The Tameness of Quantum Field Theories

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Based on:

2302.04275 Part II
2210.10057 Part I
2112.08383 + work in progress

with Mike Douglas, Lorenz Schlechter

Finiteness as swampland principle

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- Suggested finiteness principle: Tameness ('o-minimality')

  $\rightarrow$ finiteness conjectures are implied, but much broader concept

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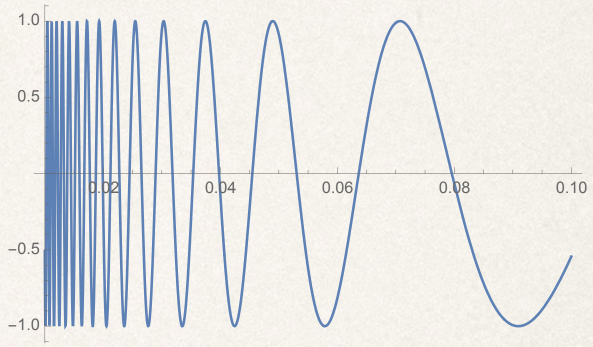

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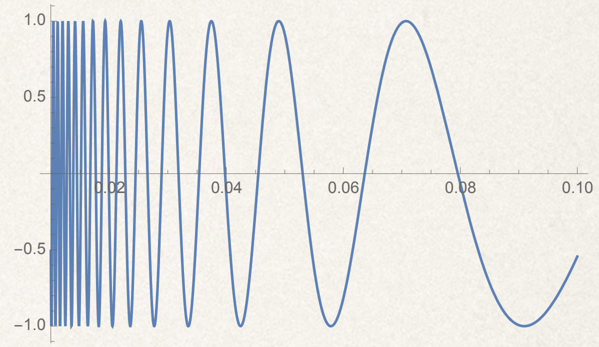

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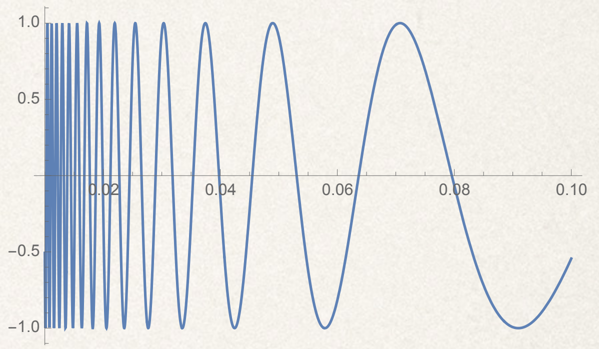

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- Grothendieck's dream to develop math. framework for geometry:
  → tame topology [Esquisse d'un programme]

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- tame/o-minimal structure $\mathcal{S}$: only subsets of $\mathbb{R}$ that are in $\mathcal{S}$ are finite unions of points and intervals

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sets in o-minimal structure $\mathcal{S}$: tame sets

functions with graph being a tame set: tame functions

$\rightarrow$ tame manifold, tame bundles… a tame geometry

- ta
- ta

finite unions of points and intervals

[van den Dries]

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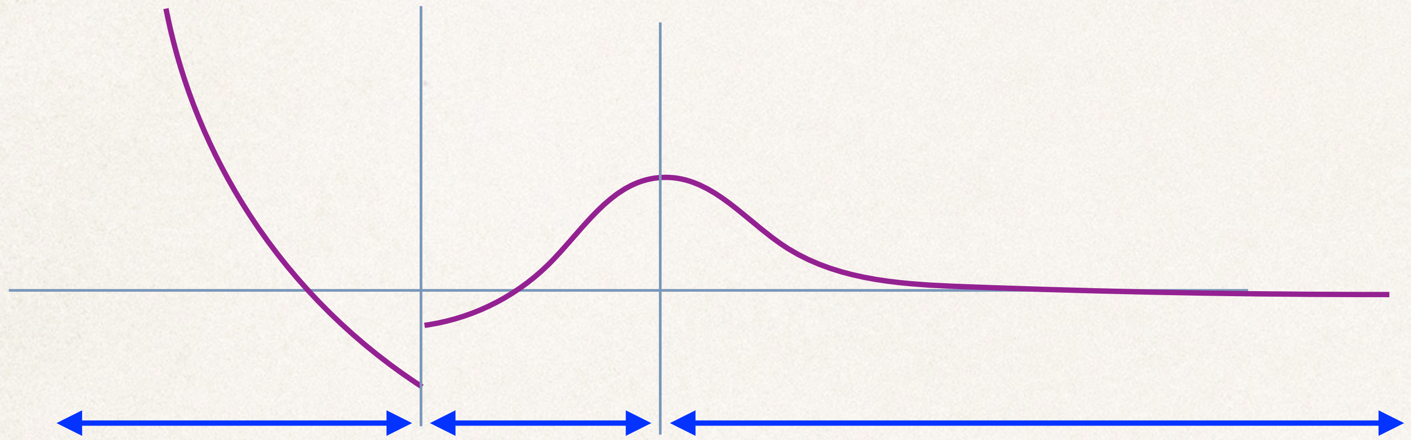

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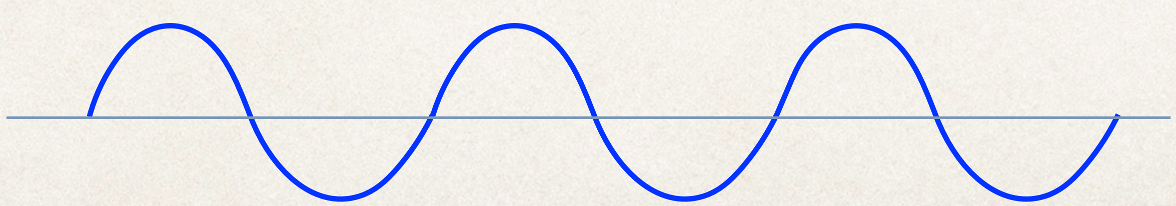
- Periodic functions $f(x+n) = f(x)$ are never tame (when not constant)

$\sin(x), x \in \mathbb{R}$



Examples of o-minimal structures

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- Logic perspective: $\mathbb{R}_{\mathcal{F}} = \langle \mathbb{R}; +, \cdot, -, \mathcal{F} \rangle$ $\qquad \mathcal{F} = \{f_1, f_2, \ldots\}$

all formulas using these symbols and ∧, ∨, ¬, ∃, ∀

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  - Pfaffian extension: $\mathcal{P}(\tilde{\mathbb{R}})$ includes solutions to $\partial_{x_i} f = F_i(x, f(x))$
    F_i functions in o-minimal structure $\tilde{\mathbb{R}}$ [Speissegger '97]

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    ⋮

  - structure including $\Gamma(x)|_{(0,\infty)}$ and $\zeta(x)|_{(1,\infty)}$ [Rolin,Servi,Speissegger '22]

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- Tameness used in many recent proofs of deep mathematics conjectures:
    - Ax-Schanuel conjecture for Hodge structures [Bakker,Tsimerman '17]
      several subsequent generalizations, e.g. to mixed Hodge structures
    - Griffiths' conjecture [Bakker,Brunebarbe,Tsimerman '18]
    - André-Oort conjecture … [Pila,Shankar,Tsimerman '21]
    - Geometric André-Grothendieck Period Conjecture [Bakker,Tsimerman '22]

- Finiteness of self-dual integral classes (inspired by physics finiteness conjecture)
  [Bakker,TG,Schnell,Tsimerman '21]

Building Structures for physical theories

Structures from physical theories

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Example:

$\mathcal{T}$: polynomial Lagrangians with real valued parameters

$\mathcal{S}$: spacetimes $\mathbb{R}^d, T^d, S^d$ with standard metric

$\longrightarrow \quad \mathbb{R}^{\text{def}}_{\mathcal{T},\mathcal{S}} = \mathbb{R}_{\text{alg}}$

<u>However</u>: physical constraints on $\mathcal{T}$ might require to go beyond $\mathbb{R}_{\text{alg}}$

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- Extend structure $\mathbb{R}^{\text{def}}_{\mathcal{T},\mathcal{S}}$ by physical observables:
  add correlation/partition functions:

$$f_\alpha(y, \lambda) = \langle \mathcal{O}_1(y_1) ... \mathcal{O}_1(y_n) \rangle_\lambda \longrightarrow \text{new structure } \mathbb{R}_{\mathcal{T},\mathcal{S}}$$

$\rightarrow$ complicated function on $\Sigma \times ... \times \Sigma$
and parameter space

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Example: harmonic oscillator in quantum mechanics (Euclidean)

$$\mathbb{R}_{\mathcal{T},\mathcal{S}} = \mathbb{R}_{\exp}$$

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We consider: $\mathbb{R}_{\text{QFT}d},\ \mathbb{R}_{\text{EFT}d},\ \mathbb{R}_{\text{CFT}d},\ \mathbb{R}_{\text{QG}},\ \ldots$

Tameness of general QFTs?

$$\mathbb{R}_{\mathrm{QFT}d}$$

$\mathbb{R}_{\mathrm{QFT}d}$ - structures for general QFTs

- Perturbative QFT:

  Theorem: For any (renormalizable) QFT with finitely many particles and interactions all finite-loop amplitudes are tame functions in $\mathbb{R}_{\mathrm{an,exp}}$ of masses, external momenta, coupling constants. [Douglas,TG,Schlechter - Part I]

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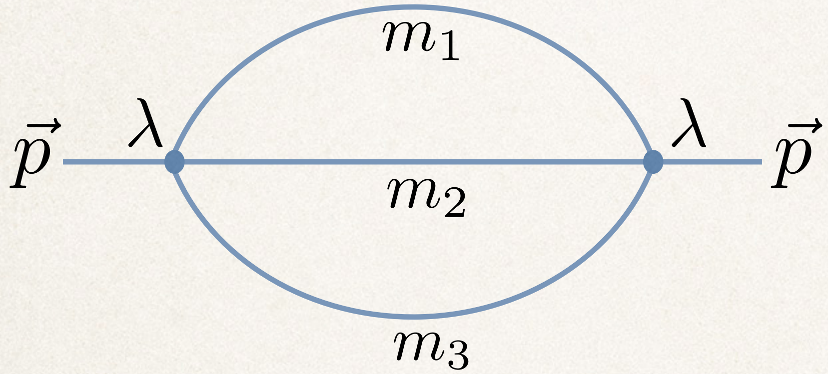

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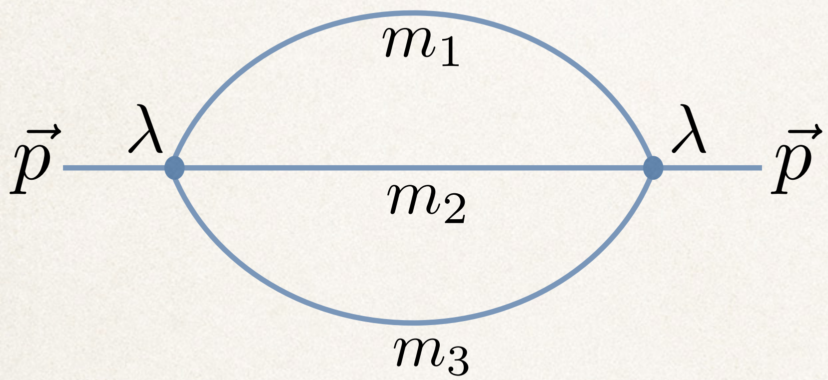

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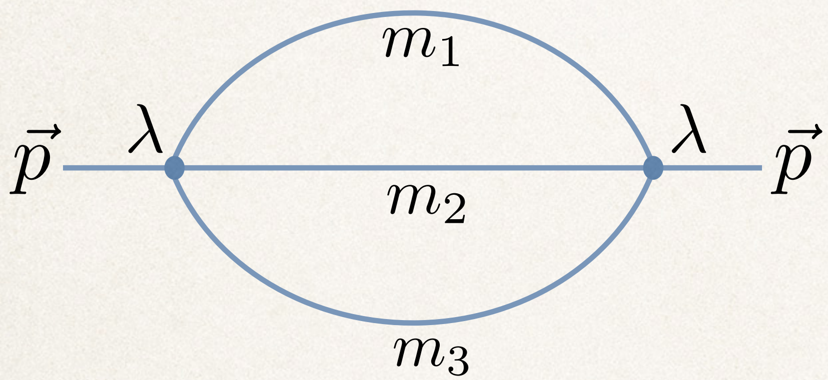

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Remarks: - proof uses relation of Feynman integrals and period integrals

tameness of periods [Bakker,Klingler,Tsimermann]…[Bakker,Mullane '22]

- theorem is non-trivial: interesting implications for Feynman amplitudes (symmetry $\longleftrightarrow$ relations) [in progress]

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  - 1d theory: harmonic oscillator (finite temperature partition function)

    $$Z(\beta, m) = \frac{1}{2\sinh \beta/(2m)} \quad \to \text{tame in } \beta, m$$

  - 2d free Yang-Mills: $SU(2)$ example $\quad Z_{SU(2)} = e^{\frac{A\lambda}{16}}(\theta_3(e^{-\frac{A\lambda}{16}}) - 1)$

    $\to$ tame in λ, A, theta tame on fundamental domain [Peterzil,Starchenko]

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General statements about 0d QFTs (QFTs on points)
→ path integrals reduce to ordinary integrals

Simplest example

- Consider in 0d: $S = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \quad \rightarrow \quad Z = \sqrt{\frac{3}{\lambda}} e^{\frac{3m^4}{4\lambda}}\, m\, K_{\frac{1}{4}}\left(\frac{3m^4}{4\lambda}\right)$

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$\to$ after WHCGP colloquium: van den Dries sent a proof that $K_\alpha(x)$ is tame in Pfaffian structure but not in $\mathbb{R}_{\text{an,exp}}$

Challenges in mathematics

- QFTs on points

recently e.g. [Gasparotto,Rapakoulias,Weinzierl]

correlation functions in 0d are ordinary integrals

$$\langle \mathcal{O}_1 ... \mathcal{O}_n \rangle_\lambda = \int d\phi_1 ... d\phi_k \, \mathcal{O}_1 ... \mathcal{O}_n \, e^{-S^{(0)}(\phi,\lambda)}$$

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Conjecture [van den Dries][Kaiser]: Given a real-valued tame function $f(\lambda, \phi)$ (in some o-minimal structure $\mathcal{S}$) the integral

$$g(\lambda) = \int d\phi_1 ... d\phi_k f(\phi, \lambda)$$

is also a tame function (in some o-minimal structure $\widetilde{\mathcal{S}}$).

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Conjecture [van den Dries][Kaiser]: Given a real-valued tame function $f(\lambda, \phi)$ (in some o-minimal structure $\mathcal{S}$) the integral

$$g(\lambda) = \int d\phi_1 ... d\phi_k f(\phi, \lambda)$$

is also a tame function (in some o-minimal structure $\widetilde{\mathcal{S}}$).

Note: Theorem for $\mathcal{S} = \mathbb{R}_{\text{an}} \rightarrow \widetilde{\mathcal{S}} = \mathbb{R}_{\text{an,exp}}$. [Comte,Lion,Rolin]

However, for non-perturbative results, we need exponential to be in $\mathcal{S}$.

Challenges in mathematics

- QFTs on points

recently e.g. [Gasparotto,Rapakoulias,Weinzierl]

correlation functions in 0d are ordinary integrals

$$\langle \mathcal{O}_1 ... \mathcal{O}_n \rangle_\lambda = \int d\phi_1 ... d\phi_k \, \mathcal{O}_1 ... \mathcal{O}_n \, e^{-S^{(0)}(\phi,\lambda)}$$

$\Rightarrow$ math. conjecture implies:

[Douglas,TG,Schlechter - Part II]

in order that physical observables $\langle \mathcal{O}_1 ... \mathcal{O}_n \rangle_\lambda$ are tame functions of parameters λ if one needs to require:

$S^{(0)}(\phi, \lambda)$ is tame function of λ, ϕ

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$\rightarrow$ tameness requires that all such symmetries are gauged or eventually broken in full Z

$\rightarrow$ Fits with best understood conjectures about Quantum Gravity: 'No global symmetries in QG'

[Banks,Dixon][Banks,Seiberg]

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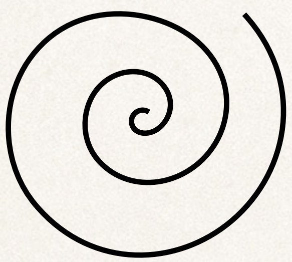

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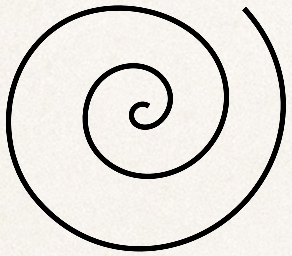

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More Fancy: $W_\xi = Y P_\xi(X_1, \dots, X_k)^2 + \sum_a Z_a(\sin 2\pi i X_a)^2$ [Tachikawa]

Existence of supersymmetric vacua is undecidable!

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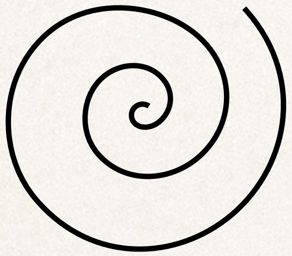

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Existence of supersymmetric vacua is undecidable!

- in general: $\mathbb{R}_{\mathrm{QFT}d}$ not o-minimality / tame
  $\rightarrow$ tameness depends on the UV origin of the theory

Effective field theories compatible with quantum gravity

$\mathbb{R}_{\mathrm{EFT}d}$

A tameness conjecture

Conjecture [TG '21]:

All effective theories valid below a fixed finite energy cut-off scale Λ that can be coupled to QG are labelled by a tame parameter space and have scalar field spaces and Lagrangians that are tame in an o-minimal structure.

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Conjecture [Douglas,TG,Schlechter - Part II]:

$\mathbb{R}_{\mathrm{EFT}d}[\Lambda]$ are o-minimal structures, i.e. also EFT observables are tame.

Tameness of CFTs

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Alternative: Structure $\mathbb{R}_{\mathcal{T},\mathcal{S}}$ for such theories is o-minimal.

$\rightarrow \mathbb{R}_{\mathrm{CFT}d}$

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evidence from considering expansion into conformal partial waves

implications: conditions on gaps for operators
finite radius of convergence of conformal perturbation

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Conjecture 2 (Tame theory spaces):

Theory space $\mathcal{T}$ of CFTs in d=2 is tame set if

- central charge is bounded by $\hat{c}$
- lowest operator dimensions bounded from below by Δ_{min}

implies conjectures by [Douglas,Acharya][Kontsevich,Soibelman]

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- many challenging cases: e.g. 3d Chern-Simons matter theories
  $\rightarrow$ show that there are no infinite discrete families

Outlook: Complexity

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Proposal: Tameness in physics is definability in a sharply o-minimal structure.

$\rightarrow$ physical observables have finite complexity

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Much left to be explored:

implications of tameness (computational + understanding QFTs)
relation to other QG conjectures,…,
connection with complexity/information

Thanks!