosseini
10. a lot of talks in the gong show

Non-invertible Symmetries in $d > 3$:

In the context of QFTs in $d > 3$ within the last year

[Heidenreich, McNamara, Monteiro, Reece, Rudelius, Valenzuela]

[Koide, Nagoya, Yamaguchi]

[Kaidi, Ohmori, Zheng][2]

[Choi, Cordova, Hsin, Lam, Shao]

[Bhardwaj, Bottini, SSN, Tiwari][3]

[Roumpedakis, Seifnashri, Shao]

[Antinucci, Galati, Rizi]

[Choi, Cordova, Hsin, Lam, Shao]

[Kaidi, Zafrir, Zheng]

[Choi, Lam, Shao]

[Cordova, Ohmori]

[Bhardwaj, SSN, Wu]

[Bartsch, Bullimore, Ferrari, Pearson]

[Bashmakov, del Zotto, Hasan, Kaidi]

[Antinucci,Benini, Copetti, Galati, Rizi]

[Cordova, Hong, Koren, Ohmori]

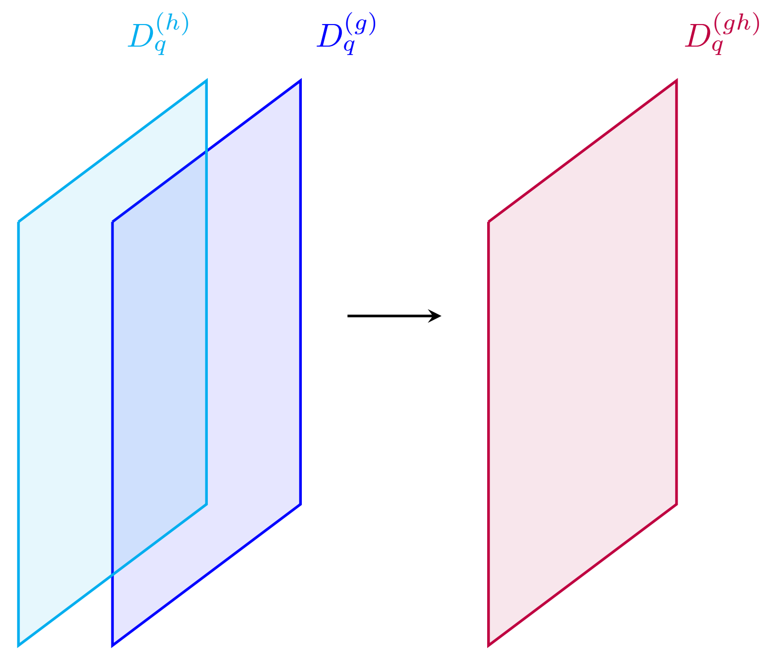
. . .

Symmetries from Topological Operators

Any topological operator in a QFT is a symmetry generator.

1. Higher-form symmetries $\Gamma^{(p)}$: [Gaiotto, Kapustin, Seiberg, Willett, 2014]
   charged objects are p-dimensional defects, whose charge is measured by codimension $p+1$ topological operators $D^{g}_{q=d-(p+1)}$, $g \in \Gamma^{(p)}$:

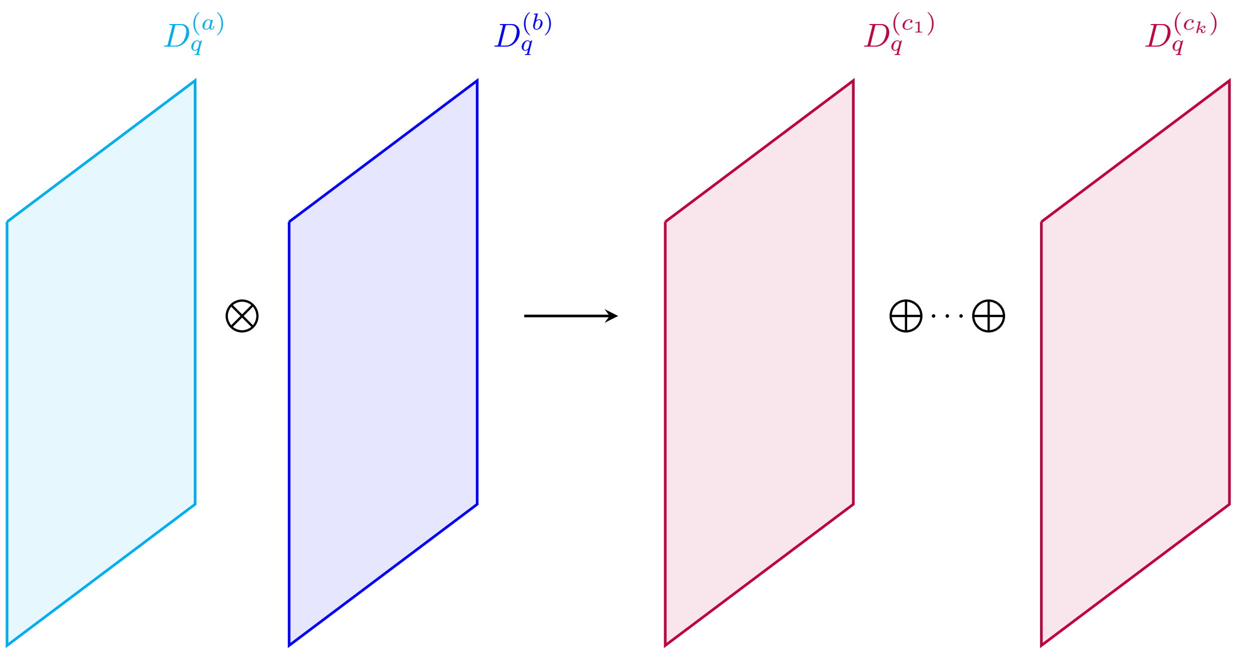
$$D_q^g \otimes D_q^h \;=\; D_q^{gh}\,, \qquad g,h \in \Gamma^{(p)}$$



2. Higher-group symmetries:
   $\{p$-form symmetries$\}$ might not form product groups

3. Non-invertible symmetries:

group ⇒ algebra



Perhaps surprisingly:

These are ubiquitous in higher dimensional QFTs, e.g. 4d pure Yang Mills.

Science-sociological bonus: Provides a really exciting connection between hep-th, hep-ph, cond-mat, and math.

Examples:

2d: Verlinde lines in a 2d rational conformal field theory (RCFT)

$$D_1^i \otimes D_1^j = \bigoplus_k N_k^{ij} D_1^k$$

N_{ij}^k = RCFT fusion coefficients obtained by the Verlinde formula

3d: Modular tensor categories: classification of topological order:

4d: By now many examples and methods of construction: e.g.

(i) $O(2) = U(1) \rtimes \mathbb{Z}_2$ [Heidenreich, McNamara, Montero,Reece, Rudelius, Valenzuela]

(ii) Outer automorphism gauging, e.g. $\text{Pin}^+(4N)$ theory. [Bhardwaj, Bottini, SSN, Tiwari]: Non-invertible 1-form symmetry

(iii) Duality defects: [Choi, Cordova, Hsin, Lam, Shao][Kaidi, Ohmori, Zheng]: Non-invertible 0-form symmetry

Utility of Non-Invertible Symmetries

Many applications – but clearly only scratching the surface:

1. 2d: constraints on existence of gapped phases, and number of vacua
2. Confinement/Deconfinement: in 4d QFT and holography constrained by non-invertible defects in $\mathcal{N} = 1$ pure Yang-Mills
3. Applications to hep-ph: e.g. neutrino mass generation from non-invertible symmetry breaking etc.
4. Swampland/No Global Symmetry conjecture

Non-Invertible to Categorical

Consider more generally a QFT in d-dimensions, with not-necessarily invertible fusion of topological defects of various dimensions.

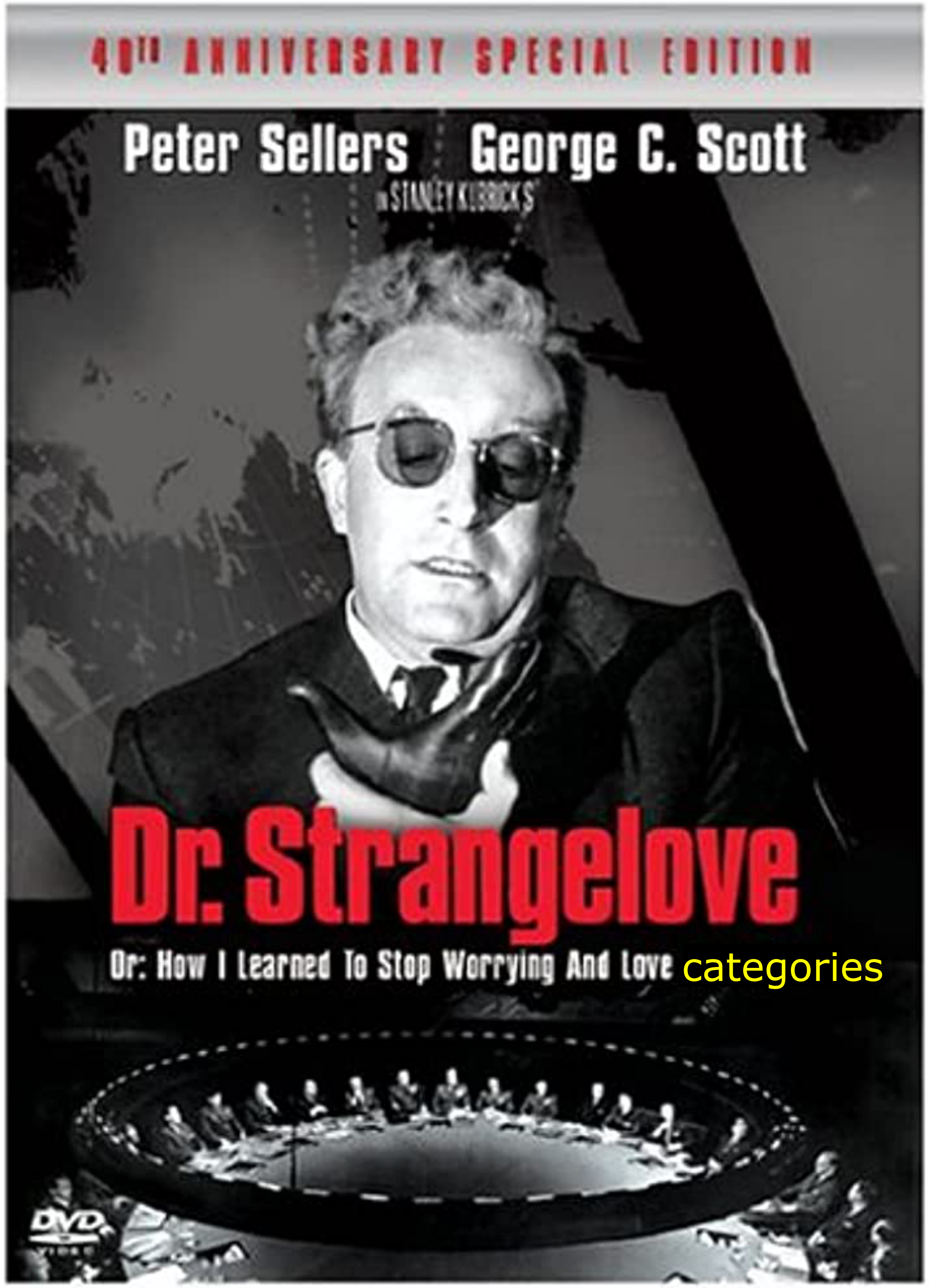
What is the proper framework for characterizing such symmetries?

- For 0-form (and p-form) symmetry groups:
  Obviously, Group Theory and Representations

  Historic note: this was not always so obvious. According to Wigner (1981), Erwin Schrödinger coined the expression "Gruppenpest" and stated it ought to be abandoned.

- For non-invertible symmetries: topological operators of dimensions $0, \cdots, d-1$, with non-invertible fusion:
  Higher-Fusion Categories

$\Rightarrow$ (Higher) Categorical Symmetries


40TH ANNIVERSARY SPECIAL EDITION
Peter Sellers
George C. Scott
in STANLEY KUBRICK'S
Dr. Strangelove
Or: How I Learned To Stop Worrying And Love categories
DVD

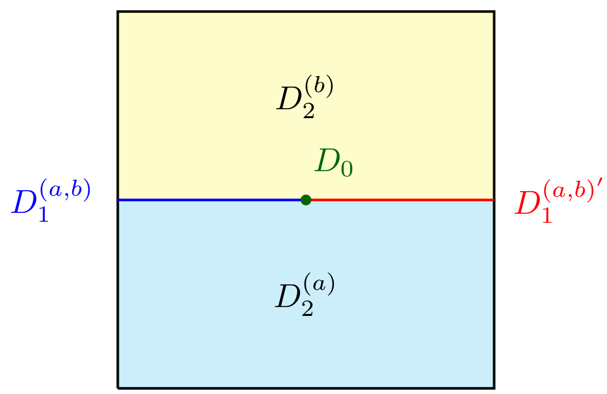
Categorical Symmetries

Higher-categories play a similar role to groups and group representations.

A fusion p-category has the following structure:

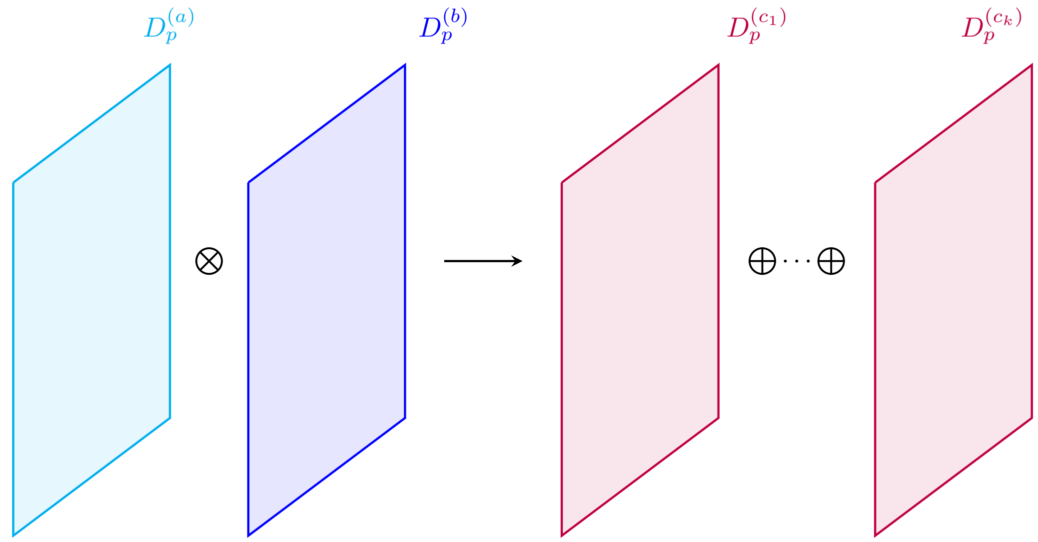
1. A set of objects: D_p of p-dimensional topological defects
2. 1-Morphisms: maps between two defects D_p and D'_p, i.e. D_{p-1} defects.
3. 2-Morphisms: maps between two 1-morphisms, i.e. D_{p-2} defects.
4. $\cdots$

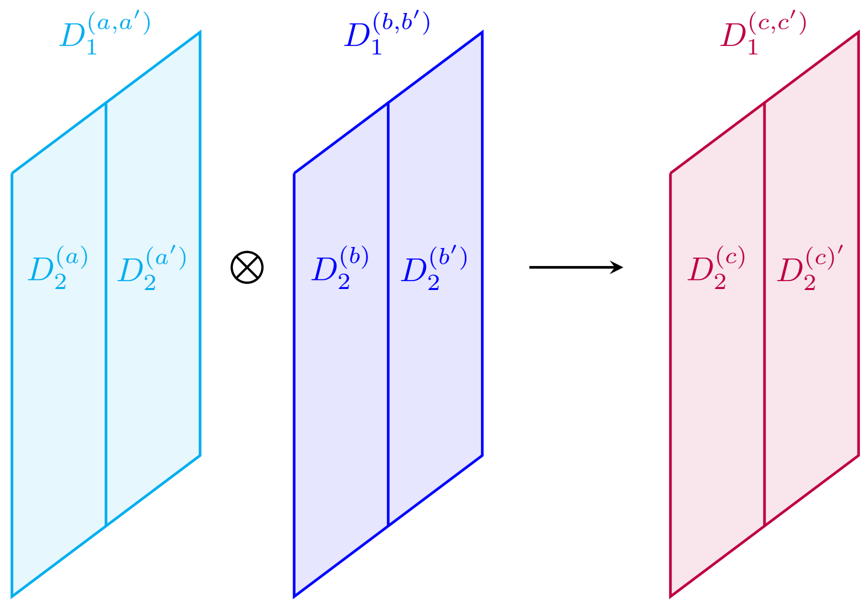
Example: 2-fusion categories [Douglas, Reutter; 2018]



Objects: topological surface operators
1-morphisms: topological lines
2-morphisms: local operators

Monoidal structure: loosely speaking, there is a fusion between objects, 1-morphisms, etc:





$$D_1^{(c,c')} = D_1^{(a,a')} \otimes D_1^{(b,b')}\,, \qquad D_2^{(a)} \otimes D_2^{(b)} \supset D_2^{(c)}$$

Theta-Defects:
Universal Construction of Non-Invertibles

[Bhardwaj, SSN, Wu][Bhardwaj, SSN, Tiwari]

Theta-Defects

Lets start with 4d Maxwell:

$$\mathcal{L}_{U(1)} = \frac{1}{g^2}\int F \wedge \star F + \theta \int F \wedge F$$

We can think of this theory as follows:
Consider a 4d trivial theory with a trivial $U(1)$ global symmetry, background A, but a symmetry protected phase (SPT)

$$\mathcal{L}_T = \mathcal{L}_{\text{trivial}} + \text{SPT}\,, \qquad \text{SPT} = \theta \int F \wedge F$$

Gauging $U(1)$ we obtain Maxwell and the SPT becomes the theta-angle

$$\mathcal{L}_{T/U(1)} = \frac{1}{g^2}\int F \wedge \star F + \theta \int F \wedge F$$

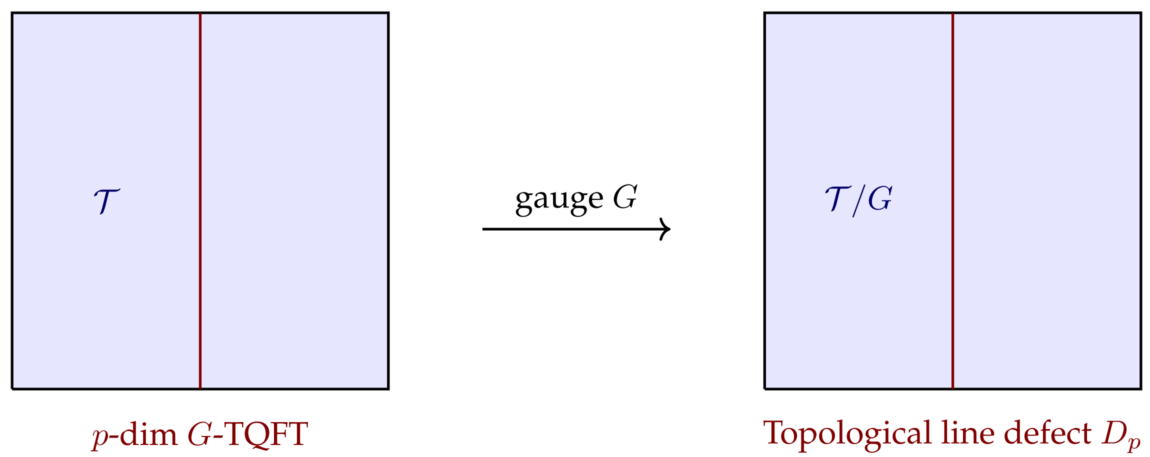
This can be generalized to any theory with $U(1)$ global symmetry that can be gauged:

Stacking with $U(1)$-SPT and gauging adds a θ-angle.

Theta Defects

[Bhardwaj, SSN, Wu][Bhardwaj, SSN, Tiwari]

Let $\mathcal{T}$ be a d-dim QFT with a $G^{(0)}$ symmetry.
Consider a G-symmetric p-dimensional TQFT. Gauge the diagonal G:



In the gauged theory, the TQFT is now a topological defect of the theory.
Generically: the fusion of these defects is non-invertible.

Example: Dual Symmetry in 2d

Let $\mathcal{T}$ be a 2d theory with a 0-form symmetry G, generated by topological lines $D_1^{(g)}$, which fuse according to the group multiplication in G

$$D_1^{(g)}\,,\; g\in G\,, \qquad D_1^{(g)}\otimes D_1^{(h)} = D_1^{(gh)}$$

This defines a fusion category: $\mathcal{C} = \text{Vec}(G)$.

Gauging G means introducing a dynamical G gauge field:

- There is a dynamical G gauge field a and Wilson lines in G-representations $\mathbf{R}$

$$D_1^{(\mathbf{R})} = \text{Tr}_{\mathbf{R}}\, e^{\int a}$$

- These Wilson lines fuse according to the representations of G, $\text{Rep}(G)$:

$$D_1^{(\mathbf{R}_1)}\otimes D_1^{(\mathbf{R}_2)} = \bigoplus_{\mathbf{R}_3} N_{\mathbf{R}_3}^{\mathbf{R}_1\mathbf{R}_2}\, D_1^{(\mathbf{R}_3)}$$

For abelian groups, e.g.

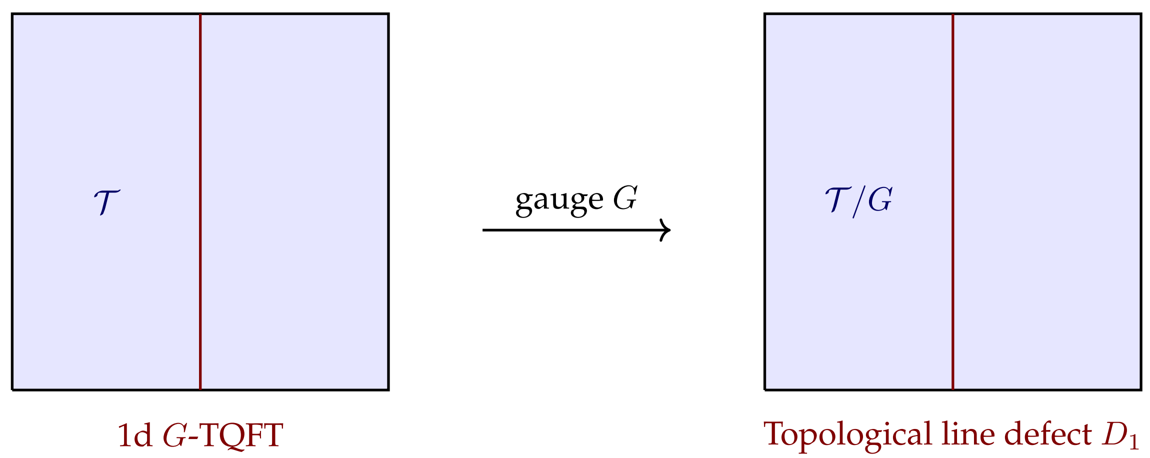
$$\text{Rep}(\mathbb{Z}_N) \equiv \text{Hom}(\mathbb{Z}_N, U(1)) \cong \mathbb{Z}_N$$

Theta-Defects

Theta-defects are a complementary perspective, which naturally generalizes to higher dims.

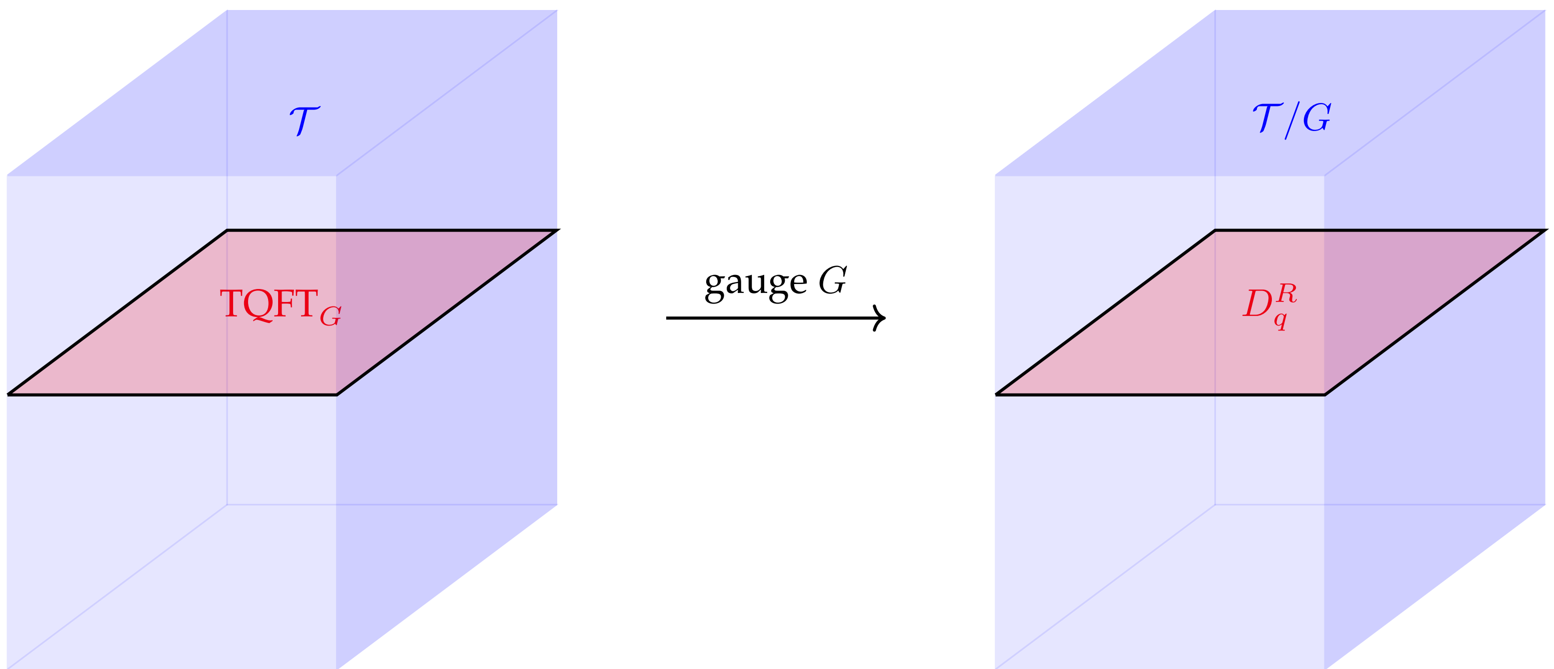
Consider a 2d theory $\mathcal{T}$, finite 0-form symmetry G: $\mathcal{C}_\mathcal{T} = \text{Vec}(G)$.

Stacking a 1d TQFT with G-symmetry, and gauging the diagonal G results in topological lines $D_1^{(\mathbf{R})}$, $\mathbf{R}$ rep of G in the gauged theory:



1d G-TQFTs:
Characterized by the number of vacua and G action on them, i.e. a G-representation. They form a subset $\text{Rep}(G)$ of the symmetry of $\mathcal{T}/G$.

Theta-Defects: higher dimensions



2d G-TQFTs to 2d Defects

Let $\mathcal{T}$ be a 3d theory with G 0-form symmetry:
$D_2^{(g)}$ with group fusion

$$D_2^{(g)} \otimes D_2^{(h)} = D_2^{(gh)}$$

and trivial lines on D_2^g. This is $2\mathrm{Vec}(G)$.

Gauging G using Theta-defects:

What are 2d G-TQFTs?

- Spontaneous Symmetry Breaking (SSB) to a subgroup H in G.
- 2d H-SPTs, which are classified by

$$\alpha \in H^2(H, U(1))$$

Example: $G = \mathbb{Z}_2$

$G = \mathbb{Z}_2$ then $H = 1$ or $\mathbb{Z}_2$ and α trivial.

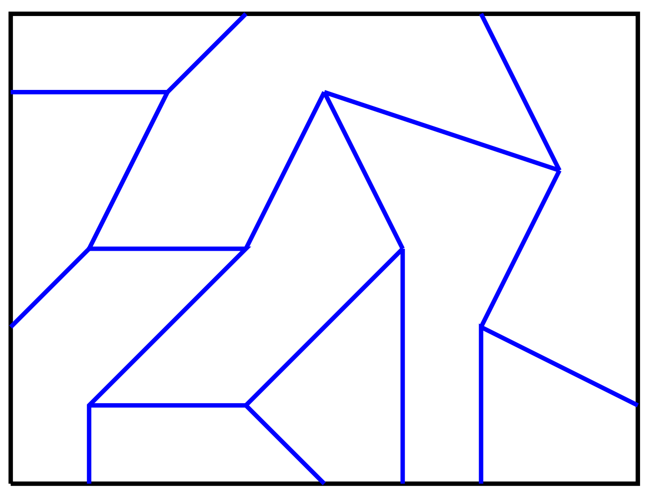
Objects:

- $D_2^{(H=1)} \equiv D_2^{(-)}$: TQFT with two vacua $|\pm\rangle$, which is a non-trivial defect.
- $D_2^{(H=\mathbb{Z}_2)} \equiv D_2^{(\mathrm{id})}$: TQFT with 1 vacuum $|0\rangle$, trivial defect (identity).

$$D_2^{(-)} \otimes D_2^{(-)} = 2D_2^{(-)}$$

This is known as the 2-fusion category: $2\mathrm{Rep}(\mathbb{Z}_2)$.

The defects $D_2^{(-)}$ are precisely condensation defects, [Roumpedakis, Saifnashri, Shao] obtained by inserting a fine mesh of lines of the dual symmetry $D_1^{(-)}$



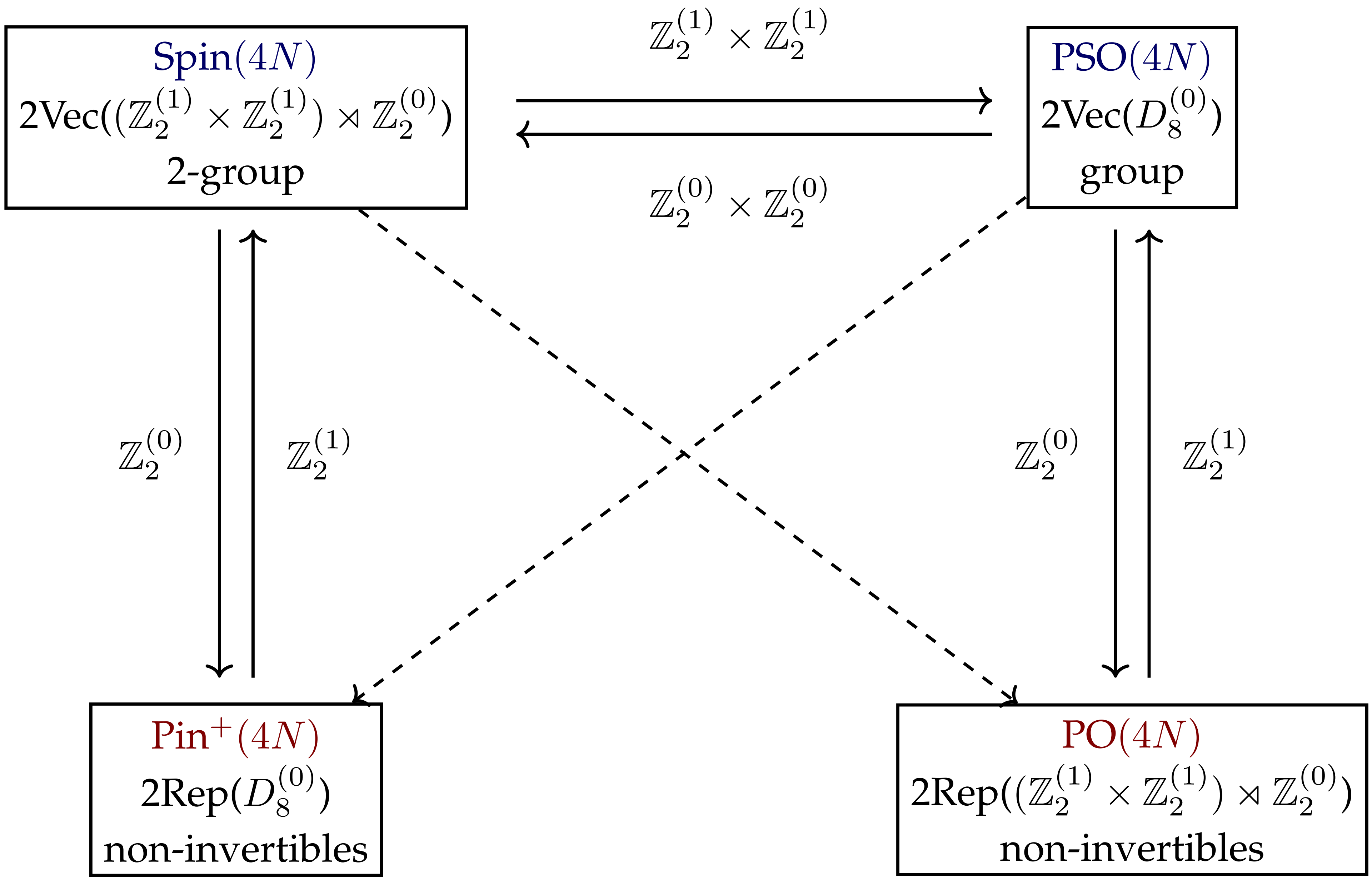
Theta-Defects: $\mathbb{Z}_2^{(0)}$

1. 1d TQFTs with $\mathbb{Z}_2$ symmetry:
   we have seen these form theta-defects that are
   $$D_1^{(\pm)}$$
   $\Rightarrow \text{Rep}(\mathbb{Z}_2) \cong \mathbb{Z}_2$
2. 2d TQFTs with $\mathbb{Z}_2$ symmetry:
   $$\begin{aligned} \text{Trivial TQFT (no SSB)}: &\quad D_2^{(\text{id})} \\ \mathbb{Z}_2 \text{ gauge theory (SSB)}: &\quad D_2^{(-)} \end{aligned}$$
   $\Rightarrow 2\text{Rep}(\mathbb{Z}_2)$, which has non-invertible fusion
   $$D_2^{(-)} \otimes D_2^{(-)} = 2D_2^{(-)}$$
3. Theta defects for $G^{(0)}$ (not necessarily abelian)
   $\Rightarrow 2\text{Rep}(G)$

Are there QFTs with such symmetries? Plenty!

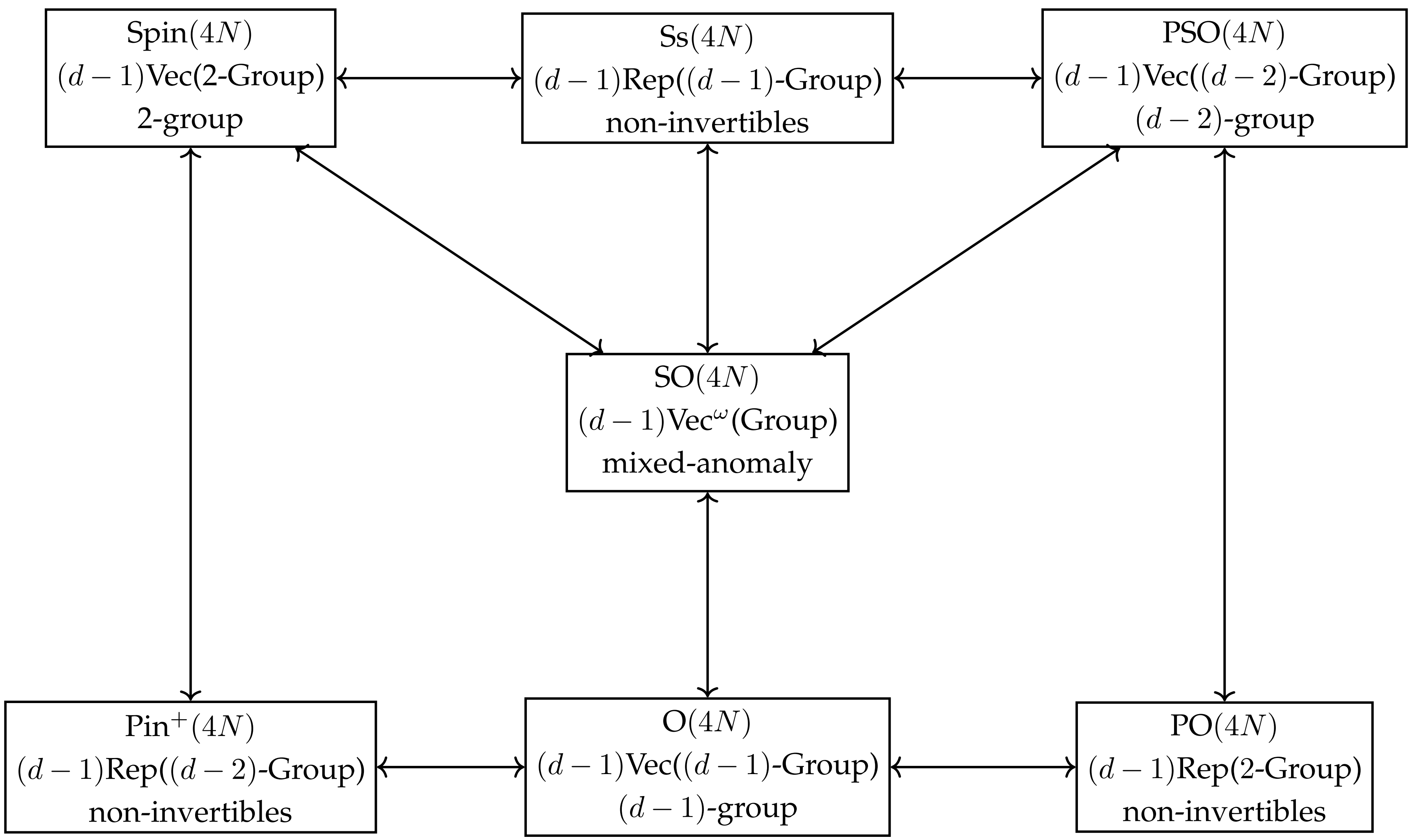
Categorical Symmetry Webs: 3d $\mathfrak{so}(4N)$

[Bhardwaj, Bottini, SSN, Tiwari]



d-dim Categorical Symmetry Web

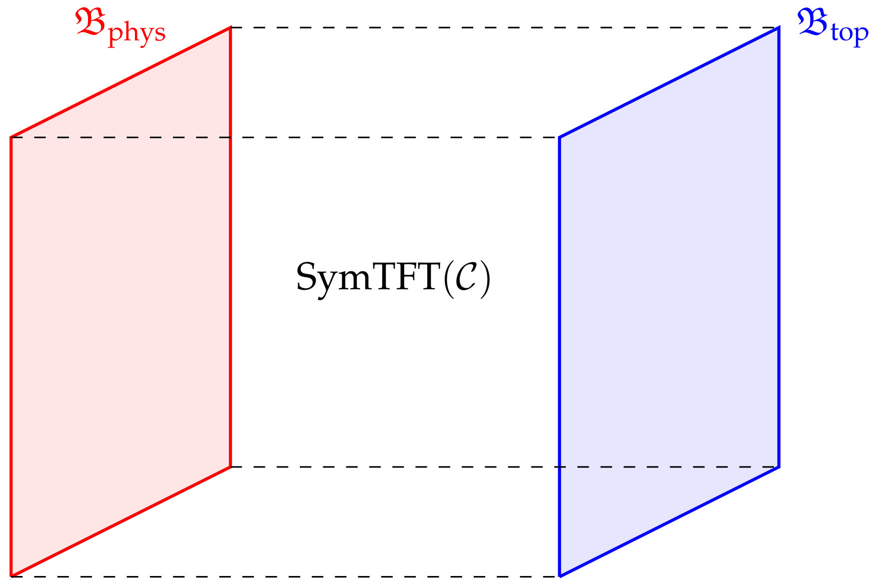
[Bhardwaj, Bottini, SSN, Tiwari]



Symmetry TFT:
The "Everything Everywhere All at Once" of Symmetries

[Gaiotto, Kulp][Apruzzi, Bonetti, Garcia-Extebarria, Hosseini, SSN] [Freed, Moore, Teleman]

Different choices of global forms (related by gauging) correspond to different b.c. on the so-called Symmetry TFT, which is a $(d+1)$dim TQFT that admits gapped boundary conditions:

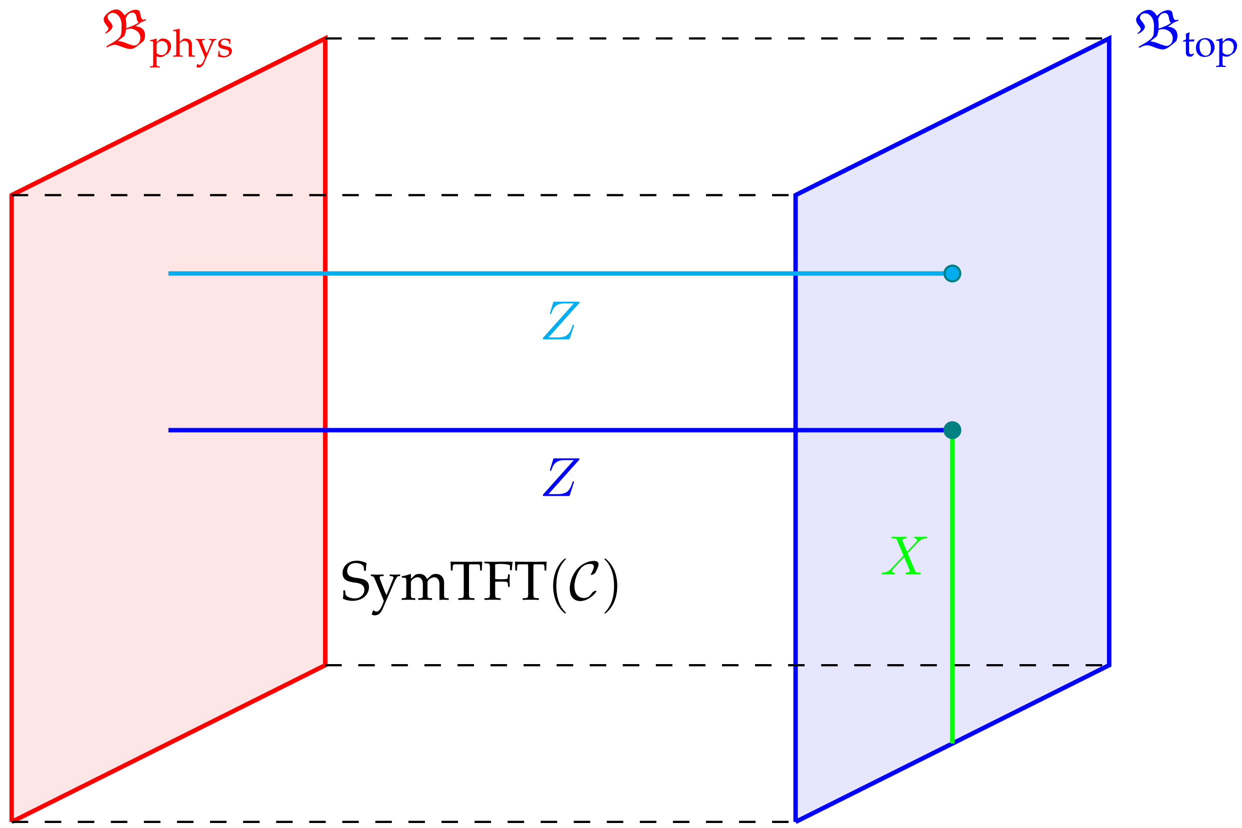


B.c. result in different "global forms".

Example: Turaev-Viro $\text{TV}_\mathcal{C}$ for fusion symmetry $\mathcal{C}$.

More generally: Drinfeld center of the symmetry category. (2-categories [Bhardwaj, SSN, to appear])

Symmetries and Charged Operators from SymTFT



Defects that can end on the boundary resulting charged defects.
Defects that cannot end, result in symmetry generators $X \in \mathcal{C}$.

$\mathcal{C}$ can be any higher fusion category (not necessarily invertible). Some 2d and 4d examples in [Kaidi, Ohmori, Zheng, 2] and general structure will appear in [Bhardwaj, SSN]

Twisted Theta-Defects

- Theta-defects exist for any theory with G global symmetry: stack with a p-dimensional G-TQFT and gauge. These are ″universal″ defects.
- There are Theta-defects, which are theory-dependent: e.g. if there is an obstruction to gauging on a defect, such as a ′t Hooft anomaly: Twisted Theta Defects.
- For 3d topological defects D_3 we can also consider G-TQFTs which do not necessarily admit gapped boundary conditions.

Example: (see Ibou's talk)

$G^{(0)} = \mathbb{Z}_{2M}$, $\Gamma^{(1)} = \mathbb{Z}_M$, with mixed anomaly.

$$\mathcal{A} = -\frac{2\pi}{M}\int A_1 \cup \frac{(B_2 \cup B_2)}{2}$$

The chiral symmetry generator $D_3^{(g)}$ transforms as [Kaidi, Ohmori, Zheng]

$$D_3^{(g)}(M_3) \to D_3^{(g)}(M_3)\exp\left(\int_{M_4} -\frac{2\pi i}{M}\frac{(B_2 \cup B_2)}{2}\right)$$

To gauge $\mathbb{Z}_M^{(1)}$, requires twisted theta-defect: stack with

$$\mathcal{A}^{M,1} = U(1)_M$$

which also does not admit gapped boundary conditions.
The Twisted Theta-defect is

$$\mathcal{N}_3^{(1)} = D_3^{(1)} \otimes \mathcal{A}^{M,1}$$

with fusion

$$\mathcal{N}_3^{(1)} \otimes \mathcal{N}_3^{(1)} = \mathcal{A}^{M,2}\mathcal{N}_3^{(2)}$$

$$\mathcal{N}_3^{(1)} \otimes \mathcal{N}_3^{(1)\dagger} = \mathcal{C}_{\mathbb{Z}_M^{(1)}}(M_3) = \sum_{M_2 \in H_2(M_3,\mathbb{Z}_M)} \frac{(-1)^{Q(M_2)}D_2(M_2)}{|H^0(M_3,\mathbb{Z}_M)|}$$

Connections to Geometry/Strings

Of course many of these theories that have categorical symmetries have realization in string theory. See talks by Ibou, Max, Inaki, Saghar.

Much of the defects/symmetries/etc is realizable in terms of topological operators in some Lagrangian for the SymTFT, and the interpretation in terms of branes [Apruzzi, Bah, Bonetti, SSN][Garcia-Extebarria].

Key open question:
determine the categorical structures directly from string theory/holography, without reverting to the effective descriptions of branes.

Example: Holographic dual of $\mathcal{N} = 1$ SYM.

Branes realizing Categories

[Apruzzi, Bah, Bonetti, SSN]

Recall branes realizing non-invertibles in 4d $\mathcal{N} = 1$ SYM:

$$\mathcal{N}_3^{(1)} \otimes \mathcal{N}_3^{(1)} = \mathcal{A}^{M,2} \mathcal{N}_3^{(2)}$$

translates into (Myers effect)

$$D5(S^3) \otimes D5(S^3) = D7(T^{1,1})$$

$$\mathcal{N}_3^{(1)} \otimes \mathcal{N}_3^{(1)\dagger} = \mathcal{C}_{\mathbb{Z}_M^{(1)}}(M_3) = \sum_{M_2 \in H_2(M_3, \mathbb{Z}_M)} \frac{(-1)^{Q(M_2)} D_2(M_2)}{|H^0(M_3, \mathbb{Z}_M)|}$$

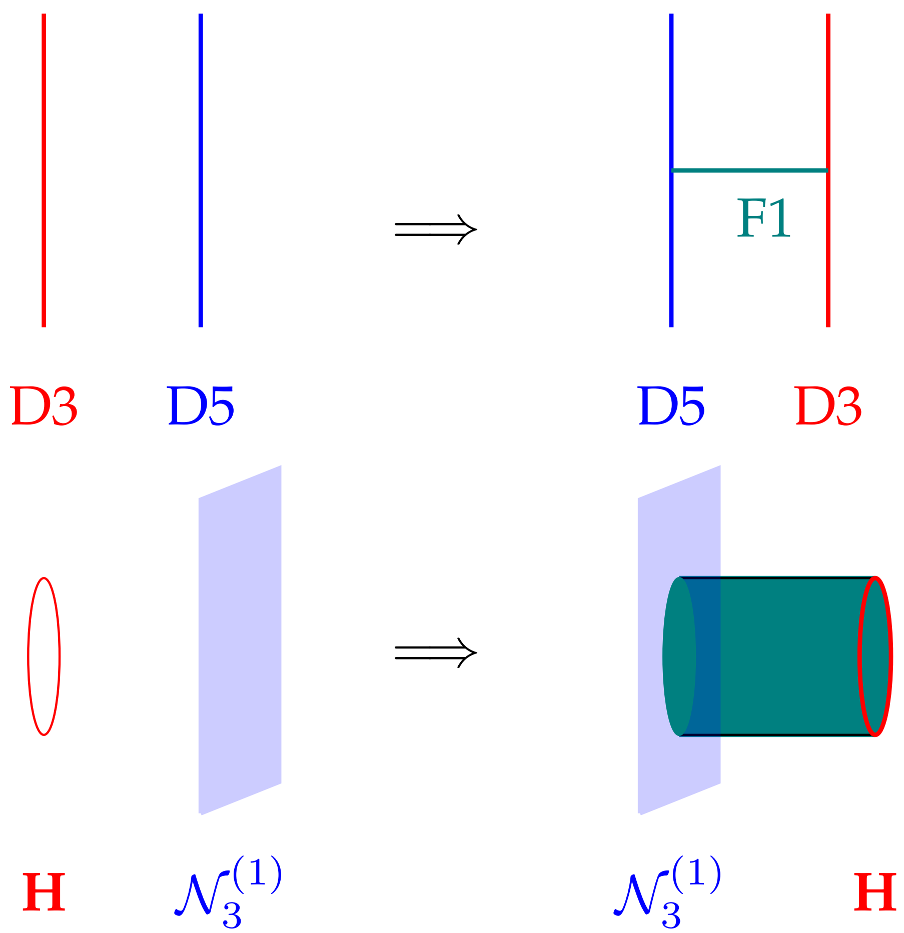
has realization in terms of D5-branes [Bah, Leung, Waddleton]

$$D5(S^3) \otimes \overline{D5}(S^3) = \sum_{M_2} D3(M_2)$$

Hanany Witten as Action of Symmetries on Defects

[Apruzzi, Bah, Bonetti, SSN][Apruzzi, Gould, Bonetti, SSN – wip]

How do the non-invertible symmetry generators act on ′t Hooft lines?
Hanany-Witten:



′t Hooft loop gets flux attachment when it crosses the non-invertible defect – similar to disorder operator in Kramers-Wannier duality.

Categories from Geometry: Some open questions

1. Branes placed at infinity in holography and in geometric engineering realize symmetry generators.
   How is the higher-fusion category structure (objects, morphisms, higher morphisms) encoded in branes? (general theory)

2. How is the Symmetry TFT realized in the brane-picture?

3. Action of branes on branes (a la Hanany-Witten) as action of non-invertible symmetries on charged objects. Can this be sharpened and mapped to a representation-theoretic statement?