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# When is market incompleteness irrelevant for the price of aggregate risk (and when is it not)? <sup>☆</sup>

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## Abstract

In a standard incomplete markets model with a continuum of households that have constant relative risk aversion (CRRA) preferences, the absence of insurance markets for idiosyncratic labor income risk has no effect on the premium for aggregate risk if the distribution of idiosyncratic risk is independent of aggregate shocks and aggregate consumption growth is independent over time. In equilibrium, households only use the stock market to smooth consumption; the bond market is inoperative. Furthermore, the cross-sectional distributions of wealth and consumption are not affected by aggregate shocks. These results hold regardless of the persistence of idiosyncratic shocks, even when households face tight solvency constraints. A weaker irrelevance result survives when we allow for predictability in aggregate consumption growth.

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## 1. Introduction

This paper provides general conditions under which closing down insurance markets for idiosyncratic risk (our definition of market incompleteness) does not increase the risk premium that stocks command over bonds. We study a standard incomplete markets endowment economy populated by a continuum of agents who have CRRA preferences and who can trade a risk-free bond and a stock, subject to potentially binding solvency constraints. In the benchmark version of our model the growth rate of the aggregate endowment is uncorrelated over time, that is, the logarithm of aggregate income is a random walk. Under these assumptions, the presence of *uninsurable idiosyncratic risk* is shown to lower the equilibrium risk-free rate, but it has no effect on the price of aggregate risk in equilibrium if the distribution of idiosyncratic shocks is statistically independent of aggregate shocks. Consequently, in this class of models, the representative agent Consumption-CAPM (CCAPM) developed by Rubinstein [33], Breeden [7] and Lucas [28], prices the excess returns on the stock correctly. Therefore, as long as idiosyncratic shocks are distributed independently of aggregate shocks, the extent to which households manage to insure against idiosyncratic income risk as well as the tightness of the borrowing constraints they face are irrelevant for risk premia. These results deepen the equity premium puzzle, because we show that Mehra and Prescott's [31] statement of the puzzle applies to a much larger class of incomplete market models.<sup>1</sup> We also show that uninsurable income risk does not contribute to any variation in the *conditional* market price of risk.

The asset pricing implications of our model follow from a crucial result about equilibrium consumption and portfolio allocations. We show that in this class of incomplete market models with idiosyncratic and aggregate risk, the equilibrium allocations and prices can be obtained from the allocations and interest rates of an equilibrium in a stationary model with only idiosyncratic risk (as in Bewley [6], Huggett [21] or Aiyagari [1]). Specifically, we demonstrate that scaling up the consumption allocation of the Bewley equilibrium by the aggregate endowment delivers an equilibrium for the model with aggregate risk. In this equilibrium, there is *no* trade in bond markets, only in stock markets. This result is the key to why the history of aggregate shocks has no bearing on equilibrium allocations and prices. Households in equilibrium only trade the stock (i.e. the market portfolio), regardless of their history of idiosyncratic shocks, even in the presence of binding solvency constraints.

This finding has several important consequences beyond the asset pricing irrelevance result. First, it implies that the wealth and consumption distribution in the model with aggregate risk (normalized by the aggregate endowment) coincides with the stationary wealth distribution of the Bewley equilibrium. Aggregate shocks therefore have no impact on these equilibrium distributions.<sup>2</sup>

Second, our findings have strong implications for household portfolios. In the Bewley model, due to the absence of aggregate risk, a bond and a stock have exactly the same return charac-

<sup>1</sup> Weil's [37] statement of the risk-free rate puzzle, on the contrary, does not.

<sup>2</sup> Since stationary equilibria in Bewley models are relatively straightforward to compute, our result implies an algorithm for computing equilibria in this class of models which appears to be simpler than the auctioneer algorithm devised by Lucas [27] and its extension to economies with a continuum of agents. There is also no need for computing a law of motion for the aggregate wealth distribution, or approximating it by a finite number of moments, as in Krusell and Smith [23,24]. This result also implies the existence of a recursive competitive equilibrium with only asset holdings in the state space. Kubler and Schmedders [25] establish the existence of such an equilibrium in more general models, but only under very strong conditions. Miao [32] relaxes these conditions, but he includes continuation utilities in the state space.

teristics in equilibrium and thus one asset is redundant. Households need not use the bond to smooth consumption. This portfolio implication carries over to the model with aggregate risk: at equilibrium prices households do not need to trade the uncontingent bond and only use the stock, i.e. a claim to aggregate consumption, to transfer resources over time and to self-insure against idiosyncratic labor earnings risk. In equilibrium, all households bear the same share of aggregate risk and they all hold the same portfolio, regardless of their history of idiosyncratic shocks and the implied financial wealth holdings. Hence, this class of models does not generate any demand for bonds at the equilibrium interest rate, if bonds are in zero net supply.

In our benchmark model, aggregate consumption growth is not predictable (that is, the growth rate of the aggregate endowment is i.i.d.). This is a standard assumption in the asset pricing literature (e.g., Campbell and Cochrane [8]), because the empirical evidence for predictability is rather weak (see, e.g., Heaton and Lucas [20]). It is also a natural benchmark, because in this case all the dynamics in the conditional market price of risk, if any, come from the model itself rather than the exogenously specified aggregate consumption growth process. However, if there is predictability in aggregate income growth in our model, we show that agents want to hedge their portfolio against interest rate shocks, creating a role for trade in a richer menu of assets than just the stock. The risk premium irrelevance result, however, still applies as long as households can trade a full set of aggregate state-contingent claims and the solvency constraints do not bind in equilibrium.

The key assumptions underlying our irrelevance result are (i) a continuum of agents, (ii) CRRA utility, (iii) idiosyncratic labor income risk that is independent of aggregate risk, (iv) a constant capital share of income, and (v) solvency constraints or borrowing constraints on total financial wealth that are proportional to aggregate income. We now discuss each of these assumptions in detail to highlight and explain the differences with existing papers in the literature.

First, we require the economy to be populated by a continuum of agents. As forcefully pointed out by DenHaan [13], in an economy with a finite number of agents, each idiosyncratic shock is by construction an aggregate shock because it changes the wealth distribution and, through these wealth effects, asset prices.

Second, our results rely on the homogeneity property of the CRRA utility function. Third, in our model labor income grows with the aggregate endowment, as is standard in this literature.<sup>3</sup> In addition, for our results to go through, idiosyncratic income shocks must be distributed independently of aggregate shocks. This explicitly rules out that the variance of idiosyncratic shocks is higher in recessions (henceforth we refer to this type of correlation as counter-cyclical cross-sectional variance of labor income shocks, or CCV).

Fourth, we require the assumption that the capital share of income is independent of the aggregate state of the economy. This assumption is standard in macroeconomics, and it is not obviously at odds with the data (see, e.g., Cooley and Prescott [12]).

Finally, in the main section of the paper, households face either constraints on total net wealth today or state-by-state solvency constraints on the value of their portfolio in each state tomorrow. These solvency constraints have to be proportional to aggregate income. If this were not the case, the tightness of the constraints would depend on the aggregate history of shocks, invalidating our map from equilibria of the Bewley model into equilibria of the model with stochastic aggregate

<sup>3</sup> By contrast, Weil [38] derives a positive effect of background risk on the risk premium in a two-period model in which labor income is not proportional to the aggregate endowment. This non-homotheticity invalidates our mapping from the growing to the stationary economy.

endowment growth.<sup>4</sup> Apart from the proportionality condition the solvency constraints we can permit are very general.

Our irrelevance result is robust along many empirically important dimensions. It does not hinge on mean reversion in labor income as it also goes through in the case of permanent idiosyncratic shocks. If the idiosyncratic part of household income follows a logarithmic random walk, equilibria still feature trade in general. In fact, it is easy to show that in this case autarky is an equilibrium if and only if all assets are in zero net supply, that is if and only if there is no net wealth in the economy.

Finally, the irrelevance result survives cyclicity in the aggregate capital share and preference heterogeneity, but only in special cases. In the case of stochastic variation in the capital share households can keep their consumption in line with aggregate consumption by taking positions in the contingent bond market, but only if the variation in the aggregate labor income share does not lead to variation in the distribution of *idiosyncratic* labor income shocks and activate the CCV mechanism. If this condition is satisfied, the irrelevance result survives. In the case of preference heterogeneity, a weaker version of the irrelevance result remains valid if less risk averse households have labor income that is more exposed to aggregate risk. In both cases, trade in a richer menu of assets that include aggregate state-contingent claims is required.

We characterize equilibria in the model with aggregate risk that are derived from Bewley equilibria<sup>5</sup> by scaling allocations by the aggregate endowment. While we cannot formally rule out the existence of other types of equilibria, we conjecture that the equilibria that we characterize are the only ones.

*Related literature.* In the quest towards the resolution of the equity premium puzzle identified by Hansen and Singleton [19] and Mehra and Prescott [31], uninsurable idiosyncratic income risk has been introduced into standard dynamic general equilibrium models.<sup>6</sup> The main finding, derived numerically for various parameterizations, is that uninsurable labor income risk has only small effects on the equity premium (see, e.g., Telmer [35]). The main contribution of our paper is to establish theoretically that, under a set of fairly general conditions spelled out above and further discussed below, adding uninsurable idiosyncratic income risk to the standard consumption-based asset pricing model does not alter the asset pricing implications of the model with respect to excess returns at all. Our theoretical result holds regardless of how tight the solvency constraints are, how persistent the income process is and how much agents discount the future.

While earlier work exploring autarkic equilibria by Constantinides and Duffie [11] (henceforth CD) is suggestive of the irrelevance of idiosyncratic (consumption) risk, our work derives the exact conditions for irrelevance in non-autarkic equilibria for economies with standard labor income processes while allowing for binding borrowing constraints. We show that these conditions depend critically on the properties of aggregate consumption growth once trade of financial

<sup>4</sup> This assumption is less restrictive than it appears. Constraints arising from limited commitment, as, e.g., in Alvarez and Jermann [3] satisfy this restriction, as do the tight borrowing constraints used by Krusell and Smith [23] and many others.

<sup>5</sup> For our equivalence and asset pricing results there is no need to assume or show that the Bewley equilibrium is unique. Thus our results imply that there is at least one equilibrium in the incomplete markets model with idiosyncratic and aggregate risk that has the same market price of risk as the standard representative agent model.

<sup>6</sup> For examples, see the work of Aiyagari and Gertler [2], Telmer [35], Lucas [27], Heaton and Lucas [20], Krusell and Smith [23,24], Marcet and Singleton [30]. Also see Kocherlakota [22] for an overview of this literature.

assets occurs in equilibrium. In contrast, in the autarkic equilibria examined by CD, their irrelevance result does not depend on the properties of aggregate consumption growth.

CD consider an environment in which agents face only permanent idiosyncratic income shocks and in which they can trade stocks and bonds. They choose a (somewhat non-standard) stochastic household income process where household labor income depends on aggregate dividends that delivers an autarkic equilibrium even in the case of positive net wealth. CD's results also imply that if the cross-sectional variance of consumption growth is orthogonal to returns, then the equilibrium risk premium is equal to the one in the representative agent model. We show that this characterization of risk premia is indeed the correct one in equilibrium in a large class of incomplete market models with standard household labor income processes, including those with permanent idiosyncratic shocks, provided that aggregate consumption growth is not predictable. Relative to CD, our paper adds potentially binding solvency or borrowing constraints. Moreover, our equilibrium does feature trade in financial markets, and we do not make any assumptions about the distribution of the underlying idiosyncratic shocks. The consumption growth distribution across households in our model is the endogenous, equilibrium outcome of these trades, but we can still fully characterize equilibrium asset prices.<sup>7</sup> However, we also show that only a weaker irrelevance result survives when we allow for predictability (serial correlation) in aggregate consumption growth, because different households have different hedging demands to insure against interest rate shocks and the hedging demands give rise to trade in the contingent bond markets. In CD's autarkic equilibrium, idiosyncratic risk is not priced regardless of the properties of aggregate consumption growth. Allowing for equilibrium trade overturns this result. This also means that contingent (on aggregate shocks) claims markets are needed and borrowing constraints cannot bind in order to obtain aggregation in models with long-run risk, like the one considered by Bansal and Yaron [5].

Most of the work on incomplete markets and risk premia focuses on the moments of model-generated data for particular calibrations of the model, but there are few general theoretical results. Levine and Zame [26] show that in economies populated by agents with infinite horizons, the equilibrium allocations converge in the limit, as their discount factors approach one, to the complete markets allocations. We provide a qualitatively similar equivalence result that applies only to the risk premium. Our result, however, does not depend on the time discount factor of households.

The result of our paper that equilibria in the model with aggregate risk can be obtained from equilibria of the stationary Bewley model also makes contact with the literature on aggregation. Rubinstein [33] obtains an aggregation result without assuming complete markets, but he does not allow for non-traded assets such as risky labor income. Our findings extend these aggregation results to a large class of incomplete market models with idiosyncratic income shocks, but the result only applies to excess returns.

*Road map.* The paper is structured as follows. In Section 2, we lay out the physical environment of our model. Before analyzing the infinite horizon economy, Section 3 considers a two-period example, to develop some intuition for our main results in a version of the model that permits

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<sup>7</sup> In a separate class of continuous-time diffusion models, Grossman and Shiller [17] have demonstrated that idiosyncratic risk has no effect on risk premia, simply because the cross-sectional variance of consumption growth is locally deterministic. Our irrelevance result is obtained in a different class of discrete-time incomplete market models in which uninsurable idiosyncratic background risk does potentially matter, as shown by Mankiw [29], Constantinides and Duffie [11] and others.

a full analytical solution. Section 4 then analyzes the general infinite horizon model. This section demonstrates how to transform a model where aggregate income grows stochastically into a stationary model with a constant aggregate endowment. In Section 4.1 we study this stationary model, called the *Bewley* model henceforth. The next section, Section 4.2, introduces the Arrow model, a model with aggregate uncertainty and a full set of Arrow securities whose payoffs are contingent on the realization of the aggregate shock. We show that any equilibrium of the Bewley model can be mapped into an equilibrium of the Arrow model just by scaling up allocations by the aggregate endowment. In Section 4.3 we argue that the same result emerges in an economy in which agents only trade a one-period risk-free bond and the stock. We call this the *Bond* model. We show that the risk premia in the representative agent *Breeden–Lucas (BL)* model and the Arrow model (and by implication, in the Bond model) coincide. Section 5 investigates the robustness of our results with respect to the assumptions about the underlying stochastic income process, and shows in particular that most of our results can be extended to the case in which the aggregate shocks are correlated over time and to the case of permanent household income shocks. We also explain how cyclicalities in the capital income share weakens our result. Finally, Section 6 concludes; all proofs are contained in Appendix A.

## 2. Environment

Our exchange economy is populated by a continuum of individuals of measure 1. There is a single nonstorable consumption good. The aggregate endowment of this good is stochastic. Each individual's endowment depends, in addition to the aggregate shock, also on the realization of an idiosyncratic shock. Thus, the model we study is identical to the one described by Lucas [27], except that ours is populated by a continuum of agents (as in Bewley [6], Aiyagari and Gertler [2], Huggett [21] and Aiyagari [1]), instead of just two agents.

### 2.1. Representation of uncertainty

We denote the current aggregate shock by  $z_t \in Z$  and the current idiosyncratic shock by  $y_t \in Y$ . For simplicity, both  $Z$  and  $Y$  are assumed to be finite. Furthermore, let  $z^t = (z_0, \dots, z_t)$  and  $y^t = (y_0, \dots, y_t)$  denote the history of aggregate and idiosyncratic shocks. As shorthand notation, we use  $s_t = (y_t, z_t)$  and  $s^t = (y^t, z^t)$ . We let the economy start at an initial aggregate node  $z_0$ . Conditional on an idiosyncratic shock  $y_0$  and thus  $s_0 = (y_0, z_0)$ , the probability of a history  $s^t$  is given by  $\pi_t(s^t | s_0)$ . We assume that shocks follow a first order Markov process with transition probabilities given by  $\pi(s^t | s)$ .

### 2.2. Preferences and endowments

Consumers rank stochastic consumption streams  $\{c_t(s^t)\}$  according to the following homothetic utility function:

$$U(c)(s_0) = \sum_{t=0}^{\infty} \sum_{s^t \geq s_0} \beta^t \pi(s^t | s_0) \frac{c_t(s^t)^{1-\gamma}}{1-\gamma}, \quad (1)$$

where  $\gamma > 0$  is the coefficient of relative risk aversion and  $\beta \in (0, 1)$  is the constant time discount factor. We define  $U(c)(s^t)$  to be the continuation expected lifetime utility from a consumption allocation  $c = \{c_t(s^t)\}$  in node  $s^t$ . This utility can be constructed recursively as follows:

$$U(c)(s^t) = u(c_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) U(c)(s^t, s_{t+1}),$$

where we made use of the Markov property the underlying stochastic processes.

The economy's aggregate endowment process  $\{e_t\}$  depends only on the aggregate event history; we let  $e_t(z^t)$  denote the aggregate endowment at node  $z^t$ . The stochastic growth rate of the endowment of the economy is denoted by  $\lambda(z^{t+1}) = e_{t+1}(z^{t+1})/e_t(z^t)$ . We assume that aggregate endowment growth only depends on the current aggregate state.

**Condition 2.1.** *Aggregate endowment growth is a function of the current aggregate shock only*  $\lambda(z^{t+1}) = \lambda(z_{t+1})$ .

There is a Lucas tree that yields a share  $\alpha(z)$  of the total aggregate endowment as capital income, so that the total dividends of the tree are given by  $\alpha(z_t)e_t(z^t)$  in each period. The remaining fraction of the total endowment accrues to individuals as labor income, so that  $1 - \alpha(z)$  denotes the labor income share.

Each agent draws a 'labor income' share  $\eta(y_t, z_t)$ , as a fraction of the aggregate endowment in each period. Her labor income share only depends on the current individual and aggregate event. We denote the resulting individual labor income process by  $\{\eta_t\}$ , with

$$\eta_t(s^t) = (1 - \alpha(z_t))\eta(y_t, z_t)e_t(z^t), \quad (2)$$

where  $s^t = (s^{t-1}, y_t, z_t)$ . We assume that  $\eta(y_t, z_t) > 0$  in all states of the world.

We assume that a Law of Large Numbers holds,<sup>8</sup> so that  $\pi(s^t|s_0)$  is not only a household's individual probability of receiving income  $\eta_t(s^t)$ , but also the fraction of the population receiving that income.

By construction, the labor shares sum to one:  $\sum_{y_t \in Y} \Pi_{z_t}(y_t)\eta(y_t, z_t) = 1$ , for all  $z_t$ , where  $\Pi_{z_t}(y_t)$  represents the cross-sectional distribution of idiosyncratic shocks  $y_t$ , conditional on the aggregate shock  $z_t$ . By the law of large numbers, the fraction of agents who draw  $y$  in state  $z$  only depends on  $z$ .<sup>9</sup>

At time 0, the agents are endowed with initial wealth  $\theta_0$ . This wealth represents the value of an agent's share of the Lucas tree producing the dividend flow in units of time 0 consumption, as well as the value of her labor endowment at date 0. We use  $\Theta_0$  to denote the initial joint distribution of wealth and idiosyncratic shocks  $(\theta_0, y_0)$ .

This concludes our description of the physical environment. We start our analysis by studying a two-period example of our model in which prices and allocations can be characterized analytically to develop some intuition for the irrelevance result, when it holds and when it breaks down. We use the model to illustrate the importance of the conditions that we have listed as sufficient for our result: (i) a continuum of agents, (ii) CRRA utility, (iii) idiosyncratic labor income risk that is independent of aggregate risk, (iv) a constant capital share of income, and (v) solvency constraints or borrowing constraints on total financial wealth that are proportional to aggregate income.

<sup>8</sup> See Hammond and Sun [18] for conditions under which a LLN holds with a continuum of random variables.

<sup>9</sup> Our setup nests the baseline model of Heaton and Lucas [20], except for the fact that they allow for the capital share  $\alpha$  to depend on  $z$ .

### 3. A two-period example

We consider a simple example in which the economy only lasts for two periods,  $t = 0, 1$ , and there are only two possible realizations of the aggregate state. In the first period, there is no uncertainty and the initial aggregate state of the economy is fixed at  $z_0 \in Z = \{z_l, z_h\}$ . There is no ex ante heterogeneity in initial endowments or income realizations among agents in period 0. In the second period, an aggregate shock is realized  $z_1 \in Z = \{z_l, z_h\}$  with  $z_1 \sim \phi(z_1|z_0)$ , as well as an idiosyncratic shock  $y_1 \in Y$ , with  $y_1 \sim \pi(y_1|z_1)$ . We use the following notation:  $z^1 = (z_0, z_1)$  and  $s^1 = (s_0, s_1)$  with  $s_0 = z_1$  and  $s_1 = (y_1, z_1)$ . Initial shares in the Lucas tree are given by  $\sigma_{-1} = 1$  for all households; initial financial assets excluding shares in the Lucas tree are given by  $a_{-1} = 0$ .

The aggregate endowment in period 0,  $e_0(z_0)$ , is not subject to randomness, but the aggregate endowment in period 1 depends on the realization of the aggregate shock:  $e_1(z^1) = e_0(z_0)\lambda(z_1)$ . We simply denote the first period aggregate endowment as  $e_0$ .

In period 1, the Lucas tree yields a fraction  $\alpha(z_1)$  of the aggregate endowment in the form of dividends, the remainder,  $1 - \alpha(z_1)$  consists of labor income. The individual labor endowments in the first and second period are given by:  $\eta_0(s_0) = (1 - \alpha(z_0))e_0$  and  $\eta_1(s^1) = (1 - \alpha(z_1))\eta(y_1, z_1)e_0\lambda(z_1)$ .

*Household problem.* In our baseline model (we will refer to this as the Arrow model), households trade two different contingent bonds with payoffs contingent on the two aggregate shocks as well as shares in the Lucas tree. Here, households trade a full set of Arrow securities against aggregate risk, but not against idiosyncratic risk. We use  $a_0(s_0, z_1)$  to denote the holdings of bond that pays off one unit of consumption if  $z_1$  is realized and we use  $q_0(z_0, z_1)$  to denote the price of this security. We refer to these as Arrow prices.

In the remainder, we leave out the obvious dependence on initial wealth  $\theta_0$ , which is trivial in this example since there is no heterogeneity across households in  $\theta_0$ . The household chooses a consumption allocation  $\{c_0(s_0), c_1(s^1)\}$ , contingent bond positions  $\{a_0(s_0, z_1 = z_l), a_0(s_0, z_1 = z_h)\}$  and shares in the Lucas tree  $\sigma_0$  to maximize its expected utility (where for simplicity we have abstracted from time discounting):

$$\max \left\{ u(c_0(s_0)) + \sum_{z_1 \in Z} \sum_{y_1 \in Y} \pi(y_1|z_1)\phi(z_1|z_0)u(c_1(s^1)) \right\}$$

subject to the budget constraint in period 0:  $c_0(s_0) + \sum_{z_1} a_0(s_0, z_1)q_0(z_0, z_1) + \sigma_0(s_0)v_0(z_0) = \theta_0(s_0)$ , and the budget constraint in period 1:  $c_1(s^1) \equiv \eta(s^1) + a_0(s_0, z_1) + \sigma_0(s_0)\alpha e_1(z^1)$ . The market clearing conditions for stock markets, contingent bond markets and goods markets are straightforward:

$$\begin{aligned} \int \sigma_0(s_0) d\Theta_0 &= 1, \\ \int a_0(s_0, z_1) d\Theta_0 &= 0 \quad \text{for all } z_1, \\ \int c_0(s_0) d\Theta_0 &= e_0, \\ \int \sum_{y_1 \in Y} c_1(s^1)\pi(y_1|z_1) d\Theta_0 &= e_1(z_1) \quad \text{for all } z_1. \end{aligned}$$



3.1. Irrelevance result

In this special example, we conjecture that the equilibrium features no-trade. In equilibrium all households simply hold on to their initial shares in the Lucas tree ( $\sigma_0(s_0) = 1$ ), and contingent bond markets are inoperative ( $a_0(s_0, z_1) = 0$  for all  $z_1$ ). In the first period, households consume the average endowment  $c_0(s_0) = e_0$ , and in the second period they consume  $c_1(s^1) = [(1 - \alpha(z_1))\eta(y_1, z_1) + \alpha(z_1)]e_1(z^1)$  for all  $z_1$ .

In this no-trade equilibrium, we can analytically characterize the ratio of the Arrow prices  $q_0(z_0, z_h)/q_0(z_0, z_l)$  in the two different aggregate states. We use  $m(c_1(s^1), c_0(s_0)) = u'(c_1(s^1))/u'(c_0(s_0))$  to denote the household's intertemporal marginal rate of substitution (MRS henceforth) as a function of consumption in the first and second period.

**Proposition 3.1.** *There exists a no-trade equilibrium of the two-period economy without trade in the contingent bond and stock markets. The ratio of the prices of the Arrow securities in the two aggregate states in period 1 is given by:*

$$\frac{q_0(z_0, z_h)}{q_0(z_0, z_l)} = \frac{\phi(z_h|z_0)}{\phi(z_l|z_0)} * \frac{\sum_{y_1} \pi(y_1|z_h)m([(1 - \alpha(z_h))\eta(y_1, z_h) + \alpha(z_h)]\lambda(z_h)e_0, e_0)}{\sum_{y_1} \pi(y_1|z_l)m([(1 - \alpha(z_l))\eta(y_1, z_l) + \alpha(z_l)]\lambda(z_l)e_0, e_0)}. \tag{3}$$

In this equilibrium, all households choose the same exposure to aggregate consumption growth risk by holding one share in the Lucas tree and no contingent bonds. The key to the irrelevance result that idiosyncratic risk does not affect risk premia is that the ratio of Arrow prices in the two aggregate states is the same in the representative agent model and the model with idiosyncratic risk. This being the case, market incompleteness only raises the price of a unit of consumption by the same amount in the good and the bad aggregate state, but it does not change the premium for bearing aggregate risk. In the representative agent model the ratio of state prices is given by:

$$\frac{q_0^{RE}(z_0, z_h)}{q_0^{RE}(z_0, z_l)} = \frac{\phi(z_h|z_0)}{\phi(z_l|z_0)} * \frac{m(\lambda(z_h)e_0, e_0)}{m(\lambda(z_l)e_0, e_0)}.$$

Hence, by comparing this expression with the incomplete markets analogue in Eq. (3), it becomes apparent that the necessary and sufficient condition for irrelevance is given by:

$$\frac{m(\lambda(z_h)e_0, e_0)}{m(\lambda(z_l)e_0, e_0)} = \frac{\sum_{y_1} \pi(y_1|z_h)m([(1 - \alpha(z_h))\eta(y_1, z_h) + \alpha(z_h)]\lambda(z_h)e_0, e_0)}{\sum_{y_1} \pi(y_1|z_l)m([(1 - \alpha(z_l))\eta(y_1, z_l) + \alpha(z_l)]\lambda(z_l)e_0, e_0)}. \tag{4}$$

Three conditions need to be satisfied in order to enforce this equality:

- (i) homogeneity of the period utility function; the CRRA utility function is homogeneous of degree  $1 - \gamma$ ;
- (ii) independence of idiosyncratic shocks and aggregate shocks; this rules out CCV (counter-cyclical cross-sectional variance of income shocks);
- (iii) constant capital income share.

When independence (condition (ii)) is satisfied and when the capital income share is constant (condition (iii)), household consumption in the second period is proportional to the aggregate endowment ( $c_1(s^1) = [(1 - \alpha)\eta(y_1) + \alpha]e_1(z^1)$ ), and the constant of proportionality does not

depend on the aggregate shock, only the idiosyncratic shock. As a result, the ratio of state prices in the incomplete markets economy simplifies to:

$$\frac{\sum_{y_1} \pi(y_1) m([(1-\alpha)\eta(y_1) + \alpha]\lambda(z_h)e_0, e_0)}{\sum_{y_1} \pi(y_1) m([(1-\alpha)\eta(y_1) + \alpha]\lambda(z_l)e_0, e_0)},$$

but we still cannot separate the idiosyncratic shocks  $y_1$  from the aggregate shocks  $z_1 \in \{z_l, z_h\}$ . To do that, we need homogeneity of the utility function (condition (i)). Since  $u$  is homogeneous of degree  $1 - \gamma$ , it follows that the lifetime utility function is homogeneous of degree  $1 - \gamma$  and therefore also homothetic.<sup>10</sup> Furthermore it follows that marginal utility  $u'$  is homogeneous of degree  $-\gamma$ .

From the homotheticity of the lifetime utility function and the homogeneity of degree  $-\gamma$  of the marginal utility function it follows that the ratio of state prices in the representative agent economy, the left-hand side of Eq. (4) can be written as

$$\frac{m(\lambda(z_h)e_0, e_0)}{m(\lambda(z_l)e_0, e_0)} = \frac{m(\lambda(z_h), 1)}{m(\lambda(z_l), 1)} = \left(\frac{\lambda(z_h)}{\lambda(z_l)}\right)^{-\gamma}.$$

Similarly, given the same properties, the ratio of state prices in the incomplete markets economy simplifies to:

$$\begin{aligned} & \frac{\sum_{y_1} \pi(y_1) m([(1-\alpha)\eta(y_1) + \alpha]\lambda(z_h)e_0, e_0)}{\sum_{y_1} \pi(y_1) m([(1-\alpha)\eta(y_1) + \alpha]\lambda(z_l)e_0, e_0)} \\ &= \frac{\lambda(z_h)^{-\gamma} \sum_{y_1} \pi(y_1) m([(1-\alpha)\eta(y_1) + \alpha], 1)}{\lambda(z_l)^{-\gamma} \sum_{y_1} \pi(y_1) m([(1-\alpha)\eta(y_1) + \alpha], 1)} = \left(\frac{\lambda(z_h)}{\lambda(z_l)}\right)^{-\gamma}. \end{aligned}$$

Thus, the ratio of Arrow prices is the same in the representative agent model and the model with idiosyncratic risk. To derive this result, we used the homogeneity of the lifetime utility function, not the time-separability. This suggests that the irrelevance results carries over to Epstein–Zin recursive preferences. We demonstrate in Section A.6 of the separate appendix (see section Supplementary material) that this is indeed the case.

The proportionality of household consumption to the aggregate endowment, combined with the homogeneity assumption on preferences, are key for the asset pricing irrelevance result: market incompleteness raises the price of a unit of consumption by the same proportion in the good and in the bad aggregate state, but leaves their ratio unchanged. These results carry over exactly to the general model in which there is trade in equilibrium.

*Market structure.* To demonstrate our results in the general model we proceed as in the simple example, by first arguing that the equilibrium consumption allocation of households can be decomposed into an idiosyncratic part and an aggregate part that is proportional to the aggregate endowment, and then showing that this characterization of allocations implies the asset pricing results. In order to do so, we use four models, whose main characteristics are summarized in Table 1. The first three models are endowment models with aggregate shocks. The models differ along two dimensions, namely whether agents can trade a full set of Arrow securities against aggregate shocks, and whether agents face idiosyncratic risk, in addition to aggregate risk. Idiosyncratic risk, if there is any, is never directly insurable.

<sup>10</sup> The degree of homogeneity is of course irrelevant, all we need is that  $u$  is homogeneous of some degree, which reduces the class of utility functions we can handle to the CRRA class.

Table 1  
Summary of four models.

Model	Environment		Trading	
	Idiosyncratic shocks	Aggregate shocks	Arrow securities	Bond and stock
Arrow	Yes	Yes	Yes	Yes
Bond	Yes	Yes	No	Yes
<i>BL</i>	No	Yes	Yes	Yes
Bewley	Yes	No	N/A	Yes

Our primary goal is to analyze asset prices in the first model in the table, the Arrow model. This model has idiosyncratic and aggregate risk, as well as incomplete markets (against idiosyncratic shocks). Agents can only insure against aggregate risk. The second model, the Bond model, differs from the Arrow model only in the lack of a full set of aggregate state-contingent claims. We analyzed a two-period version of the Arrow model to derive the irrelevance result above, but we found that there is no-trade in the contingent bond markets. This implies that our irrelevance results for the two-period economy immediately carries over to two-period Bond economy, in which agents can only trade a stock and a non-contingent bond.<sup>11</sup> Because of the result that idiosyncratic household consumption is proportional to aggregate consumption, excess returns in the Bond and Arrow model are priced as in the standard representative agent complete markets Breeden [7]–Lucas [28] (*BL*) model, which does not have idiosyncratic risk.

These results carry over to the infinite horizon economy. However, in the two-period economy, the equilibrium allocations are autarkic. They are not in the infinite horizon economy. The fundamental result underlying our asset pricing findings is that equilibrium allocations in both the Bond and the Arrow model can be found by first determining equilibria in a model with *only* idiosyncratic risk (the *Bewley* model henceforth, the fourth row in the table) and then by simply scaling consumption allocations in that model by the stochastic aggregate endowment. As in the simple model, household consumption is therefore proportional to aggregate consumption, and the asset pricing results immediately follow.

### 3.2. Robustness

We briefly consider the robustness of this irrelevance result in the two-period economy to solvency constraints, cyclical in the labor income share and preference heterogeneity. Basically, as long as households can keep their consumption proportional to aggregate consumption (or a levered version thereof), the irrelevance result goes through.

*Solvency constraints.* The households face solvency constraints in the second period that put a lower bound on net wealth in each state  $z_1$ :

$$a_0(s_0, z_1) + \sigma_0(s_0)\alpha e_1(z^1) \geq K_1(z_1).$$

If we assume that the lower bound is proportional to the aggregate endowment,  $K_1(z_1) = \hat{K}_1 e_1(z^1)$ , then the solvency constraint simplifies to:

<sup>11</sup> With two shocks a bond and a stock complete the markets, so restricting the number of aggregate shocks to two automatically generates the equivalence between the Arrow model and the Bond model. In the general model aggregate shocks can take an arbitrary (finite) number of values, and thus the two models do not trivially deliver the same results.

$$\frac{a_0(s_0, z_1)}{e_1(z^1)} + \sigma_0(s_0)\alpha \geq \hat{K}_1.$$

Since the right-hand side of the constraint does not depend on the aggregate shock realization, households can still choose  $a_0(s_0, z_1) = 0$  in equilibrium, as before. Thus, the irrelevance result survives solvency constraints that are proportional to the aggregate endowment. However, if  $\hat{K}_1(z_1)$  depends on the realization of the aggregate shock, this no-trade equilibrium may not be implementable without violating the solvency constraints in some aggregate state.

*Counter-cyclical labor income share.* If the capital share  $\alpha$  depends on  $z_1$ , then the no-trade equilibrium still obtains but the ratio of state prices in the representative agent and the Arrow models are not identical, as can be seen from Eq. (3), simply because second period consumption is no longer simply proportional to the aggregate endowment; the fraction of the aggregate endowment consumed by households depends on the realization of the aggregate shock.<sup>12</sup> There is one important exception. Suppose the individual labor endowments are given by the following additive specification:

$$\begin{aligned}\eta_0(s_0) &= (1 - \alpha)e_0 + (\alpha - \alpha(z_0))e_0, \\ \eta_1(s^1) &= (1 - \alpha)\eta(y_1, z_1)e_0\lambda(z_1) + (\alpha - \alpha(z_1))e_0\lambda(z_1).\end{aligned}$$

This additive specification effectively eliminates CCV. If idiosyncratic labor income would be affected multiplicatively by the aggregate shock to the labor income share, the cross-sectional distribution of labor income and its risk is affected by the aggregate shock, in effect violating assumption (ii).

In this additive case, the realization of the aggregate shock does not affect the ratio of state prices and we obtain the same equality of the ratios in the representative agent economy and the economy with idiosyncratic risk, because the labor income and capital income share effect offset each other if the households all hold the same amount of shares in the Lucas tree. A less trivial version of this result carries over to the infinite horizon economy, but it requires households to trade in the contingent bond market to offset the labor income fluctuations induced by the term  $\alpha(z_0)e_0$ . The crucial feature of this additive specification is that the cross-sectional distribution of labor income shares remains independent of the aggregate shocks, as can easily be verified.

*Preference heterogeneity.* Finally, we explore what happens if households have different risk attitudes. Suppose there are two classes of agents. The first type is less risk averse than the second type. Then the first type of agents will seek more exposure to aggregate risk in equilibrium than the second type. In order for the irrelevance result to hold we need to assume that the less risk averse group of agents has labor income processes that are more exposed to aggregate risk:  $\eta_1^1(s^1) = (1 - \alpha)\eta(y_1, z_1)\chi_1(z^1)e_1(z^1)$ , and  $\eta_1^2(s^1) = (1 - \alpha)\eta(y_1, z_1)(1 - \chi_1(z^1))e_1(z^1)$  where the properties of  $\chi_1(z^1)$  will be discussed below. Schulhofer-Wohl (unpublished manuscript, 2008) shows that, empirically, risk-tolerant workers hold jobs whose earnings carry more aggregate risk. This is plausible because their shadow valuations of these earnings is higher. In the first period labor income for both groups is given by  $\eta_0^1(s_0) = \chi_0(z_0)e_0$  and  $\eta_0^2(s_0) = (1 - \chi_0(z_0))e_0$ .

<sup>12</sup> Households cannot fully hedge against these aggregate shocks to their labor income share because the contingent bond payoff has to depend on both the idiosyncratic shock  $y$  and the aggregate shock  $z$  to neutralize the effect of the shock to the aggregate labor income share  $z$ .

With this income process the equilibrium consumption allocations in the first period are given by  $c_0^1(s_0) = \chi_0(z_0)e_0$ , and in the second period

$$c_1^1(s^1) \equiv [(1 - \alpha)\eta(y_1, z_1) + \alpha(z_1)]\chi_1(z^1)e_1(z^1),$$

where  $\chi_1(z_0, z_l) < 1 < \chi_1(z_0, z_h)$  is chosen such that the MRS of the stand-in agent who consumes the group consumption lines up with that of the other group:

$$\left(\lambda(z_1)\frac{\chi_1(z^1)}{\chi_0(z_0)}\right)^{-\gamma_1} = \left(\lambda(z_1)\frac{1 - \chi_1(z^1)}{1 - \chi_0(z_0)}\right)^{-\gamma_2}$$

in each aggregate history  $z^1$ . While household consumption for the less risk averse group is not proportional to the aggregate endowment, it is proportional to a levered up version thereof. Hence, the ratio of state prices in the economy without idiosyncratic risk (and thus a representative agent in each household group) simplifies to:

$$\frac{m^1(\chi_1(z_0, z_h)\lambda(z_h)e_0, \chi_0(z_0)e_0)}{m^1(\chi_1(z_0, z_l)\lambda(z_l)e_0, \chi_0(z_0)e_0)} = \left(\frac{\lambda(z_h)\chi_1(z_0, z_h)}{\lambda(z_l)\chi_1(z_0, z_l)}\right)^{-\gamma_1},$$

for agents of type 1. Similarly, the ratio of state prices in the incomplete markets economy with idiosyncratic risk simplifies to:

$$\frac{\sum_{y_1} \pi(y_1)m^1([(1 - \alpha)\eta(y_1) + \alpha]\chi_1(z_0, z_h)\lambda(z_h)e_0, \chi_0(z_0)e_0)}{\sum_{y_1} \pi(y_1)m^1([(1 - \alpha)\eta(y_1) + \alpha]\chi_1(z_0, z_l)\lambda(z_l)e_0, \chi_0(z_0)e_0)} = \left(\frac{\lambda(z_h)\chi_1(z_0, z_h)}{\lambda(z_l)\chi_1(z_0, z_l)}\right)^{-\gamma_1},$$

for agents of type 1, and thus the ratio of state prices  $q_0(z_0, z_h)/q_0(z_0, z_l)$  is the same in the model without and the model with idiosyncratic risk. This does not mean that preference heterogeneity leaves risk premia unaffected; but it implies that it affects risk premia in the same way in both the model with idiosyncratic risk and the model with a representative agent per household group, as first analyzed by Dumas [14].<sup>13</sup>

We conjecture that the same results go through in the infinite horizon economy, as long we can tune the aggregate risk exposure of the labor income processes for each group of households in the same way as in the two-period example. Under these conditions, the least risk averse households as a group are more exposed to aggregate risk than the most risk averse households, but the within group wealth dynamics are invariant with respect to the history of aggregate shocks.<sup>14</sup>

*Summary.* This simple model conveys the main intuition. In spite of the idiosyncratic shocks, households simply want to keep their individual consumption lined up with aggregate consumption, and if conditions (i) and (ii) are satisfied, they can do so. If condition (iii) holds as well, this also implies that risk premia are unaffected.

What is missing in the two-period model? In the no-trade equilibrium of the two-period version of our economy, securities markets are inoperative, and the allocations are autarkic. However, we will show that the basic irrelevance result carries over to equilibria which do feature equilibrium trade in an infinite horizon version of this economy: The zero (contingent) bond trade result, which holds by construction in this simple economy, turns out to carry over to the infinite horizon economy, if aggregate shocks are i.i.d., but the zero stock trade result does not, in

<sup>13</sup> Recent papers that explore preference heterogeneity as a vehicle for time variation in risk premia include Chan and Kogan [9] and Garleanu and Panageas [15].

<sup>14</sup> This being the case, the excess returns on any asset can be priced off the consumption of each group.

that households in the general model will trade the stock for consumption-smoothing purposes. As a result, in the infinite horizon version of this economy, we need to solve a Bewley model to compute the equilibrium allocations (and the risk-free rate).

In addition, the simple two-period model with its autarkic equilibrium *by construction* does not display cross-sectional dispersion or variation in the wealth distribution. Therefore this model is silent about what is needed for the cross-sectional distribution of wealth and consumption to be nonstochastic and how wealth dispersion affects asset prices. In the general model a continuum of agents is needed to obtain an equilibrium with a wealth distribution that is nonstochastic, which is key for the asset pricing irrelevance result since prices depend on the wealth distribution. In the two-period example the irrelevance result can be established even with a finite number of agents, whereas in the general model with finitely many households household-specific shocks move the wealth distribution and thus asset prices. Second, when aggregate shocks are non-i.i.d., interest rates are stochastic. Households in the infinite horizon economy want to hedge against this risk. Different households have different hedging demands depending on their net financial wealth position. This wealth heterogeneity creates trade in the contingent bond markets. In contrast, the two-period model cannot capture these intertemporal hedging motives, and the importance the wealth distribution has for these motives.

We therefore now turn to the theoretical analysis of the general model.

#### 4. General results in the infinite horizon economy

Most of our results for the infinite horizon economy are derived in a de-trended version of our model. This de-trended model features a constant aggregate endowment and a growth-adjusted transition probability matrix. The agents in this de-trended model, discussed now, have stochastic time discount factors.

*Transformation of the growth model into a stationary model.* We transform our growing model into a stationary model with a stochastic time discount rate and a growth-adjusted probability matrix, following Alvarez and Jermann [3]. First, we define growth-deflated consumption allocations (or consumption shares) as

$$\hat{c}_t(s^t) = \frac{c_t(s^t)}{e_t(z^t)} \quad \text{for all } s^t. \quad (5)$$

Next, we define *growth-adjusted* probabilities and the growth-adjusted discount factor as:

$$\hat{\pi}(s_{t+1}|s_t) = \frac{\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\gamma}}{\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\gamma}} \quad \text{and} \quad \hat{\beta}(s_t) = \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\gamma}.$$

Note that  $\hat{\pi}$  is a well-defined Markov matrix in that  $\sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t) = 1$  for all  $s_t$  and that  $\hat{\beta}(s_t)$  is stochastic as long as the original Markov process is not i.i.d. over time. For future reference, we also define the time zero discount factor applied to utils at time  $t$ :  $\hat{\beta}(s^t) = \hat{\beta}(s_0) \cdots \hat{\beta}(s_t)$ , and we note that  $\hat{\beta}(s^t)/\hat{\beta}(s^{t-1}) = \hat{\beta}(s_t)$ . Finally, we let  $\hat{U}(\hat{c})(s^t)$  denote the lifetime expected continuation utility in node  $s^t$ , under the new transition probabilities and discount factor, defined over consumption shares  $\{\hat{c}_t(s^t)\}$

$$\hat{U}(\hat{c})(s^t) = u(\hat{c}_t(s^t)) + \hat{\beta}(s_t) \sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t) \hat{U}(\hat{c})(s^t, s_{t+1}). \quad (6)$$

In the separate appendix we prove that this transformation does not alter the agents' ranking of different consumption streams.

**Proposition 4.1.** *Households rank consumption share allocations in the de-trended model in exactly the same way as they rank the corresponding consumption allocations in the original model with growth: for any  $s^t$  and any two consumption allocations  $c, c'$*

$$U(c)(s^t) \geq U(c')(s^t) \iff \hat{U}(\hat{c})(s^t) \geq \hat{U}(\hat{c}')(s^t)$$

where the transformation of consumption into consumption shares is given by (5).

This result is crucial for demonstrating that equilibrium allocations  $c$  for the stochastically growing model can be found by solving for equilibrium allocations  $\hat{c}$  in the transformed model.

To derive the irrelevance result in the infinite horizon economy, we need to impose more structure on the environment, specifically on the interaction between aggregate and idiosyncratic risk.

*Independence of idiosyncratic shocks from aggregate conditions.* We assume that idiosyncratic shocks are independent of the aggregate shocks. This assumption is crucial for most of the results in this paper.

**Condition 4.1.** Individual endowment shares  $\eta(y_t, z_t)$  are functions of the current idiosyncratic state  $y_t$  only, that is  $\eta(y_t, z_t) = \eta(y_t)$ . Also, transition probabilities of the shocks can be decomposed as  $\pi(z_{t+1}, y_{t+1}|z_t, y_t) = \varphi(y_{t+1}|y_t)\phi(z_{t+1}|z_t)$ .

That is, individual endowment *shares* and the transition probabilities of the idiosyncratic shocks are independent of the aggregate state of the economy  $z$ . In this case, the growth-adjusted probability matrix  $\hat{\pi}$  and the re-scaled discount factor is obtained by adjusting only the transition probabilities for the *aggregate* shock,  $\phi$ , but not the transition probabilities for the idiosyncratic shocks:

$$\hat{\pi}(s_{t+1}|s_t) = \varphi(y_{t+1}|y_t)\hat{\phi}(z_{t+1}|z_t), \quad \text{and} \quad \hat{\phi}(z_{t+1}|z_t) = \frac{\phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}}.$$

Furthermore, the growth-adjusted discount factor only depends on the aggregate state  $z_t$ :

$$\hat{\beta}(z_t) = \beta \sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}. \quad (7)$$

We assume that  $\max_{z_t} \hat{\beta}(z_t) < 1$  in order to insure that lifetime utility remains finite. Evidently this jointly restricts the time discount factor  $\beta$ , the endowment growth process and the coefficient of relative risk aversion  $\gamma$ .

The first part of our analysis (Section 4.4 included) also assumes that the aggregate shocks are independent over time:

**Condition 4.2.** Aggregate endowment growth is i.i.d.:  $\phi(z_{t+1}|z_t) = \phi(z_{t+1})$ .

In this case the growth rate of aggregate endowment is uncorrelated over time, so that the logarithm of the aggregate endowment follows a random walk with drift.<sup>15</sup> As a result, the growth-adjusted discount factor is a constant:  $\hat{\beta}(z_t) = \hat{\beta}$ , since  $\hat{\phi}(z_{t+1}|z_t) = \hat{\phi}(z_{t+1})$ . There are two competing effects on the growth-adjusted discount rate: consumption growth itself makes agents more impatient, while the consumption risk makes them more patient.

Finally, we rule out cyclicity in the labor income share:

**Condition 4.3.** The labor income share  $(1 - \alpha)$  is constant.

We will relax some of these assumption in Section 5. We start our analysis of the infinite horizon economy in Section 4.1 by characterizing equilibria for the Bewley model, a stationary model with a constant aggregate endowment in which agents trade a single discount bond and a stock.<sup>16</sup> This model merely serves as a device to compute equilibrium allocations and prices. In the infinite horizon economy, household consumption will still be proportional to aggregate consumption in that the consumption share does not depend on aggregate shocks, like in the two-period example, but now the consumption shares depend on the household's history of idiosyncratic shocks in potentially complicated ways. The Bewley economy's only purpose is to allow us to compute these consumption shares. Then we turn to the stochastically growing economy (with different market structures), the one whose asset pricing implications we are interested in, and we show that equilibrium consumption allocations from the Bewley model can be implemented as equilibrium allocations in the stochastically growing Arrow model in Section 4.2 and the Bond model in Section A.2, simply by scaling them by the aggregate endowment.

#### 4.1. The Bewley model

In this model the aggregate endowment is constant and equal to 1. Households face idiosyncratic shocks  $y$  that follow a Markov process with transition probabilities  $\varphi(y'|y)$ . The household's preferences over consumption shares  $\{\hat{c}(y^t)\}$  are defined in Eq. (6), with the constant time discount factor  $\hat{\beta}$  as defined in Eq. (7).

##### 4.1.1. Market structure

Agents trade only a riskless one-period discount bond and shares in a Lucas tree that yields safe dividends of  $\alpha$  in every period. The price of the Lucas tree at time  $t$  is denoted by  $v_t$ .<sup>17</sup> The riskless bond is in zero net supply. Each household is indexed by an initial condition  $(\theta_0, y_0)$ , where  $\theta_0$  denotes its wealth (including period 0 labor income) at time 0.

The household chooses consumption  $\{\hat{c}_t(\theta_0, y^t)\}$ , bond positions  $\{\hat{a}_t(\theta_0, y^t)\}$  and share holdings  $\{\hat{\sigma}_t(\theta_0, y^t)\}$  to maximize its normalized expected lifetime utility  $\hat{U}(\hat{c})(s^0)$ , subject to a standard budget constraint<sup>18</sup>:

$$\hat{c}_t(y^t) + \frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t = (1 - \alpha)\eta(y_t) + \hat{a}_{t-1}(y^{t-1}) + \hat{\sigma}_{t-1}(y^{t-1})(\hat{v}_t + \alpha).$$

<sup>15</sup> In Section 5 we show that most of our results survive the introduction of persistence in the growth rates if a complete set of contingent claims on aggregate shocks is traded.

<sup>16</sup> One of the two assets will be redundant for the households, so that this model is a standard Bewley model, as studied by Bewley [6], Huggett [21] or Aiyagari [1].

<sup>17</sup> The price of the tree is nonstochastic due to the absence of aggregate risk.

<sup>18</sup> We suppress dependence on  $\theta_0$  for simplicity whenever there is no room for confusion.



Finally, each household faces one of two types of borrowing constraints. The first one restricts household wealth at the end of the current period. The second one restricts household wealth at the beginning of the next period<sup>19</sup>:

$$\frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t \geq \hat{K}_t(y^t) \quad \text{for all } y^t, \tag{8}$$

$$\hat{a}_t(y^t) + \hat{\sigma}_t(y^t)(\hat{v}_{t+1} + \alpha) \geq \hat{M}_t(y^t) \quad \text{for all } y^t. \tag{9}$$

#### 4.1.2. Equilibrium in the Bewley model

The definition of equilibrium in this model is standard.

**Definition 4.1.** For an initial distribution  $\Theta_0$  over  $(\theta_0, y_0)$ , a competitive equilibrium for the Bewley model consists of trading strategies  $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ , and prices  $\{\hat{R}_t, \hat{v}_t\}$  such that (i) given prices, trading strategies solve the household maximization problem, (ii) the markets clear in all periods  $t$ : the goods market,  $\int \sum_{y^t} \varphi(y^t|y_0)\hat{c}_t(\theta_0, y^t) d\Theta_0 = 1$ , the bonds markets,  $\int \sum_{y^t} \varphi(y^t|y_0)\hat{a}_t(\theta_0, y^t) d\Theta_0 = 0$ , and the stock markets,  $\int \sum_{y^t} \varphi(y^t|y_0)\hat{\sigma}_t(\theta_0, y^t) d\Theta_0 = 1$ .

In the absence of aggregate risk, the bond and the stock are perfect substitutes for households, and no-arbitrage implies that the stock return equals the risk-free rate:  $\hat{R}_t = \frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t}$ . In addition, at these equilibrium prices, household portfolios are indeterminate. Without loss of generality one can therefore focus on trading strategies in which households only trade the stock, but not the bond:  $\hat{a}_t(\theta_0, y^t) \equiv 0$ .

A *stationary equilibrium* in the Bewley model consists of a constant interest rate  $\hat{R}$ , a share price  $\hat{v}$ , optimal household allocations and a time-invariant measure  $\Phi$  over income shocks and financial wealth.

#### 4.2. The Arrow model

We now turn to our main object of interest, the model with aggregate risk. We first consider the Arrow market structure in which households can trade shares of the stock and a complete menu of contingent claims on aggregate shocks. Idiosyncratic shocks are still uninsurable. We demonstrate in this section that the allocations and prices of a stationary Bewley equilibrium can be transformed into equilibrium allocations and prices in the Arrow model with aggregate risk.

##### 4.2.1. Trading

Let  $a_t(s^t, z_{t+1})$  denote the quantity purchased of a security that pays off one unit of the consumption good if aggregate shock in the next period is  $z_{t+1}$ , irrespective of the idiosyncratic shock  $y_{t+1}$ . Its price today is given by  $q_t(z^t, z_{t+1})$ . In addition, households trade shares in the Lucas tree. We use  $\sigma_t(s^t)$  to denote the number of shares a household with history  $s^t = (y^t, z^t)$  purchases today and we let  $v_t(z^t)$  denote the price of one share.

<sup>19</sup> This distinction is redundant in the Bewley model, but it will become meaningful in our models with aggregate risk.

An agent starting period  $t$  with initial wealth  $\theta_t(s^t)$  buys consumption commodities in the spot market and trades securities subject to the usual budget constraint:

$$c_t(s^t) + \sum_{z_{t+1}} a_t(s^t, z_{t+1})q_t(z^t, z_{t+1}) + \sigma_t(s^t)v_t(z^t) \leq \theta_t(s^t). \tag{10}$$

If next period's state is  $s^{t+1} = (s^t, y_{t+1}, z_{t+1})$ , her wealth is given by her labor income, the payoff from the contingent claim purchased in the previous period as well as the value of her position on the stock, including dividends:

$$\begin{aligned} \theta_{t+1}(s^{t+1}) &= (1 - \alpha)\eta(y_{t+1}, z_{t+1})e_{t+1}(z_{t+1}) + a_t(s^t, z_{t+1}) \\ &\quad + \sigma_t(s^t)[v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z_{t+1})]. \end{aligned} \tag{11}$$

In addition to the budget constraints, the households' trading strategies are subject to solvency constraints of one of two types. The first type of constraint imposes a lower bound on the value of the asset portfolio at the end of the period today,

$$\sum_{z_{t+1}} a_t(s^t, z_{t+1})q_t(z^t, z_{t+1}) + \sigma_t(s^t)v_t(z^t) \geq K_t(s^t), \tag{12}$$

while the second type imposes state-by-state lower bounds on net wealth tomorrow,

$$a_t(s^t, z_{t+1}) + \sigma_t(s^t)[v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z_{t+1})] \geq M_t(s^t, z_{t+1}) \quad \text{for all } z_{t+1}. \tag{13}$$

We assume these solvency constraints are at least tight enough to prevent Ponzi schemes. In addition, we impose restrictions on the solvency constraints that make them proportional to the aggregate endowment in the economy:

**Condition 4.4.** We assume that the borrowing constraints only depend on the aggregate history through the level of the aggregate endowment. That is, we assume that:  $K_t(y^t, z^t) = \hat{K}_t(y^t)e_t(z^t)$ , and  $M_t(y^t, z^t, z_{t+1}) = \hat{M}_t(y^t)e_{t+1}(z^{t+1})$ .

If the constraints did not have this feature in a stochastically growing economy, the constraints would become more or less binding as the economy grows. The borrowing constraints that are typically used in the incomplete markets literature satisfy this condition (see, e.g., [20]). The definition of an equilibrium is completely standard (see Section A.1 of the separate appendix).

Instead of working with the model with aggregate risk, we transform the Arrow model into a stationary model. As we are about to show, the equilibrium allocations and prices in the de-trended model are the same as the allocations and prices in a stationary Bewley equilibrium.

4.2.2. *Equilibrium in the de-trended Arrow model*

We use hatted variables to denote the variables in the stationary model. Households rank consumption shares  $\{\hat{c}_t\}$  in exactly the same way as original consumption streams  $\{c_t\}$ . Dividing the budget constraint (10) by  $e_t(z^t)$  and using Eq. (11) yields the deflated budget constraint:

$$\begin{aligned} \hat{c}_t(s^t) + \sum_{z_{t+1}} \hat{a}_t(s^t, z_{t+1})\hat{q}_t(z^t, z_{t+1}) + \sigma_t(s^t)\hat{v}_t(z^t) \\ \leq (1 - \alpha)\eta(y_t) + \hat{a}_{t-1}(s^{t-1}, z_t) + \sigma_{t-1}(s^{t-1})[\hat{v}_t(z^t) + \alpha], \end{aligned} \tag{14}$$

where we have defined the deflated Arrow positions as  $\hat{a}_t(s^t, z_{t+1}) = a_t(s^t, z_{t+1})/e_{t+1}(z^{t+1})$  and prices as  $\hat{q}_t(z^t, z_{t+1}) = q_t(z^t, z_{t+1})\lambda(z_{t+1})$ . The deflated stock price is given by  $\hat{v}_t(z^t) =$

$v_t(z^t)/e_t(z^t)$ . Similarly, by deflating the solvency constraints (12) and (13), using condition (4.4), yields:

$$\sum_{z_{t+1}} \hat{a}_t(s^t, z_{t+1}) \hat{q}_t(z^t, z_{t+1}) + \sigma_t(s^t) \hat{v}_t(z^t) \geq \hat{K}_t(y^t), \tag{15}$$

$$\hat{a}_t(s^t, z_{t+1}) + \sigma_t(s^t) [\hat{v}_{t+1}(z^{t+1}) + \alpha] \geq \hat{M}_t(y^t) \quad \text{for all } z_{t+1}. \tag{16}$$

The conditional probabilities simplify due to condition (4.1). The asset market clearing conditions are exactly the same as before. In the stationary model, the household maximizes  $\hat{U}(\hat{c})(s_0)$  by choosing consumption, Arrow securities and shares of the Lucas tree, subject to the budget constraint (14) and the solvency constraint (15) or (16) in each node  $s^t$ . The definition of a competitive equilibrium in the de-trended Arrow model is straightforward.

**Definition 4.2.** For initial aggregate state  $z_0$  and distribution  $\Theta_0$  over  $(\theta_0, y_0)$ , a competitive equilibrium for the de-trended Arrow model consists of trading strategies  $\{\hat{a}_t(\theta_0, s^t, z_{t+1})\}$ ,  $\{\hat{\sigma}_t(\theta_0, s^t)\}$ ,  $\{\hat{c}_t(\theta_0, s^t)\}$  and prices  $\{\hat{q}_t(z^t, z_{t+1})\}$ ,  $\{\hat{v}_t(z^t)\}$  such that (i) given prices, trading strategies solve the household maximization problem, (ii) the goods market clears, that is  $\int \sum_{y^t} \phi(y^t|y_0) \hat{c}_t(\theta_0, s^t) d\Theta_0 = 1$  holds for all  $z^t$ , (iii) the stock market clears  $\int \sum_{y^t} \phi(y^t|y_0) \hat{\sigma}_t(\theta_0, s^t) d\Theta_0 = 1$ , and the bond market clears:  $\int \sum_{y^t} \phi(y^t|y_0) \hat{a}_t(\theta_0, s^t, z_{t+1}) d\Theta_0 = 0$  for all  $z_{t+1} \in Z$ .

The first order conditions and complementary slackness conditions, together with the appropriate transversality condition, are listed in the separate appendix in Section A.1.1. These are necessary and sufficient conditions for optimality on the household side. Now we are ready to establish the equivalence between equilibria in the Bewley model and in the Arrow model.

4.2.3. *Equivalence results*

The equilibria in the Bewley model can be mapped into equilibria of the stochastically growing Arrow model.

**Theorem 4.1.** *If  $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$  and  $\{\hat{R}_t, \hat{v}_t\}$  is an equilibrium of the Bewley economy, then  $\{a_t(\theta_0, s^t, z_{t+1})\}$ ,  $\{\sigma_t(\theta_0, s^t)\}$ ,  $\{c_t(\theta_0, s^t)\}$  and  $\{q_t(z^t, z_{t+1})\}$ ,  $\{v_t(z^t)\}$ , is an equilibrium of the stochastically growing Arrow economy with*

$$\begin{aligned} c_t(\theta_0, s^t) &= \hat{c}_t(\theta_0, y^t) e_t(z^t), \\ \sigma_t(\theta_0, s^t) &= \hat{\sigma}_t(\theta_0, y^t), \\ a_t(\theta_0, s^t, z_{t+1}) &= \hat{a}_t(\theta_0, y^t) e_{t+1}(z^{t+1}), \\ v_t(z^t) &= \hat{v}_t e_t(z^t), \\ q_t(z^t, z_{t+1}) &= \frac{1}{\hat{R}_t} * \frac{\hat{\phi}(z_{t+1})}{\lambda(z_{t+1})} = \frac{1}{\hat{R}_t} * \frac{\phi(z_{t+1}) \lambda(z_{t+1})^{-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}) \lambda(z_{t+1})^{1-\gamma}}. \end{aligned} \tag{17}$$

The proof is given in Appendix A, but here we provide its main intuition. Conjecture that the equilibrium prices of Arrow securities in the de-trended Arrow model are given by:

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t}. \tag{18}$$

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An unconstrained household’s Euler equation for the Arrow securities is given by (see Section A.1.1 in the appendix)

$$1 = \frac{\hat{\beta}(s_t)}{\hat{q}_t(z^t, z_{t+1})} \sum_{s^{t+1}|s^t, z_{t+1}} \hat{\pi}(s_{t+1}|s_t) \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))}.$$

But, in light of conditions (4.1) and (4.2) and given our conjecture that consumption allocations in the de-trended Arrow model only depend on idiosyncratic shock histories  $y^t$  and not on  $s^t = (y^t, z^t)$ , this Euler equation reduces to

$$\begin{aligned} 1 &= \frac{\hat{\beta}\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})} \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \\ &= \hat{\beta}\hat{R}_t \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}, \end{aligned} \tag{19}$$

where we used the conjectured form of prices in (18). This is exactly the Euler equation for bonds in the Bewley model. Since Bewley equilibrium consumption allocations satisfy this condition, they therefore satisfy the Euler equation in the de-trended Arrow model, if prices are of the form (18). The proof in Appendix A shows that a similar argument applies for the Euler equation with respect to the stock (under the conjectured stock prices), and also shows that for agents whose solvency constraints binds the Lagrange multipliers on the constraints in the Bewley equilibrium are also valid Lagrange multipliers for the constraints in the de-trended Arrow model. Once one has established that allocations and prices of a Bewley equilibrium are an equilibrium in the de-trended Arrow model, one simply needs to scale up allocation and prices by the appropriate growth factors to obtain the equilibrium prices and allocations in the stochastically growing Arrow model, as stated in the theorem.

By summing over aggregate states tomorrow on both sides of Eq. (18), we find that the risk-free rate in the de-trended Arrow model coincides with that of the Bewley model:

$$\hat{R}_t^A = \frac{1}{\sum_{z_{t+1}} \hat{q}_t(z^t, z_{t+1})} = \hat{R}_t. \tag{20}$$

Once we have determined risk-free interest rates for the de-trended model,  $\hat{R}_t^A = \hat{R}_t$ , we can back out the implied interest rate for the original growing Arrow model, using (17) in the previous theorem.<sup>20</sup>

**Corollary 4.1.** *If equilibrium risk-free interest rates in the de-trended Arrow model are given by (20), equilibrium risk-free interest rates in the Arrow model with aggregate risk are given by*

$$R_t^A = \frac{1}{\sum_{z_{t+1}} q_t(z^t, z_{t+1})} = \hat{R}_t \frac{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{-\gamma}}. \tag{21}$$

The theorem implies that we can solve for an equilibrium in the Bewley model of Section 4.1 (and, in particular, a stationary equilibrium), including risk-free interest rates  $\hat{R}_t$ , and we can

<sup>20</sup> The dependence of  $\hat{R}_t^A$  on time  $t$  is not surprising since, for an arbitrary initial distribution of assets  $\Theta_0$ , we cannot expect the equilibrium to be stationary.

deduce the equilibrium allocations and prices for the Arrow model from those in the Bewley model, using the mapping described in Theorem 4.1. The key to this result is that households in the Bewley model face exactly the same Euler equations as the households in the de-trended version of the Arrow model.

This theorem has several important implications. First, we will use this equivalence result to show below that asset prices in the Arrow model are identical to those in the representative agent model, except for a lower risk-free interest rate (and a higher price/dividend ratio for stocks).<sup>21</sup> Second, the existence proofs in the literature for stationary equilibria in the Bewley model directly carry over to the stochastically growing model.<sup>22</sup> Third, the moments of the wealth distribution vary over time but proportionally to the aggregate endowment. If the initial wealth distribution in the de-trended model corresponds to an invariant distribution in the Bewley model, then for example the ratio of the mean to the standard deviation of the wealth distribution is constant in the Arrow model with aggregate risk as well. Finally, an important result of the previous theorem is that, in the Arrow equilibrium, the trade of Arrow securities is simply proportional to the aggregate endowment:  $a_t(\theta_0, s^t, z_{t+1}) = \hat{a}_t(\theta_0, y^t)e_{t+1}(z^{t+1})$ , or, equivalently, in the de-trended Arrow model households choose not to make their contingent claims purchases contingent on next period's aggregate shock:  $\hat{a}_t(\theta_0, s^t, z_{t+1}) = \hat{a}_t(\theta_0, y^t)$ .

Furthermore, since in the Bewley model, without loss of generality, we could focus only on equilibria without trade in bonds,  $\hat{a}_t(\theta_0, y^t) = 0$ , we can focus on the situation where Arrow securities are not traded at all:  $a_t(\theta_0, s^t, z_{t+1}) = 0$ . This no-trade result for contingent claims suggests that our equivalence result will carry over to environments with more limited asset structures. That is what we show in the next section.

### 4.3. The Bond model

We now discuss how our main irrelevance result extends to the Bond model in which households can only trade a stock and a single one-period bond  $b_t(\cdot)$ ; the markets for contingent claims are shut down. In the de-trended Arrow model, we demonstrated that contingent claims positions were in fact uncontingent and equal to the bond position in the Bewley equilibrium:  $\hat{a}_t(\theta_0, s^t, z_{t+1}) = \hat{a}_t(\theta_0, y^t)$ . That immediately implies that we can implement the Arrow equilibrium allocations in the Bond model as well. In this equilibrium, there is no-trade in the non-contingent bond market:  $b_t(\theta_0, s^t) = 0$ .<sup>23</sup> Section A.2 in the separate appendix formally establishes the equivalence of equilibria in the Bond model and the Bewley model by showing that the optimality conditions in the de-trended Arrow and the de-trended Bond model are identical.

Our results show that one can solve for equilibria in a standard Bewley model and then map this equilibrium into an equilibrium for both the Arrow model and the Bond model *with* aggregate risk. The risk-free interest rate and the price of the Lucas tree are the same in the stochastic Arrow and Bond models. Finally, without loss of generality, we can restrict attention to equilibria in

<sup>21</sup> The fact that the risk-free interest rate is lower comes directly from the fact that interest rates in the Bewley model are lower than in the corresponding representative agent model without aggregate risk.

<sup>22</sup> See, e.g., Huggett [21], Aiyagari [1] or Miao [32] for (elements of) existence proofs. Uniqueness of a stationary equilibrium is much harder to establish. Our equivalence result shows that for any stationary Bewley equilibrium there exists a corresponding Arrow equilibrium in the model with aggregate risk. Our result does not rule out other Arrow equilibria.

<sup>23</sup> We also argued in the Bewley model that we could focus, without loss of generality, on equilibria in which  $\hat{a}_t(\theta_0, y^t) \equiv 0$ . In the Bond model, the bond positions have to be zero for the result to go through.

which bonds are not traded; consequently transaction costs in the bond market would not change our results. Transaction costs in the stock market of course would (see Section 5). In addition, this implies that our result is robust to the introduction of short-sale constraints imposed on stocks and bonds separately, because agents choose not to trade bonds in equilibrium, as long as these short-sale constraints are not tighter than the solvency constraints.

In both the Arrow and the Bond model, households do not have a motive for trading bonds, unless there are tighter short-sale constraints on stocks than on bonds. We do not deal with this case. In addition, the no-trade result depends critically on the i.i.d. assumption for aggregate shocks, as we will show in Section 5. If the aggregate shocks are not i.i.d., agents want to hedge against the implied shocks to interest rates. We will show in Section 5 that these interest rate shocks look like taste shocks in the de-trended model. But, first, we compare the asset pricing implications of the equilibria just described in the Arrow and the Bond models to those emerging from the *BL* (standard representative agent) model.

#### 4.4. Asset pricing implications

This section uses these results to show that the multiplicative risk premium on a claim to aggregate consumption in the Arrow model—and thus the Bond model—equals the risk premium in the representative agent model. Uninsurable idiosyncratic income risk only lowers the risk-free rate, because the ratio of any two equilibrium state prices is the same as in the representative agent *BL* economy, exactly as in the two-period example.

*Consumption-CAPM.* The benchmark model of consumption-based asset pricing is the representative agent *BL* model. The representative agent owns a claim to the aggregate ‘labor’ income stream  $\{(1 - \alpha)e_t(z^t)\}$  and she can trade a stock (a claim to the dividends  $\alpha e_t(z^t)$  of the Lucas tree), a bond and a complete set of Arrow securities.<sup>24</sup>

First, we show that the Breeden–Lucas Consumption-CAPM also prices excess returns on the stock in the Bond model and the Arrow model. Let  $R^s$  denote the return on a claim to aggregate consumption.

**Lemma 4.1.** *The BL Consumption-CAPM prices excess returns in the Arrow model and the Bond model. In equilibrium, in both models, the following Euler equation is satisfied:*

$$E_t[(R_{t+1}^s - R_t)\beta(\lambda_{t+1})^{-\gamma}] = 0.$$

Appendix A shows that this result follows directly from subtracting the Euler equations for the bond from that for the stock in both models.

*Risk premia.* The equilibrium risk premium in the Arrow and the Bond model is identical to the one in the representative agent model. While the risk-free rate is higher in the representative agent model than in the Arrow and Bond model, and consequently the price of the stock is correspondingly lower, the multiplicative risk premium is the same in all three models and it is constant across states of the world.

<sup>24</sup> See separate appendix available online for a complete description of the model.

In order to demonstrate our main result we first show that the stochastic discount factors that price stochastic payoffs in the representative agent model and the Arrow (and thus the Bond) model only differ by a non-random multiplicative term, equal to the ratio of (growth-deflated) risk-free interest rates in the two models. In what follows we use the superscript *RE* to denote equilibrium variables in the representative agent model.

**Proposition 4.2.** *The equilibrium stochastic discount factor in the Arrow and the Bond model given by  $m_{t+1}^A = m_{t+1}^{RE} \kappa_t$  where the non-random multiplicative term is given  $\kappa_t = \frac{\hat{R}_t^{RE}}{\hat{R}_t} \geq 1$ , and  $m_{t+1}^{RE}$  is the stochastic discount factor in the representative agent model.*

$\kappa_t$  is straightforward to compute, because it only involves the equilibrium risk-free interest rates from the stationary version of the representative agent model,  $\hat{R}_t^{RE}$ , and the equilibrium interest rates from the Bewley model,  $\hat{R}_t$ .

The proof that risk premia are identical in the representative agent model and the Arrow as well as the Bond model follows directly from the previous decomposition of the stochastic discount factor. Let  $R_{t,j}[\{d_{t+k}\}]$  denote the  $j$ -period holding return on a claim to the endowment stream  $\{d_{t+k}\}_{k=0}^{\infty}$  at time  $t$ . Consequently  $R_{t,1}[1]$  is the gross risk-free rate and  $R_{t,1}[\alpha e_{t+k}]$  is the one-period holding return on a  $k$ -period strip of the aggregate endowment (a claim to  $\alpha$  times the aggregate endowment  $k$  periods from now). Thus  $R_{t,1}[\{\alpha e_{t+k}\}]$  is the one-period holding return on an asset (such as a stock) that pays  $\alpha$  times the aggregate endowment in all future periods. Finally, we define the multiplicative risk premium as the ratio of the expected return on stocks and the risk-free rate:  $1 + v_t = E_t R_{t,1}[\{\alpha e_{t+k}\}] / R_{t,1}[1]$ . With this notation in place, we can state now our main result.

**Theorem 4.2.** *The multiplicative risk premium in the Arrow model and Bond model equals that in the representative agent model  $1 + v_t^A = 1 + v_t^B = 1 + v_t^{RE}$ , and is constant across states of the world.*

Thus, the extent to which households smooth idiosyncratic income shocks in the Arrow model or in the Bond model has absolutely no effect on the size of risk premia; it merely lowers the risk-free rate. Market incompleteness does not generate any dynamics in the conditional risk premia either: the conditional risk premium is constant.

## 5. Extensions of the general results

In Section 5.1, we investigate how robust our results are to the assumption that aggregate shocks are i.i.d. over time, which implies that the growth rate of the aggregate endowment is i.i.d. over time. In Section 5.2, we show that our results go through in the case of permanent idiosyncratic shocks. These extensions are important, especially because the two-period model does not allow us to analyze these cases. Section 5.3 shows that the results we derived in the case of cyclical capital shares still hold in the infinite horizon economy.

The intuition that we built in the two-period example applies here too. In all of these cases, the irrelevance result goes through if households can manage to keep their consumption proportional to aggregate consumption (i.e. keep their consumption shares independent of aggregate shocks) by trading the limited menu of available securities.

5.1. Non-i.i.d. aggregate shocks

Assume that the aggregate shock  $z$  follows a first order Markov chain characterized by the transition matrix  $\phi(z'|z) > 0$ . So far we studied the special case in which  $\phi(z'|z) = \phi(z')$ . Recall that the growth-adjusted Markov transition matrix and time discount factor are given by

$$\hat{\phi}(z'|z) = \frac{\phi(z'|z)\lambda(z')^{1-\gamma}}{\sum_{z'} \phi(z'|z)\lambda(z')^{1-\gamma}} \quad \text{and} \quad \hat{\beta}(z) = \beta \sum_{z'} \phi(z'|z)\lambda(z')^{1-\gamma}.$$

Thus if  $\phi$  is serially correlated,  $\hat{\phi}$  is too, and the discount factor  $\hat{\beta}$  depends on the current aggregate state of the world. In the de-trended model, aggregate endowment shocks act as a common taste shock that renders all households more or less impatient. Since this shock affects all households in the same way, they will not be able to insure against it. As a result, this shock affects the price/dividend ratio and the interest rate, but it leaves the risk premium unaltered. In contrast to our previous results, however, now there is trade in Arrow securities in equilibrium, so the equivalence between equilibria in the Arrow and the Bond model breaks down.

5.1.1. Stationary Bewley model

We adhere to the same strategy, and we show that the equilibrium allocations and prices from a stationary version of the model, the Bewley model can be implemented, after appropriate scaling by the aggregate endowment, as equilibria in the stochastically growing model. Since the time discount factors are subject to aggregate shock, we first have to choose an appropriate nonstochastic time discount factor for this Bewley model. We choose a sequence of non-random time discount factors that assures that Bewley equilibrium allocations satisfy the time zero budget constraint in the model with aggregate shocks when the initial wealth distribution  $\Theta_0$  in the two models coincide. Let

$$\hat{\beta}_{0,\tau}(z^\tau|z_0) = \hat{\beta}(z_0)\hat{\beta}(z_1)\cdots\hat{\beta}(z_\tau)$$

denote the time discount factor between period 0 and period  $\tau + 1$ , given by the product of one-period time discount factors. We define the average (across aggregate shocks) time discount factor between period 0 and  $t$  as:

$$\tilde{\beta}_t = \sum_{z^{t-1}|z_0} \hat{\phi}(z^{t-1}|z_0)\hat{\beta}_{0,t-1}(z^{t-1}|z_0), \quad t \geq 1, \tag{22}$$

where  $\hat{\phi}(z^{t-1}|z_0)$  is the probability distribution over  $z^{t-1}$  induced by  $\phi(z'|z)$ . If aggregate shocks are i.i.d., then we have that  $\tilde{\beta}_t = \hat{\beta}^t$ , as before. Since  $z_0$  is a fixed initial condition, we chose not to index  $\tilde{\beta}_t$  by  $z_0$  to make sure it is understood that  $\tilde{\beta}_t$  is nonstochastic.

In order to construct equilibrium allocations in the stochastically growing model, we will show that equilibrium allocations and interest rates in the Bewley model with a sequence of non-random time discount factors  $\{\tilde{\beta}_t\}_{t=1}^\infty$  can be implemented as equilibrium allocations and interest rates for the *actual* Arrow model with stochastic discount factors. The crucial adjustment in this mapping is to rescale the risk-free interest rate in proportion to the taste shock  $\hat{\beta}(z)$ .

To understand the effect of these aggregate taste shocks on the time discount rate to be used in the Bewley model, we use a simple example:



**Example 5.1.** Suppose that  $\hat{\beta}(z) = e^{-\hat{\rho}(z)}$  is lognormal and i.i.d., where  $\hat{\rho}(z)$  has mean  $\hat{\rho}$  variance  $\sigma^2$ . Define the average  $t$ -period time discount rate  $\tilde{\rho}_t$  by  $\tilde{\beta}_t = e^{-\tilde{\rho}_t}$ . Then the average one-period discount rate used in the Bewley model is given by:

$$\frac{\tilde{\rho}_t}{t} = \hat{\rho} - \frac{1}{2}\sigma^2 \quad \text{for any } t \geq 1.$$

Thus the presence of taste shocks ( $\sigma^2 > 0$ ) in the de-trended Arrow model induces a discount rate  $\tilde{\rho}$  to be used in the Bewley model that is lower than the mean discount rate  $\hat{\rho}$  because of the risk associated with the taste shocks.

This example suggests that these taste shocks lower the risk-free interest rates compared to those in model without taste shocks (which originate from aggregate endowment shocks in the stochastically growing model).

As before, we denote the Bewley equilibrium by  $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$  and  $\{\hat{R}_t, \hat{v}_t\}$  Bewley equilibrium allocations and prices, for a given sequence of time discount factors  $\{\tilde{\beta}_t\}$ . The risk-free rate has to equal the stock return in each period, to rule out arbitrage:

$$\hat{R}_t = \frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t}.$$

Only the total wealth positions in the Bewley model are uniquely pinned down. Without loss of generality, we focus on the case where  $\hat{a}_t(\theta_0, y^t) = 0$  for all  $y^t$ . We now argue that the allocation  $\{\hat{c}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$  can be made into an Arrow equilibrium, and in the process show why we need to choose the specific discount factor sequence in (22) for the Bewley model. We need some additional notation for state prices.

$$\tilde{Q}_{t,\tau} = \prod_{j=0}^{\tau-t-1} \hat{R}_{t+j}^{-1} = \frac{1}{\hat{R}_{t,\tau}} \tag{23}$$

denotes the Bewley equilibrium price of one unit of consumption to be delivered at time  $\tau$ , in terms of consumption at time  $t$ . By convention  $\tilde{Q}_\tau = \tilde{Q}_{0,\tau}$  and  $\tilde{Q}_{\tau,\tau} = 1$  for all  $\tau$ .  $\hat{R}_{t,\tau}$  is the gross risk-free interest rate between period  $t$  and  $\tau$  in the Bewley equilibrium.

5.1.2. Arrow model

In contrast to the Bewley model, the de-trended Arrow model features aggregate shocks to the time discount factor  $\hat{\beta}$ . These need to be reflected in prices. We therefore propose state-dependent equilibrium prices for the de-trended Arrow model, then we show that the Bewley equilibrium allocations, in turn, satisfy the Euler equations when evaluated at these prices, and they also satisfy the intertemporal budget constraint in the de-trended Arrow model. This implies that, absent binding solvency constraints, the Bewley equilibrium can be made into an equilibrium of the de-trended Arrow model, and thus, after the appropriate scaling, into an equilibrium of the original Arrow model. Finally, we discuss potentially binding solvency constraints and the Bond model.

We conjecture that the Arrow–Debreu prices in the deflated Arrow model are given by

$$\hat{Q}_t(z^t | z_0) = \hat{\phi}(z^t | z_0) \tilde{Q}_t \frac{\hat{\beta}_{0,t-1}(z^{t-1} | z_0)}{\tilde{\beta}_t} = \hat{\phi}(z^t | z_0) \frac{\hat{\beta}_{0,t-1}(z^{t-1} | z_0)}{\tilde{\beta}_t \hat{R}_{0,t}}, \tag{24}$$

where  $\tilde{Q}_t$  was defined above as the time 0 price of consumption in period  $t$  in the Bewley model. The prices of the (one-period ahead) Arrow securities are then given by:

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{Q}_{t+1}(z^{t+1}|z_0)}{\hat{Q}_t(z^t|z_0)} = \hat{\beta}(z_t)\hat{\phi}(z_{t+1}|z_t) \frac{1}{\hat{R}_t} \frac{\tilde{\beta}_t}{\tilde{\beta}_{t+1}} = \hat{q}_t(z_{t+1}|z_t), \tag{25}$$

where we used the fact that  $\frac{1}{\hat{R}_t} = \frac{\hat{R}_{0,t}}{\hat{R}_{0,t+1}}$ . Arrow prices are Markovian in  $z_t$ , since  $\hat{R}_t$  and  $(\tilde{\beta}_t, \tilde{\beta}_{t+1})$  are all deterministic. Eq. (25) implies that interest rates in the de-trended Arrow model are given by

$$\hat{R}_t^A(z_t) \equiv \sum_{z_{t+1}} \hat{q}_t(z_{t+1}|z_t) = \hat{R}_t \frac{\tilde{\beta}_{t+1}}{\hat{\beta}(z_t)\tilde{\beta}_t}. \tag{26}$$

Interest rates now depend on the current aggregate state of the world  $z_t$ . Finally, we also conjecture that the stock price in the de-trended Arrow model satisfies:

$$\hat{v}_t(z^t) = \hat{v}_t(z_t) = \sum_{z_{t+1}} \hat{\phi}(z_{t+1}|z_t) \left( \frac{\hat{v}_{t+1}(z_{t+1}) + \alpha}{\hat{R}_t^A(z_t)} \right). \tag{27}$$

Armed with these conjectured prices we can now prove the following result.

**Lemma 5.1.** *Absent solvency constraints, the household Euler equations are satisfied in the Arrow model at the Bewley allocations  $\{\hat{c}_{t+1}(y^t, y_{t+1})\}$  and Arrow prices  $\{\hat{q}_t(z_{t+1}|z_t)\}$  given by (25).*

*Trading.* Next, we spell out which asset trades support the Bewley equilibrium consumption allocations in the de-trended Arrow model, and we show that the implied contingent claims positions clear the market for Arrow securities.

At any point in time and any node of the event tree, the position of Arrow securities at the beginning of the period, plus the value of the stock position cum dividends, has to finance the value of excess demand from today into the infinite future. Thus, the Arrow securities position implied by the Bewley equilibrium allocation  $\{\hat{c}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$  is given by<sup>25</sup>:

$$\begin{aligned} \hat{a}_{t-1}(\theta_0, y^{t-1}, z^t) &= \hat{c}_t(\theta_0, y^t) - (1 - \alpha)\eta(y_t) \\ &+ \sum_{\tau=t+1}^{\infty} \sum_{z^\tau, y^\tau} \hat{Q}_\tau(z^\tau|z_t)\varphi(y^\tau|y^t)(\hat{c}_\tau(\theta_0, y^\tau) - (1 - \alpha)\eta(y_\tau)) \\ &- \hat{\sigma}_{t-1}(\theta_0, y^{t-1})[\hat{v}_t(z_t) + \alpha]. \end{aligned} \tag{28}$$

**Proposition 5.1.** *The contingent claims positions implied by the Bewley allocations in (28) clear the Arrow securities markets, that is*

$$\int \sum_{y^{t-1}} \varphi(y^{t-1}|y_0)\hat{a}_{t-1}(\theta_0, y^{t-1}, z^t) d\Theta_0 = 0 \quad \text{for all } z^t.$$

<sup>25</sup> We will verify below that the price of the stock in the de-trended Arrow model satisfies  $\hat{v}_t(z^t) = \hat{v}_t(z_t)$ .

Since the wealth from stock holdings at the beginning of the period

$$\hat{\sigma}_{t-1}(\theta_0, y^{t-1})[\hat{v}_t + \alpha] = \alpha \hat{\sigma}_{t-1}(\theta_0, y^{t-1}) \sum_{\tau=t}^{\infty} \tilde{Q}_{t,\tau}$$

has to finance future excess consumption demand in the Bewley equilibrium, we can state the contingent claims positions as:

$$\begin{aligned} \hat{a}_{t-1}(\theta_0, y^{t-1}, z_t) &= \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} (\hat{Q}_\tau(z^\tau | z_t) - \tilde{Q}_{t,\tau}) \sum_{y^\tau} \varphi(y^\tau | y^t) (\hat{c}_\tau(\theta_0, y^\tau) \\ &\quad - (1 - \alpha)\eta(y_\tau)) - \hat{\sigma}_{t-1}(\theta_0, y^{t-1})\alpha \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} (\hat{Q}_\tau(z^\tau | z_t) - \tilde{Q}_{t,\tau}). \end{aligned}$$

The Arrow securities positions held by households are used to hedge against the interest rate shocks that govern the difference between the stochastic  $\hat{Q}_\tau(z^\tau | z_t)$  and the deterministic  $\tilde{Q}_{t,\tau}$ . If aggregate endowment growth is i.i.d., there are no taste shocks in the de-trended Arrow model, and from (26) we see that the interest rates are deterministic. The gap between  $\hat{Q}_\tau(z^\tau | z_t)$  and  $\tilde{Q}_{t,\tau}$  is zero and no Arrow securities are traded in equilibrium, confirming the results in Section 4.2.

In order to close our argument, we need to show that no initial wealth transfers between individuals are required for this implementation. In other words, the initial Arrow securities position  $\hat{a}_{-1}(\theta_0, y^{-1}, z_0)$  implied by (28) at time 0, is zero for all households. To do so we proceed in two steps. First, we show that the average time zero state prices in the Arrow model coincide with the state prices in the Bewley model. For this result to hold our particular choice of time discount factors  $\{\hat{\beta}_t\}$  for the Bewley model is crucial.

**Lemma 5.2.** *The conjectured prices for the Arrow model in (24) and the prices in the Bewley model defined in (23) satisfy  $\sum_{z^\tau} \hat{Q}_\tau(z^\tau | z_0) = \tilde{Q}_\tau$ .*

Finally, using this result we can establish that no wealth transfers are necessary to implement the Bewley equilibrium as equilibrium in the de-trended Arrow model.

**Lemma 5.3.** *The Arrow securities position at time 0 given in (28) is zero:  $\hat{a}_{-1}(\theta_0, y^{-1}, z_0) = 0$ .*

Having established that the Bewley equilibrium is an equilibrium for the de-trended Arrow model with the same initial wealth distribution  $\Theta_0$ , the following theorem obviously results.

**Theorem 5.2.** *If  $\{\hat{c}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$  and  $\{\hat{R}_t, \hat{v}_t\}$  is an equilibrium of the Bewley economy in which households have a sequence of time discount factors  $\{\tilde{\beta}_t\}$ , then  $\{a_t(\theta_0, s^t, z_{t+1})\}$ ,  $\{\sigma_t(\theta_0, s^t)\}$ ,  $\{c_t(\theta_0, s^t)\}$  and  $\{q_t(z^t, z_{t+1})\}$ ,  $\{v_t(z^t)\}$ , is an equilibrium of the stochastically growing Arrow economy with*

$$\begin{aligned} c_t(\theta_0, s^t) &= \hat{c}_t(\theta_0, y^t) e_t(z^t), \\ \sigma_t(\theta_0, s^t) &= \hat{\sigma}_t(\theta_0, y^t), \\ a_t(\theta_0, s^t, z_{t+1}) &= \hat{a}_t(\theta_0, y^t, z_{t+1}) e_{t+1}(z^{t+1}) \quad \text{with } \hat{a}_t \text{ defined in (28),} \end{aligned}$$

$$v_t(z_t) = \sum_{z_{t+1}} \frac{\hat{\phi}(z_{t+1}|z_t)}{\lambda(z_{t+1})} \left[ \frac{v_{t+1}(z_{t+1}) + \alpha e_{t+1}(z_{t+1})}{\hat{R}_t^A(z_t)} \right],$$

$$\hat{R}_t^A(z_t) = \frac{\hat{R}_t \tilde{\beta}_{t+1}}{\hat{\beta}(z_t) \tilde{\beta}_t},$$

$$q_t(z^t, z_{t+1}) = \frac{\hat{q}_t(z^t, z_{t+1})}{\lambda(z_{t+1})} = \frac{1}{\hat{R}_t^A(z_t)} * \frac{\phi(z_{t+1}|z_t) \lambda(z_{t+1})^{-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}|z_t) \lambda(z_{t+1})^{1-\gamma}}.$$

*Risk premia.* Of course, this implies that our baseline irrelevance result for risk premia survives the introduction of non-i.i.d. aggregate shocks, provided that a complete menu of aggregate state-contingent securities is traded.<sup>26</sup> These aggregate taste shocks only affect interest rates and price/dividend ratios, not risk premia. When agents in the transformed model become more impatient, the interest rises and the price/dividend ratio decreases, but the conditional expected excess return is unchanged.

*Solvency constraints.* So far, we have abstracted from binding solvency constraints. Previously, we assumed that the solvency constraints satisfy  $K_t(s^t) = \hat{K}_t(y^t) e_t(z^t)$  and  $M_{t+1}(s^{t+1}) = \hat{M}_t(y^t) e_t(z^{t+1})$ . The allocations computed in the stationary model using  $\hat{K}_t(y^t)$  and  $\hat{M}_t(y^t)$  as solvency constraints, satisfy a modified version of the solvency constraints  $K_t(s^t)$  and  $M_{t+1}(s^{t+1})$ .

**Proposition 5.2.** *The allocations from Theorem 5.2 satisfy the modified solvency constraints:*

$$K_t^*(s^t) = K_t(s^t) + \sum_{z_{t+1}} q_t(z^t, z_{t+1}) a_t(\theta_0, s^t, z_{t+1}) + \sigma_t(\theta_0, s^t) [v_t(z_t) - \hat{v}_t e_t(z^t)],$$

$$M_{t+1}^*(s^{t+1}) = M_{t+1}(s^{t+1}) + a_t(\theta_0, s^t, z_{t+1}) + \sigma_t(\theta_0, s^t) [v_{t+1}(z_{t+1}) - \hat{v}_{t+1} e_{t+1}(z^{t+1})],$$

where  $\hat{v}_t$  is the (deterministic) Bewley equilibrium stock price.

If the allocations satisfy the constraints in the stationary Bewley model, they satisfy the modified solvency constraints in the actual Arrow model, but not the ones we originally specified, because of the nonzero state-contingent claims positions.<sup>27</sup> Nevertheless, in principle, one could reverse-engineer a sequence of auxiliary solvency constraints such that in the actual equilibrium the modified version of the auxiliary constraints coincides with the actual constraints we want to impose,  $K_t(s^t)$ .

Finally, in the Bond model, our previous equivalence result no longer holds, since with predictability in aggregate consumption growth households trade state-contingent claims in the equilibrium of the Arrow model. Unless the aggregate shock can only take two values, the market

<sup>26</sup> Note that

$$\kappa_t(z_t) = \frac{\hat{R}_t^{RE}(z_t)}{\hat{R}_t^A(z_t)} = \frac{\tilde{\beta}_t}{\hat{R}_t \tilde{\beta}_{t+1}} = \kappa_t$$

is still deterministic, and thus the proofs of Section 4.4 go through unchanged.

<sup>27</sup> These violations of the original constraints are completely due to the impact of interest rate shocks on the value of the asset portfolio.

structure of the Bond model prevents them from doing so, and thus our implementation and irrelevance results in this model are not robust to the introduction of non-i.i.d. aggregate endowment growth.

### 5.2. Permanent idiosyncratic shocks

Our results do not hinge on mean reversion in household income. In fact, we can also consider a version of our model with permanent idiosyncratic income shocks. In this economy, household income in logs follows a random walk:

$$\log \eta_t = \log \eta_{t-1} + y_t, \quad (29)$$

where  $y_t \sim N(-\frac{1}{2}\sigma_y^2, \sigma_y^2)$  is normally distributed and independent over time and across households. Also, we assume that  $\eta_{-1} = 1$  for all households. This implies that the level of household labor income at  $t$  is given by  $\eta_t(y^t) = \exp(\sum_{s=0}^t y_s)$ , and thus, at any time  $t$ , we know that  $E(\eta_t(y^t)) = 1$ , where  $E$  is the expectation at time 0 for a single household. By the law of large numbers this expression is also equal to the cross-sectional expectation. As before, the stock pays a fraction  $\alpha$  of the aggregate endowment  $e_t(z^t)$  as dividends and the shares are in constant unit supply. In this environment, our aggregation and irrelevance result survives, under the same assumptions as in the paper. This follows because the implementation in the proof we provided above does not rely on mean reversion in labor income at all.<sup>28</sup>

In addition, one can show that in the case of permanent shocks as modeled above, an autarkic equilibrium in which agents do not trade and simply consume their labor income and dividends exists if and only if  $\alpha = 0$ , that is, if all assets are in zero net supply. Therefore, even in the case of permanent shocks, equilibria in our environment need to feature trade in financial markets, unless  $\alpha = 0$ . How does this result relate to the autarky equilibrium in CD? To generate such an equilibrium with  $\alpha > 0$  they reverse-engineer the aggregate dividend process and the household labor income process. The resulting labor income process depends on the aggregate dividend process and it is *not* consistent with the standard random walk specification in Eq. (29).<sup>29</sup>

Finally, we conclude by describing the extension of our irrelevance result to an environment with a counter-cyclical labor income share. It turns out that the results we derived in the two-period economy carry over.

### 5.3. Counter-cyclical aggregate labor income share

The relevant question is whether households can keep their own consumption in line with the aggregate endowment in spite of the cyclical variation in the aggregate labor and capital income share by taking positions in the contingent bond markets. If they can do so, then the variation in the aggregate labor income share will have no bearing on allocations and prices. The intuition that we derived in the two-period economy extends to the infinite horizon economy: In the standard multiplicative case ( $\eta(y, z) = \eta(y)(1 - \alpha(z))$ ), the Bewley equilibrium that we

<sup>28</sup> In the case of permanent shocks the existence of (stationary) equilibrium in the Bewley model becomes an issue. This problem can be circumvented by assuming that households have exponentially distributed lifespans, as in Yaari [39].

<sup>29</sup> As explained by CD, their specification of the household labor income process is not empirically motivated (see bottom of p. 228); the sole focus is tractability. Their household income process in logs features some mean reversion whose magnitude depends on the size of aggregate dividend payments.

have derived cannot be implemented in the de-trended Arrow economy, because the required contingent bond payoff depends on the idiosyncratic shock realization  $y_{t+1}$  at  $t + 1$ :

$$a_t(\theta_0, s^t, y_{t+1}, z_{t+1}) = \hat{a}_t(\theta_0, y^t)e_{t+1}(z^{t+1}) - (\alpha - \alpha(z_{t+1}))\eta(y_{t+1}) \\ + \hat{\sigma}_t(\theta_0, y^t)(\alpha - \alpha(z_{t+1})).$$

Hence, the irrelevance result breaks down. To replicate the payoffs of a claim to aggregate consumption, households need to issue bonds with payoffs that are contingent on the idiosyncratic shocks. However, if we assume that the household's labor income as a share of the aggregate endowment is given by the standard component  $\eta(y)$ , which depends only on the realization of the idiosyncratic event  $y$  and a second component which depends on the realization of the aggregate shock  $z$ :

$$\eta(y, z) = \alpha - \alpha(z) + (1 - \alpha)\eta(y),$$

then the required contingent bond payoff does not depend on the idiosyncratic shock  $y_{t+1}$  at  $t + 1$ :

$$a_t(\theta_0, s^t, z_{t+1}) = \hat{a}_t(\theta_0, y^t)e_{t+1}(z^{t+1}) - (\alpha - \alpha(z_{t+1})) + \hat{\sigma}_t(\theta_0, y^t)(\alpha - \alpha(z_{t+1})).$$

Households with more than average financial wealths short contingent consumption claims for high aggregate consumption growth states and go long in contingent consumption claims for low aggregate consumption growth states. Households with financial wealth below the average do the opposite. Section A.5 in the separate appendix spells out the details. This means that we have implemented the same Bewley equilibrium in the de-trended Arrow economy with stochastic capital shares. Of course, the implementation of the Bewley equilibrium now requires trade in the contingent claim markets. The irrelevance result breaks down in the Bond economy, except in the case with only two aggregate states. This is not surprising. In the standard multiplicative case, the cross-sectional distribution of the (demeaned) household income shares is *not independent of aggregate shocks*, whereas in the additive case it is. And our results go through in the additive case, but not in the multiplicative case. Essentially, in the multiplicative case, a large labor share (small  $\alpha(z_t)$ ) increases the variance of the idiosyncratic shocks, exactly like CCV.

Finally, in Section A.4 of the separate appendix, we also show that the endowment economy we considered is isomorphic to a production economy with a single input, namely labor. All of the results go through.

## 6. Conclusion

We have derived conditions under which the history of a household's idiosyncratic shocks has no effect on his portfolio choice, even in the presence of binding solvency constraints. This portfolio irrelevance result directly implies the risk premium irrelevance result. Since all households bear the same amount of aggregate risk in equilibrium, the history of aggregate shocks does not affect equilibrium prices and allocations. However, only a weaker version of this irrelevance result survives in economies with a predictable component in aggregate consumption growth (see, e.g., Bansal and Yaron [5]), because this component creates heterogeneous hedging demands that do depend on the household's history of idiosyncratic shocks. This deserves to be explored further.

Other than predictability in aggregate consumption growth shocks, there are two main ways around our results. One approach is to concentrate aggregate risk by forcing some households

out of the stock market altogether. This is the approach adopted in the literature on limited participation (see, e.g., Guvenen (unpublished manuscript, 2003), Vissing-Jorgensen [36], Gomes and Michaelides [16], Chien, Cole, and Lustig [10] and Attanasio, Banks, and Tanner [4]). The second approach consists of concentrating labor income risk in recessions. Recently Krusell and Smith [23] and Storesletten, Telmer, and Yaron [34] have argued that models with idiosyncratic income shocks and incomplete markets can generate an equity premium that is substantially larger than the CCAPM if there is counter-cyclical cross-sectional variance (CCV) in labor income shocks. Storesletten, Telmer, and Yaron [34] argue that this condition is satisfied in the data, although it is not clear the CCV in the data is strong enough to explain equity risk premia at reasonable levels of risk aversion. Our paper demonstrates analytically that CCV in labor income or limited stock market participation is not only sufficient, but *necessary* to make uninsurable idiosyncratic income shocks potentially useful for explaining the equity premium, even in the presence of binding solvency constraints.

**Appendix A**

**Proof of Proposition 3.1.** We briefly derive this result. We can construct an equilibrium in the two-period economy from the household’s first order conditions. These are necessary and sufficient. We use  $\lambda_0(s_0)$  and  $\lambda_1(s^1)$  to denote the Lagrangian multipliers on the budget constraints at time 0 and time 1. The first order conditions are given by the first order condition for consumption at time 0, at time 1, and for contingent bond purchases in each aggregate state of the world:

$$\begin{aligned} \lambda_0(s_0) &= u'(c_0(s_0)), \\ \lambda_1(s^1) &= u'(c_1(s^1))\beta\pi(y_1|y_0)\phi(z_1|z_0), \\ q_0(z_0, z_1 = z_l)\lambda_0(s_0) &= \sum_{y_1} \lambda_1(s_0, (y_1, z_1 = z_l)), \\ q_0(z_0, z_1 = z_h)\lambda_0(s_0) &= \sum_{y_1} \lambda_1(s_0, (y_1, z_1 = z_h)), \\ \lambda_0(s_0)v_0(z_0) &= \sum_{z_1} \sum_{y_1} \lambda_1(s^1)\alpha e_1(z^1). \end{aligned}$$

By combining these conditions, we obtain the following expression for the contingent bond prices and for the stock price:

$$\begin{aligned} q_0(z_0, z_1 = z_l) &= \phi(z_1 = z_l|z_0) \sum_{y_1} \pi(y_1|z_1)m(c_1(s^1), c_0(s_0)), \\ q_0(z_0, z_1 = z_h) &= \phi(z_1 = z_h|z_0) \sum_{y_1} \pi(y_1|z_1)m(c_1(s^1), c_0(s_0)), \\ v_0(z_0) &= \sum_{z_1} \sum_{y_1} \pi(y_1|z_1)\phi(z_1|z_0)m(c_1(s^1), c_0(s_0))\alpha(z_1)e_1(z^1), \end{aligned}$$

where  $m$  denotes the households intertemporal marginal rate of substitution:

$$m(c_1(s^1), c_0(s_0)) = \frac{u'(c_1(s^1))}{u'(c_1(s^1))}.$$

Finally, given that we have no initial heterogeneity, all households choose the same asset allocation in period zero, and so we get zero bond holdings, identical stock holdings, and each household consumes the aggregate endowment in period zero, while they consume the labor endowment plus the capital endowment in period 1:

$$\begin{aligned}\sigma_0(s_0) &= 1, \\ a_0(s_0, z_1) &= 0, \\ c_0(s_0) &= e_0, \\ c_1(s^1) &\equiv [(1 - \alpha(z_1))\eta(y_1, z_1) + \alpha(z_1)]e_1(z^1).\end{aligned}$$

Inserting this into the asset prices for Arrow securities yields the following expression for the Arrow prices and for the stock price:

$$\begin{aligned}q_0(z_0, z_1) &= \phi(z_1|z_0) \sum_{y_1} \pi(y_1|z_1) m([(1 - \alpha(z_1))\eta(y_1, z_1) + \alpha(z_1)]\lambda(z_1)e_0, e_0), \\ v_0(z_0) &= \sum_{z_1} \sum_{y_1} \pi(y_1|z_1) \phi(z_1|z_0) \alpha(z_1) e_1(z^1) \\ &\quad \times m([(1 - \alpha(z_1))\eta(y_1, z_1) + \alpha(z_1)]\lambda(z_1)e_0, e_0).\end{aligned}$$

Hence, there is no equilibrium trade in securities markets in this simple two-period economy.  $\square$

**Proof of Proposition 4.1.** We use  $U(c)(s^t)$  to denote the continuation utility of an agent from consumption stream  $c$ , starting at history  $s^t$ . This continuation utility follows the simple recursion

$$U(c)(s^t) = u(c_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) U(c)(s^t, s_{t+1}),$$

where it is understood that  $(s^t, s_{t+1}) = (z^t, z_{t+1}, y^t, y_{t+1})$ . Divide both sides by  $e_t(s^t)^{1-\gamma}$  to obtain

$$\frac{U(c)(s^t)}{e_t(z^t)^{1-\gamma}} = u(\hat{c}_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \frac{e_{t+1}(z^{t+1})^{1-\gamma}}{e_t(z^t)^{1-\gamma}} \frac{U(c)(s^t, s_{t+1})}{e_{t+1}(z^{t+1})^{1-\gamma}}.$$

Define a new continuation utility index  $\hat{U}(\cdot)$  as follows:

$$\hat{U}(\hat{c})(s^t) = \frac{U(c)(s^t)}{e_t(z^t)^{1-\gamma}}.$$

It follows that

$$\begin{aligned}\hat{U}(\hat{c})(s^t) &= u(\hat{c}_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \lambda(z_{t+1})^{1-\gamma} \hat{U}(\hat{c})(s^t, s_{t+1}) \\ &= u(\hat{c}_t(s^t)) + \hat{\beta}(s_t) \sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t) \hat{U}(\hat{c})(s^t, s_{t+1}).\end{aligned}$$

Thus it follows, for two consumption streams  $c$  and  $c'$ , that

$$U(c)(s^t) \geq U(c')(s^t) \quad \text{if and only if} \quad \hat{U}(\hat{c})(s^t) \geq \hat{U}(\hat{c}')(s^t),$$

i.e., the household orders original and growth-deflated consumption streams in exactly the same way.  $\square$



**Proof of Theorem 4.1.** The proof consists of two parts. In a first step, we argue that the Bewley equilibrium allocations and prices can be transformed into an equilibrium for the de-trended Arrow model, and in a second step, we argue that by scaling the allocations and prices by the appropriate endowment (growth) factors, we obtain an equilibrium of the stochastically growing Arrow model. Step 1: Take allocations and prices from a Bewley equilibrium,  $\{\hat{c}_t(y^t), \hat{a}_t(y^t), \hat{\sigma}_t(y^t)\}$ ,  $\{\hat{R}_t, \hat{v}_t\}$  and let the associated Lagrange multipliers on the solvency constraints be given by

$$\hat{\beta}^t \varphi(y^t | y_0) u'(\hat{c}_t(y^t)) \hat{\mu}(y^t) \geq 0,$$

for the constraint in (8) and

$$\hat{\beta}^t \varphi(y^t | y_0) u'(\hat{c}_t(y^t)) \hat{\kappa}_t(y^t) \geq 0,$$

for the constraint in (9). The first order conditions (which are necessary and sufficient for household optimal choices together with the complementary slackness and transversality conditions) in the Bewley model, once combined to the Euler equations, are given by:

$$1 = \hat{R}_t \hat{\beta} \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} + \hat{\mu}_t(y^t) + \hat{R}_t \hat{\kappa}_t(y^t) \tag{30}$$

$$= \hat{\beta} \left[ \frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t} \right] \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} + \hat{\mu}_t(y^t) + \left[ \frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t} \right] \hat{\kappa}_t(y^t). \tag{31}$$

The corresponding Euler equations for the de-trended Arrow model, evaluated at the Bewley equilibrium allocations and Lagrange multipliers  $\hat{\mu}(y^t)$  and  $\hat{\kappa}_t(y^t) \hat{\phi}(z_{t+1})$ , read as (see (43) and (44)):

$$1 = \frac{\hat{\beta}(s_t)}{\hat{q}_t(z^t, z_{t+1})} \sum_{s^{t+1}|s^t, z_{t+1}} \hat{\pi}(s_{t+1}|s_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} + \hat{\mu}_t(y^t) + \frac{\hat{\kappa}_t(y^t) \hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})} \quad \forall z_{t+1},$$

$$1 = \hat{\beta}(s_t) \sum_{s^{t+1}|s^t} \hat{\pi}(s_{t+1}|s_t) \left[ \frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} \right] \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} + \hat{\mu}_t(y^t) + \hat{\kappa}_t(y^t) \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[ \frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} \right].$$

Evaluated at the conjectured prices,

$$\hat{v}_t(z^t) = \hat{v}_t, \tag{32}$$

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t}, \tag{33}$$

and using the independence and i.i.d. assumptions, which imply

$$\hat{\pi}(s_{t+1}|s_t) = \varphi(y_{t+1}|y_t) \hat{\phi}(z_{t+1}),$$

$$\hat{\beta}(s_t) = \hat{\beta}$$

these Euler equations can be restated as follows:

$$1 = \frac{\hat{\beta}\hat{R}_t}{\hat{\phi}(z_{t+1})} \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t)\hat{\phi}(z_{t+1}) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} + \hat{\mu}_t(y^t) + \hat{R}_t\hat{\kappa}_t(y^t), \quad (34)$$

$$1 = \hat{\beta} \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t) \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[ \frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t} \right] \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} + \hat{\mu}_t(y^t) + \hat{\kappa}_t(y^t) \left[ \frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t} \right] \sum_{z_{t+1}} \hat{\phi}(z_{t+1}), \quad (35)$$

which are, given that  $\sum_{z_{t+1}} \hat{\phi}(z_{t+1}) = 1$ , exactly the Euler conditions (30) and (31) of the Bewley model and hence satisfied by the Bewley equilibrium allocations. A similar argument applies to the complementary slackness conditions, which for the Bewley model read as

$$\hat{\mu}_t(y^t) \left[ \frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t - \hat{K}_t(y^t) \right] = 0, \quad (36)$$

$$\hat{\kappa}_t(y^t) [\hat{a}_t(y^t) + \hat{\sigma}_t(y^t)(\hat{v}_{t+1} + \alpha) - \hat{M}_t(y^t)] = 0, \quad (37)$$

and for the de-trended Arrow model, evaluated at Bewley equilibrium allocations and conjectured prices, read as

$$\hat{\mu}_t(y^t) \left[ \frac{\hat{a}_t(y^t)}{\hat{R}_t} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) + \hat{\sigma}_t(y^t) - \hat{K}_t(y^t) \right] = 0,$$

$$\hat{\kappa}_t(y^t) [\hat{a}_t(y^t) + \hat{\sigma}_t(y^t)(\hat{v}_{t+1} + \alpha) - \hat{M}_t(y^t)] = 0/\hat{\phi}(z_{t+1}).$$

Again, the Bewley equilibrium allocations satisfy the complementary slackness conditions in the de-trended Arrow model. The argument is exactly identical for the transversality conditions. Finally, we have to check whether the Bewley equilibrium allocation satisfies the de-trended Arrow budget constraints. Plugging in the allocations yields:

$$\hat{c}_t(s^t) + \frac{\hat{a}_t(y^t)}{\hat{R}_t} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) + \hat{\sigma}_t(y^t)\hat{v}_t \leq \eta(y_t) + \hat{a}_{t-1}(y^{t-1}) + \sigma_{t-1}(y^{t-1})(\hat{v}_t + \alpha), \quad (38)$$

which is exactly the budget constraint in the Bewley model. Thus, given the conjectured prices, the Bewley equilibrium allocations are optimal in the de-trended Arrow model.

Since the market clearing conditions for assets and consumption goods coincide in the two models, Bewley allocations satisfy the market clearing conditions in the de-trended Arrow model. Thus we conclude that the Bewley equilibrium allocations, together with prices (32) and (33) are an equilibrium in the de-trended Arrow model. Step 2: Now, we need to show that an equilibrium of the de-trended Arrow model is, after appropriate scaling, an equilibrium in the stochastically growing model, but this was established in Section 4.2.2, in which we showed that by with the transformations  $\hat{c}_t(s^t) = c_t(s^t)/e_t(z^t)$ ,  $\hat{a}_t(s^t, z_{t+1}) = a_t(s^t, z_{t+1})/e_{t+1}(z^{t+1})$ ,  $\hat{\sigma}_t(s^t) = \sigma_t(s^t)$ ,  $\hat{q}_t(z^t, z_{t+1}) = q_t(z^t, z_{t+1})\lambda(z_{t+1})$ ,  $\hat{v}_t(z^t) = v_t(z^t)/e_t(z^t)$  household problems and market clearing conditions in the de-trended and the stochastically growing Arrow model coincide. □

**Proof of Lemma 4.1.** The stock return is defined as:

$$R_{t+1}^s(z^{t+1}) = \frac{v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z_{t+1})}{v_t(z^t)}$$

Subtracting the two Euler equations (34)–(35) in the Arrow model and (53)–(54) in the Bond model yields, in both cases

$$\hat{\beta} \sum_{z^{t+1}|z^t} \hat{\phi}(z_{t+1}) \left[ \frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} - \hat{R}_t(z^t) \right] = 0.$$

Using the fact that  $\hat{v}_{t+1}(z^{t+1}) = v_{t+1}(z^{t+1})/e_{t+1}(z_{t+1})$  and the definition of  $\hat{\phi}(z_{t+1})$  and  $\hat{\beta}$ , as well as (20) yields

$$\beta \sum_{z^{t+1}|z^t} \phi(z_{t+1}) \lambda(z_{t+1})^{-\gamma} [R_{t+1}^s(z^{t+1}) - R_t(z^t)] = 0$$

or in short

$$E_t \{ \beta \lambda(z_{t+1})^{-\gamma} [R_{t+1}^s - R_t] \} = 0.$$

Thus the representative agent stochastic discount factor  $\beta \lambda(z_{t+1})^{-\gamma}$  prices the excess return of stocks over bonds in both the Arrow and the Bond model. Note that in the Arrow model (but not in the Bond model) this stochastic discount factor any excess return  $R_{t+1}^i - R_t$  as long as the returns only depend on the aggregate state  $z_{t+1}$ . □

**Proof of Proposition 4.2.** From Theorem 4.1 we know that in the Arrow model equilibrium prices for Arrow securities are given by:

$$q_t^A(z^t, z_{t+1}) = \frac{\hat{q}_t^A(z^t, z_{t+1})}{\lambda(z_{t+1})} = \frac{\hat{\phi}(z_{t+1})}{\lambda(z_{t+1}) \hat{R}_t^A}$$

whereas in the representative agent model equilibrium prices for Arrow securities are given by:

$$q_t(z^t, z_{t+1}) = \frac{\hat{q}_t(z^t, z_{t+1})}{\lambda(z_{t+1})} = \hat{\beta} \frac{\hat{\phi}(z_{t+1})}{\lambda(z_{t+1})}$$

so that

$$\frac{q_t^A(z^t, z_{t+1})}{q_t(z^t, z_{t+1})} = \frac{1}{\hat{\beta} \hat{R}_t^A} = \frac{\hat{R}^{RE}}{\hat{R}_t^A} = \kappa_t \geq 1 \tag{39}$$

where

$$\hat{R}_t^{RE} = \hat{R}^{RE} = \frac{1}{\sum_{z_{t+1}} \hat{q}_t(z^t, z_{t+1})} = \frac{1}{\sum_{z_{t+1}} \hat{\beta} \hat{\phi}(z_{t+1})} = \frac{1}{\hat{\beta}}$$

is the risk-free interest rate in the de-trended representative agent model. Note that the multiplicative factor  $\kappa_t$  may depend on time since  $\hat{R}_t^A$  may, but is nonstochastic, since  $\hat{R}_t^A = \hat{R}_t$  (the risk-free interest rate in the de-trended Arrow model equals that in the Bewley model, which is evidently nonstochastic). Since interest rates in the Bewley model are (weakly) smaller than in the representative agent model,  $\kappa_t \geq 1$ . Eq. (39) implies that the stochastic discount factor in the Arrow model equals the SDF in the representative agent model, multiplied by  $\kappa_t$ :

$$m_{t+1}^A(z^{t+1}) = m_{t+1}^{RE}(z^{t+1}) \kappa_t.$$

Finally, since the stochastic discount factor for the Arrow model is also a valid stochastic discount factor in the Bond model (although not necessarily the unique valid stochastic discount factor), the previous result also applies to the Bond model.  $\square$

**Proof of Theorem 4.2.** Remember that we defined the multiplicative risk premium in the main text as:

$$1 + v_t = \frac{E_t R_{t,1}[\{e_{t+k}\}]}{R_{t,1}[1]}.$$

We use  $m_{t,t+k} = m_{t+1} \cdot m_{t+2} \cdots m_{t+k}$  to denote the  $k$ -period ahead pricing kernel (with convention that  $m_{t,t} = 1$ ), such that  $E_t(d_{t+k}m_{t,t+k})$  denotes the price at time  $t$  of a random payoff  $d_{t+k}$ . Note that whenever there is no room for confusion we suppress the dependence of variables on  $z^t$ . First, note that the multiplicative risk premium on a claim to aggregate consumption can be stated as a weighted sum of risk premia on strips (as shown by Alvarez and Jermann [3]). By definition of  $R_{t,1}[\{e_{t+k}\}]$  we have

$$\begin{aligned} R_{t,1}[\{e_{t+k}\}] &= \frac{\sum_{k=1}^{\infty} E_{t+1}m_{t+1,t+k}\alpha e_{t+k}}{\sum_{k=1}^{\infty} E_t m_{t,t+k}\alpha e_{t+k}} \\ &= \frac{1}{\sum_{j=1}^{\infty} E_t m_{t,t+j} e_{t+j}} * \sum_{k=1}^{\infty} \frac{E_{t+1}m_{t+1,t+k} e_{t+k}}{E_t m_{t,t+k} e_{t+k}} * E_t m_{t,t+k} e_{t+k} \\ &= \sum_{k=1}^{\infty} \frac{E_{t+1}m_{t+1,t+k} e_{t+k}}{E_t m_{t,t+k} e_{t+k}} \frac{E_t m_{t,t+k} e_{t+k}}{\sum_{j=1}^{\infty} E_t m_{t,t+j} e_{t+j}} \\ &= \sum_{k=1}^{\infty} \omega_k R_{t,1}[e_{t+k}] \end{aligned}$$

where the non-random weights  $\omega_k$  are given by

$$\omega_k = \frac{E_t m_{t,t+k} e_{t+k}}{\sum_{j=1}^{\infty} E_t m_{t,t+j} e_{t+j}}.$$

Thus

$$1 + v_t = \frac{E_t R_{t,1}[\{e_{t+k}\}]}{R_{t,1}[1]} = \sum_{k=1}^{\infty} \omega_k \frac{E_t R_{t,1}[e_{t+k}]}{R_{t,1}[1]} \tag{40}$$

and it is sufficient to show that the multiplicative risk premium  $E_t R_{t,1}[e_{t+k}]/R_{t,1}[1]$  on all  $k$ -period strips of aggregate consumption (a claim to the Lucas tree’s dividend in period  $k$  only, not the entire stream) is the same in the Arrow model as in the representative agent model. First, we show that the one-period ahead conditional strip risk premia are identical:

$$\frac{E_t \frac{E_{t+1}e_{t+1}}{E_t[m_{t+1}^A e_{t+1}]}}{\frac{1}{E_t[m_{t+1}^A]}} = \frac{E_t \frac{E_{t+1}\lambda_{t+1}}{E_t[m_{t+1}^{\lambda_{t+1}}]}}{\frac{1}{E_t[m_{t+1}^A]}} = \frac{E_t \frac{E_{t+1}\lambda_{t+1}}{E_t[m_{t+1}^{RE} \lambda_{t+1}]}}{\frac{1}{E_t[m_{t+1}^{RE}]}}.$$

The first equality follows from dividing through by  $e_t$ . The second equality follows from the expression for  $m^A$  in Proposition 4.2:  $m_{t+1}^A = m_{t+1}^{RE} \kappa_t$ . Next we repeat the argument for the risk premium of a  $k$ -period strip:

$$\frac{E_t R_{t,1}^A [e_{t+k}]}{E_t R_{t,1}^A [1]} = \frac{E_t \frac{E_{t+1}[m_{t+1,t+k}^A e_{t+k}]}{E_t [m_{t+1,t+k}^A e_{t+k}]}}{\frac{1}{E_t [m_{t+1}^A]}} = \frac{E_t \frac{\kappa_{t+1} \cdots \kappa_{t+k-1}}{\kappa_t \kappa_{t+1} \cdots \kappa_{t+k-1}} \frac{E_{t+1}[m_{t+1,t+k}^{RE} e_{t+k}]}{E_t [m_{t+1,t+k}^{RE} e_{t+k}]}}{\frac{1}{\kappa_t E_t [m_{t+1}^{RE}]}} = \frac{E_t R_{t,1}^{RE} [e_{t+k}]}{R_{t,1}^{RE} [1]}$$

and thus risk premia on all  $k$ -period consumption strips in the Arrow model coincide with those in the representative agent model. But then (40) implies that the multiplicative risk premium in the two models coincide as well.  $\square$

**Proof of Lemma 5.1.** Absent binding solvency constraints the Euler equation in the Bewley model read as

$$\begin{aligned} 1 &= \frac{\tilde{\beta}_{t+1}}{\tilde{\beta}_t} \hat{R}_t \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \\ &= \frac{\tilde{\beta}_{t+1}}{\tilde{\beta}_t} \frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t} \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \end{aligned}$$

while in the Arrow model the Euler equations for Arrow securities are given by

$$1 = \frac{\hat{\beta}(z_t) \hat{\phi}(z_{t+1}|z_t)}{\hat{q}_t(z^t, z_{t+1})} \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}.$$

With conjectured Arrow securities prices  $\hat{q}_t(z^t, z_{t+1}) = \hat{\beta}(z_t) \hat{\phi}(z_{t+1}|z_t) \frac{1}{\hat{R}_t} \frac{\tilde{\beta}_t}{\tilde{\beta}_{t+1}}$  these equations obviously coincide with the bond Euler equation in the Bewley model, and thus the Bewley equilibrium allocation satisfies the Euler equations for Arrow securities. A similar argument applies to the Euler equation for stocks:

$$\begin{aligned} 1 &= \hat{\beta}(z_t) \sum_{z^{t+1}|z^t} \hat{\phi}(z_{t+1}|z_t) \left[ \frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} \right] \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \\ &= \frac{\tilde{\beta}_{t+1}}{\tilde{\beta}_t} \hat{R}_t \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}. \end{aligned}$$

The state-contingent interest rate in this model is given by:

$$\frac{1}{\hat{R}_t^A(z_t)} = \hat{\beta}(z_t) \frac{\tilde{\beta}_t}{\hat{R}_t \tilde{\beta}_{t+1}}$$

which can easily be verified from Eq. (25).  $\square$

**Proof of Proposition 5.1.** We need to check that Arrow securities positions defined in (28) satisfy the market clearing condition

$$\int \sum_{y^{t-1}} \varphi(y^{t-1}|y_0) \hat{a}_{t-1}(\theta_0, y^{t-1}, z^t) d\Theta_0 = 0 \quad \text{for all } z^t$$

for each  $z_t$ . By the goods market clearing condition in the Bewley model we have, since total labor income makes up a fraction  $1 - \alpha$  of total income

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$$\begin{aligned} & \int \sum_{y^{t-1}} \varphi(y^{t-1}|y_0) (\hat{c}_t(y^t, \theta_0) - (1 - \alpha)\eta(y_t)) d\Theta_0 \\ &= \int \sum_{y^{t-1}} \varphi(y^{t-1}|y_0) \sum_{y^t|y^{t-1}} \varphi(y^t|y^{t-1}) (\hat{c}_t(y^t, \theta_0) - (1 - \alpha)\eta(y_t)) d\Theta_0 \\ &= \sum_{y^t} \varphi(y^t|y_0) (\hat{c}_t(y^t, \theta_0) - (1 - \alpha)\eta(y_t)) d\Theta_0 = \alpha. \end{aligned}$$

Similarly

$$\int \sum_{y^{t-1}} \varphi(y^{t-1}|y_0) \sum_{y^\tau} \varphi(y^\tau|y^t) [\hat{c}_\tau(\theta_0, y^\tau) - (1 - \alpha)\eta(y_\tau)] d\Theta_0 = \alpha \quad \text{for all } \tau > t.$$

Since the stock market clears in the Bewley model we have

$$\int \sum_{y^{t-1}} \varphi(y^{t-1}|y_0) \hat{\sigma}_{t-1}(\theta_0, y^{t-1}) d\Theta_0 = 1.$$

Since the stock is a claim to  $\alpha$  times the aggregate endowment in all future periods its (ex-dividend) price has to satisfy

$$\hat{v}_t(z_t) = \alpha \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \hat{Q}_\tau(z^\tau|z_t).$$

Combining these results implies that

$$\begin{aligned} & \int \sum_{y^t} \varphi(y^t|y_0) \hat{a}_{t-1}(\theta_0, y^{t-1}, z^t) d\Theta_0 \\ &= \alpha + \alpha \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \hat{Q}_\tau(z^\tau|z_t) - 1 \left( \alpha + \alpha \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \hat{Q}_\tau(z^\tau|z_t) \right) = 0 \end{aligned}$$

for each  $z_t$ . Thus, each of the Arrow securities markets clears if households hold portfolios given by (28).  $\square$

**Proof of Lemma 5.2.** By (24)

$$\hat{Q}_t(z^t|z_0) = \hat{\phi}(z^t|z_0) \tilde{Q}_t \frac{\hat{\beta}_{0,t-1}(z^{t-1}|z_0)}{\tilde{\beta}_t}$$

and thus

$$\begin{aligned} \sum_{z^t} \hat{Q}_t(z^t|z_0) &= \frac{\tilde{Q}_t}{\tilde{\beta}_t} \sum_{z^t|z_0} \hat{\phi}(z^t|z_0) \hat{\beta}_{0,t-1}(z^{t-1}|z_0) \\ &= \frac{\tilde{Q}_t}{\tilde{\beta}_t} \sum_{z^t|z^{t-1}} \hat{\phi}(z^t|z^{t-1}) \sum_{z^{t-1}|z_0} \hat{\phi}(z^{t-1}|z_0) \hat{\beta}_{0,t-1}(z^{t-1}|z_0) = \tilde{Q}_t \end{aligned}$$

by definition of  $\tilde{\beta}_t$  in Eq. (22) and the fact that  $\sum_{z^t|z^{t-1}} \hat{\phi}(z^t|z^{t-1}) = 1$ .  $\square$

**Proof of Lemma 5.3.** The Arrow securities position at time zero needed to finance all future excess consumption mandated by the Bewley equilibrium is given by

$$\hat{a}_{-1}(\theta_0, y_0, z_0) = \hat{c}_0(\theta_0, y_0) - (1 - \alpha)\eta(y_0) + \sum_{\tau=1}^{\infty} \sum_{z^\tau, y^\tau} \hat{Q}_\tau(z^\tau | z_0) (\hat{c}_\tau(\theta_0, y^\tau) - (1 - \alpha)\eta(y_\tau)) - \hat{\sigma}_0(\theta_0, y_0) [\hat{v}_0(z_0) + \alpha],$$

where we substituted indexes  $_{-1}$  by 0 to denote initial conditions. In particular,  $\hat{\sigma}_0(\theta_0, y_0)$  is the initial share position of an individual with wealth  $\theta_0$ . But

$$\begin{aligned} \hat{a}_{-1}(\theta_0, y_0, z_0) &= \hat{c}_0(\theta_0, y_0) - (1 - \alpha)\eta(y_0) \\ &\quad + \sum_{\tau=1}^{\infty} \sum_{z^\tau, y^\tau | z_0, y_0} \hat{Q}_\tau(z^\tau | z_0) \varphi(y^\tau | y_0) (\hat{c}_\tau(\theta_0, y^\tau) - (1 - \alpha)\eta(y_\tau)) \\ &\quad - \hat{\sigma}_0(\theta_0, y_0) [\hat{v}_0(z_0) + \alpha] \\ &= \hat{c}_0(\theta_0, y_0) - (1 - \alpha)\eta(y_0) \\ &\quad + \sum_{\tau=1}^{\infty} \sum_{y^\tau | y_0} \varphi(y^\tau | y_0) (\hat{c}_\tau(\theta_0, y^\tau) - (1 - \alpha)\eta(y_\tau)) \sum_{z^\tau | z_0} \hat{Q}_\tau(z^\tau | z_0) \\ &\quad - \hat{\sigma}_0(\theta_0, y_0) \alpha \left[ 1 + \sum_{\tau=1}^{\infty} \sum_{z^\tau | z_0} \hat{Q}_\tau(z^\tau | z_0) \right] \\ &= \hat{c}_0(\theta_0, y_0) - (1 - \alpha)\eta(y_0) + \sum_{\tau=1}^{\infty} \tilde{Q}_\tau \sum_{y^\tau | y_0} \varphi(y^\tau | y_0) (\hat{c}_\tau(\theta_0, y^\tau) \\ &\quad - (1 - \alpha)\eta(y_\tau)) - \hat{\sigma}_0(\theta_0, y_0) \alpha \left[ \sum_{\tau=0}^{\infty} \tilde{Q}_\tau \right] \\ &= 0, \end{aligned}$$

where the last equality comes from the intertemporal budget constraint in the standard incomplete markets Bewley model and the fact that the initial share position in that model is given by  $\hat{\sigma}_0(\theta_0, y_0)$ .  $\square$

**Proof of Proposition 5.2.** The stationary Bewley allocation  $\{\hat{a}_t^B(y^t) = 0, \hat{\sigma}_t(y^t)\}$  satisfies the constraint

$$\frac{\hat{a}_t^B(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t) \hat{v}_t \geq \hat{K}_t(y^t). \tag{41}$$

Using the fact that  $\hat{a}_t^B(y^t) = 0$  and adding

$$\sum_{z_{t+1}} \hat{q}_t(z_{t+1} | z_t) \hat{a}_t(y^t, z_{t+1}) + \hat{\sigma}_t(y^t) \hat{v}_t(z_t)$$

to both sides of (41) yields

$$\sum_{z_{t+1}} \hat{q}_t(z_{t+1} | z_t) \hat{a}_t(y^t, z_{t+1}) + \hat{\sigma}_t(y^t) \hat{v}_t(z_t)$$

$$\begin{aligned} &\geq \hat{K}_t(y^t) + \sum_{z_{t+1}} \hat{q}_t(z_{t+1}|z_t) \hat{a}_t(y^t, z_{t+1}) + \hat{\sigma}_t(y^t) [\hat{v}_t(z_t) - \hat{v}_t] \\ &\equiv \hat{K}_t^*(y^t, z_t) \end{aligned}$$

where  $\hat{K}_t^*(y^t)$  is the modified constraint for the de-trended Arrow model. Multiplying both sides by  $e_t(z^t)$  gives the modified constraint for the Arrow model with growth stated in the main text. For the alternative constraint, we know that the Bewley equilibrium allocation satisfies

$$\hat{a}_t^B(y^t) + \hat{\sigma}_t(y^t) [\hat{v}_{t+1} + \alpha] \geq \hat{M}_t(y^t). \quad (42)$$

Again using  $\hat{a}_t^B(y^t) = 0$  and adding

$$\hat{a}_t(y^t, z_{t+1}) + \hat{\sigma}_t(y^t) [\hat{v}_{t+1}(z_{t+1}) + \alpha]$$

to both sides of (41) yields

$$\begin{aligned} &\hat{a}_t(y^t, z_{t+1}) + \hat{\sigma}_t(y^t) [\hat{v}_{t+1}(z_{t+1}) + \alpha] \\ &\geq \hat{M}_t(y^t) + \hat{a}_t(y^t, z_{t+1}) + \hat{\sigma}_t(y^t) [\hat{v}_{t+1}(z_{t+1}) - \hat{v}_{t+1}] \\ &\equiv \hat{M}_t^*(y^t, z_{t+1}). \end{aligned}$$

Multiplying both sides by  $e_{t+1}(z^{t+1})$  again gives rise to the constraint stated in the main text.  $\square$

## Supplementary material

The online version of this article contains additional separate appendix.  
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