Neoclassical Growth with Long-Term One-Sided Commitment Contracts*

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Abstract

This paper characterizes the stationary equilibrium of a continuous-time neoclassical production economy with capital accumulation in which households seek to insure against idiosyncratic income risk through long-term insurance contracts. Insurance companies operating in perfectly competitive markets can commit to future contractual obligations, whereas households cannot. For the case in which household labor productivity takes two values, one of which is zero, and where households have logarithmic utility we provide a complete analytical characterization of the optimal consumption insurance contract as well as the stationary consumption distribution. Under parameter restrictions, we show that there is a unique stationary equilibrium with partial consumption insurance. We also demonstrate analytically that the stationary consumption distribution has a Pareto form, truncated by an upper mass point. The unique equilibrium interest rate (capital stock) is strictly decreasing (increasing) in income risk. Thus the paper provides an analytically tractable alternative to the standard incomplete markets general equilibrium model developed in Aiyagari (1994) by

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retaining its physical structure, but substituting the assumed incomplete asset markets structure with one in which limits to consumption insurance emerge endogenously, as in Krueger and Uhlig (2006).

1 Introduction

This paper develops a new, analytically tractable general equilibrium macroeconomic model with idiosyncratic income risk and endogenous financial contracts, and therefore, inequality in household incomes, consumption and wealth. To do so, long-term contractual arrangements between risk-averse households and risk-neutral competitive financial intermediaries are embedded into a stationary version of the neoclassical growth model. We seek to integrate two foundational strands of the literature on macroeconomics with household heterogeneity. The first strand has developed and applied the standard incomplete markets model with uninsurable idiosyncratic income shocks and neoclassical production, as Bewley (1986), Imrohoroglu (1989), Huggett (1993) and Aiyagari (1994). In that model, households can trade assets to self-insure against income fluctuations, but these assets are not permitted to pay out contingent on a household’s individual income realization, thereby ruling out explicit insurance against income risk.

The second branch is the broad literature on recursive contracts and endogenously incomplete markets which permits explicit insurance, but whose scope is limited by informational or contract enforcement frictions. More specifically, we incorporate dynamic insurance contracts offered by competitive financial intermediaries (as analyzed previously in Krueger and Uhlig, 2006) into a neoclassical production economy. Financial intermediaries can commit to long term financial contracts, whereas households can not. The project thereby seeks to provide the macroeconomics profession with a novel, fully micro founded, analytically tractable model of neoclassical investment, production and the cross-sectional consumption and wealth distribution, where the limits to cross-insurance are explicitly derived from first principles of contractual frictions.

We aim to make two contributions, one substantive and one methodological in nature. On the substantive side, we provide a model that links the accumulation of the aggregate capital stock in the economy to the extent to which financial intermediaries can provide insurance to households against their idiosyncratic income risk, and their resulting demand for assets fund their contractual obligations. On the methodological side we construct and analytically (as well as numerically) characterize a dynamic optimal insurance model with
one-sided limited commitment and production as well as capital accumulation.

In a seminal paper, Aiyagari (1994) analyzed an economy in which households self-insure against idiosyncratic income fluctuations by purchasing shares of the aggregate capital stock. His model has become the canonical macro model with household heterogeneity. Variants of the model differ in the precise set of assets that households can trade, but the common assumption is that agents do not have access to financial instruments that provide direct insurance against the idiosyncratic income risk, despite the fact that such insurance would be mutually beneficial, given the underlying physical environment. There is now a large literature building on that model to link microeconomic inequality to macroeconomic performance, including applied policy (reform) analysis.¹ Any analysis of welfare in such models then necessarily comes with the caveat, that households may already be able to do better for themselves, if only the model builder allowed them to do so. As parameters or policies change, one may be concerned that these missing gains from trade shift too. Alternative general equilibrium workhorse models are therefore needed, in which households are allowed to pursue all contractual possibilities, limited only by informational or commitment constraints. The purpose of this paper to provide such an alternative model.

The contractual friction in our model arises from the inability of households to commit to future obligations implied by full insurance, risk sharing contracts. More precisely, we postulate financial markets in which perfectly intermediaries offer long-term insurance contracts to households. These financial intermediaries receive all incomes from a customer that has signed a contract, and can commit perfectly to future state-contingent consumption payments. Competition among intermediaries implies that the present discounted value of profits from these contracts is zero at the time of contract signing. The crucial friction that prevents perfect consumption insurance in the model is that households, at any moment, can costlessly switch to another intermediary, signing a new contract there. That is, we model relationships between financial intermediaries and private households as long-term contracts with one-sided limited commitment (the intermediary is fully committed, the household is not). This structure of financial markets is identical to the one assumed in the discrete-time, partial equilibrium model of Krueger and Uhlig (2006), which in turn builds on the seminal work of Harris and Holmstrom (1986), Thomas and Worrall (1989), Kehoe and Levine (1993) and Kocherlakota (1996).

In that paper, we showed that the one-sided limited commitment induces contracts with payments from the household to the intermediary that are front-loaded: when income is

¹See recent surveys by Heathcote, Storesletten and Violante (2011) and Krueger, Mitman and Perri (2016).
high, the household effectively builds up a stock of savings with the intermediary, which then finances the insurance offered by the intermediary against low income realizations down the road. In this paper we embed these contracts into a dynamic production economy, as in Aiyagari (1994). Now these contractual savings implied by back-loaded insurance contracts finance the aggregate capital stock of the economy. Effectively, financial intermediaries buy shares of the capital stock to fund their future liabilities from the insurance contracts they have signed with households. As in a standard neoclassical growth model, aggregate capital itself is accumulated linearly and used together with inelastically supplied in an aggregate Cobb-Douglas production function by a competitive sector of production firms.

Households supply their labor inelastically to these firms, but as in Bewley (1986), Imrohoroglu (1989), Huggett (1993) and Aiyagari (1994) their labor productivity and thus earnings are subject to idiosyncratic risk. This risk induces household insurance needs and thus generates a savings motive, which in turn finances the capital stock. Our model therefore provides a third (and intermediate) alternative neoclassical production economy with capital, relative to the self-insurance framework of Aiyagari (1994) and the full insurance framework (a.k.a. the standard neoclassical growth model with complete markets and implied full consumption insurance).

As a methodological innovation to the limited commitment general equilibrium literature we describe our model in continuous time. This is useful since, as we demonstrate, the optimal insurance contract is akin to an optimal stopping problem, and the use of continuous time avoids integer problems (the optimal stopping time falling in between two period) that arise in a discrete time setting. In order to obtain a sharp analytical characterization of the equilibrium we focus on the case where households have logarithmic utility and labor productivity can take only two values, one of which is zero. For this case, we provide a complete analytical characterization of the optimal consumption insurance contract as well as the stationary consumption distribution. Under mild restrictions on the parameters, we show that there is a unique equilibrium. We provide explicit formulas for this equilibrium, including the steady state level and return on the capital stock. We can also analytically calculate the stationary consumption distribution, and show that this distribution has a Pareto in shape, truncated by an upper mass point. Comparative statics with respect to the deep parameters of the model (and specifically, the parameters determining income risk, preferences and production technologies) deliver unambiguous results. We submit that this full analytical characterization of a stationary equilibrium is an additional, attractive benefit of
our model, and a welcome methodological advance, noting that Aiyagari-type models (as standard limited commitment economies with a continuum of households, as in Krueger and Perri, 2006) typically require numerical solutions. We therefore hope that our model structure can serve as an analytically tractable framework for applied work in macroeconomics that connects idiosyncratic risk to aggregate phenomena such as consumption distributions and the size of the economy.

1.1 Relation to the Literature

As discussed above, our broad aim in this paper is to connect the dynamic contracting literature with income risk and limited commitment to the quantitative, general equilibrium literature in macroeconomics with household heterogeneity discussed above. Our dynamic limited commitment risk sharing contract model builds on the theoretical work characterizing optimal contracts in such environments. Especially relevant is the subset of the literature that has done so in continuous time.

Specifically, Zhang (2013) studies a consumption insurance model with limited commitment similar to that in Krueger and Uhlig (2006), but permits the income process to be serially correlated finite state Markov chain, rather than a sequence of iid random variables. He also allows the outside option of the household to be a general function of the current income state, rather than simply autarky. The author derives the optimal consumption insurance contract. Grochulski and Zhang (2012) characterize the optimal contract in continuous time, under the assumption that the market return equals the discount rate, the outside option is autarky, and the income process follows a general geometric Brownian motion. The work by Miao and Zhang (2013) contains related results. Turning to general equilibrium treatments, Hellwig and Lorenzoni (2009) consider an endowment economy, in which two agents optimally share their risky income stream over time, subject to contractual constraints. The market return in their economy is shown to be zero under appropriate assumptions. Gottardi and Kubler (2015) study an endowment economy with finitely many (types of) agents and complete markets, but under the assumption that the short sales of the Arrow securities have have to be collateralized. Default on debt results in the loss of the collateral, but as in our work there is no additional punishment. The main focus of their work is to study existence and efficiency properties of equilibria in their model without capital. Although our focus is very different, we conjecture that the long-term risk sharing consumption allocations we characterize and then embed in a neoclassical production
economy with capital accumulation could also be decentralized as competitive equilibria in a model where households trade a full set of Arrow securities and physical capital, and where the short sales of the Arrow securities have to be collateralized by capital, akin to the market structure in Gottardi and Kubler (2015).

Perhaps most related to our work is the paper by Abdou et al. (2017). The authors study the existence, uniqueness and analytical characterization of a stationary equilibrium in a continuous time economy with idiosyncratic risk as well, but follow the standard incomplete markets literature in assuming that households can only trade a one-period risk-free bond, subject to a potentially binding borrowing constraint (and abstract from production and capital accumulation). As in our paper, equilibrium is characterized by two differential equations, one governing the optimal solution of the consumption insurance problem, and one characterizing the associated stationary distribution. Thus, methodologically, the papers complement each other by characterizing equilibria in the same physical environment, but under two fundamentally different market structures.

Returning to the substantive contribution of the paper, one interpretation of the contractual arrangements is that of firms who provide workers with long-term employment-wage contracts. A recent literature, building on the early work of Harris and Holmstrom (1986), emphasizes that firms provide insurance to its workers and their productivity fluctuations. Lamadon (2016) has calculated the optimal within-firm insurance mechanism, in the presence of a variety of sources of risk, including firm-specific risk, worker productivity risk and unobservable effort. Guiso, Pistaferri and Schivardi (2005) also argue, empirically, that the insurance of worker productivity by firms is an important mechanism to insulate workers from idiosyncratic shocks. Finally, Saporta-Eksten (2014) has shown that wages are lower after a spell of unemployment, which he interprets as a loss in productivity. In the context of our model this observation can alternatively be rationalized as part of the optimal consumption insurance contract, in the event productivity of the worker has dropped temporarily.

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2Thus they study a continuous time version of the Huggett (1993) economy.
2 The Model

2.1 Preferences and Endowments

Time is continuous. There is a population of a continuum of infinitely lived agents of mass 1. Agents have the period utility function

\[ u(c) = \log(c) \]

and discount the future at rate \( \rho > 0 \), so that the expected utility of a household born in period \( t \) is given by

\[ E \left[ \int_t^\infty e^{-\rho(\tau-t)} \log(c(\tau))d\tau \right]. \]

Labor productivity \( z_{it} \) of an individual agent \( i \) at time \( t \) is assumed to follow two-state Markov process that is independent across agents. More precisely, productivity can either be high, \( z_{it} = z > 0 \) or zero \( z_{it} = 0 \). Let \( Z = \{0, z\} \). The transition from high to low productivity occurs at rate \( \xi > 0 \), whereas the transition from low to high productivity occurs at rate \( \nu > 0 \). Since labor income will equal labor productivity times a common wage \( w \) for each household, we will use the terms (labor) productivity and income interchangeably in this paper. Households with low labor productivity also have some nontradable endowment \( \chi > 0 \) that they can consume if they do not sign up for a consumption risk sharing contract. Denote the utility from consuming the nontradable endowment by \( u = u(\chi) > -\infty \).

Given the stochastic structure of the endowment process the share of households with low and high income is equal to

\[ (\psi_l, \psi_h) = \left( \frac{\xi}{\xi + \nu}, \frac{\nu}{\xi + \nu} \right) \]

We assume that newborn households draw their productivity from the stationary income distribution and that the average labor productivity in the economy is equal to 1. Thus we assume that

\[ \frac{\nu}{\xi + \nu} z = 1. \]

\(^3\)This assumption avoids the complication that individuals that have not yet received the high income realization at least once and thus won’t be provided with consumption insurance (as we will show) are forced to consume 0.
For future reference we note that this assumption implies

\[ \nu(z - 1) = \xi \quad (1) \]

### 2.2 Technology

There is a competitive sector of production firms which uses labor and potentially capital to produce the final output good according to the production function

\[ F(K, L) = AK^\theta L^{1-\theta}. \]

where \( \theta \in (0, 1) \) denotes the capital share. The capital depreciates at a constant rate \( \delta \geq 0 \). Production firms seek to maximize profits, taking as given the market spot wage \( w \) per efficiency unit of labor and the market rental rate per unit of capital. Capital accumulation is linear and depreciates at rate \( \delta \). There is a resulting equilibrium rate of return or interest rate \( r \) for investing in capital. We dropped the subscript \( t \) to economize on notation, since we shall concern ourselves only with stationary equilibria and aggregate variables will be constant.

There is a competitive sector of intermediaries, who seek to maximize profits. Agents attempt to insure themselves against these income fluctuations with financial intermediaries. However, the commitment is one-sided only: while the intermediary can commit to the contract for all future, agents can leave the contract at any time they please and sign up with the next intermediary. Intermediaries compete for agents, and do not have resources on their own. Similar to Krueger-Uhlig (2006), newborn agents wait until their first time that they receive the high income. They then provide their chosen intermediary with a stream of “insurance premium payments”, while in the high income state, to finance subsequent payments for a potential “dry spell” of low productivity, until they transit to high income again. We assume that the law of large numbers applies at each individual intermediary or, alternatively, that there is full mutual insurance among intermediaries, so that intermediaries are only exposed to aggregate risk. We only examine stationary equilibria in this paper, thus rendering these intermediaries risk neutral. The intermediaries invest the premium payments in capital and therefore discount future streams of payments and incomes at the rate of return \( r \) on capital.
2.3 Timing of Events

In each instant of time, birth and death occur first. A newborn household draws labor productivity $z$ from the stationary income distribution and then signs a long-term consumption insurance contract with one of the many competing financial intermediaries, delivering lifetime utility $U^{\text{out}}(z)$. For surviving households, the current labor productivity $z$ is realized from the household-level Markov process. The household has the option of sticking with the previous intermediary or signing up with another intermediary, in the latter case receiving a contract delivering lifetime utility $U^{\text{out}}(z)$. Consumption is then allocated to the household according to the consumption insurance contract this household has signed in the past.

2.4 Equilibrium

Intermediary contracts promise some lifetime utility $U$ for the household per delivering a stochastic stream of future consumption. Given $U$ and given the current labor productivity $z$ of the household, the profit maximization objective of intermediaries is equivalent to minimizing the net present value $V(z, U)$ of the contract costs, i.e. to minimize the net present value of the difference between the stream of consumption of the household and its income. The income is given by the labor productivity $z(\tau)$ at future dates $\tau$ multiplied with the wage $w$. It will likewise be convenient to scale consumption by the wage level. In slight abuse of notation, let $c(\tau)w$ denote the consumption of the household at date $\tau$. In designing the contract, the intermediary needs to take into account that the household will depart, should the residual lifetime utility drop below promises the outside option $U^{\text{out}}(z)$ of promises, available when signing a new contract with some other intermediary.

Definition 1. For fixed outside options $U^{\text{out}}(z)$, with $z \in Z$, and a fixed wage $w$ and rate of return on capital or interest rate $r$, an optimal consumption insurance contract $c(\tau; z, U)$, $V(z, U)$ solves

$$V(z, U) = \min_{\langle c(\tau) \rangle \geq 0} \mathbb{E} \left[ \int_{t}^{\infty} e^{-r(\tau-t)} [wc(\tau) - wz(\tau)] d\tau \right] z(t) = z$$
subject to

\[ E \left[ \int_t^\infty e^{-\rho(\tau-t)} u(c(\tau)) d\tau \mid z(t) = z \right] \geq U \]
\[ E \left[ \int_s^\infty e^{-\rho(\tau-s)} u(c(\tau)) d\tau \mid z(s) \right] \geq U_{\text{out}}(z(s)) \text{ for all } s > t \]

for all \( t \geq 0 \), for all \( z \in Z \) and all \( U \in \left[ U_{\text{out}}(z), \frac{\bar{u}}{\rho} \right] \).

Note that the stationary structure of the model insures that the optimal consumption insurance contract does not depend on calendar time, but rather only on the income \( z \) the household is born with.

**Definition 2.** A Stationary Equilibrium consists of outside options \( \{U_{\text{out}}(z)\}_{z \in Z} \), consumption insurance contracts \( c(\tau, z, U) : \mathbb{R}_+ \times Z \times \left[ U_{\text{out}}(z), \frac{\bar{u}}{\rho} \right] \to \mathbb{R}_+ \) and \( V : Z \times \left[ U_{\text{out}}(z), \frac{\bar{u}}{\rho} \right] \to \mathbb{R} \), an equilibrium wage \( w \) and interest rate \( r \) and a stationary consumption probability density function \( \phi(c) \) such that

1. Given \( \{U_{\text{out}}(z)\}_{z \in Z} \) and \( r \), the consumption insurance contract \( c(\tau, z, U), V(z, U) \) is optimal in the sense of definition (1).

2. The outside options lead to zero profits of the financial intermediaries: for all \( z \in Z \)

\[ V(z, U_{\text{out}}(z)) = 0. \]

3. The interest rate and wage \((r, w)\) satisfy

\[ r = AF_K(K, 1) - \delta \quad (2) \]
\[ w = AF_L(K, 1) \quad (3) \]

4. The goods market clears

\[ \int wc\phi(c)dc + \delta K = AF(K, 1). \quad (4) \]

5. The capital market clears

\[ \frac{w \left[ \int c\phi(c)dc - 1 \right]}{r} = K \quad (5) \]
6. The stationary consumption probability density function is consistent with the dynamics of the optimal consumption contract as well as the stochastic structure of birth and death in the model.

2.4.1 Discussion of Equilibrium Definition

Several elements of this definition are noteworthy. The first two items formalize the notion that financial intermediaries compete for households by offering optimal consumption insurance contracts (item 1), and that their profits are driven to zero by perfect competition (item 2). These equilibrium requirements are identical to those in the endowment economy of Krueger and Uhlig (2006), accounting for the fact that the current model is cast in continuous time. Whereas item 3 contains the standard optimality conditions of the representative production firm, the statement of the capital market clearing condition in 5, as well as the inclusion of both the goods market clearing and the capital market clearing condition require further discussion.

In item 5, the right hand side $K = K^d$ is the demand for capital by the representative firm. The numerator on the left hand side is the excess consumption, relative to labor income, of all households, that is, the capital income required to finance the consumption that exceeds labor income. Dividing by the return to capital $r$ gives the capital stock households, or financial intermediaries on behalf of households, need to own to deliver the required capital income. Thus we can think of

$$K^s = \frac{w \left[ \int c\phi(c)dc - 1 \right]}{r}$$

(6)

as the supply of capital by the household sector, intermediated by the financial intermediaries. By restating the capital market clearing condition as

$$K^s(r) = K^d(r)$$

where $K^s(r)$ is defined in (6) and $K^d(r)$ is defined through (2) we will be able to provide a graphical analysis of existence and uniqueness of stationary equilibrium in $(K, r)$ space, analogously to the well-known figure contained in Aiyagari (1994) for the standard incomplete markets model.

Finally, we note that as long as $r \neq 0$, the usual logic of Walras law applies and one of the two market clearing conditions is redundant. To see this, note that the right hand side
of equation (4) can be written as

\[ AF(K, 1) = AF_L(K, 1) + AF_K(K, 1)K \]

and from equations (2) and (3) it follows that

\[ AF(K, 1) = w + (r + \delta)K. \]

Using this in equation (4) and rearranging implies, for \( r \neq 0 \), the capital market clearing condition (5). Thus for all \( r \neq 0 \) we can use either of the market clearing conditions in our analysis. The case \( r = 0 \), however, will require special attention, and we will argue in section 5 that even though the goods market clears for \( r = 0 \) under fairly general conditions, the capital market generically does not, indicating that a) \( r = 0 \) is generically not a stationary equilibrium interest rate and b) at \( r = 0 \) we need to study both the goods and the capital market clearing condition when analyzing a stationary equilibrium.

In order to do so, in the next sections we now aim to characterize the entire steady state equilibrium, including the stationary consumption distribution whose cumulative distribution function we denote by \( \Phi \) (with associated probability density function \( \phi \)). First we characterize the optimal consumption contract under various assumptions on the relationship between the constant interest rate \( r \) and the constant time discount rate \( \rho \) of the household. Then we discuss aggregation and the equilibrium determination of interest rates.

### 3 The Optimal Risk-Sharing Contract

The nature of the optimal consumption insurance contract depends crucially on the relationship between the subjective time discount factor \( \rho \) and the endogenous stationary equilibrium interest rate \( r \). We discuss the relevant cases in turn. First we discuss the case \( r = \rho \) which will deliver a sharp and very simple characterization of the optimal consumption contract that features full consumption insurance of the household after the first instance of having received high income. We then analyze the case \( r < \rho \) which will result in a partial consumption insurance, the relevant case for the general equilibrium in a wide range of model parameterizations. Finally, in our model we cannot exclude the possibility of equilibria in which the real interest rate exceeds the household time discount factor, and thus we
conclude by analyzing the optimal consumption contract under the assumption that \( r > \rho \), a case we call superinsurance.

### 3.1 Full Insurance in the Long Run: \( \rho = r \)

We first characterize the optimal consumption insurance contract for \((z, U^\text{out}(z))\) and then discuss how it looks for other promised lifetime \( U > U^\text{out}(z) \).

**Proposition 1.** Suppose that \( \rho = r \). In that case, the consumption contract has constant consumption \( c_l = 0 \) as long as \( z(t) = 0 \), and then consumption jumps up to some level \( c_h \) the instant income rises to \( z \) and remains there forever. Households born with income \( z \) instead consume \( c_h \) forever.

The proof is in the appendix. As shorthand, denote as

\[
V_l = V(0, U^\text{out}(0)) \\
V_h = V(z, U^\text{out}(z))
\]

and let \( V_{hl} \) denote the cost of a contract for the financial intermediary in which the household had high income in some previous periods (and thus currently consumes \( c_h \)) but now has no income 0. In what follows we characterize the net cost of this contract deflated by the wage level, and the let the wage-deflated cost be denoted by \( v = V/w \).

These cost levels of the financial intermediary satisfy the Hamilton-Jacoby-Bellman equations

\[
rv_l = c_l + \nu(v_h - v_l) \\
rv_h = c_h - z + \xi(v_{hl} - v_h) \\
rv_{hl} = c_h + \nu(v_h - v_{hl})
\]

Due to perfect competition of financial intermediaries (item 2 of the equilibrium condition) \( v_h = v_l = 0 \). Using these conditions in the equations above yields

\[
c_l = 0 \\
\xi v_{hl} = z - c_h \\
(r + \nu)v_{hl} = c_h
\]
and, solving the last two equations explicitly, and evaluating at \( r = \rho \), delivers

\[
    c_h = \frac{\rho + \nu}{\rho + \nu + \xi} z = c_h(\rho) \tag{8}
\]

\[
    v_{hl} = \frac{c_h}{\rho + \nu} > 0 \tag{9}
\]

Thus the optimal risk sharing contract collects a net insurance premium

\[
    z - c_h = \frac{\xi z}{\rho + \nu + \xi}
\]

from households with high income realizations and uses it to pay consumption insurance

\[
    c_h = \frac{(\rho + \nu) z}{\rho + \nu + \xi}
\]

to those households that have obtained insurance (those with previously high income realizations) and have currently low income. The expected net present discount value of this insurance, recognizing that with Poisson intensity \( \nu \) the household receives high income and leaves the current insurance spell is given by \( v_{hl} \) in equation (9).

### 3.2 Partial Insurance: \( \rho > r \)

We denote the expected discounted net cost to the financial intermediary from a consumption contract by \( v \). With two income levels 0 and \( z \) and Poisson arrival probabilities of switching down (\( \xi \)) and up (\( \nu \)) we can write the consumption dynamics and profit value functions as follows. The consumption dynamics is such that whenever a household has the high income, she consumes \( c_h \), and when income switches to 0 consumption drifts down according to the full insurance Euler equation

\[
    \frac{\dot{c}(t)}{c(t)} = -(\rho - r) = -g < 0
\]

where we have defined the growth rate of consumption as

\[
    g = \rho - r > 0.
\]
This consumption dynamics implies that

\[ c(t) = c_h e^{-gt} \]  \hspace{1cm} (10)

Furthermore, by perfect competition expected profits when entering the consumption contract with high income, \( v_h = 0 \), and similarly for entering the consumption contract with low income, \( v_l = 0 \).

Denote by \( t \) the time elapsed since having had the high income. Asymptotically, as \( t \to \infty \), consumption converges to \( c_l = 0 \). The Hamilton-Jacobi-Bellman equations read as

\[ rv_h = c_h - z + \xi (v(0) - v_h) \]  \hspace{1cm} (11)
\[ rv_l = c_l + \nu (v_h - v_l) \]  \hspace{1cm} (12)
\[ rv(t) = c(t) + \nu (v_h - v(t)) + \dot{v}(t) \]  \hspace{1cm} (13)

with terminal condition

\[ v(\infty) = v_l = 0. \]

Here \( v(t) \) is the cost of the consumption contract of an agent that had high income last \( t \) periods ago, and had low income for time interval \( t \) since.

Simplifying equations (11) to (13) again delivers \( c_l = 0 \). As before \( c_l = 0 \), and individuals with initially low income do not obtain any consumption insurance in the risk sharing contract. Insurance would require prepayment by the insurance company, and perfect competition plus limited commitment on the household side imply that this prepayment cannot be recouped later. The other two equations simplify to

\[ \xi v(0) = z - c_h \]  \hspace{1cm} (14)
\[ (r + \nu) v(t) = c(t) + \dot{v}(t) \]  \hspace{1cm} (15)

The first equation states that in the case of high income the household pays an insurance premium \( z - c_h \) which has to compensate the financial intermediary for the cost incurred during the low spell in which the losses for the intermediary amount to \( v(0) \). This equation relates the two endogenous variables \( c_h \) and \( v(0) \) to each other.

Equation (15) can be integrated (for details see appendix A.1.2), using the fact that
\[ c(t) = c_h e^{-gt} \] to obtain

\[ v(t) = \int_t^\infty e^{-(r+\nu)(\tau-t)} c_h e^{-\gamma \tau} d\tau = c_h e^{-gt} \int_t^\infty e^{-(r+\nu+\gamma)(\tau-t)} d\tau = \frac{e^{-gt}}{r + \nu + g} c_h \quad (16) \]

We can evaluate (42) at \( t = 0 \) to obtain\(^4\)

\[ v(0) = \frac{c_h}{r + \nu + g} \quad (17) \]

The optimal consumption contract has consumption driving down at rate \(-g = r - \rho\) from \( c_h \) towards \( c_l = 0 \), and asymptotically it reaches \( c_l = 0 \). Thus the consumption level \( c_h \) fully characterizes the consumption contract. Using equation (14) to substitute out \( v(0) \) in equation (43) yields

\[ \frac{c_h}{r + \nu + g} = \frac{z - c_h}{\xi} \]

or

\[ c_h = \frac{r + \nu + g}{r + \nu + g + \xi} z = \frac{\rho + \nu}{\rho + \nu + \xi} z \quad (18) \]

We summarize the optimal consumption contract in the following

**Proposition 2.** If \( \rho > r \), there exists a unique consumption level \( c_h \), defined in (44), characterizing the optimal consumption risk sharing contract

\[ c(t) = c_h e^{-gt} \]

where \( g = \rho - r \). Households that never have had high income consume \( c_l = 0 \) until the first time they receive high income if they were to sign a consumption risk sharing contract. The upper bound \( c_h \) is strictly increasing in \( z \), independent of the interest rate \( r \) and

\[ c_h = \frac{\rho + \nu}{\rho + \nu + \xi} z = c_h(\rho) \]

where \( c_h(\rho) \) is the full insurance consumption level.

\(^4\)Note that this cost \( v(0) \) is the counterpart to the insurance cost in equation (9) for the full insurance case; if \( r = \rho \) and thus \( g = 0 \), \( v(0) = v_{hi} \) in (9).
3.3 Super-Insurance: $\rho < r$

Optimal insurance contracts can also be characterized for the case in which the interest rate exceeds the time preference rate are a possibility. Off constraints, as in the partial insurance case consumption grows at a constant rate,

$$c_h(t) = c_h(0)e^{-gt}$$

but now $g = \rho - r < 0$, that is, consumption grows at the positive rate $r - \rho > 0$. As in the full and partial insurance case, households born with the low income cannot obtain insurance until their income switches to $z$, at which it jumps to $c_h(0)$, as in the partial and full insurance case. From that point on the household obtains income insurance (as in the full insurance case), but now consumption grows at rate $-g$ (rather than remains constant), until the household dies. The level $c_h(0)$ is determined by the zero profit condition of the intermediary, equating the expected revenue from the household’s income stream with the expected cost of the consumption contract. In Appendix A.1.3 we exploit the zero profit condition to determine the entry level of consumption $c_h(0)$ as

$$c_h(0) = \frac{r + g}{r} \cdot \frac{r + \nu}{r + \nu + \xi} z \quad (19)$$

As in the full insurance case, upon income increasing, so does consumption, but not as strongly as income. The household pays an insurance premium $z - c_h(0)$ in exchange for future consumption insurance and consumption growth. Note that since $g < 0$, the insurance premium is larger than in the full insurance case to finance future consumption growth, and as the interest rate $r$ converges to the time discount rate $\rho$ from above, the entry consumption level as well as the insurance premium converge to the full insurance consumption level $c_h(\rho)$ from below.

4 The Invariant Consumption Distribution

In the previous section we have shown that the optimal consumption insurance contract depends on the relationship between the endogenous market interest rate $r$ and the subjective discount factor $\rho$, which determines whether the contract is characterized by full or partial consumption insurance. The risk sharing contract in turn determines the stationary consumption distribution, which we now derive.
4.1 **Full Insurance in the Long Run:** $\rho = r$

In this case, the optimal consumption contract has only two consumption levels, $c_l = 0$ and $c_h$, as characterized in Section 3.1. Since individuals flow out of $c_l$ at positive rate $\nu$ and there is no inflow to this consumption level, the stationary consumption distribution places unit mass $\phi_h = 1$ on $c_h$.

4.2 **Partial Insurance:** $\rho > r$

In section 3.2 we characterized the optimal consumption contract under the parametric restriction that $r < \rho$. We showed that all households with high income consume $c_h(r)$, which is a function of the real interest rate, to be determined in equilibrium. Thus the stationary consumption distribution has a mass point at $c_h$ with mass $\phi(c_h) = \frac{\nu}{\nu + \xi}$.

Households with currently low income have a consumption process that satisfies

$$\dot{c}(t) = -gc(t)$$

with

$$g = \rho - r > 0$$

Finally, households that have not yet received high income consume $c_l = 0$, and the invariant consumption distribution possibly has a second mass point at $c_l$ equal to $\phi(c_l)$ whose size yet needs to be determined.

In $c \in (0, c_h)$ the consumption process follows a diffusion process with drift $-g$ (and no variance) and thus on $(0, c_h)$ the stationary consumption distribution satisfies the Kolmogorov forward equation (for the case of Poisson jump processes):

$$0 = -\frac{d[-gc\phi(c)]}{dc} - \nu\phi(c)$$

where the second term comes from the fact that with Poisson intensity $\nu$ the household has a switch to high income. Since

$$-\frac{d[-gc\phi(c)]}{dc} = -[-g\phi(c) - gc\phi'(c)] = g[\phi(c) + c\phi'(c)]$$
we find that on $c \in (0, c_h)$ the stationary distribution satisfies

$$g [\phi(c) + c\phi'(c)] = \nu \phi(c)$$

and thus

$$\frac{c\phi'(c)}{\phi(c)} = \frac{\nu}{g} - 1$$

and thus on this interval the stationary consumption distribution is Pareto with tail parameter $\frac{\nu}{g} - 1$, that is

$$\phi(c) = \phi_1 c^{(\frac{\nu}{g} - 1)}$$

where $\phi_1$ is a constant that is determined by the requirement that the stationary consumption distribution integrates to 1. In Appendix A.2.2 we determine this constant and thus prove:

**Proposition 3.** For any given $r \in (-\nu, \rho)$, the stationary consumption distribution is given by a mass point at and $c_h$ and a Pareto density below this mass point:

$$\phi_r(c) = \begin{cases} 
\xi \nu (c_h)^{-\frac{\nu}{\rho - r}(\nu + \xi)} c^{\frac{-\nu}{\rho - r} - 1} & \text{if } c \in (0, c_h) \\
\frac{\nu}{\nu + \xi} & \text{if } c = c_h
\end{cases}$$

Thus, for a given interest rate $r$ the invariant consumption distribution is completely characterized by the upper bound $c_h = \frac{\rho + \nu}{\rho + \nu + \xi} z$. Note that the shape of the consumption probability function in between the two mass points depends on the relative size of $\nu$, which governs the hazard rate of moving out of this part of the distribution (either through death or a positive income shock), and $g$, the speed at which consumption drifts down. The growth rate of the pdf is given by

$$\frac{d \log \phi_r(c)}{d \log c} = \frac{\nu}{g} - 1 = \frac{\nu}{\rho - r} - 1$$

We therefore have the following

**Corollary 1.** If $\nu < g$, then the pdf is strictly decreasing in $c$. If $\nu > g$ then the pdf is strictly increasing in $c$. If $\nu \in (g, 2g)$, then the pdf is strictly increasing and strictly concave in $c$. Finally, if $\nu > 2g$ then the pdf is strictly increasing and strictly convex in $c$.  

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4.3 Superinsurance: \( \rho < r \)

Although we can characterize the optimal consumption insurance contract in this case, since conditional on having received the high income once consumption of all individuals continues to drift up, there is no stationary consumption distribution for the case \( r > \rho \), and thus we can discard this case as a possibility for a stationary equilibrium.

5 General Equilibrium: The Market Clearing Interest Rate

In the previous sections we derived, as a function of the interest rate \( r \), the optimal consumption risk sharing contract as well as the associated invariant consumption distribution. Denote by

\[
C(r) = \int c \phi_r(c) dc
\]

aggregate consumption (scaled down by the aggregate wage) implied by these two entities. Recall that the goods market clearing condition (4) reads as

\[
w(r)C(r) + \delta K = AF(K, 1)
\]

or

\[
C(r) = \frac{AF(K, 1) - \delta K}{w(r)} := G(r)
\]

and the capital market clearing condition (5) can be written as

\[
\kappa^s(r) := \frac{K^s(r)}{w(r)} = \frac{C(r) - 1}{r} = \frac{K^d(r)}{w(r)} := \kappa^d(r)
\]

5.1 Supply of Consumption Goods and Demand for Capital

From (2) and (3), as in Aiyagari (1994) we can express the aggregate capital stock and wage as a function of the interest rate: \( K = K(r), w = w(r) \). Thus the aggregate net supply of goods is given by

\[
G(r) = \frac{AF(K(r), 1) - \delta K(r)}{AF_L(K(r), 1)} = 1 + \left[ \frac{AF_K(K(r), 1) - \delta}{AF_L(K(r), 1)} \right] K(r)
\]

and the aggregate demand for capital \( K^d(r) \) is implicitly defined by the marginal product of capital equation (2). The following result then immediately follows:
**Proposition 4.** Let the production function be of the form

\[ Y = AK^\theta L^{1-\theta}. \]

Then

\[ G(r) = 1 + \frac{\theta r}{(1-\theta)(r+\delta)} \]  
(22)

\[ \kappa^d(r) := \frac{K^d(r)}{w(r)} = \frac{\theta}{(1-\theta)(r+\delta)} \]  
(23)

The functions \( G(r), \kappa^d(r) \) are continuously differentiable on \( r \in (-\delta, \infty) \), and \( G(r) \) is strictly increasing, with \( \lim_{r \to -\delta} G(r) = -\infty \), \( G(r = 0) = 1 \) and \( \lim_{r \to \infty} G(r) = 1 + \frac{\theta}{1-\theta} \) and \( \kappa^d(r) \) is strictly decreasing, with \( \lim_{r \to -\delta} \kappa^d(r) = \infty \), \( \kappa^d(r = 0) = \frac{\theta}{(1-\theta)\delta} \) and \( \lim_{r \to \infty} \kappa^d(r) = 0 \).

Having very sharply characterized goods supply and capital demand from the firm side, the question of existence, uniqueness and characterization of a stationary equilibrium thus rests with the analysis of how aggregate consumption demand \( C(r) \) and thus (normalized) aggregate capital supply \( \kappa^s(r) = \frac{K^s(r)}{w(r)} \) depends upon the interest rate \( r \).

### 5.2 Demand of Consumption Goods and Supply of Capital

In order to characterize stationary equilibria we now have to characterize the demand for consumption good in a stationary equilibrium, as given by the demand function \( C(r) \)

\[ C(r) = \int c\phi_r(c)dc \]

and the associated supply of capital, which is, for \( r \neq 0 \),

\[ \kappa^s(r) := \frac{K^s(r)}{w(r)} = \frac{\int c\phi_r(c)dc - 1}{r} \]  
(24)

For \( r = 0 \), we need to determine \( \kappa^s(r = 0) \) through an application of L’Hopital’s rule as we will show below that

\[ \lim_{r \to 0} \int c\phi_r(c)dc - 1 = 0. \]

Depending on the relationship between the interest rate and the discount rate, the al-
location and corresponding invariant consumption distribution $\phi_r(c)$ features full ($r = \rho$) or partial ($r < \rho$) insurance. Given that the consumption allocation and invariant distribution differs qualitatively in the two cases we will discuss both cases in turn, and provide necessary and sufficient conditions for the existence of each type of equilibrium.

5.3 Existence and Uniqueness of Equilibrium with Full Insurance and $r = \rho$: A Knife-Edge Case

In this section we will provide conditions under which a stationary general equilibrium with an interest rate $r = \rho > 0$ exists, and thus provides full insurance. The invariant consumption distribution puts unit mass on consumption $c_h$ given by (8), so that, conditional on $r = \rho$ aggregate consumption demand is given by:

$$C(r = \rho) = c_h(r = \rho) = \frac{\rho + \nu}{\rho + \nu + \xi} z = 1 + \frac{\rho \xi}{\nu(\rho + \nu + \xi)}$$

(see appendix A.4.1 for the detailed calculations showing the second equality). Thus conditional on full insurance and $\rho = r$, the demand and supply of goods read as

$$C' = 1 + \frac{\rho \xi}{\nu(\rho + \nu + \xi)}$$

$$\kappa^s(\rho) : \kappa^{FI} = \frac{\xi}{\nu(\rho + \nu + \xi)}$$

Recall that from equation (23)

$$\kappa^d(\rho) = \frac{\theta}{(1 - \theta)(\rho + \delta)}$$

Thus there is a unique knife-edge time discount factor $\rho^{FI}$ such that $C(\rho^{FI}) = G(\rho^{FI})$ and $\kappa^s(\rho^{FI}) = \kappa^d(\rho^{FI})$, and it satisfies

$$\frac{\theta}{(1 - \theta)(\rho^{FI} + \delta)} = \frac{\xi}{\nu(\rho^{FI} + \nu + \xi)}$$

We now make the following assumption, which will insure the existence of a unique equilibrium:
Assumption 1. Let the exogenous parameters of the model satisfy $\theta, \nu, \xi, \rho > 0$ and

\[
\frac{\theta}{(1-\theta)(\rho+\delta)} \leq \frac{\xi}{\nu(\nu+\xi)}
\]  

(29)

Theorem 1. Let assumption 1 be satisfied. Then there exists a unique stationary equilibrium. If $\rho = \rho^{FI}$ then the equilibrium features full insurance. If $\rho \neq \rho^{FI}$, then the equilibrium features partial insurance. In contrast, if assumption 1 is violated, then no stationary equilibrium exists.

Proof: If assumption 1 holds with equality, the previous discussion showed that in this knife edge case the unique stationary equilibrium satisfies full insurance with $r^* = \rho$. If assumption 1 is violated, then the next section will show that any stationary equilibrium will require $r^* > \rho$, but section 4.3 has argued that in this case no stationary consumption distribution (and thus no stationary equilibrium) can exist. Finally, if assumption 1 holds with strict inequality, the next section will show there exists a unique equilibrium with interest rate $r^* < \rho$ and partial consumption insurance.

5.4 Unique Equilibrium with Partial Insurance and $r < \rho$

Now assume that assumption 1 is satisfied with strict inequality. We now show that under this assumption there exists a unique equilibrium with $r^* < \rho$ and a consumption allocation that features partial insurance. To do so we now derive aggregate consumption demand for the partial insurance case. Recall that

\[
c_h = \frac{\nu + \rho}{\xi + \nu + \rho} z
\]

Thus aggregate consumption demand and capital supply are given by (see appendix A.4.2 for the detailed derivations)

\[
C(r) = \frac{\nu}{\nu + \xi} c_h + \int_0^{c_h} c_{\xi} \frac{\nu (c_h)^{\xi}}{g(\nu + \xi)} c_s^{-1} dc = 1 + \frac{r\xi}{(\xi + \nu + \rho)(\nu + \rho - r)}
\]

(30)

\[\kappa^s(r) = \frac{\xi}{(\xi + \nu + \rho)(\nu + \rho - r)}
\]

(31)

The next proposition is proved in appendix A.4.2 and summarizes useful properties of the capital supply function.
Proposition 5. The capital supply function \( \kappa^s(r) \) is continuously differentiable and strictly increasing on \( r \in [-\delta, \rho) \), with \( \kappa^s(r = -\delta) = \frac{\xi}{(\xi + \nu + \rho)(\nu + \rho + \delta)} > 0 \) and

\[
\lim_{r \to \rho^-} \kappa^s(r) = \frac{\xi}{\nu (\xi + \nu + \rho)} = \kappa^{FI}
\]

We now use propositions 4 and 5 to complete the proof of theorem 1. Since \( \kappa^s(r), \kappa^d(r) \) are continuous on \(( -\delta, \rho)\) and \( \kappa^s(r) \) is strictly increasing, \( \kappa^d(r) \) is strictly decreasing, and \( \kappa^s(r = -\delta) < \lim_{r \to -\delta} \kappa^d(r) \), by the intermediate value theorem there exists a unique \( r^* \in (-\delta, \rho) \) such that

\[
\kappa^s(r^*) = \kappa^d(r^*)
\]
as long as

\[
\lim_{r \to \rho^-} \kappa^s(r) = \kappa^{FI} > \kappa^d(\rho)
\]
Assumption 1 guarantees precisely that this is the case. Furthermore, if instead assumption 1 is violated, then

\[
\lim_{r \to \rho^-} \kappa^s(r) = \kappa^{FI} < \kappa^d(\rho)
\]
and thus any stationary equilibrium must satisfy \( r^* > \rho \). However, for any \( r > \rho \), as argued in Section 4.3 there is no stationary distribution associated with the optimal consumption insurance contract (which does exist), and thus no stationary equilibrium exists.

5.4.1 Characterization of the Equilibrium and Comparative Statics

The unique equilibrium interest rate satisfies \( \kappa^s(r^*) = \kappa^d(r^*) \), or, exploiting equations (23) and (31)

\[
r^* = \frac{\theta(\xi + \nu + \rho)(\nu + \rho)}{\xi + \theta(\nu + \rho) - \xi(1 - \theta)}
\]

(32)

Associated with this interest rate is a stationary consumption distribution with mass point at \( c_h \) and a truncated Pareto distribution below \( c_h \)

\[
\phi_{r^*}(c) = \begin{cases} 
\frac{\xi c_h^\theta (r^*)^\theta}{(r^*)^{1-\theta} (\nu + \xi)} c_h^\nu (r^*)^{-1} & \text{if } c \in (0, c_h) \\
\frac{\nu}{(\nu + \xi)} & \text{if } c = c_h = \frac{\nu + \rho}{\xi + \nu + \rho} \ z
\end{cases}
\]

with Pareto coefficient \( \kappa = \frac{\nu}{(\nu + \rho)} - 1 \). The comparative statics of this unique equilibrium are immediate and summarized in the following
Proposition 6. Assume that assumption 1 is satisfied with strict inequality. Then the unique equilibrium interest rate $r^* \in (-\delta, \rho)$ is a strictly increasing function of $\rho + \nu, \theta$ and a strictly decreasing function of $\xi, \delta$. The associated equilibrium capital stock $K^* > K^{FI}$ is a strictly increasing function of $\xi$ and a strictly decreasing function of $\rho + \nu, \delta$.

5.4.2 Graphical Depiction of Unique Stationary Equilibrium

The unique equilibrium can be represented graphically, as in the standard incomplete markets models. Aiyagari (1994) plots asset demand and supply in $(r, K)$ space. We do the same here, in Figure 1a for a specific parameterization chosen in the welfare analysis conducted in the next section. As shown above, there is a unique equilibrium with a positive interest rate that clears the capital market.

Figure 1: This Figure Depicts the Equilibrium in the Capital and Goods Market

(a) Capital Demand and Supply as a Function of the Interest Rate $r$
(b) Goods Demand $w(r)C(r)$ and Net Supply $Y(r) - \delta K(r)$

Note that, even though at $r = 0$ the goods market clears (see Figure 1b which plots consumption demand and production net of depreciation), capital demand by firms exceeds capital supplied by households through the financial intermediaries, and thus $r = 0$ can only be implemented as a stationary equilibrium if the government (or some other outside entity) owns part of the capital $K^d(r = 0) - K^s(r = 0) > 0$. Thus, a stationary recursive equilibrium as defined in this paper with an interest rate $r = 0$ does in general not exist, despite the fact that the goods market clears at that interest rate. We further clarify this issue in the next section.
5.4.3 Why no Equilibrium with $r = 0$?  

We should again emphasize that for any interest rate $r \neq 0$, we can be sure that whenever the goods market condition, as stated in the equilibrium condition, clears, the capital market also clears. However, for $r = 0$, this is not necessarily the case. In this section we discuss why $r = 0$ is not a stationary equilibrium despite the fact that

$$C(r = 0) = 1 = G(r = 0)$$

The problem is that at $r = 0$ goods market clearing, as stated in the equilibrium definition, does not necessarily imply that the capital market clears. To see this, consider the capital market clearing condition (21)

$$K^s(r) := \frac{w(r)[C(r) - 1]}{r} = K^d(r)$$

Note that

$$\frac{w(r)}{r} = \frac{(1 - \theta)AK(r)^\theta}{r} = \frac{(1 - \theta)A}{r} \left( \frac{\theta A}{r + \delta} \right)^{\frac{\theta}{1 - \theta}}$$

and thus we can state the capital market clearing condition (21) as

$$K^s(r) := (1 - \theta)A^{\frac{1}{1 - \theta}} \left( \frac{\theta}{r + \delta} \right)^{\frac{\theta}{1 - \theta}} \frac{[C(r) - 1]}{r} = \left( \frac{\theta A}{r + \delta} \right)^{\frac{1}{1 - \theta}} := K^d(r)$$

(33)

Evidently, as stated $K^s(r = 0)$ is not well-defined, and has to be determined by L’Hopital’s rule. In Appendix A.5 we show that after application of L’Hopital’s rule the capital market clears only if the following knife edge condition needs to be satisfied:

$$\frac{\xi}{(\xi + \nu + \rho)(\nu + \rho)} = \frac{\theta}{(1 - \theta)\delta}$$

However, this condition is explicitly ruled out by assumption 1. Furthermore, under this assumption

$$K^s(r = 0) < K^d(r = 0)$$

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5We thank Marcus Hagedorn and Matt Rognlie for very helpful discussions leading to this section. Auclert and Rognlie (2018) show that the same argument applies to the standard incomplete markets model (as originally described in Aiyagari, 1994). Our discussion here is simply an adaptation of their argument.
and thus at $r = 0$ the goods market clears, but the household sector, through the financial intermediaries, supply insufficient assets to the capital market.

Note that one way to implement $r = 0$ as an equilibrium is to have the government own just the right amount of capital $K^g > 0$ such that

$$K^s(r = 0) + K^g = K^d(r = 0)$$

Since $r = 0$, the government does not collect any revenue from this ownership that would need to be distributed, and thus a simple adjustment of the equilibrium definition that has the government own just the right amount of the capital stock would implement $r = 0$ as a second equilibrium, with associated partial insurance consumption allocation.\(^6\)

### 5.4.4 Comparison to the Standard Incomplete Markets Model

Our model presents an alternative general equilibrium model with idiosyncratic income shocks to the canonical standard incomplete markets model. It is therefore instructive to compare stationary equilibria in both models. In Figure 2a we plot the (normalized by the wage) capital demand by firms and the capital supply, and display the market-clearing real interest rate, both in our model as well as the standard incomplete markets model,\(^6\)

\(^6\)This discussion appears to suggest that Walras’ law breaks down at $r = 0$. Since the issue emerges in an identical manner in the standard incomplete markets economy, and there households face a standard period budget constraint, it is more transparent if we clarify the issue in the context of that model. Using the notation of this paper, the household budget constraint in the standard incomplete markets model (in a stationary equilibrium) reads as

$$wc + a' = wy + (1 + r)a.$$ Integrating over all individuals and imposing stationarity yields

$$wC = w + rA.$$ where $A = A'$ are the constant aggregate asset holdings by all households (and we have imposed that aggregate labor efficiency units sum to 1.). Using Euler’s theorem, this equation can be rewritten as

$$wC = w + (F_K(K, 1) - \delta)K + (F_K(K, 1) - \delta)(A - K)$$

$$wC + \delta K = F(K, 1) + r(A - K)$$

We observe that asset market clearing $A = K$ always implies goods market clearing $wC + \delta K = F(K, 1)$. If $r \neq 0$ the reverse is also true: $wC + \delta K = F(K, 1)$ implies $A = K$. But for $r = 0$ goods market clearing does not imply asset market clearing, in the same way that Walras law $p* z(p) = 0$ only implies the statement that if $N-1$ markets clear (excess demand $d_i(p) = 0$ for all $i = 1, \ldots N - 1$), then the $N$-th market clears if the price vector $p$ has only non-zero elements. Thus the discussion in this section does not mean that Walras’ law is violated, but it simply does not follow from Walras law, at $r = 0$, that one market clearing condition implies the other.
as pioneered by Aiyagari (1994), and characterized in continuous time with two income shocks by Achdou et al. (2020). Figure 2b displays the associated stationary (normalized) consumption distributions of both models.

Figure 2: This Figure Compares Equilibria in Our Economy and the Aiyagari Model

We observe that for every interest rate, asset accumulation from the household sector is higher in the standard incomplete markets model than in the limited commitment model with endogenous consumption insurance contracts. Consequently the equilibrium interest rate is lower in that model. The associated stationary distribution is more unequal, reflecting explicit insurance in our model, and the fact that off constraints consumption drifts down faster in the standard incomplete markets model due to a larger wedge between the subjective time discount factor and the equilibrium interest rate in that model.

5.5 Multiple Partial Insurance Equilibria when the Elasticity of Substitution is not Unity

In the previous sections we have shown that with log-utility at most one stationary recursive competitive equilibrium exists. We now argue that deviating from a unit elasticity of substitution raises the possibility of multiple stationary recursive competitive equilibria. Thus, assume now that the period utility function is given by

\[ u(c) = \frac{c^{1-\sigma}}{1 - \sigma} \]

\(^7\text{Sargent et al. (2020)}\)
where $\frac{1}{\sigma} \neq 1$ is the intertemporal elasticity of substitution. All other model elements remain intact completely unchanged.

Evidently, the normalized capital demand function $\kappa^d(r)$ is unaffected since it is determined purely from the production side of the economy. The argument that there cannot be a stationary equilibrium with $r > \rho$ and the condition for a full insurance equilibrium remain intact completely unchanged. Thus, we continue to assume that Assumption 1 holds with strict inequality, so that we can focus on partial insurance equilibria with $r < \rho$. In Appendix B we show that optimal consumption insurance contract has exactly the same properties as in the log-case (consumption jumps up upon receiving high productivity and drifts down at rate $-g = -\frac{\rho - r}{\sigma}$) and that the stationary consumption distribution is still characterized by a mass point at the top and a truncated Pareto distribution below the top.

There we also derive that the normalized supply of capital is now given by

$$\kappa^s(r) = \frac{\xi}{(\xi + \nu + \frac{\rho - r}{\sigma} + r)(\nu + \frac{\rho - r}{\sigma})}$$

which of course specializes to the log-case analyzed above for $\sigma = 1$. The capital market clearing condition now reads as

$$\kappa^s(r) = \frac{\xi}{(\xi + \nu + \frac{\rho - r}{\sigma} + r)(\nu + \frac{\rho - r}{\sigma})} = \frac{\theta}{(1 - \theta)(r + \delta)} = \kappa^d(r) \quad (34)$$

The characterization of equilibrium remains fully analytically tractable since any equilibrium interest rate is a solution $r$ to this equation, which is a quadratic equation in $r$ for all $\sigma \in (0, \infty)$. We will now demonstrate that as long as $\sigma$ is sufficiently small (the IES is sufficiently large and the substitution effect is strong relative to the income effect), the capital supply function $\kappa^s(r)$ is upward-sloping in the interest rate and the equilibrium remains unique. On the other hand, for large enough $\sigma$ (small enough IES and thus small enough substitution effect), capital supply might be downward sloping, and is downward sloping if $\sigma = \infty$ and the lifetime utility function is of Leontieff form. This opens up the possibility of multiple stationary equilibria. We now discuss these results more formally.

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8Note that the total cost of the optimal consumption contract is only finite, and thus the capital supply function $\kappa^s(r)$ for the partial insurance case is only well-defined for interest rates satisfying $\nu + \frac{\rho - r}{\sigma} + r > 0$. For $\sigma \leq 1$, this imposes no further restrictions and $\kappa^s(r)$ is well-defined on $(-\infty, \rho)$. However, for $\sigma > 1$ the domain of $\kappa^s(r)$ is restricted since the interest rate cannot be too negative. Concretely, it is given by $\left(-\frac{\sigma\nu + \rho}{\sigma - 1}, \rho\right)$.
5.5.1 Intertemporal Elasticity of Substitution Equal to Zero: Leontieff Preferences

In this case $\sigma = \infty$, households are not willing to intertemporally substitute, the optimal consumption contract resembles that of the full-insurance case (consumption jumps upon the first high income receipt and stays constant thereafter), the stationary consumption distribution has unit mass at this consumption level, and the equilibrium interest rate satisfies the linear equation

$$\frac{\xi}{\nu (\xi + \nu + r)} = \frac{\theta}{(1 - \theta)(r + \delta)}$$

(35)

Proposition 7. Let $\sigma = \infty$ and thus the lifetime utility function is Leontieff. Then $\kappa^s(r)$ is well-defined and strictly decreasing in the interest rate on the interval $(-\nu, \infty)$.

1. Generically, there is a unique equilibrium interest rate $r^* > \max\{-\delta, -\nu\}$.

2. For the knife-edge case $\delta = \xi + \nu$, there is either no equilibrium, if in addition $\frac{\xi}{\nu} \neq \frac{\theta}{1 - \theta}$, or there is a continuum of equilibria $r^* \in (-\delta, \infty)$ if $\frac{\xi}{\nu} = \frac{\theta}{1 - \theta}$.

5.5.2 General IES $\sigma \neq 1$

The possibility that normalized capital demand is downward sloping in the interest rate for sufficiently large values of $\sigma$ (sufficiently weak substitution effects) admits the possibility of multiple equilibria. The next proposition shows that the equilibrium remains unique for $\sigma \leq 2$, but the possibility of exactly two stationary equilibria emerges for larger $\sigma$. These equilibrium interest rates are solution to a quadratic equation, and thus the characterization of equilibrium remains fully analytically tractable even for $\sigma \neq 1$.

Proposition 8. Assume that assumption 1 is satisfied with strict inequality.

1. If $\sigma < 1$, then $\kappa^s(r)$ is well-defined, continuous and strictly increasing on $r \in [-\delta, \rho]$. There exists a unique stationary equilibrium with interest rate $r \in (-\delta, \rho)$.

2. Let $\sigma > 1$ and $\frac{\alpha + \rho}{\sigma - 1} > \delta$. Then $\kappa^s(r)$ is well-defined and continuous on $r \in [-\delta, \rho]$.

   (a) Suppose $\sigma \in (1, 2]$ and $\xi \geq \delta$. Then $\kappa^s(r)$ is increasing on $r \in [-\delta, \rho]$ and there exists a unique stationary equilibrium with interest rate $r \in (-\delta, \rho)$.

   (b) If the condition in part (a) is violated and $\sigma < \infty$, then $\kappa^s(r)$ might have decreasing parts on $[-\delta, \rho)$. There exists at least one stationary equilibrium with $r \in (-\delta, \rho)$, but there might be two stationary equilibria with $r \in (-\delta, \rho)$. They satisfy the quadratic equation (34).
Proof. See Appendix B.3.1

This proposition shows that for wide parameter combinations uniqueness of equilibrium can be guaranteed, and identifies the part of the parameter space where multiple equilibria can emerge. Prior to exploring this issue numerically we observe that equilibrium interest rate(s) scale in the parameters representing rates per unit of time, i.e. the time discount rate, the income transition rates and the depreciation rate. Cutting each of these rates in half will cut down the equilibrium interest rate in half, and also preserves the number of equilibria.

**Corollary 2.** Suppose that \( r^* \in (\delta, \rho) \) is an equilibrium interest rate for parameters \( \rho, \delta, \xi, \nu, \sigma, \theta, A \). Let \( \alpha > 0 \). Then \( r = \alpha r^* \in (-\alpha \delta, \alpha \rho) \) is an equilibrium interest rate for parameters \( \alpha \rho, \alpha \delta, \alpha \xi, \alpha \nu, \sigma, \theta, A \).

**Proof.** See Appendix B.3.2

Since all equilibrium interest rates are solutions to the quadratic equation we could in principle characterize regions of the six-dimensional parameter space \( (\sigma, \theta, \delta, \nu, \xi, \rho) \) for which multiple equilibria emerge. Rather than doing so, in Figure 3 we display an example parameter combination with \( \sigma = 10 \) that exhibits two stationary equilibria. This example is not meant to be empirically realistic, but rather simply demonstrate that our model can indeed have multiple stationary equilibria.

Figure 3a plots normalized capital demand \( \kappa^d(r) \) and supply \( \kappa^s(r) \) against the interest rate \( r \). as shown in the proposition above, since \( \sigma > 2 \), both capital demand and supply are downward-sloping in the interest rate, and thus can intersect more than once. For the example the two equilibrium interest rates are \( r^*_1 = -2.4\% \) and \( r^*_2 = 13.6\% \). Note that while these interest rates appear to be large, by Corollary 2 we can always scale \( \delta, \rho, \nu, \xi \) to obtain two equilibrium interest rates closer to 0.

Figure 3b displays the consumption (normalized by the wage) distributions \( \phi_r(c) \) associated with the two equilibrium interest rates. The blue circled line corresponds to the low equilibrium interest rate, the red x-ed line to the high equilibrium interest rate. Both distributions have a mass point equal to \( \psi_h = \frac{\nu}{\xi + \nu} = 0.05 \), and a truncated power distribution below this mass point. The consumption mass point \( c_h(r) \) is increasing in the (equilibrium) interest rate as long as the IES \( 1/\sigma \) is less than 1, and thus the remaining probability mass spreads out over a larger support of the consumption distribution in the high interest rate equilibrium, falling more rapidly as consumption approaches zero. Thus, the consumption distribution has fewer individuals with very low consumption in the equilibrium with the
Figure 3: Two equilibria with partial insurance when $\sigma > 2$.

This figure plots an example of two equilibria, both with partial insurance, under parameter values $\sigma = 10$, $\theta = 0.25$, $\delta = 0.16$, $\nu = 0.05$, $\xi = 0.02$, $\rho = 0.4$. The two equilibrium interest rates are given by $r^*_1 = -0.0246$, $r^*_2 = 0.1357$. Left panel: solid line represents the capital supply curve $k^s(r)$, dashed line represents the capital demand curve $k^d(r)$. The right panel displays the two equilibrium consumption distributions.

high interest rate, and therefore better consumption insurance. Note that by Proposition 4 aggregate normalized normalized consumption $C(r) = \int c \phi_r dc$ is increasing in the interest rate $r$, a fact clearly visible when comparing the two consumption distributions. In Section D of the appendix we show that expected lifetime utility from the equilibrium consumption contract is indeed higher in the high-interest rate stationary equilibrium.

However, since the aggregate wage is lower in that equilibrium, this conclusion is premature, since aggregate consumption $w(r)C(r)$ can be higher in the low-interest rate equilibrium (and will be higher if the low interest rate equilibrium has a positive interest rate).\(^9\) In Appendix D we give a necessary and sufficient condition for the wage effect to dominate the consumption contract effect, so that steady state expected welfare is higher in the low-interest rate equilibrium.

Finally, in Figure 4 we display how the set of equilibrium interest rates changes as we change parameters. Specifically, we vary the depreciation rate and keep all other parameters constant. The figure shows that the example above with two stationary equilibria is not a knife-edge case, but rather emerges for a large set of parameter values, as long as $\sigma$ is sufficiently large, and therefore the intertemporal elasticity of substitution is sufficiently

\(^9\)As always in this neoclassical production economy, steady state aggregate consumption is maximized at the golden rule interest rate $r = 0$ and thus the closer the equilibrium interest rate is to the golden rule, the larger is aggregate consumption.
small and the income effect sufficiently potent relative to the substitution effect. The figure also shows that in many of these examples both equilibrium interest rates are positive. Finally, note that as the deprecation rate $\delta$ becomes too large, one part of the sufficient condition for existence of equilibrium in proposition 8, namely

$$\frac{\sigma \nu + \rho}{\sigma - 1} > \delta,$$

fails and a stationary equilibrium ceases to exist.

6 Conclusion

In this paper we have analytically characterized stationary equilibria in a neoclassical production economy with idiosyncratic income shocks and long-term one-sided limited commitment contracts. For an important special case (log-utility, two income state, zero income in the lower state) the equilibrium is unique and can be given in closed form, with complete comparative statics results. Given these findings, we would identify three immediately relevant next questions. First, on account of our use of continuous time, the endogenous optimal contract length is analytically tractable even outside the special case we have focused on thus far, and it will be important to generalize our findings about the stationary equilibria to the more general case.
Second, thus far we have focused on stationary equilibria, thereby sidestepping the question whether this stationary equilibrium is reached from a given initial aggregate stock, and what are the qualitative properties of the associated transition path. This also raises the conceptual question what is the appropriate initial condition for the distribution of outstanding insurance contracts.

Finally, thus far we have focused on an environment that has idiosyncratic, but no aggregate shocks, rendering the macro economy deterministic. Given our sharp analytical characterization of the equilibrium in the absence of aggregate shocks, we conjecture that the economy with aggregate shocks might be at least partially analytically tractable as well. We view these questions as important topics for future research.

References


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A Details of the Theoretical Results for Log-Utility

In this section we provide further details of the mathematical derivations in the paper. These
are straightforward but tedious manipulations which were therefore excluded from the main
text.

A.1 Optimal Consumption Insurance Contract

A.1.1 Full Insurance: \( r = \rho \)

Details are given in the main text. The optimal consumption contract is fully characterized
by the constant consumption level and associated insurance premium charged from high-
income households:

\[
\begin{align*}
    c_h(\rho) &= \frac{\rho + \nu}{\rho + \nu + \xi} \\
    v_{hl} &= \frac{c_h}{\rho + \nu} > 0
\end{align*}
\]

A.1.2 Partial Insurance: \( r < \rho \)

The differential equation determining the cost function is given by

\[
rv(t) = c(t)(0 - v(t)) + \nu(v_h - v(t)) + \dot{v}(t)
\]

and thus

\[
\dot{v}(t) = -(c(t)) - (r + \nu)v(t)
\]

Then, integrating the differential equation, we have

\[
v(t) = \int_t^\infty e^{-(r+\nu)(\tau-t)} (c(t)) \, d\tau = \int_t^\infty e^{-(r+\nu)(\tau-t)} \left(c_h e^{-gt}\right) \, d\tau
\]
One can of course check this by differentiating the solution to obtain back the differential equation. Solving the integral yields

\[ v(t) = c_h e^{-gt} \int_t^\infty e^{-(r+\nu+g)(\tau-t)} d\tau = \frac{c_h e^{-gt}}{r+\nu+g} \]

which is the equation in the main text. Evaluating at \( t = 0 \) yields

\[ v(0) = \frac{c_h}{r+\nu+g} \]

which is equation (43) in the main text.

### A.1.3 Superinsurance: \( r > \rho \)

First, characterize the cost for a household that starts with high income and consumes a profile

\[ c_h(t) = c_h(0)e^{-gt} \]

where \( g = \rho - r < 0 \). For transparency, split the net cost into the gross cost \( \kappa(t) \) and the revenue \( a(t) \) from the contract. The gross cost satisfies

\[ r\kappa(t) = c_h(t)(0 - \kappa(t)) + \dot{\kappa}(t) \]

or

\[ \kappa(t) = \int_t^\infty e^{-(r-t)\tau}c_h(\tau)d\tau \]

Note that Leibniz rule implies that

\[ \dot{\kappa}(t) = -c_h(t) + (r) \int_t^\infty e^{-(r-t)\tau}c_h(\tau)d\tau \]

\[ = -c_h(t) + (r)\kappa(t) \]
Solving the integral yields

\[
\kappa(t) = ch(0)e^{-gt} \int_t^\infty e^{-(r-g)(\tau-t)} d\tau = ch(0)e^{-gt} \int_0^\infty e^{-(r-g)\tau} d\tau = \frac{ch(0)e^{-gt}}{r+g}
\]

The revenue satisfies

\[
ra_h(t) = z(0 - a_h(t)) + \xi(a_l(t) - a_h(t)) + \dot{a}_h(t) \\
ra_l(t) = \nu(a_h(t) - a_l(t)) + \dot{a}_l(t)
\]

Evidently these two functions do not depend on time and solve

\[
ra_h = z(0 - a_h) + \xi(a_l - a_h) \\
ra_l = \nu(a_h - a_l)
\]

or

\[
(ra)_h = z + \xi(a_l - a_h) \\
(ra)_l = \nu(a_h - a_l)
\]

and subtracting one from the other

\[
(r + \nu + \xi)(a_h - a_l) = z \\
a_l = a_h - \frac{z}{(r + \nu + \xi)}
\]

and therefore

\[
a_h = \frac{r + \nu}{(r)(r + \nu + \xi)}z \\
a_l = \frac{\nu}{(r)(r + \nu + \xi)}z
\]
Therefore the net cost function satisfies

\[ v_h(t) = \kappa(t) - a_h \]
\[ v_h(0) = \kappa(0) - a_h = \frac{c_h(0)}{r + g} - \frac{(r + \nu)z}{(r + \nu + \xi)(r)} = 0 \]

which determines the entry level of consumption as

\[ c_h(0) = \frac{(r + g)}{(r)} \cdot \frac{(r + \nu)}{(r + \nu + \xi)} z < z \]
\[ = \frac{(r + g)}{(r)} \cdot \frac{(r + \nu)(\xi + \nu)}{(r + \nu + \xi)\nu} \]

since \( g < 0 \). Also note that since \( \kappa(t) \) is strictly increasing in \( t \), so is \( v(t) \). Therefore the household eventually becomes a liability; initially the intermediary collects (in expectation) positive contributions from the household, and these pay for insurance and a rising consumption profile later on. Finally, for \( r \searrow \rho \) we obtain

\[ \lim_{r \searrow \rho} c_h(0) = \frac{(\rho + \nu)}{(\rho + \nu + \xi)} z \]

which is identical to \( c_h \) from the full insurance case \( (r = \rho) \).

It remains to characterize the consumption dynamics and cost of the consumption contract for a household starting with income 0. Since consumption cannot fall below zero, prepayment for a rising consumption profile conditional on continuing with zero income cannot happen, and thus, as in the full insurance and partial insurance case households receive \( c_l = 0 \) until they switch to high income.

### A.2 Stationary Consumption Distribution

#### A.2.1 Full Insurance: \( r = \rho \)

The invariant consumption distribution satisfies

\[ \phi_h = \phi_h + \Delta \nu(1 - \phi_h) \]

Thus \( \phi_h = 1 \).
A.2.2 Partial Insurance: $r < \rho$

In the main text we showed that on the interval $(0, c_h)$ the consumption density has the form of a (truncated) Pareto distribution

$$\phi(c) = \phi_1(c)^\kappa$$

where the scale parameter $\phi_1 > 0$ has yet to be determined, and the Pareto parameter $\kappa = \frac{\nu}{g} - 1 = \frac{\nu - (\rho - r)}{\rho - r}$. Now we need to determine the constant $\phi_1$. Because of the mass point at $c_h$ it is easier to think of the cdf for consumption on $(0, c_h)$ given by $\Phi(c) = \frac{\phi_1(c)^{\kappa + 1}}{\kappa + 1}$. The inflow mass into this range is given by the mass of individuals at $c_h$ given by $\phi(c_h) = \frac{\nu}{\nu + \xi}$ times the probability $\xi$ of switching to the low income state, whereas the outflow is due to death and due to receiving the high income shock, and thus the stationary cdf has to satisfy

$$\nu \Phi(c_h) = \frac{\xi \nu}{\nu + \xi}$$

and therefore

$$\frac{\phi_1(c_h)^{\kappa + 1}}{\kappa + 1} = \frac{\xi \nu}{\nu + \xi}$$

Exploiting the fact that $\kappa + 1 = \frac{\nu}{g}$ we find

$$\phi_1 g(c_h)^{\frac{\nu}{g}} = \frac{\xi \nu}{\nu + \xi}$$

and thus

$$\phi_1 = \frac{\xi \nu}{g(\nu + \xi)}$$

and therefore the density on $(0, c_h)$ is given by

$$\phi(c) = \frac{\xi \nu}{g(\nu + \xi)} c_h^{\frac{\nu}{g} - 1}.$$ 

Finally we can investigate whether there is a potential mass point $\phi_l$ at 0. From the requirement that the consumption distribution needs to integrate to 1 we have

$$\phi_l + \phi_h + \int_0^{c_h} \phi(c) dc = 1.$$
which determines the Dirac mass of people at 0. Solving the integral yields
\[ \int_0^{c_h} \phi(c) dc = \frac{1}{\nu} \frac{\xi \nu (c_h)^{-\frac{\nu}{\sigma}}}{\nu (\nu + \xi) c_h^{\nu}} \bigg|_0^{c_h} = \frac{\xi \nu}{\nu (\nu + \xi)} \left( 1 - \left( \frac{0}{c_h} \right)^{\frac{\nu}{\sigma}} \right) = \frac{\xi}{\nu + \xi} \]

and therefore \( \phi_l = 0 \), which is intuitive since households leave \( c_l = 0 \) with positive probability and there is no inflow into this point. Thus the stationary distribution places zero mass on \( c_l \). Thus the analytical characterization of the stationary consumption distribution summarized in proposition 3 of the main text follows.

**A.2.3 Superinsurance: \( r > \rho \)**

Here we argue that a stationary consumption distribution cannot exist with \( r > \rho \). All households experiencing a jump to high income jump to \( c_h(0) \) and immediately drift up in the consumption distribution, so there is no mass point at \( c_h(0) \). Instead, there is a continuous consumption density on \([c_h(0), \infty)\) with power and scale parameters that need to determined in the same way as we did for the \( r < \rho \) case.

In \( c \in [c_h(0), \infty) \) the consumption process follows a diffusion process with drift \(-g = \frac{r - \rho}{\sigma} > 0\) (and no variance) and thus on this interval the stationary consumption distribution satisfies the Kolmogorov forward equation (for the case of Poisson jump processes):

\[ 0 = - \frac{d[-g c \phi(c)]}{dc} \]

Since
\[ - \frac{d[-g c \phi(c)]}{dc} = - [ -g \phi(c) - g c \phi'(c) ] = g [ \phi(c) + c \phi'(c) ] \]

we find that on \( c \in (c_h(0), \infty) \) the stationary distribution satisfies
\[ 0 = g [ \phi(c) + c \phi'(c) ] \]

and thus the tail parameter of the truncated Pareto distribution is given by:

\[ \kappa := \frac{c \phi'(c)}{\phi(c)} = -1 \]

But this implies that
\[ \int_{c_h}^{\infty} c \phi(c) dc = \infty \]
and thus no stationary consumption distribution with finite aggregate consumption can exist in the case of $\rho < r$, ruling out the existence of a stationary equilibrium in this case.

A.3 Goods Supply and Capital Demand

The (normalized by wages) supply of consumption goods in the stationary equilibrium is given by

$$G(r) = \frac{AF(K(r), 1) - \delta K(r)}{AF_L(K(r), 1)} = 1 + \frac{[AF_K(K(r), 1) - \delta] K(r)}{AF_L(K(r), 1)}$$

Calculating the capital stock and wages for Cobb-Douglas production from the optimality condition yields

$$K(r) = \left(\frac{\theta A}{r + \delta}\right)^{\frac{1-\theta}{\theta}}$$
$$w(r) = (1 - \theta)AK^\theta$$

and thus

$$\frac{[AF_K(K(r), 1) - \delta] K(r)}{AF_L(K(r), 1)} = \frac{r}{(1 - \theta)AK^\theta} = \frac{r\theta}{(1 - \theta)(r + \delta)}$$

and thus

$$G(r) = 1 + \frac{r\theta}{(1 - \theta)(r + \delta)}$$
$$\kappa^d(r) = \frac{K^d(r)}{w(r)} = \frac{[K^d(r)]^{1-\theta}}{(1 - \theta)A} = \frac{\theta}{(1 - \theta)(r + \delta)}$$

A.4 Capital Supply and Consumption Demand

A.4.1 Full Insurance: $r = \rho$

Here we expand on the algebra in the main text for the calculation of aggregate consumption demand in the case of $\rho = r$. The only requirement is that $\rho = r > -(\nu + \xi)$. 
\[ C(r) = \phi_h c_h(r) \]
\[ = \left(1 - \frac{\nu r}{\nu (r + \nu + \xi)}\right) \left(1 - \frac{\nu z}{\xi + \nu}\right) + \left(\frac{\xi + \nu}{\nu} \right) \left(\frac{r + \nu}{\xi + \nu}\right) \frac{\nu z}{\xi + \nu} \]
\[ = 1 - \frac{\nu r}{\nu (r + \nu + \xi)} + \left(\frac{(\nu + \xi)(r + \nu) + \nu r - (\nu)(r + \nu + \xi)}{\nu (r + \nu + \xi)}\right) \nu z \frac{1}{\xi + \nu} \]
\[ = 1 - \frac{\nu r}{\nu (r + \nu + \xi)} + \frac{\nu z}{\nu (r + \nu + \xi)} \]
\[ = 1 + \frac{\nu z (r + 1)}{\nu (r + \nu + \xi)} \]
and using equation (1) we have
\[ C(r) = \rho = 1 + \frac{\rho \xi}{\nu (\rho + \nu + \xi)} \]
\[ \kappa^s(r) = \rho = \frac{\xi}{\nu (\rho + \nu + \xi)} \]

A.4.2 Partial Insurance: \( r < \rho \)

In this section we collect the details of the derivations about the properties of the capital supply function \( \kappa^s(r) \) in the partial insurance case. Direct calculations reveal that aggregate consumption demand and capital supply are given by:
\[ C(r) = \frac{\nu}{\nu + \xi} c_h + \int_0^{c_h} c \xi \nu (c_h)^{-\frac{\nu}{\nu + \xi}} c^{\xi - 1} dc = \frac{\nu}{\nu + \xi} \frac{\xi + \nu + g}{\nu + g} c_h \]
\[ = \frac{\xi + \nu + g}{\nu + g} \frac{r + \nu + g}{\xi + r + \nu + g} \]
\[ = \left(1 + \frac{\xi}{\nu + g}\right) \left(1 - \frac{\xi}{\xi + \nu + g(r) + r}\right) \]
\[ = 1 + \frac{\xi}{\nu + g} - \frac{\xi}{\xi + \nu + g + r} - \frac{\xi^2}{(\xi + \nu + g + r)(\nu + g)} \]
\[ = 1 + \frac{\nu}{\nu + g} \frac{\xi}{\xi + \nu + g + r} \]
\[ = 1 + \frac{\nu}{\xi + \nu + g + r} \frac{\xi}{\nu + g} \]
\[ = 1 + \frac{\nu}{\xi + \nu + g + r} \frac{\xi}{\nu + g} \]
\[ = 1 + \frac{\nu}{\xi + \nu + g + r} \frac{\xi}{\nu + g} \]
\[ = 1 + \frac{\nu}{\xi + \nu + g + r} \frac{\xi}{\nu + g} \]
\[ \kappa^s(r) = \frac{\xi}{\xi + \nu + \rho} \frac{\xi}{\nu + \rho - r} \]

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It follows immediately that on \([-\delta, \rho]\) the function \(\kappa^s(r)\) is continuously differentiable and strictly increasing. Finally we show that \(C(r)\) and \(\kappa^s(r)\) are continuous at \(r = \rho\). From the full-insurance case we recall that consumption demand and capital supply for the \(r = \rho\) full insurance case were given by

\[
C(r = \rho) = 1 + \frac{\rho \xi}{\nu(\rho + \nu + \xi)} \quad (37)
\]

\[
\kappa^s(r = \rho) = \frac{\xi}{\nu(\xi + \nu + \rho)} \quad (38)
\]

and it follows that aggregate consumption demand and capital supply are continuous in the interest rate at \(r = \rho\) since

\[
\lim_{r \to \rho} C(r) = 1 + \frac{\rho \xi}{(\rho + \nu + \xi)\nu} = C(r = \rho) \quad (39)
\]

\[
\lim_{r \to \rho} \kappa^s(r) = \frac{\xi}{(\xi + \nu + \rho)\nu} = \kappa^s(r = \rho) \quad (40)
\]

### A.5 Equilibrium with \(r = 0\)? Details

For \(r = 0\) the supply of capital has to be determined by L’Hôpital’s rule

\[
K^s(r = 0) = 0 = \lim_{r \to 0} \frac{w(r) [C(r) - 1]}{r} = \lim_{r \to 0} \frac{[C(r) - 1]}{r/w(r)} = \frac{[C'(r = 0)]}{\lim_{r \to 0} \frac{dr/w(r)}{dr} |_{r=0}} = \frac{(1 - \theta)A^{1/\alpha} \theta^{\frac{\alpha}{\alpha - \theta}}}{(\delta)^{\frac{\alpha}{\alpha - \theta}}} C'(r = 0) = \frac{(1 - \theta)A^{1/\alpha} (\theta/\delta)^{\frac{\alpha}{\alpha - \theta}} \xi}{(\xi + \nu + \rho)(\nu + \rho)}
\]

since

\[
\lim_{r \to 0} \frac{dr/w(r)}{dr} = [(r + \delta)^{\frac{\alpha}{\alpha - \theta}} + r \frac{\theta}{1 - \theta}(r + \delta)^{\frac{\alpha}{\alpha - \theta} - 1}] * \frac{1}{(1 - \theta)A(\theta A)^{\frac{\alpha}{\alpha - \theta}}}
\]

\[
\lim_{r \to 0} \frac{dr/w(r)}{dr} \bigg|_{r=0} = \frac{(\delta)^{\frac{\alpha}{\alpha - \theta}}}{(1 - \theta)A(\theta A)^{\frac{\alpha}{\alpha - \theta}}}
\]

Thus

\[
K^s(r = 0) = \frac{(1 - \theta)A^{1/\alpha} \theta^{\frac{\alpha}{\alpha - \theta}}}{(\delta)^{\frac{\alpha}{\alpha - \theta}}} C'(r = 0)
\]
and capital market clearing at \( r = 0 \) then requires

\[
\frac{C'(r = 0)}{(\delta A \theta (\theta A)^{\frac{1}{\gamma}})} = \left( \frac{\theta A}{\delta} \right)^{\frac{1}{1-\gamma}}
\]

and thus

\[
C'(r = 0) = \frac{\theta}{(1-\theta)\delta} = G'(r = 0),
\]

in addition to

\[
C(r = 0) = G(r = 0)
\]

But for \( C'(r = 0) = G'(r = 0) \) to hold the knife edge condition stated in the main text needs to be satisfied:

\[
\frac{\xi}{(\xi + \nu + \rho)(\nu + \rho)} = \frac{\theta}{(1-\theta)\delta}
\]

B  General CRRA Utility

The analysis for the full insurance case goes through completely unchanged, since at \( r = \rho \) the growth rate of consumption and thus the aggregate consumption demand and capital supply is unaffected by the intertemporal elasticity of substitution \( 1/\sigma \). Here we focus on the case \( \rho > r \).

B.1  Optimal Consumption Contract

As in the log-case, whenever a household has the high income, she consumes \( c_h \), and when income switches to 0, consumption drifts down according to the full insurance Euler equation

\[
\frac{\dot{c}(t)}{c(t)} = \frac{\rho - r}{\sigma} = -g < 0
\]

where the growth rate of consumption is now defined as

\[
g = \frac{\rho - r}{\sigma} > 0.
\]

The log-utility case is of course just a special case where \( \sigma = 1 \) and thus \( g = \rho - r \).

The steps of deriving the optimal consumption contract and associated cost then pro-
ceeds completely in parallel to the log-case. Consumption is given as

$$c(t) = c_h e^{-gt} \quad (41)$$

and the cost of the contract is given by

$$v(t) = \frac{c_h e^{-gt}}{r + \nu + g} \quad (42)$$

Evaluating (42) at $t = 0$ gives

$$v(0) = \frac{c_h}{r + \nu + g} \quad (43)$$

Using equation (14), which continues to hold unchanged, to substitute out $v(0)$ in equation (43) yields

$$\frac{c_h}{r + \nu + g} = \frac{z - c_h}{\xi}$$

or

$$c_h(r) = \frac{r + \nu + g}{r + \nu + g + \xi} z = \frac{1}{1 + \frac{\xi}{r(1 - \frac{1}{2}) + \nu + \xi}} z \quad (44)$$

### B.2 Invariant Consumption Distribution

As in the log-case, on $c \in (0, c_h)$ the consumption process follows a diffusion process with drift $-g$ (and no variance) and thus on $(0, c_h)$ the stationary consumption distribution satisfies the Kolmogorov forward equation (for the case of Poisson jump processes):

$$0 = -\frac{d}{dc} \left[-gc\phi(c)\right] - \nu\phi(c)$$

where the second term comes from the fact that with Poisson intensity $\nu$ the household has a switch to high income. Since

$$-\frac{d}{dc} \left[-gc\phi(c)\right] = -\left[-g\phi(c) - gc\phi'(c)\right] = g \left[\phi(c) + c\phi'(c)\right]$$

we find that on $c \in (0, c_h)$ the stationary distribution satisfies

$$\frac{c\phi'(c)}{\phi(c)} = \frac{\nu}{g} - 1$$
and thus on this interval the stationary consumption distribution is Pareto with tail parameter \( \kappa = \frac{\nu}{g} - 1 \), that is

\[
\phi(c) = \phi_1 c^{\left(\frac{\nu}{g} - 1\right)}
\]

where \( \phi_1 \) is a constant that needs to be determined. Now we need to determine the constant \( \phi_1 \). Because of the mass point at \( c_h \) is is easier to think of the cdf for consumption on \((0, c_h)\) given by \( \Phi(c) = \frac{\phi_1(c)^{\kappa+1}}{\kappa+1} \). The inflow mass into this range is given by the mass of individuals at \( c_h \) given by \( \phi(c_h) = \frac{\nu}{\nu+\xi} \) times the probability \( \xi \) of switching to the low income state, whereas the outflow is due to receiving the high income shock, and thus the stationary cdf has to satisfy

\[
\nu \Phi(c_h) = \frac{\xi \nu}{\nu + \xi}
\]

and therefore

\[
\nu \frac{\phi_1(c_h)^{\kappa+1}}{\kappa+1} = \frac{\xi \nu}{\nu + \xi}
\]

Exploiting the fact that \( \kappa + 1 = \frac{\nu}{g} \) we find

\[
\phi_1 g(c_h)^{\frac{\nu}{g}} = \frac{\xi \nu}{\nu + \xi}
\]

and thus

\[
\phi_1 = \frac{\xi \nu (c_h)^{-\frac{\nu}{g}}}{g(\nu + \xi)}
\]

and therefore the density on \((0, c_h)\) is given by

\[
\phi(c) = \frac{\xi \nu (c_h)^{-\frac{\nu}{g}}}{g(\nu + \xi)} c^{\frac{\nu}{g} - 1}.
\]

Therefore the stationary consumption distribution is now given by:

\[
\phi_r(c) = \begin{cases} 
\frac{\xi \nu (c_h(r))^{\frac{\nu}{g}}}{g(\nu + \xi)} c^{\frac{\nu}{g} - 1} & \text{if } c \in (0, c_h) \\
\frac{\nu}{\nu + \xi} & \text{if } c = c_h
\end{cases}
\]

Thus, for a given interest rate \( r \) the invariant consumption distribution is completely characterized by the upper bound \( c_h(r) = \frac{r+\nu+g}{r+\nu+g+\xi} \).
B.3 Equilibrium

It remains to determine the aggregate consumption demand \( C(r) \) and the normalized capital supply function \( \kappa^s(r) \). Direct calculations reveal that aggregate consumption demand and capital supply are given by:

\[
C(r) = \frac{\nu}{\nu + \xi} c_h(r) + \int_0^{c_h(r)} \frac{\xi \nu (c_h(r))^{-\frac{\nu}{\sigma}} c^{-\frac{\nu}{\sigma}}} {g(\nu + \xi)} \, dc = \frac{\nu}{\nu + \xi} \frac{\xi + \nu + g(r)} {\nu + g(r)} c_h(r)
\]

\[
= \frac{\xi + \nu + g(r)} {\nu + g(r)} \frac{r + \nu + g(r)} {\xi + r + \nu + g(r)}
\]

\[
= \left( 1 + \frac{\xi}{\nu + g(r)} \right) \left( 1 - \frac{\xi}{\xi + \nu + g(r) + r} \right)
\]

\[
= 1 + \frac{\xi}{\nu + g(r)} - \frac{\xi}{\xi + \nu + g(r) + r} - \frac{\xi^2}{(\xi + \nu + g(r) + r)(\nu + g(r))}
\]

\[
= 1 + \frac{\xi}{r \xi} = \frac{\xi}{\xi + \nu + g(r)}
\]

\[
\kappa^s(r) = \frac{\xi}{(\xi + \nu + \frac{\rho - r}{\sigma} + r)(\nu + \frac{\rho - r}{\sigma})}
\]

\[
= \frac{\xi}{(\xi + \nu + \frac{\rho - r}{\sigma} + (1 - \frac{1}{\sigma}) r)(\nu + \frac{\rho - r}{\sigma} - \xi)}
\]

where we have repeatedly used \( g(r) = \frac{\rho - r}{\sigma} \), as stated in the main text.

B.3.1 Proof of Proposition 8

Proof. The first step of the proof is to establish that the normalized capital supply function is well-defined and continuous on \( r \in [-\delta, \rho] \). The previous section gave \( \kappa^s(r) \) in closed form, and it is evidently continuous and well-defined on \( [-\delta, \rho] \) as long as both terms of the denominator are strictly positive. Since \( r \leq \rho \), the second term in the denominator of equation (45) is always strictly positive.

The total cost of the optimal consumption contract is only finite for interest rates satisfying \( \nu + \frac{\rho - r}{\sigma} + r > 0 \). The derivation of the entry consumption level \( c_h(r) \) required a division by this term, and thus we need to make sure it is strictly positive. For \( \sigma \leq 1 \), this is automatically satisfied, but for \( \sigma > 1 \) this requires \( -\frac{\alpha \rho + \beta}{\sigma - 1} < -\delta \), so that \( c_h(r = -\delta) \) and thus \( \kappa^s(r = -\delta) \) are well-defined. This is the condition imposed in part 2.a of the proposition, and is also sufficient for the first term in the denominator of (45) to be strictly positive on \( [-\delta, \rho] \). Thus \( \kappa^s(r) \) is well-defined and continuous on \( r \in [-\delta, \rho] \).
Since by Assumption 1 we have $\kappa^s(r = \rho) > \kappa^d(r = \rho)$, that $\kappa^s(r = -\delta) < \kappa^d(r = -\delta) = \infty$, it follows that $\kappa^s$ and $\kappa^d$ intersect at least once in $(-\delta, \rho)$. This establishes the existence of a stationary equilibrium.

The uniqueness of equilibrium follows if $\kappa^s(r)$ is increasing (given that $\kappa^d(r)$ is strictly decreasing). The derivative of $\kappa^s(r)$ is given by
\[
\frac{d\kappa^s(r)}{dr} = \xi \left[ \frac{2}{\sigma} - 1 \right] \left[ \frac{\nu - r}{\sigma} + \nu \right] + \frac{\xi + r}{\sigma} \left[ \frac{r}{\xi + \nu + \frac{\nu - r}{\sigma} + r} \left( \nu + \frac{\nu - r}{\sigma} \right) \right]^2
\]

A sufficient condition for this expression to be positive is $\sigma < 1$ (part 1 of the proposition) or $\sigma \in (1, 2]$ and $\xi \geq \delta$ (part 2a of the proposition). Part 2b follows from the fact that equation (34) is a quadratic equation, and thus has at most two solutions (and we have already established that under the assumptions made it has at least one solution. The numerical example in the main text shows that the statement in 2b of the proposition is not vacuous.

**B.3.2 Proof of Corollary 2**

**Proof.** In general equilibrium interest rates are real-valued solutions to the quadratic equation
\[
0 = F(r) \equiv A_2 r^2 + A_1 r + A_0
\]
where
\[
A_0 = (\sigma - 1)^2 \left[ \xi \delta - \theta \nu^2 - \xi \theta (\delta + \nu) \right] + (\sigma - 1) \left[ -2\theta \nu (\nu + \rho) - \xi \left( 2\delta (\theta - 1) + \theta (2\nu + \rho) \right) \right]
- \theta \left( \nu + \rho \right)^2 + \xi \left( \delta (\theta - 1) + \theta (\nu + \rho) \right)
\]
\[
A_1 = - (\sigma - 1)^2 \left( \theta (\nu + \xi) - \xi \right) - (\sigma - 1) \left( \theta (\rho + \xi) - 2\xi \right) + \theta (\rho + \nu) + \xi
\]
\[
A_2 = \theta (\sigma - 1)
\]

The coefficients $A_0, A_1, A_2$ defined above are functions of the parameters. Note that
\[
A_0 (\alpha \rho, \alpha \delta, \alpha \xi, \alpha \nu; \sigma, \theta) = \alpha^2 A_0 (\rho, \delta, \xi, \nu; \sigma, \theta)
\]
\[
A_1 (\alpha \rho, \alpha \xi, \alpha \nu; \sigma, \theta) = \alpha A_1 (\rho, \xi, \nu; \sigma, \theta)
\]
and $A_2(\sigma, \theta)$ does not depend on $\rho, \delta, \xi, \nu$. Define

$$F(r; \alpha) = A_2(\sigma, \theta) r^2 + A_1(\alpha \rho, \alpha \xi, \alpha \nu; \sigma, \theta) r + A_0(\alpha \rho, \alpha \delta, \alpha \xi, \alpha \nu; \sigma, \theta)$$

Then

$$\alpha^2 F(r; 1) = F(\alpha r; \alpha)$$

Hence, if $\bar{r}$ solves $F(\bar{r}; 1) = 0$, then $r = \alpha \bar{r}$ solves $F(r; \alpha) = 0$. \hfill \Box

### C \quad \text{Steps to find an example for multiple equilibria}

We take the following steps to find two partial insurance equilibria:

1. Fix $A = 1, \gamma = 0$, find a set of parameters $\sigma, \theta, \delta, \nu, \xi, \rho$ such that
   \begin{itemize}
   \item $\theta \in (0, 1), \sigma > 1, \delta > 0, \nu > 0, \xi > 0, \rho > 0$
   \item $\xi + \nu < \delta$
   \item $-\frac{A_1}{2A_2} \in (- (\xi + \nu), \rho)$
   \item $F\left(-\frac{A_1}{2A_2}\right) > 0$
   \end{itemize}

2. Decrease $\delta$ and keep all other parameters fixed, until $F\left(-\frac{A_1}{2A_2}\right)$ is slightly below 0. Since $A_2, A_1$ does not depend on $\delta$, as $\delta$ decreases, the function $F(r)$ shifts downward in parallel.

3. Check whether $F\left(- (\xi + \nu)\right) > 0, F(\rho) > 0$. If yes, then we have found two partial insurance equilibria. If not, go back to step 1.
D Welfare in Stationary Equilibrium for IES $\sigma \neq 1$

The equilibrium allocation is given by

$$c(t) = c_h(r)e^{-g(r)t}$$
$$g(r) = \frac{\rho - r}{\sigma}$$
$$c_h(r) = \frac{1}{1 + \frac{\xi}{r + \nu + g(r)}} z = \frac{1}{1 + \frac{\xi}{r + \nu + \frac{\rho - r}{\sigma}}} z = \frac{1}{1 + \frac{\xi}{\rho + \nu + (\rho - r)\left(\frac{1}{\sigma} - 1\right)}} z$$

D.1 Lifetime Utility for Given Interest rate $r$

Expected lifetime utility is the weighted sum of lifetime utility from being born with low (no) income and being born with high income $z$. It is given, for interest rate $r$, by

$$EU(r) = \frac{\xi U_l(r) + \nu U_h(r)}{\xi + \nu}$$

where $U_i(r)$ is lifetime utility being born with income $i = l, h$. For the low income state lifetime utility is given by

$$\rho U_l(r) = u + \nu(U_h(r) - U_l(r))$$

and thus

$$U_l(r) = \frac{u + \nu U_h(r)}{\rho + \nu}$$

Thus

$$EU(r) = \frac{\xi u + \nu U_h(r)}{\rho + \nu} + \nu U_h(r) = \frac{\xi}{(\xi + \nu)(\rho + \nu)} u + \frac{(\xi + \rho + \nu)\nu}{(\xi + \nu)(\rho + \nu)} U_h(r)$$

and lifetime utility is linear in lifetime utility conditional on being borne with high income.

For being borne with high income (for now suppressing dependence on $r$) lifetime utility is given by

$$\rho U_h = u(w(r)c_h(r)) + \xi(U(0) - U_h)$$

where $U(t)$ is the lifetime continuation utility from the consumption contract after having
had low income for $t$ units of time. It is given by the differential equation

$$\rho U(t) = u(w(r)c_h(r)e^{-g(r)t}) + \nu(U_h - U(t)) + \dot{U}(t)$$

Now define

$$u(t) = \frac{U(t)}{w(r)^{1-\sigma}}$$
$$u_h(r) = \frac{U_h(r)}{w(r)^{1-\sigma}}$$

as wage-deflated lifetime utility. Lifetime utility can be decomposed in this way since the period utility function is CRRA (and thus lifetime utility is homothetic), and the aggregate wage is constant over time in a stationary equilibrium. The so-defined deflated lifetime utility function follows the Hamilton-Jacobi-Bellman equation:

$$\rho u_h(r) = u(c_h(r)) + \xi(u(0) - u_h(r))$$
$$\rho u(t) = u(c_h(r))e^{-(1-\sigma)g(r)t} + \nu(u_h - u(t)) + \dot{u}(t)$$

or rewriting the second equation

$$\dot{u}(t) = (\rho + \nu)u(t) - u(c_h(r))e^{-(1-\sigma)g(r)t} - \nu u_h$$

Solving the differential equation (one can differentiate with respect to time $t$ using Leibnitz’ rule to check that the solution is correct) yields, for now suppressing the dependence of $u_h(r)$ on $r$:

$$u(t) = \int_{t}^{\infty} e^{-(\rho+\nu)(s-t)} \left[ \nu u_h + u(c_h(r))e^{-(1-\sigma)g(r)s} \right] ds.$$ 

Evaluating at $t = 0$ one obtains

$$u(0) = \int_{0}^{\infty} e^{-(\rho+\nu)s} \left[ \nu u_h + u(c_h(r))e^{-(1-\sigma)g(r)s} \right] ds$$

$$= \nu u_h \int_{0}^{\infty} e^{-(\rho+\nu)s} ds + u(c_h(r)) \int_{0}^{\infty} e^{-(\rho+\nu+(1-\sigma)g(r)s)} ds$$

$$= -\frac{\nu u_h}{\rho + \nu} \bigg|_0^{\infty} + \frac{u(c_h(r))}{\rho + \nu + (1-\sigma)g(r)} e^{-[\rho+\nu+(1-\sigma)g(r)]s} \bigg|_0^{\infty}$$

$$= \frac{\nu u_h}{\rho + \nu} + \frac{u(c_h(r))}{\rho + \nu + (1-\sigma)g(r)}$$
and thus the two equations

\[
\begin{align*}
    u(0) &= \frac{\nu u_h}{\rho + \nu} + \frac{u(c_h(r))}{\rho + \nu + (1 - \sigma)g(r)} \\
    (\rho + \xi)u_h &= u(c_h(r)) + \xi u(0)
\end{align*}
\]

can be solved for \( u_h, u(0) \). This delivers

\[
\begin{align*}
    (\rho + \xi)u_h &= u(c_h(r)) + \frac{\xi u_h}{\rho + \nu} + \frac{\xi u(c_h(r))}{\rho + \nu + (1 - \sigma)g(r)} \\
    \left[1 + \frac{\xi}{\rho + \nu}\right]\rho u_h &= \left[1 + \frac{\xi}{\rho + \nu + (1 - \sigma)g(r)}\right]u(c_h(r))
\end{align*}
\]

and thus

\[
\begin{align*}
    u_h(r) &= \frac{1 + \frac{\xi}{\rho + \nu + (1 - \sigma)g(r)}}{1 + \frac{\xi}{\rho + \nu}} \frac{u(c_h(r))}{\rho} \\
    &= \frac{1 + \frac{\xi}{\rho + \nu + (1 - \sigma)g(r)}}{1 + \frac{\xi}{\rho + \nu}} \left(\frac{1}{1 + \frac{\xi}{\rho + \nu + (1 - \sigma)g(r)}}\right)^{1-\sigma} \\
    &= \frac{1 + \frac{\xi}{\rho + \nu + (1 - \sigma)g(r)}}{1 + \frac{\xi}{\rho + \nu}} \left(\frac{1}{1 + \frac{\xi}{\rho + \nu + (1 - \sigma)g(r)}}\right)^{1-\sigma} \\
    &= \frac{1 + \frac{\xi}{\rho + \nu + (1 - \sigma)g(r)}}{1 + \frac{\xi}{\rho + \nu}} \left(\frac{1}{1 + \frac{\xi}{\rho + \nu + (1 - \sigma)g(r)}}\right)^{1-\sigma} \\
    &= \left(1 + \frac{\xi}{\rho + \nu + (1 - \sigma)g(r)}\right)^{\sigma} \left(1 + \frac{\xi}{\rho + \nu}\right)^{1-\sigma} \\
    &= \left(\frac{1 + \frac{\xi}{\rho + \nu + (1 - \sigma)g(r)}}{1 + \frac{\xi}{\rho + \nu}}\right)^{\sigma} \frac{z^{1-\sigma}}{\rho(1 - \sigma)} \\
    &= \frac{z}{1 + \frac{\xi}{\rho + \nu + (1 - \sigma)g(r)}}^{1-\sigma} \frac{z^{1-\sigma}}{\rho(1 - \sigma)}
\end{align*}
\]
D.2 Comparing Welfare Across two Equilibria with Interest Rates \( r_1 \) and \( r_2 \)

From the above calculations we note that for any two (equilibrium) interest rates \( r_1 < r_2 \)

\[
\frac{u_h(r_1)}{u_h(r_2)} = \left[ \frac{1 + (\frac{\xi}{\rho + \nu + (1-\sigma)g(r_1)})}{1 + (\frac{\xi}{\rho + \nu + (1-\sigma)g(r_2)})} \right]^{\sigma}
\]

and thus deflated lifetime utility is higher in the high interest equilibrium. For \( \sigma < 1 \) this follows directly from the above ratio and the fact that \( g(r_1) > g(r_2) \). For \( \sigma > 1 \) both lifetime utilities are negative and thus \( \frac{u_h(r_1)}{u_h(r_2)} > 1 \) (because of \( (1-\sigma)g(r_1) < (1-\sigma)g(r_2) < 0 \)) implies that \( u_h(r_2) > u_h(r_1) \).

Now we provide a complete comparison of expected lifetime utilities in two equilibria with interest rates \( r_1 < r_2 \). On one hand \( u_h(r_2) > u_h(r_1) \), on the other hand a higher interest rate implies a lower capital stock and thus lower wage. Which effect dominates depends on parameter values, and below we provide an explicit condition on the exogenous parameters of the model under which each effect dominates.

Note that

\[
EU(r_2) - EU(r_1) = \frac{(\xi + \rho + \nu)\nu}{(\xi + \nu)(\rho + \nu)} [U_h(r_2) - U_h(r_1)]
\]

\[
= \frac{(\xi + \rho + \nu)\nu U_h(r_2)}{(\xi + \nu)(\rho + \nu)} \left[ 1 - \frac{U_h(r_1)}{U_h(r_2)} \right]
\]

\[
= \frac{(\xi + \rho + \nu)\nu \left( \frac{1}{\rho + \nu + (1-\sigma)g(r_2)} \right)^{\sigma} \left( \frac{w(r_2)}{w(r_1)} \right)^{1-\sigma}}{U_h(r_2)}
\]

\[
= \frac{1}{1 - \sigma} \left[ 1 - \left( \frac{w(r_1)}{w(r_2)} \right)^{1-\sigma} \left[ \frac{c_h(r_1)}{c_h(r_2)} \right]^{-\sigma} \right]
\]

\[
= \frac{\kappa}{1 - \sigma} \left[ 1 - \left( \frac{r_2 + \delta}{r_1 + \delta} \right)^{\frac{\theta(1-\sigma)}{1-\sigma}} \left[ \frac{1 + \frac{(1-\sigma)g(r_1)}{g(r_2)}}{1 + \frac{\xi}{\rho + \nu + (1-\sigma)g(r_2)}} \right]^{\sigma} \right]
\]

where \( \kappa = \frac{(\xi + \rho + \nu)\nu \left( \frac{1}{\rho + \nu + (1-\sigma)g(r_2)} \right)^{\sigma} \left( \frac{w(r_2)}{w(r_1)} \right)^{1-\sigma}}{(\xi + \nu)(\rho + \nu)} > 0 \) is a constant and we have substituted in

\[
\frac{w(r_1)}{w(r_2)} = \left( \frac{r_2 + \delta}{r_1 + \delta} \right)^{\frac{\theta}{1-\sigma}} > 1
\]

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Thus the sign of the expected utility differential between a high and a low interest equilibrium is the product between two counterveiling effects. A higher interest rate is associated with a lower capital stock and thus a lower wage. This is the term \( \frac{w(r_2)}{w(r_1)} = \left( \frac{r_2 + \delta}{r_1 + \delta} \right)^{\frac{\theta}{\theta - 1}} > 1 \). On the other hand, a higher interest rate leads to a better consumption contract, with higher initial normalized consumption and slower decay rate, \( g(r_2) < g(r_1) \). For the sake of clarity we distinguish two cases.

**Case 1.** \( \sigma < 1 \). Then \( 1 - \sigma > 0 \) and the terms satisfy

\[
\left( \frac{r_2 + \delta}{r_1 + \delta} \right)^{\frac{\theta(1-\sigma)}{\theta - 1}} > 1
\]

\[
0 < \left[ 1 + \frac{\xi}{\rho + \nu + (1-\sigma)g(r_1)} \right]^{\sigma} < 1
\]

and the sign of \( EU(r_2) - EU(r_1) \) is determined by the product of the two terms, i.e. whether the wage effect or the consumption contract effect dominates. That is

\[
EU(r_2) - EU(r_1) > 0 \iff \left( \frac{r_2 + \delta}{r_1 + \delta} \right)^{\frac{\theta(1-\sigma)}{\theta - 1}} \left[ 1 + \frac{\xi}{\rho + \nu + (1-\sigma)g(r_1)} \right]^{\sigma} < 1
\]

**Case 2.** \( \sigma > 1 \). Then \( 1 - \sigma < 0 \) and the terms satisfy

\[
\left[ 1 + \frac{\xi}{\rho + \nu + (1-\sigma)g(r_2)} \right]^{\sigma} > 1
\]

\[
0 < \left( \frac{r_2 + \delta}{r_1 + \delta} \right)^{\frac{\theta(1-\sigma)}{\theta - 1}} < 1
\]

and again the sign of \( EU(r_2) - EU(r_1) \) is again determined by the relative size of the two effects

\[
EU(r_2) - EU(r_1) > 0 \iff \left( \frac{r_2 + \delta}{r_1 + \delta} \right)^{\frac{\theta(1-\sigma)}{\theta - 1}} \left[ 1 + \frac{\xi}{\rho + \nu + (1-\sigma)g(r_2)} \right]^{\sigma} > 1
\]

Note that the condition to rank welfare in the two equilibria is fully analytical, that is, can readily be checked upon knowing the two equilibrium interest rates.