Cobordism and a Modified Gauss Law

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Based on:
• 1909.10355 with Cumrun Vafa
• 2212.00039 with Matthew Reece
Motivating Example: D-Branes in NS-NS Flux

Textbook Example: The D3-brane worldvolume supports a dynamical $U(1)$ gauge field, under which the ends of D1-strings are magnetically charged particles.

Suppose the D3-brane worldvolume is $\mathbb{R}^1 \times M^3$, and we turn on $k$ units of NS-NS flux through $M^3$:

$$\int_{M^3} H_3 = k.$$

Then we have a modified Gauss Law: there must be exactly $k$ D1-strings ending at points in $M^3$. [Maldacena, Moore, Seiberg '01]

Due to anomaly inflow: the $U(1)$ gauge field is charged under $B_2$.

Goal Today: Replace everything in sight with the spacetime manifold.
Overview

Dictionary:

Magnetic Flux ↔ Geometric Flux
Monopoles ↔ Cobordism defects
NS-NS flux ↔ Normal bundle

Outline:

1. Review the Swampland Cobordism Conjecture
2. Examples: Nelson-Barr Models and F-theory
3. Connection to Adams Spectral Sequence
Swampland Cobordism Conjecture

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Topological Charges and Cobordism

Topological charges of the spacetime manifold are classified by cobordism.

Fix structure $\mathcal{X}$ (orientation, spin structure, etc.), study $k$-dimensional $\mathcal{X}$-manifolds up to $\mathcal{X}$-preserving topology changes. Set of equivalence classes forms an abelian group $\Omega^\mathcal{X}_k$ under disjoint union.

Defines a global symmetry of semiclassical gravity.
Swampland Cobordism Conjecture

Cobordism Conjecture: Any UV complete theory of quantum gravity must have trivial cobordism groups, $\Omega_k^{QG} = 0$, once all effects and objects are included.

Suppose the semiclassical cobordism groups are nontrivial, $\Omega_k^X \neq 0$. Then the UV complete theory must include defects that trivialize the cobordism classes.

Examples: Orientifold planes, etc. See also Miguel and Markus’s talks.
Gauge Charge of Cobordism Defects

Defects carry a **topological gauge charge** valued in $\Omega^X_k$. Can be detected inside a black brane by measuring the cobordism class of the horizon manifold.

**Gauss Law:** Net number of cobordism defects transverse to closed slice must vanish.

$$\sum_i [M_i] = [\bigcup_i M_i] = [\partial W] = 0.$$  

**Hidden Assumption:** Normal bundle is trivial.
Example: Nelson-Barr Models

Based on 2212.00039 with Matthew Reece
Review: Nelson-Barr Models

**Strong CP Problem:** Our universe has order-one CP violation, yet $\bar{\theta}_{QCD} \approx 10^{-10}$.

**Nelson-Barr Models:** Promote CP or parity to exact symmetry, spontaneously broken at low energies. [Nelson ’84, Barr ‘84]

In terms of cobordism:

- Low-energy theory is chiral, depends on orientation ($\mathcal{X} = SO$).
- High-energy theory makes sense on non-orientable manifolds ($\mathcal{X} = O$).

Nelson-Barr models contain a cobordism defect, much discussed in pheno community: the **parity domain wall (PDW)**.
Parity Domain Walls as Cobordism Defects

Radiates nontrivial cobordism class: \( 2 [\text{pt}] \in \Omega_0^{SO} \). Could detect PDW inside black domain wall!

\[ 
\begin{array}{ccc}
- & + & + \\
\bullet & \bullet & \bullet \\
& & PDW
\end{array}
\]

**Gauss Law:** Every 1-manifold is orientable, so net number of PDWs on abstract 1-manifold must vanish mod 2.

**Pheno Lesson:** PDWs are exactly stable. Big problem for cosmology!
Modified Gauss Law

**Modified Gauss Law**: Odd number of PDWs transverse to circle $C$ with non-orientable normal bundle:

![Diagram showing PDWs transverse to circle C](image)

**Anomaly Inflow**: The cobordism class of a point in an ambient oriented manifold suffers an ambiguity: it flips sign under a normal reflection, $[pt] \to -[pt]$. 
Example: F-theory

Based on 1909.10355 with Cumrun Vafa
Type IIB and F-theory

Two descriptions (see Miguel and Markus’s talks for more refined version):

• Perturbative Type IIB requires a spin structure \((\mathcal{X} = Spin)\).

• Non-perturbatively, F-theory base requires spin\(^c\) structure \((\mathcal{X} = Spin^c)\).

F-theory contains a cobordism defect relative to perturbative Type IIB: a spin vortex, around which fermions pick up an additional minus sign. Radiates \([S_p^1] \in \Omega^\text{Spin}_1\). Realized by 12 singular fibers (deficit angle \(2\pi\)). [Green, Shapere, Vafa, Yau ’89]
**Modified Gauss Law**

**Gauss Law:** Elliptic fibration over a Riemann surface $\Sigma$ has a multiple of 24 singular fibers. **Ex:** F-theory on $K3 \to \mathbb{P}^1$ has 24 singular fibers, comprising 2 spin vortices.

**Modified Gauss Law:** Consider elliptic fibration over a $(-1)$-curve $\Sigma$ in the base. Normal bundle is non-spin. Only 12 singular fibers: a single spin vortex.

**Anomaly Inflow:** The cobordism class of a circle in an ambient spin manifold suffers an ambiguity: it flips under twisting the normal framing, $[S_p^1] \to [S_{ap}^1]$. 
Adams Spectral Sequence

Based on 2212.00039 with Matthew Reece
Gaunting in Cobordism

In both examples considered, the cobordism defects are measured by a characteristic class:

- Parity domain walls are defects in the orientation, measured by $w_1$.
- Spin vortices are defects in the spin structure, measured by $w_2$.

In both cases, the characteristic class is nontrivial in the fundamental description ($\mathcal{X} = 0$ or $\mathcal{X} = Spin^c$) but vanishes on abstract manifolds of the same dimension:

- All unoriented 1-manifolds are orientable, $w_1 = 0$.
- All spin$^c$ 2-manifolds are spin, $w_2 = 0$.

While the characteristic classes are nontrivial, they fail to appear in cobordism groups of the same dimension.
The Adams Spectral Sequence

Our motivating example (D-branes in NS-NS flux) is mathematically described by a **spectral sequence** (AHSS) which takes the modified Gauss Law into account.

The same is true for cobordism: our story is described by the **Adams spectral sequence**, a tool for computing cobordism groups given characteristic classes.

Both of our examples are captured by a **differential** on the $E_1$ page:

$$\chi = 0$$

$$\chi = \text{Spin}^c$$
Thank You For Listening!