Competitive Risk Sharing Contracts with One-Sided Commitment: Technical Appendix

Dirk Krueger
Goethe University Frankfurt, CEPR and NBER

Harald Uhlig
Humboldt University zu Berlin, CentER, Bundesbank and CEPR

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Abstract

This appendix provides details of some of the proofs in the main paper

Proposition 4: *Let outside options $(U^{Out}(y))_{y \in Y} \in [\underline{w}, \bar{w}]$ and $\beta < 1 < R$ by given. Further suppose that condition 1 in the main text is satisfied. Then, an optimal contract for $\left((U^{Out}(y))_{y \in Y}, \underline{w}, \bar{w}\right)$ exists and has the following properties.*

1. *$V(y, w)$ is strictly convex, strictly increasing, continuous and differentiable in w.*
2. *The decision rules are unique and continuous.*
3. *The decision rules and the solution to the dynamic programming problem satisfy the first order conditions and the envelope condition*

$$(1-\beta)\lambda = \left(1 - \frac{1}{R}\right) C'(h) \tag{1}$$

$$\lambda\beta = \frac{1}{R}\frac{\partial V}{\partial w}(y', w'(y, w; y')) - \mu(y') \tag{2}$$

$$\lambda = \frac{\partial V}{\partial w}(y, w) \tag{3}$$

$$\lambda \geq 0 \tag{4}$$

$$\mu(y') \geq 0, \text{ for all } y' \in Y \tag{5}$$

   where λ and $\mu(y')$ are the Lagrange multipliers on the first and second constraints.

4. *The decision rule $h(y, w)$ is strictly increasing in w. The decision rule $w'(y, w; y')$ is weakly increasing in w, and strictly so, if the continuing participation constraint $w'(y, w; y') \geq U^{Out}(y') - \nu(y')$ is not binding.*

5. *If the income process is iid, then $V(y,w)$ depends on w alone, $V(y,w) \equiv V(w)$. If additionally $U^{Out}(y') - \nu(y')$ is weakly increasing in y', then $w'(y,w;y')$ is weakly increasing in y'.*

Proof. While the proof draws on fairly standard techniques as in Stokey, Lucas with Prescott (1989), some points require particular attention.

1. Condition 1 assures that the equation

$$w = (1-\beta)h + \beta \sum_{y' \in Y} \pi(y'|y)w'(y')$$

subject to $w'(y') \geq U^{\text{Out}}(y') - \nu(y')$ always has a solution. Existence and convexity of V follows from a standard contraction argument. The contraction argument also shows that $V(y,\cdot)$ is strictly increasing.

To prove that $V(y,\cdot)$ is strictly convex we make use of the fact that under our assumptions the value function from the sequential and the recursive problem of the principal coincide. Now consider some y and $w_1 \neq w_2$ and, for each w_i, consider a stochastic sequence of optimal choices $(y_{t,i}, w_{t,i}, h_{t,i})_{t=0}^{\infty}$ from iterating the solution to the dynamic programming problem forward, starting with $y_{0,i} = y$ and $w_{0,i} = w_i$. Note that we do not require at this point that there is a unique solution. Note that $\underline{w} - \beta\bar{w} \leq h_t \leq \bar{w} - \beta\underline{w}$. This implies convergence of discounted sums of the $h_{t,i}$ and, per iteration,

$$w_i = (1-\beta)E\left[\sum_{j=0}^{\infty} \beta^t h_{t,i}\right]$$

A similar iteration for $V(y,w_i)$ and observing the upper bound $V(y,w) \leq V(y,\bar{w})) \leq C(\bar{w}) < \infty$ yields

$$V(y,w_i) = E\left[\sum_{j=0}^{\infty} \frac{1}{R^t} C(h_{t,i})\right]$$

Consider now the convex combination $w_\lambda = \lambda w_1 + (1-\lambda)w_2$. A feasible plan is given by the convex combination of $(w_{t,i}, h_{t,i})_{t=0}^{\infty}$, with costs given by

$$V_\lambda = E\left[\sum_{j=0}^{\infty} \frac{1}{R^t} C(h_{t,\lambda})\right] < \lambda V(y,w_1) + (1-\lambda)V(y,w_2)$$

where $h_{t,\lambda} = \lambda h_{t,1} + (1-\lambda)h_{t,2}$ and the inequality follows from the strict convexity of the cost function C. Obviously $V(y,w_\lambda) \leq V_\lambda < \lambda V(y,w_1) + (1-\lambda)V(y,w_2)$.

2. Now we want to show that the value function is differentiable. Rewrite the Bellman equation in the main text in operator form

$$
\begin{aligned}
T\left[V(y,w)\right] &= \min_{h\in D,\{w'(y')\in[\underline{w},\bar{w}]\}_{y'\in Y}} \left(1-\frac{1}{R}\right)C(h)+\frac{1}{R}\sum_{y'\in Y}\pi(y'|y)V(y',w'(y')) \\
\text{s.t. } w &= (1-\beta)h+\beta\sum_{y'\in Y}\pi(y'|y)w'(y') \qquad (6)
\end{aligned}
$$

We want to show that the fixed point V^* of the operator T is differentiable, conditional on having established that it is unique (contraction mapping theorem), and that it is strictly convex. Define the sequence of functions $\{V_n\}_{n=1}^{\infty}$ as

$$
\begin{aligned}
V_0(y,w) &= C(w) \\
V_{n+1}(y,w) &= T\left[V_n(y,w)\right]
\end{aligned}
$$

We now note the following

- $V_n \to V^*$ uniformly (by the contraction mapping theorem)
- Since by assumption C is strictly convex, all V_n are strictly convex (this follows from simple induction).
- For each n, the associated policy correspondences $h_n, w_n'(y')$ are single-valued, continuous functions, since the V_n are strictly convex.
- Again by induction one shows that each V_n is continuously differentiable: clearly V_0 is, then by the envelope theorem

  $$\frac{\partial V_1(y,w)}{\partial w}=\frac{\left(1-\frac{1}{R}\right)}{1-\beta}C'(h_1(y,w))$$

  and thus V_1 is continuously differentiable (since h_1 is continuous). Continue the induction to show that all V_n are continuously differentiable, with

  $$\frac{\partial V_n(y,w)}{\partial w}=\frac{\left(1-\frac{1}{R}\right)}{1-\beta}C'(h_n(y,w))$$

- V^* is strictly convex
- $w\in[\underline{w},\bar{w}]$, a compact set. Now we can apply Stokey, Lucas with Prescott (1989), Theorem 3.8., which guarantees that $h_n, w_n'(y')$ converge uniformly to $(h, w'(y'))$, the optimal policies associated with the fixed point V^*. This implies that

  $$\frac{\partial V_{n+1}(y,w)}{\partial w}=\frac{\partial T\left[V_n(y,w)\right]}{\partial w}\to\frac{\left(1-\frac{1}{R}\right)}{1-\beta}C'(h(y,w))$$

  uniformly. But since $V_n\to V^*$ uniformly, the expression $\frac{\left(1-\frac{1}{R}\right)}{1-\beta}C'(h(y,w))$ is the derivative of V^* at (y,w). Uniqueness of the decision rules now follows from strict convexity of the objective function in the operator equation.[1]

3. Deriving the first-order conditions is standard.

4. The monotonicity properties in w follow from the first order conditions, the fact, that $V(y,w)$ is strictly convex in w and the strict convexity of the cost function $c = C(h)$.

5. The fact that V does not depend on y in the iid case follows from the fact that y appears in the Bellman equation within the conditional probabilities $\pi(y'|y)$, which are independent of y in the iid case. The properties for $w'(w,y;y')$ follow from the first order conditions, the fact that $U^{\text{Out}}(y) - \nu(y)$ is increasing in y and strict convexity of V.

■

Proposition 5: *Let condition 1 be satisfied. Then an optimal contract and outside options* $\left\{U^{Out}(y)\right\}_{y\in Y}$ *satisfying* (13) *in the main text exist. If the income process is iid (or* $w'(y,w;y')$ *associated with* $\left\{U^{Out}(y)\right\}_{y\in Y}$ *is weakly increasing in* y*), then an equilibrium exists.*

Proof. We will prove this proposition for $\nu(y) \equiv 0$; equivalent arguments prove existence for the general case. We first prove that there exists outside options $U^{Out} = (U^{Out}(y_1), \ldots, U^{Out}(y_m))$ and associated value and policy functions $V_{U^{\mathrm{Out}}}, h_{U^{\mathrm{Out}}}, w'_{U^{\mathrm{Out}}}(y')$ of the principals solving $V_{U^{\mathrm{Out}}}(y, U^{Out}(y)) = a(y)$ for all y. Then we prove that the Markov transition function induced by π and $w'_{U^{\mathrm{Out}}}(y')$ has a stationary distribution.

For the first part define the function $f : [\underline{w}, \bar{w}]^m \to [\underline{w}, \bar{w}]^m$ by

$$f_j\left[U^{Out}\right] = \min\{\tilde{w} \in [\underline{w}, \bar{w}] : V_{U^{\mathrm{Out}}}(y_j, \tilde{w}) \geq a(y_j)\} \text{ for all } j = 1, \ldots, m$$

We need to show three things: 1) The function f is well defined on all of $[\underline{w}, \bar{w}]^m$, 2) The function f is continuous, 3) Any fixed point w^* of f satisfies $V_{w^*}(y_j, w_j^*) = a(y_j)$ for all $j = 1, \ldots, m$. We discuss each point in turn

1. If the set $\{\tilde{w} \in [\underline{w}, \bar{w}] : V_{U^{\mathrm{Out}}}(y_j, \tilde{w}) \geq a(y_j)\}$ is non-empty for all j, then the function f is well-defined since the minimization over $\tilde{w}$ is a minimization of a continuous function over a compact set. To show that the set is non-empty for all j and all $U^{Out} \in [\underline{w}, \bar{w}]^m$ it suffices to show that $V_{\underline{w}}(y_j, \bar{w}) \geq a(y_j)$ for all j, since $V_{\underline{w}}(y_j, w)$ is strictly increasing in w and $V_{\underline{w}}(y_j, w) \leq V_{U^{\mathrm{Out}}}(y_j, w)$ for all $U^{Out} \in [\underline{w}, \bar{w}]^m$ and all $w \in U^{Out} \in [\underline{w}, \bar{w}]$. Let $\hat{V}$ denote the cost function of a principal that does not face the competition constraints. But then

$$V_{\underline{w}}(y_j, \bar{w}) \geq \hat{V}(y_j, \bar{w}) = \bar{a} \equiv \max_i a(y_i) \geq a(y_j)$$

where the first equality follows from the definition of $\bar{w}$ in the main text. This proves the first point in our list.

[1] The main parts of this proof stem from Krueger (1999), which in turn adopts the proof strategy of Atkeson and Lucas (1995).

2. In order to prove that f is continuous in $U^{Out} \in [\underline{w}, \bar{w}]^m$ it is useful to rewrite f as

$$f_j \left[U^{Out}\right] \begin{cases} = \underline{w} & \text{if } V_{U^{\text{Out}}}(y_j, \underline{w}) > a(y_j) \\ \text{solves } V_{U^{\text{Out}}}(y_j, f_j \left[U^{Out}\right]) = a(y_j) & \text{if } V_{U^{\text{Out}}}(y_j, \underline{w}) \leq a(y_j) \end{cases}$$

In the first case $f_j \left[U^{Out}\right]$ is independent of U^{Out}, in the second case it moves continuously in U^{Out} as long as $V_{U^{\text{Out}}}(.,.)$ is uniformly continuous in U^{Out}. Furthermore the switching point between the two cases moves continuously with U^{Out} if $V_{U^{\text{Out}}}(.,.)$ is uniformly continuous in U^{Out}. Thus it suffices to show uniform continuity of $V_{U^{\text{Out}}}(.,.)$ in U^{Out}. Take a sequence $\{U_n^{Out}\}_{n=0}^{\infty}$ in $[\underline{w}, \bar{w}]^m$ converging to U^{Out}. Let $\|.\|$ denote the sup-norm and $T_{U_n^{\text{Out}}}$ the operator associated with the Bellman equation (7) for outside options U_n^{Out}. Let $T^k_{U_n^{\text{Out}}}$ denote the $T_{U_n^{\text{Out}}}$ operator being applied k times. Note that by the triangle inequality

$$\left\|V_{U_n^{\text{Out}}} - V_{U^{\text{Out}}}\right\| \leq \left\|V_{U_n^{\text{Out}}} - T^n_{U_n^{\text{Out}}} V_{U^{\text{Out}}}\right\| + \left\|T^n_{U_n^{\text{Out}}} V_{U^{\text{Out}}} - V_{U^{\text{Out}}}\right\|$$

Since $T_{U_n^{\text{Out}}}$ is a contraction mapping for all U_n^{Out}, $\left\|V_{U_n^{\text{Out}}} - T^n_{U_n^{\text{Out}}} V_{U^{\text{Out}}}\right\|$ converges to 0 as $n \to \infty$. Again by the triangle inequality and the fact that $T_{U_n^{\text{Out}}}$ is a contraction mapping with modulus $\frac{1}{R}$ we have

$$\begin{aligned} \left\|T^n_{U_n^{\text{Out}}} V_{U^{\text{Out}}} - V_{U^{\text{Out}}}\right\| &\leq \sum_{k=1}^{n} \left\|T^k_{U_n^{\text{Out}}} V_{U^{\text{Out}}} - T^{k-1}_{U_n^{\text{Out}}} V_{U^{\text{Out}}}\right\| \\ &\leq \sum_{k=0}^{n-1} R^{-k} \left\|T_{U_n^{\text{Out}}} V_{U^{\text{Out}}} - V_{U^{\text{Out}}}\right\| \\ &= \sum_{k=0}^{n-1} R^{-k} \left\|T_{U_n^{\text{Out}}} V_{U^{\text{Out}}} - T_{U^{\text{Out}}} V_{U^{\text{Out}}}\right\| \end{aligned}$$

where the last equality follows from the fact that $V_{U^{\text{Out}}}$ is a fixed point of $T_{U^{\text{Out}}}$. Since $\sum_{k=0}^{n-1} R^{-k}$ converges, if we can show that $\left\|T_{U_n^{\text{Out}}} V_{U^{\text{Out}}} - T_{U^{\text{Out}}} V_{U^{\text{Out}}}\right\|$ converges to 0 in n we have demonstrated that $\left\|V_{U_n^{\text{Out}}} - V_{U^{\text{Out}}}\right\|$ converges to 0 in n, that is, $V_{U_n^{\text{Out}}}$ converges to $V_{U^{\text{Out}}}$ uniformly. Consider the function

$$\psi(y, w, U^{Out}) = \min_{h, \{w'(y')\} \in \Gamma(y, wU^{\text{Out}})} \left(1 - \frac{1}{R}\right) C(h) \frac{1}{R} + \sum_{y' \in Y} \pi(y'|y) V_{U^{\text{Out}}}(y', w'(y'))$$

with constraint set

$$\Gamma(y, w, U^{Out}) = \{h, w'(y') | h \in D, w'(y') \in [\underline{w}, \bar{w}], (8) \text{ and } (9) \text{ of main text}\}.$$

on $Y \times [\underline{w}, \bar{w}]^{m+1}$. Since for all $(y, w, U^{Out}) \in Y \times [\underline{w}, \bar{w}]^{m+1}$ the objective function is continuous and the constraint set is continuous, non-empty and compact-valued, by the theorem of the maximum ψ is continuous, and thus continuous in particular with respect to U^{Out}. Uniformity follows from the compactness of the space $Y \times [\underline{w}, \bar{w}]^{m+1}$, so that $V_{U_n^{\text{Out}}}$ converges to $V_{U^{\text{Out}}}$ uniformly. From points 1. and 2. it follows that the function f defined above is well-defined and continuous on the compact space $[\underline{w}, \bar{w}]^m$. Brouwer's fixed point theorem now guaranties existence of a $w^* \in [\underline{w}, \bar{w}]^m$ such that $w^* = f(w^*)$. In point 3. we now argue that this fixed point has the desired property that $V_{w^*}(y_j, w_j^*) = a(y_j)$.

3. Suppose not. Then there is some non-empty index set $\mathcal{J}$, so that $V_{w^*}(y_j, w_j^*) = a(y_j)$ for all $j \notin \mathcal{J}$ and $V_{w^*}(y_j, w_j^*) > a(y_j)$ as well as $w_j^* = \underline{w}$ for $j \in \mathcal{J}$. We will now show that V_{w^*} cannot be the solution to its Bellman equation. Let $\Delta = \max_j(V_{w^*}(y_j, w_j^*) - a(y_j))$: by assumption in this proof by contradiction, $\Delta > 0$. Let T be large enough so that

$$\frac{1 - R^{-T}}{1 - R^{-1}}(\underline{y} - \min_j y_j) + \frac{1}{R^T}\Delta < 0$$

Such a T exists, since we have assumed in our definition of the lower bound $\underline{w}$ that $\underline{y} - \min_j y_j < 0$. We will now construct sequential allocations that attain utility promises $w^*(y_0)$ at lower cost than $V_{w^*}(y_0, w^*(y_0))$, contradicting the fact that $V_{w^*}(.,.)$ is the cost function associated with outside options w^*. Let $V_{seq}(y, w)$ denote the cost function from the sequential allocation to be constructed now. Let $\sigma_t = (w_0, j_0, \ldots, j_t)$ be the history of income states until date t, including the initial promise $w_0 \in [\underline{w}, \bar{w}]$. A sequential decision rule must specify choices $\left(h(\sigma_t), (w'(y_j'; \sigma_t))_{j=1}^m\right)$ for any t and any history σ_t. Consider the following allocation: for a given t and σ_t :

- If $t < T$ and if $j_s \in \mathcal{J}$ for all $s \leq t$ and if $w_0 = w^*(y_0)$, provide utility $h_t = \underline{h}$ (where $\underline{h}$ is specified below). Furthermore, set the promises $(w'(y_j'; \sigma_t))_{j=1}^m$ equal to the fixed point w^*.
- If $t \geq T$ or if $t < T$ and if $j_s \notin \mathcal{J}$ for some $s \leq t$ or if $w_0 \neq w^*(y_0)$, provide utility h_{j_t} and promises $(w'(y_j'; \sigma_t))_{j=1}^m$ according to the decision rule of the proposed solution $V_{w^*}(y, w)$, where $y = y_t$ and $w = w'(y_t; \sigma_{t-1})$, if $t > 0$, and $w = w_0$, if $t = 0$.

We need to check that a) the continuation promise is always at least as large as the outside option w^*, b) the realized utility $w(\sigma_t)$ to the agent is always equal to the promise $w = w'(y_t; \sigma_{t-1})$, if $t > 0$, and $w = w_0$ if $t = 0$. and c) the resulting cost function V_{seq} is nowhere higher than the candidate solution V_{w^*}, i.e. $V_{seq}(y_0, w_0) \leq V_{w^*}(y_0, w_0)$. Furthermore, the costs are not larger than $a(y_j)$ for all j and all initial states $(y_0 = y_j, w^*(y_0))$. Then, since the costs exceed $a(y_j)$ for some j in the proposed solution V_{w^*} to the Bellman equation, this then renders a contradiction that V_{w^*} is a solution to the Bellman equation. Let us check each of the items a) - c) above.

a. **Continuation promise is always is at least as large as the outside options** w^*. True by construction, since it is true for the decision rules of V_{w^*} and since it is also true at the changed decisions under the first bullet point.

b. **Realized utility** $w(\sigma_t)$ **of agent equals the promise** $w = w'(y_t; \sigma_{t-1})$, **if** $t > 0$, **and** $w = w_0$ **if** $t = 0$. Since this must be true along histories simply following the decision rules given by the candidate solution V_{w^*}, we only need to check this for the case that $t < T$, $j_s \in \mathcal{J}$ for all $s \leq t$ and $w_0 = w^*(y_0)$. But along these paths, $w_0 = \underline{w}$ for $t = 0$ and $w'(y_t; \sigma_{t-1}) = w^*(y')$ for $t > 0$. The claim follows with

$$\begin{aligned}\underline{w} &= (1-\beta)\underline{h} + \beta \sum_{y' \in Y} \pi(y'|y_t) w^*(y') \\ &> (1-\beta)\inf(\mathsf{D}) + \beta \bar{w}\end{aligned}$$

Note that condition 1 in the main text assures that the $\underline{h}$ required by this equation lies in D and that consequently there exists a $\underline{c} \in (0, \underline{y}]$ such that $\underline{h} = u(\underline{c})$. The fact that $\underline{c} \leq \underline{y}$ follows from the fact that $w^*(y') \geq \underline{w}$ and the definition of $\underline{y}$ in the main text.

c. **Resulting cost function is nowhere higher than the candidate solution** V_{w^*}, **i.e.** $V_{seq}(y_0, w_0) \leq V_{w^*}(y_0, w_0)$. **Furthermore, the costs are not larger than** $a(y_j)$ **for all** j **and all initial states** $(y_0 = y_j, w^*(y_0))$. The claim is true by construction if the sequential decisions just follows along the decision rules of the candidate solution V_{w^*}, since in particular, $V_{w^*}(y_j, w_j^*) = a(y_j)$ for all $j \notin \mathcal{J}$. Consider now $j \in \mathcal{J}$, the initial state $y_0 = y_j$ and the initial promise $w^*(y_0) = \underline{w}$. Consider all possible histories σ_T until date T and define the stopping time $\tau(\sigma_T)$ to be the earliest date $t = 1, \ldots, T$, at which $j_t \notin \mathcal{J}$, and set $\tau(\sigma_T) = T$, if all $j_t, t = 1, \ldots, T$ belong to the set $\mathcal{J}$. Let $\pi(\sigma_T, t)$ the probability of the history, truncated at date t. The costs can now be calculated directly as

$$\begin{aligned}& V_{seq}(y_0, \underline{w}) \\ = & \left(1 - \frac{1}{R}\right) \sum_{\sigma_T} \left\{ \left(\sum_{t=0}^{\tau(\sigma_T)-1} \frac{1}{R^t} \pi(\sigma_T, t) \underline{c} \right) + \frac{1}{R^{\tau(\sigma_T)}} \pi(\sigma_T, \tau(\sigma_T)) V_{w^*}(y_{j_t}, w^*(y_{j_t})) \right\} \\ \leq & \left(1 - \frac{1}{R}\right) \sum_{\sigma_T} \left\{ \left(\sum_{t=0}^{\tau(\sigma_T)-1} \frac{1}{R^t} \pi(\sigma_T, t) \underline{y} \right) + \frac{1}{R^{\tau(\sigma_T)}} \pi(\sigma_T, \tau(\sigma_T)) V_{w^*}(y_{j_t}, w^*(y_{j_t})) \right\}\end{aligned}$$

where the inequality follows from the fact that $\underline{c} \leq \underline{y}$. Now note the following. For any history σ_T, for which $j_t \notin \mathcal{J}$ for some $t \leq T$, we have $V(y_{j_t}, w^*(y_{j_t})) = a(y_{j_t})$ at $t = \tau(\sigma_T))$ and we have costs no larger than $\underline{y} < \min y_j$ for $t < \tau(\sigma_T))$, so that costs conditional on these paths are no higher than the net present value $a(y_0, \sigma_T)$ of the income, conditional on these paths, where

$$a(y_0,\sigma_T) = \left(1-\frac{1}{R}\right)\left(\left(\sum_{t=0}^{T-1}\pi(\sigma_T,t)\frac{1}{R^t}y_{j_t}\right)+\pi(\sigma_T,T)\frac{1}{R^T}a(y_{j_T})\right)$$

Finally consider the histories σ_T, for which $j_t \in \mathcal{J}$ for all $t = 0,\ldots,T$. For these paths, the contribution $\kappa(\sigma_T)$ to the costs are

$$\begin{aligned}
\kappa(\sigma_T) &= \left(1-\frac{1}{R}\right)\left(\left(\sum_{t=0}^{T-1}\frac{1}{R^t}\pi(\sigma_T,t)\underline{c}\right)+\frac{1}{R^T}\pi(\sigma_T,T)V(y_{j_t},w^*(y_{j_t}))\right)\\
&\leq \left(1-\frac{1}{R}\right)\left(\left(\sum_{t=0}^{T-1}\frac{1}{R^t}\pi(\sigma_T,t)\underline{y}\right)+\frac{1}{R^T}\pi(\sigma_T,T)V(y_{j_t},w^*(y_{j_t}))\right)\\
&\leq a(y_0,\sigma_T)+\left(1-\frac{1}{R}\right)\left(\left(\sum_{t=0}^{T-1}\frac{1}{R^t}\pi(\sigma_T,t)(\underline{y}-\min_j y_j)\right)+\frac{1}{R^T}\pi(\sigma_T,T)\Delta\right)\\
&\leq a(y_0,\sigma_T)+\left(1-\frac{1}{R}\right)\pi(\sigma_T,T)\left(\frac{1-R^{-T}}{1-R^{-1}}(\underline{y}-\min_j y_j)+\frac{1}{R^T}\Delta\right)\\
&\leq a(y_0,\sigma_T)
\end{aligned}$$

by our assumption about T. This finishes the check of point c), thus part 3. and therefore the entire proof of the existence of equilibrium outside options $U^{Out} = w^*$. It remains to be shown that a stationary distribution associated with the optimal decision rules for equilibrium outside options w^* exist.

From now on let $(h, w'(y'))$ denote the optimal policies of the principal associated with the equilibrium outside options $U^{Out} = w^*$, whose existence was established above. We know that $\{w'(y,w;y')\}_{y'\in Y}$ are continuous functions on Z. Let $Q : Z \times \mathcal{B}(Z) \to [0,1]$ denote the Markov transition function as defined in the main text. Since the policy functions are continuous and hence measurable, by theorem 9.13 of Stokey et al. (1989) Q is indeed a well-defined transition function. Furthermore, by their theorem 8.2. the operator $T^* : \Lambda(Z \times \mathcal{B}(Z)) \to \Lambda(Z \times \mathcal{B}(Z))$ mapping the space of probability measures on $(Z \times \mathcal{B}(Z))$ into itself and defined as

$$T^*\Phi(A) = \int Q(z,A)\Phi(dz)$$

is well-defined. Showing that there exists an invariant probability measure Φ associated with Q amounts to showing the existence of a fixed point of the operator T^*. Theorem 12.10 of Stockey and Lucas (1989) guarantees the existence of a fixed point if the state space is compact (which is satisfied in our case) and Q satisfies the Feller property, that is, for all bounded continuous functions f the function

$$Tf(y,w) = \int f(y',w')Q(y,w,dy' \times dw')$$

is continuous. But by definition of Q we have

$$Tf(y,w) = \int f(y',w')Q(y,w,dy' \times dw') = \sum_{y' \in Y} f(y',w'(y,w;y'))\pi(y'|y)$$

which is obviously continuous in (y,w), since $w'(y,w;y')$ is continuous in w. ■

Proposition 18: *Any contract equilibrium $V(y,w)$, $w'(y,w;y')$, $c(y,w)$ and $\{U^{Out}(y)\}_{y \in Y}$ can be implemented as a solution $W(y,b)$, $b'(y,b;y')$, $C(y,b)$ to the consumption-savings problem above with borrowing constraint given by*

$$\underline{b}(y) = \frac{R}{R-1}\left(V(y, U^{Out}(y) - \nu(y)) - a(y)\right) \tag{7}$$

Conversely, for given borrowing constraints $\underline{b}(y) \leq 0$ and associated solution to the consumption-saving problem $W(y,b), b'(y,b;y'), C(y,b)$ there exist moving costs

$$\nu(y) = W(y,0) - W(y,\underline{b}(y)) \geq 0$$

such that the solution to the consumption-savings problem can be implemented as a contract equilibrium $V(y,w), w'(y,w;y'), c(y,w)$ and $\{U^{Out}(y)\}_{y \in Y}$. The moving costs satisfy

$$U^{Out}(y) - \nu(y) = W(y,\underline{b}(y)) \text{ for all } y \in Y \tag{8}$$

Proof. In the contract economy the state variables of a contract are (y,w), in the consumption-savings problem they are (y,b). Define the mapping between state variables as

$$\begin{aligned} b(y,w) &= \frac{R}{R-1}\left(V(y,w) - a(y)\right) && (9) \\ w(y,b) &= W(y,b) && (10) \end{aligned}$$

where both functions are strictly increasing in their second arguments and thus invertible. We denote the inverse of $b(y,w)$ by $w = b^{-1}(y,b)$: it is that lifetime utility level w which requires initial bond holdings b to realize that level in the bond economy. Let $b = w^{-1}(y,w)$ be similarly defined. Furthermore define $\underline{b} = w^{-1}(y,\underline{w})$ and $\bar{b} = w^{-1}(y,\bar{w})$.[2] Finally define the map between policies and value functions in the two problems as

$$\begin{aligned} c(y,b) &= C(h(y,w(y,b))) && (11) \\ b'(y,b;y') &= b(y',w'(y,w(y,b);y')) \\ W(y,b) &= b^{-1}(y,b) \end{aligned}$$

[2] When mapping the bond economy into the contract economy bounds on bond holdings $[\underline{b},\bar{b}]$ map into utility bounds $[\underline{w},\bar{w}]$ in a similar fashion. A lower bound on bond holdings is always guaranteed via the borrowing constraints; when we start with the bond economy we assume the existence of an upper bound on bond holdings. Under fairly general conditions this upper bound can be chosen without imposing additional binding restrictions on the problem (although this is not crucial for the equivalence result).

and

$$
\begin{aligned}
h(y,w) &= u(c(y,b(y,w))) \\
w'(y,w;y') &= w(y',b'(y,b(y,w);y')) \\
V(y,w) &= w^{-1}(y,w)
\end{aligned}
\tag{12}
$$

With policies so defined it is straightforward to verify that (W,c,b') constructed from the contract problem as in (11) satisfy the Bellman equation of the consumption problem (since the underlying (V,h,w') satisfy the promise keeping constraint in the contract problem) and the budget constraint (since (V,h,w') satisfy the Bellman equation of the contract problem). Reversely, one can equally easily verify that (V,h,w') constructed from the consumption problem as in (12) satisfy the Bellman equation of the contract problem (since the underlying (W,c,b') satisfy the budget constraint in the contract problem) and the promise keeping constraint (since (W,c,b') satisfy the Bellman equation of the consumption problem).

Now we show that any feasible contract satisfies the borrowing constraint in the savings problem and that any consumption allocation satisfies the utility constraints in the contract problem. First we show that the bond holdings

$$
b'(y,b(w,y);y') = \frac{R}{R-1}\left(V(y',w'(y,b^{-1}(y,b);y')) - a(y')\right) \tag{13}
$$

derived from the contract problem satisfy the short-sale constraints

$$
\underline{b}(y') = \frac{R}{R-1}\left(V(y',U^{Out}(y') - \nu(y')) - a(y')\right)
$$

in the consumption problem. Since V is strictly increasing in its second argument and the function $w'(y,w;y')$ satisfies

$$
w'(y,w;y') \geq U^{Out}(y') - \nu(y')
$$

we have

$$
\begin{aligned}
b'(y,b(w,y);y') &= \frac{R}{R-1}\left(V(y',w'(y,b^{-1}(y,b);y')) - a(y')\right) \\
&\geq \frac{R}{R-1}\left(V(y,U^{Out}(y') - v(y')) - a(y')\right) \\
&= \underline{b}(y)
\end{aligned}
$$

Second we show that the future utility promises

$$
w'(y,w;y') = W(y',b'(y,b;y')) \tag{14}
$$

derived from the savings problem satisfy the contract enforcement constraints. We note, since

$$
b'(y,b;y') \geq \underline{b}(y')
$$

that

$$w'(y,w;y') = W(y', b'(y,b;y')) \geq W(y', \underline{b}(y')) = W(y',0) - \nu(y') \tag{15}$$

using that the function W is strictly increasing in its second argument and the definition of $v(y')$. Now we note that in the consumption problem lifetime utility $w = W(y,b)$ can be delivered at lifetime cost $b + \frac{Ra(y)}{R-1}$. Thus $V(y,w) = V[y, W(y,b)] = a(y) + \frac{R-1}{R}b$ is the cost in per-period terms. From this it follows that

$$V[y', W(y',0)] = a(y') = V(y', U^{Out}(y'))$$

and thus

$$W(y',0) = U^{Out}(y') \tag{16}$$

Combining (14)-(16) yields

$$w'(y,w;y') \geq U^{Out}(y') - \nu(y)$$

that is, the utility promises derived from the consumption problem satisfy the enforcement constraint. From $W(y',0) = U^{Out}(y')$ also (8) immediately follows.

These results imply that consumption-bond policies and value function (W, c, b') derived from the contract policies are feasible for the consumption problem and the value function satisfies the Bellman recursion. It remains to be shown that for the given value function W on the right hand side of Bellman's equation for the consumption problem, the policies (c, b') are indeed optimal. Reversely, it remains to be shown that for the given value function V on the right hand side of the contract minimization problem (h, w') are the optimal policies. Since both directions follow exactly the same logic we shall only prove the direction that (c, b') are indeed optimal.

We will do so by arguing that if (c, b') weren't optimal one can construct a contract policy from the superior consumption policy that yields lower costs than the optimal contract allocation, a contradiction. This argument is most simply made in the sequential analogue of the recursive formulation. Let $\{h_t(w_0, y^t)\}$ denote the utility allocation that solves the contract cost minimization problem for $(y_0, w_0) \in Y \times [\underline{w}, \bar{w}]$ and define the corresponding consumption allocation by

$$c_t(b_0, y^t) = C(h_t(w_0, y^t))$$

where b_0 and w_0 are related by $b_0 = w^{-1}(y_0, w_0)$. Now suppose there exist $(y_0^*, b_0^*) \in Y \times [\underline{b}, \bar{b}]$ such that $\{c_t(b_0^*, y^t)\}$ does *not* solve the consumer maximization problem; denote by $\{\tilde{c}_t(b_0^*, y^t)\}$ the allocation that beats it. Now find $w_0^* = b^{-1}(y_0^*, b_0^*)$ and define

$$\tilde{h}_t(w_0^*, y^t) = u(\tilde{c}_t(b_0^*, y^t))$$

and the new utility allocation $\hat{h}_t$ by

$$\begin{aligned} \hat{h}_t(w_0^*, y^t) &= \tilde{h}_t(w_0^*, y^t) \text{ for } t > 0 \\ \hat{h}_0(w_0^*, y_0^*) &= \tilde{h}_0(w_0^*, y_0^*) - \varepsilon \end{aligned}$$

for $\varepsilon > 0$ and small. The allocation $\{\tilde{h}_t\}$ (and thus the allocation $\{\tilde{h}_t\}$) satisfies the sequential analogues of the no-moving constraints because $\{c_t\}$ (that is, the implied asset holdings) satisfy the borrowing constraints of the consumption problem. Furthermore

$$\begin{aligned} w_0^* &= h_0(w_0^*, y_0^*) + \sum_{t\geq 1}\sum_{y^t} \pi_t(y^t) h_t(w_0^*, y^t) \\ &< \tilde{h}_0(w_0^*, y_0^*) + \sum_{t\geq 1}\sum_{y^t} \pi_t(y^t) \tilde{h}_t(w_0^*, y^t) \\ &= \hat{h}_0(w_0^*, y_0^*) + \varepsilon + \sum_{t\geq 1}\sum_{y^t} \pi_t(y^t) \hat{h}_t(w_0^*, y^t) \end{aligned}$$

where the first inequality comes from the fact that the consumption allocation $\{c_t\}$ associated with the optimal contract allocation $\{h_t\}$ is beaten by $\{\tilde{c}_t\}$ in lifetime utility terms. Thus we can choose $\varepsilon > 0$ such that

$$w_0^* = \hat{h}_0(w_0^*, y_0^*) + \sum_{t\geq 1}\sum_{y^t} \pi_t(y^t) \hat{h}_t(w_0^*, y^t)$$

Note that allocations $\{h_t\}$ and $\{\tilde{h}_t\}$ have the same expected discounted costs, since the associated $\{c_t\}$ and $\{\tilde{c}_t\}$ both satisfy the intertemporal budget constraint of the consumption problem. But then by construction $\{\hat{h}_t\}$ costs less than $\{h_t\}$ and also delivers lifetime utility w_0^*, a contradiction to the fact that $\{h_t\}$ is the optimal allocation for the contract minimization problem. ∎

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