One-Sided Limited Commitment and Aggregate Productivity Risk *

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Abstract

In this paper we study a neoclassical growth model with idiosyncratic income risk and aggregate productivity risk in which risk sharing is endogenously constrained by one-sided limited commitment. Households can trade a full set of contingent claims that pay off depending on both idiosyncratic and aggregate risk, but limited commitment rules out that households sell these assets short. The model results, under suitable restrictions of the parameters of the model, in partial consumption insurance in equilibrium. With log-utility and idiosyncratic income shocks taking two values one of which is zero (e.g., employment and unemployment) we show that the equilibrium can be characterized in closed form, despite the fact that it features a non-degenerate consumption- and wealth distribution. We use this feature of the model to study, analytically, inequality over the business cycle and asset pricing.

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Contents

1 Introduction

In this paper we study a neoclassical growth model with idiosyncratic income risk and aggregate productivity risk (as in Krusell and Smith, 1998) in which risk sharing is endogenously constrained by one-sided limited commitment, as in Krueger and Uhlig (2006). Households can trade a full set of contingent claims that pay off depending on both idiosyncratic and aggregate risk, but limited commitment rules out that households sell these assets short. The model results, under suitable restrictions of the parameters of the model, in partial consumption insurance in equilibrium. With log-utility and idiosyncratic income shocks taking two values one of which is zero (e.g., employment and unemployment) we show that the equilibrium can be characterized in closed form, despite the fact that it features a non-degenerate consumption- and wealth distribution. We use this feature of the model to study, analytically, inequality over the business cycle and asset pricing.

The paper unfolds as follows. The next section sets up the model and defines equilibrium, and Section 3 contains a general characterization of this equilibrium. In Sections 4, 5 and 6 we then analyze the stationary equilibrium, the transition path after a one-time unexpected shock and the full equilibrium with aggregate shocks of the model, respectively. Equipped with the results from our model we then relate these results to the existing literature in Section 7. Sections 8 and 9 apply the model to study, in turn, inequality over the business cycle and asset pricing. Section 10 concludes, and detailed derivations and proofs the are contained in the Appendix.

2 Model

Time is discrete, infinite and indexed by $t = 0, 1, ...$ and the economy is populated by a continuum of individuals of measure 1, a representative, competitive production firm and a representative, competitive financial intermediary.

2.1 Aggregate Risk

We consider an economy with stochastic aggregate productivity, as well as idiosyncratic labor productivity shocks. Denote by A_t current aggregate total factor productivity, and by $A^t = \{A_0, A_1, \cdots, A_t\}$ the history of productivity with probability distribution $\pi(A_{t+1}|A^t)$. We treat the initial productivity level A_0 as fixed.

For most of the paper no further restrictions on the productivity process $\{A_t\}$ need to be imposed,[1] but for a subset of the results we need to make further assumptions on

[1]We assume that A_t takes a positive real value, $A_t \in \mathbb{R}_+$, and that the number of possible states in each

the productivity process.

Assumption 1. *Let the productivity process $\{A_t\}_{t=0}^{\infty}$ satisfy one of three assumptions:*

1. *Steady state: $A_t = A_0$ with probability 1, for all $t > 0$.*

2. *MIT shock: $\{A_t\}$ is a deterministic, non-constant sequence, but households at time $t = 0$ expect that $A_t = A_0$ with probability 1.*

3. *Stochastic iid growth rates: $\frac{A_{t+1}}{A_t} \in \{1-\epsilon, 1+\epsilon\}$, with equal probability:*

$$\frac{A_{t+1}}{A_t} = \begin{cases} 1+\epsilon \text{ with probability } \frac{1}{2} \\ 1-\epsilon \text{ with probability } \frac{1}{2} \end{cases} \quad \text{for all } t \geq 0 \text{ , } iid \tag{1}$$

2.2 Technology

The production side of the economy is described by a completely standard neoclassical production function of the form

$$Y_t = K_t^{\theta}(A_t L_t)^{1-\theta} \tag{2}$$

where θ is the capital share and A denotes the (potentially time-varying) level of aggregate (labor-augmenting) productivity. Capital depreciates at rate δ. This production technology is operated by a representative and competitive firm hiring labor and capital at rental rates w_t, r_t, and the standard optimality conditions read as

$$w_t = (1-\theta)A_t \left(\frac{K_t}{A_t L_t}\right)^{\theta} \tag{3}$$

$$r_t = \theta \left(\frac{K_t}{A_t L_t}\right)^{\theta-1} - \delta \tag{4}$$

2.3 Idiosyncratic Risk, Household Endowments and Preferences

Individuals are indexed by $i \in [0,1]$ and in each period t have idiosyncratic stochastic labor productivity $z_{it} \in Z = \{0, \zeta\}$, where $\zeta > 1$ is a parameter. Since the identity of individuals is irrelevant we will suppress the index i whenever there is no scope for confusion and simply write z_t for the current idiosyncratic labor productivity as well as $z^t = (z_0, z_1, .., z_t)$ for the history of productivity realizations. The probability of a given productivity history is denoted by $\pi(z^t)$.

The idiosyncratic labor productivity process is Markov with time-invariant transition matrix:

$$\pi(z_{t+1}|z_t) = \begin{bmatrix} 1-\nu & \nu \\ \xi & 1-\xi \end{bmatrix} \tag{5}$$

period is finite. Aggregate shocks are independent of idiosyncratic shocks.

where ν is the probability of switching from productivity 0 to productivity ζ and ξ is the probability of switching from ζ to 0. The stationary distribution over labor productivity is then given by $(\psi_l, \psi_h) = \left(\frac{\xi}{\xi+\nu}, \frac{\nu}{\xi+\nu}\right)$, and households are assumed to draw their initial productivity from this stationary distribution (which is then also the cross-sectional distribution of labor productivity at all future dates $t > 0$). We normalize average labor productivity to one, which implies the parameter restriction

$$\frac{\nu}{\xi + \nu}\zeta = 1. \tag{6}$$

This assumption implies that the aggregate supply of labor is $L_t = 1$ for all t.

In addition to labor productivity a given household is endowed with initial wealth a_0, and we denote the cross-section probability measure over wealth and labor productivity by $\Phi(a_0, z_0)$.

Each household has preferences representable by a standard intertemporal utility function $u(c)$ defined over stochastic consumption streams c and given by

$$U(c) = \mathbf{E_0} \sum_{t=0}^{\infty} \beta^t \log(c_t) \tag{7}$$

with logarithmic period utility function and time discount factor β.

2.4 Financial Markets and Household Budget Constraint

Households face idiosyncratic and aggregate risk and seek to insure against that risk by trading a full set of contingent claims. Denote the price a contingent claim that pays $R_{t+1}(A^{t+1})$ units of consumption in aggregate history A^{t+1} for an indivdual with history (A^t, z^t) if and only if tomorrow's idiosyncratic state is z_{t+1} as $q_t(A_{t+1}, z_{t+1}|A^t, z^t)$. For future reference, we denote the position of these assets such for a household with initial characteristics (a_0, z_0) by $\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1})$. The budget constraint of the household then reads as

$$\mathbf{c}_t(a_0, z^t, A^t) + \sum_{A_{t+1}} \sum_{z_{t+1}} q_t(A_{t+1}, z_{t+1}|A^t, z^t)\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) = w_t(A^t)z_t + R_t(A^t)\mathbf{a}_t(a_0, z^t, A^t) \tag{8}$$

The one-sided limited commitment assumption in this context is reflected by the requirement that household contingent claim positions be non-negative, that is, by the constraint $\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) \geq 0$, as individuals can walk away, without punishment, from any negative position (i.e. from state-contingent debt) they have borrowed.[2]

[2] Alternatively, we could have introduced financial intermediaries that offer long-term consumption insurance contracts. These contracts stipulate potentially fully income-history contingent consumption payments in exchange for delivering all of labor income to the intermediaries whenever the individual

2.5 Definition of Sequential Market Equilibrium

We now define a sequential market equilibrium with aggregate shocks. Households' consumption and savings allocations depend on both the history of individual states $z^t = (z_0, z_1, \cdots, z_t)$ and the history of aggregate states $A^t = (A_0, A_1, \cdots, A_t)$. Aggregate allocations and prices depend on the history of aggregate shocks A^t.

Definition 1. *For an initial condition* $(A_0, K_0, \Phi(a_0, z_0))$*, an equilibrium is sequences of wages and interest rates* $\{w_t(A^t), R_t(A^t)\}$*, prices of contingent claims* $\{q_t(A_{t+1}, z_{t+1}|A^t, z^t)\}$*, aggregate consumption and capital* $\{C_t(A^t), K_{t+1}(A^t)\}$ *and individual consumption and asset allocations* $\{\hat{\mathbf{c}}_t(a_0, z^t, A^t), \hat{\mathbf{a}}_{t+1}(a_0, z^{t+1}, A^{t+1})\}$ *such that*

1. *Given* $\{w_t(A^t), R_t(A^t), q_t(A_{t+1}, z_{t+1}|A^t, z^t)\}_{t=0,A^t,z^t,A_{t+1},z_{t+1}}^{\infty}$*, the household consumption and asset allocation* $\{\hat{\mathbf{c}}_t(a_0, z^t, A^t), \hat{\mathbf{a}}_{t+1}(a_0, z^{t+1}, A^{t+1})\}$ *solves, for all* (a_0, z_0)*,*[3]

$$\max_{\{\mathbf{c}_t(a_0,z^t,A^t),\mathbf{a}_{t+1}(a_0,z^{t+1},A^{t+1})\}} \sum_{t=0}^{\infty} \sum_{A^t} \sum_{z^t} \beta^t \pi(A^t)\pi(z^t) \log(\mathbf{c}_t(a_0, z^t, A^t)) \tag{12}$$

is productive. The one-sided limited commitment friction implies that whereas intermediaries can fully commit to long-term contracts, individuals cannot. Specifically, in every period, after having observed current labor productivity, the individual can leave her current contract and sign up with an alternative intermediary at no punishment, obtaining in equilibrium the highest lifetime utility contract that allows a competing intermediary to break even. The lifetime utility from a newly signed contract is the key (potentially time-varying) endogenous entity in this formulation of the model. In Krueger and Uhlig (2006) we have shown that these two formulations of the one-sided limited commitment friction are equivalent, and we therefore here focus on the financial market formulation, in the spirit of Alvarez and Jermann (2000).

[3] A familiar way of defining Arrow securities is that households pay a price $q^b(A_{t+1}, z_{t+1}|A^t, z^t)$ and receive one unit of non-deflated consumption goods if the state (A_{t+1}, z_{t+1}) is realized. If we denote such an asset by $\mathbf{b}_{t+1}$, the budget constraint (8) instead reads as:

$$\mathbf{c}_t(a_0, z^t, A^t) + \sum_{A_{t+1}} \sum_{z_{t+1}} q^b(A_{t+1}, z_{t+1}|A^t, z^t)\mathbf{b}_{t+1}(a_0, z^{t+1}, A^{t+1}) = w_t(A^t)z_t + \mathbf{b}_t(a_0, z^t, A^t) \tag{9}$$

Because we define a contingent claim that yields $R_{t+1}(A^{t+1})$ at $t+1$, the relation between the two types of assets is:

$$q^b(A_{t+1}, z_{t+1}|A^t, z^t) = \frac{q^a(A_{t+1}, z_{t+1}|A^t, z^t)}{R_{t+1}(A^{t+1})}, \tag{10}$$

$$\text{and } \mathbf{b}_{t+1}(a_0, z^{t+1}, A^{t+1}) = R_{t+1}(A^{t+1})\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) \tag{11}$$

Since the return on both contingent claims is given by:

$$\frac{1}{q^b(A_{t+1}, z_{t+1}|A^t, z^t)} = \frac{R_{t+1}(A^{t+1})}{q^a(A_{t+1}, z_{t+1}|A^t, z^t)},$$

two formulations are equivalent. Our formulation of contingent claims gives a simpler and perhaps more intuitive expression for the capital market clearing condition (17), though.

subject to the budget constraints (8) *and subject to*

$$\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) \geq 0 \tag{13}$$

2. *Factor prices equal marginal products*

$$w_t(A^t) = (1-\theta)A_t \left(\frac{K_t(A^{t-1})}{A_t}\right)^{\theta} \tag{14}$$

$$R_t(A^t) = 1 + \theta \left(\frac{K_t(A^{t-1})}{A_t}\right)^{\theta-1} - \delta \tag{15}$$

3. *The goods market and capital market clear*

$$C_t(A^t) + K_{t+1}(A^t) = \left(K_t(A^{t-1})\right)^{\theta} (A_t)^{1-\theta} + (1-\delta)K_t(A^t) \tag{16}$$

$$K_{t+1}(A^t) = \int \sum_{z^{t+1}} \hat{\mathbf{a}}_{t+1}(a_0, z^{t+1}, A^{t+1})\pi(z^{t+1})d\Phi(a_0, z_0) \;\; \forall A^{t+1} \tag{17}$$

where

$$C_t(A^t) = \int \sum_{z^t} \hat{\mathbf{c}}_t(a_0, z^t, A^t)\pi(z^t)d\Phi(a_0, z_0) \tag{18}$$

3 Characterization of Equilibrium

This section characterizes a sequential equilibrium with aggregate shocks. We will first derive optimal household choices of consumption and savings for a given stochastic process for interest rates and wages, $\{R_t(A^t), w_t(A^t)\}_{t\geq 0, A^t}$ in subsection 3.1, then characterize the equilibrium asset distribution in subsection 3.2 and finally use both results to determine the aggregate law of motion of the economy in closed form in subsection 3.4.

3.1 Optimal Household Choices

We will show that equilibrium household choices take an especially simple form. As long as the real interest rate is not too high, wages do not fall too fast and households do not start life with too many assets, then they do not accumulate state-contingent assets for the high income state tomorrow and insure against sequences of low idiosyncratic income realizations through contingent asset purchases such that for these households a standard complete markets Euler equation holds. The deviation from the complete markets allocation (which would result in the equilibrium collapsing to that of a representative agent economy) stems from the fact that currently low-income individuals would want to *borrow* against the high-income state, but are prevented from doing so due to limited commitment to repay their debts. Thus, the best they can do is to set the contingent claim for the high idiosyncratic income state to zero. The following proposition makes this argument formal. It requires the following assumptions on factor prices that will be validated and can be replaced with Assumption 1 purely on the fundamentals of the economy once the equilibrium law of motion for capital has been derived.

Assumption 2 (Contingent Claims Prices)**.** *The prices of contingent claims are given by*[4]

$$q_t(A_{t+1}, z_{t+1}|A^t, z^t) = \pi(z_{t+1}|z_t)\pi(A_{t+1}|A^t). \tag{19}$$

Assumption 3 (No Savings Incentives)**.** *The equilibrium interest rate and wage rate processes,* $\{R_t(A^t), w_t(A^t)\}$ *satisfy:*

$$\beta R_0(A_0) \;<\; 1 \tag{20}$$

$$\beta R_{t+1}(A^{t+1}) \;<\; \frac{w_{t+1}(A^{t+1})}{w_t(A^t)} \text{ for all } t \geq 0 \text{ and } A^{t+1} \tag{21}$$

Assumption 4 (Initial Distribution)**.** *The initial distribution over wealth and labor productivity,* $\Phi(a_0, z_0)$*, satisfies:*

(i) $a_0 = 0$ *if* $z_0 = \zeta$ *(high-productivity households (*$z_0 = \zeta$*) have zero initial wealth).*

(ii) $0 < a_0 < \bar{a}_0 := \frac{\beta}{1-(1-\nu-\xi)\beta}\zeta$ *if* $z_0 = 0$ *(the wealth of low-productivity households is strictly positive but not too high).*

We will later show that Assumption 2 on the endogenous contingent claims prices is always satisfied in equilibrium and Assumption 3 on endogenous equilibrium prices is satisfied under Assumption 1 stated purely in terms of the exogenous fundamentals of the model. Furthermore we will demonstrate that under Assumption 1 the steady state of the model will have an associated asset distribution that satisfies Assumption 4, although the following proposition characterizing optimal household choices does not require the initial distribution to be the steady state distribution (as long as it satisfies Assumption 4).

Proposition 1 (Optimal Household Consumption and Asset Allocation)**.** *Suppose Assumption 2 on contingent claims prices is satisfied and suppose that the sequence of wages and interest rates* $\{w_t(A^t), R_t(A^t)\}_{t=0}^{\infty}$ *satisfies the no-savings Assumption 3 and that the initial wealth distribution satisfies Assumption 4. Then the optimal consumption and asset*

[4]Recall that these contingent claims pay $R_{t+1}(A^{t+1})$ units of consumption in event history A^{t+1}. Under the assumption, the price of a contingent claim that pays one state-contingent unit of consumption is then takes the perhaps more familiar form $q_t^b(A_{t+1}, z_{t+1}|A^t, z^t) = \frac{\pi(z_{t+1}|z_t)\pi(A_{t+1}|A^t)}{R_{t+1}(A^{t+1})}$.

allocation of individual household is given by

$$\mathbf{c}_t(a_0, z^t, A^t) = \begin{cases} w_t(A^t)c_0, \ \textit{where } c_0 := \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}\zeta, & \textit{if } z_t = \zeta \\ [1-(1-\nu)\beta]R_t(A^t)\mathbf{a}_t(a_0, z^t, A^t) & \textit{if } z_t = 0 \end{cases} \tag{22}$$

$$\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) = \begin{cases} 0 & \textit{if } z_{t+1} = \zeta \\ \frac{\beta}{1-(1-\nu-\xi)\beta}\zeta w_t(A^t) & \textit{if } z_t = \zeta \textit{ and } z_{t+1} = 0 \\ \beta R_t(A^t)\mathbf{a}_t(a_0, z^t, A^t) & \textit{if } z_t = 0 \textit{ and } z_{t+1} = 0 \end{cases} \tag{23}$$

where $\mathbf{a}_0(a_0, z^0, A^0) = w_0(A^0)a_0$.

Proof. See Appendix A.1.1 □

We now discuss the intuition and straightforward implications of Proposition 1. First, it is easy to verify that (22) and (23) imply:[5]

$$\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1}) = \beta R_{t+1}(A^{t+1})\mathbf{c}_t(a_0, z^t, A^t) \text{ if } z_{t+1} = 0. \tag{24}$$

that is, consumption growth between t and $t+1$ follows a standard complete markets Euler equation for those households that are unproductive in period $t+1$.[6] In contrast, those that are productive in $t+1$ and have $z_{t+1} = \zeta$ have (see Lemma 8 in Appendix A.1.1)

$$\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1}) > \beta R_{t+1}(A^{t+1})\mathbf{c}_t(a_0, z^t, A^t) \text{ if } z_{t+1} = \zeta. \tag{26}$$

Also note that households with currently positive labor income (i.e., with $z = \zeta$), consume and save a constant fraction of their current labor income, independent of the aggregate shock (history) and independent of current or (expected) future interest rates:

$$\frac{\mathbf{c}_t(a_0, z^t, A^t)}{\zeta w_t(A^t)} = \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta} \tag{27}$$

$$\frac{\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1})}{\zeta w_t(A^t)} = \frac{\beta}{1-(1-\nu-\xi)\beta}. \tag{28}$$

This result depends crucially and not surprisingly on the assumption of log-utility.

As Proposition 1 indicates, optimal consumption and asset holdings are proportional to wages (either current wages in case $s = 0$ and the household has high productivity

[5] The derivation of this result can be found in Appendix A.1.1.

[6] Another way to write is:

$$\frac{1}{\mathbf{c}_t(a_0, z^t, A^t)} = \underbrace{\pi(A_{t+1}, z_{t+1}|A^t, z^t)}_{\text{prob. of } (A_{t+1}, z_{t+1})} \underbrace{\frac{R_{t+1}(A^{t+1})}{q_t(A_{t+1}, z_{t+1}|A^t, z^t)}}_{\text{return on a contingent claim}} \underbrace{\beta \frac{1}{\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1})}}_{\text{discounted marginal utility}} \text{ if } z_{t+1} = 0. \tag{25}$$

Independence of A_{t+1} and z_{t+1} gives $\pi(A_{t+1}, z_{t+1}|A^t, z^t) = \pi(z_{t+1}|z_t)\pi(A_{t+1}|A^t)$, while Assumption 2 gives $q_t(A_{t+1}, z_{t+1}|A^t, z^t) = \pi(z_{t+1}|z_t)\pi(A_{t+1}|A^t)$. Then, equation (25) simplfies to equation (24).

today) or to the wage when she last was productive (for $s > 0$). We therefore define, for future reference and use, wage-deflated consumption and asset choices as

$$c_t(a_0, z^t, A^t) = \frac{\mathbf{c}_t(a_0, z^t, A^t)}{w_t(A^t)} \tag{29}$$

$$a_{t+1}(a_0, z^{t+1}, A^{t+1}) = \frac{\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1})}{w_{t+1}(A^t)} \tag{30}$$

3.2 Cross-Sectional Distribution

The sequential market equilibrium household consumption-asset allocation has a simple structure. Either the shortsale constraint for a given continuation history z^{t+1} is not binding, $\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) > 0$, and the standard complete-markets Euler equation

$$\frac{\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1})}{\mathbf{c}_t(a_0, z^t, A^t)} = \beta R_{t+1}(A^{t+1}) \tag{31}$$

applies or the constraint is binding, $\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) = 0$ and the Euler equation turns into an inequality.

We assert that the equilibrium consumption and asset allocation has a simple structure in which individual consumption and assets only depend on the length $s \geq 0$ of the most recent spell of low productivities (where $s = 0$ denotes an agent with currently high productivity), and potentially calendar time t. When productivity is high, the shortsale (limited commitment) constraint is binding and assets are zero. Denote this simple consumption-asset allocation by $\{\mathbf{c}_{s,t}, \mathbf{a}_{s,t}\}_{s,t=0}^{\infty}$ and note that the Markov process for individual productivity implies that the cross-sectional distribution of waiting times is time invariant and given by

$$\phi_s = \begin{cases} \frac{\nu}{\xi+\nu} & \text{if } s = 0 \\ \frac{\xi}{\xi+\nu}\nu(1-\nu)^{s-1} & \text{if } s = 1, 2, 3, ... \end{cases} \tag{32}$$

Corollary 1. *Suppose the initial consumption-asset allocation is given by $\{\mathbf{c}_{s,0}, \mathbf{a}_{s,0}\}_{s\geq 0}$ with the probability mass of s-agents given by (32). If households' consumption and saving rule follows equations (22) and (23), the consumption-asset allocation at any $t \geq 1$ is determined by:*

$$\mathbf{c}_{s,t}(A^t) = \begin{cases} \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}\zeta w_t(A^t) & \textit{if } s = 0 \\ \beta R_t(A^t)\mathbf{c}_{s-1,t-1} & \textit{if } s \geq 1 \end{cases} \tag{33}$$

$$\mathbf{a}_{s,t}(A^{t-1}) = \begin{cases} 0 & \textit{if } s = 0 \\ \frac{\beta}{1-(1-\nu-\xi)\beta}\zeta w_{t-1}(A^{t-1}) & \textit{if } s = 1 \\ \beta R_{t-1}(A^{t-1})\mathbf{a}_{s-1,t-1} & \textit{if } s \geq 2 \end{cases} \tag{34}$$

and the probability mass of s-households is time-invariant and given by (32).

Proof. Apply equation (22)–(24) to the initial allocation $\{\mathbf{c}_{s,0}, \mathbf{a}_{s,0}\}_{s\geq 0}$. □

Note that this corollary implies that even though the cross-sectional distribution of waiting times remains constant over time and across aggregate shocks, the levels of consumption and assets at these countably many mass points given by (33) and (34) varies with time and aggregate history A^t, but only through its effects on aggregate wages and interest rates. Also note that if the consumption-asset allocation is of the simple form stipulated above, aggregate consumption and assets in the goods market clearing condition and the asset market clearing condition can be written as

$$C_t(A^t) = \sum_{s=0}^{\infty} \phi_s \mathbf{c}_{s,t}(A^t) \tag{35}$$

$$K_{t+1}(A^t) = \sum_{s=0}^{\infty} \phi_s \mathbf{a}_{s,t+1}(A^t) \tag{36}$$

3.3 Confirming the Conjectured Prices of Contingent Claims on Capital Returns

Before we explicitly carry out the aggregation we can verify, with the optimal household allocations in place, that the prices of the contingent claims of capital are of the form stipulated in Assumption 2.

Fix (A^t, z^t). For all households for which the shortsale constraint for a contingent claim that pays off in aggregate state A^{t+1} is not binding (the positive mass of individuals with idiosyncratic state $z_{t+1} = 0$, i.e., those with $s > 0$ in period $t+1$), the first order conditions with respect to consumption and the state-contingent asset claim can be combined to obtain (see Appendix A.1.1, equation (114)) the standard complete markets Euler equation:

$$q_t(A_{t+1}, z_{t+1}|A^t, z^t) = \pi(z_{t+1}|z_t)\pi(A_{t+1}|A^t)\beta R_{t+1}(A^{t+1}) \left[\frac{\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1})}{\mathbf{c}_t(a_0, z^t, A^t)}\right]^{-1} \tag{37}$$

Using the fact that for each of these households (with $s > 0$) the optimal consumption allocation satisfies $\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1}) = \beta R_{t+1}(A^{t+1})\mathbf{c}_t(a_0, z^t, A^t)$ (see equation (24)). Using this result in equation (37) immediately confirms that

$$q_t(A_{t+1}, z_{t+1}|A^t, z^t) = \pi(z_{t+1}|z_t)\pi(A_{t+1}|A^t) \tag{38}$$

i.e., that Assumption 2 is satisfied in the equilibrium we are constructing. Finally, note that at these prices for households with $z_{t+1} = \zeta$ we have (see equation (26))

$$q_t(A_{t+1}, z_{t+1}|A^t, z^t) > \pi(z_{t+1}|z_t)\pi(A_{t+1}|A^t)\beta R_{t+1}(A^{t+1}) \left[\frac{\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1})}{\mathbf{c}_t(a_0, z^t, A^t)}\right]^{-1} \tag{39}$$

These households, at the posited prices, would like to reduce their consumption growth between period t and $t+1$ by borrowing against the contingency of high productivity tomorrow, but the limited commitment constraints precisely prevent these types of shortsales.

3.4 Aggregation

Now that we have characterized the cross-sectional distribution of assets, we can use equation (36) to derive the aggregate law of motion for capital. Since the optimal household consumption and asset decisions in Proposition 1 are closed-form expressions of the general equilibrium factor prices (w_t, R_t), and these in turn are functions only of the aggregate capital stock and aggregate productivity through the first-order conditions of the firm, characterizing the law of motion for the aggregate capital stock K_t is also sufficient to fully characterize the distribution of consumption and assets over time. What is special about this model with nontrivial household heterogeneity is that the model aggregates, in the sense that the capital stock in period $t+1$ can be expressed exclusively as a function of the aggregate capital stock in period t (despite the fact that the model features a non-trivial consumption and wealth distribution), and that this law of motion of capital can be characterized in closed form. The following proposition is then a straightforward consequence of the results in the previous subsection and proved in Appendix A.1.2.

Proposition 2 (The Law of Motion for Aggregate Capital). *Under the assumptions maintained in Proposition 1 and thus the household consumption and saving allocations are given by* (22) *and* (23)*, the law of motion for the aggregate capital stock is given by:*

$$\begin{aligned} K_{t+1}(A^t) &= \left[\frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta)+(1-\nu)\beta\theta\right] A_t^{1-\theta}K_t(A^{t-1})^\theta + (1-\nu)\beta(1-\delta)K_t(A^{t-1}) \\ &= \hat{s}A_t^{1-\theta}K_t(A^{t-1})^\theta + (1-\hat{\delta})K_t(A^{t-1}) \end{aligned} \tag{40}$$

where

$$\hat{s} = \frac{\xi\beta(1-\theta)}{1-(1-\nu-\xi)\beta} + (1-\nu)\beta\theta \tag{41}$$

$$\hat{\delta} = 1-(1-\nu)\beta(1-\delta). \tag{42}$$

As in the Solow model the aggregate saving rate $\hat{s}$ is a constant in this model, but in contrast to the Solow model here it is an explicit function of the fundamental parameters capturing income risk at the micro level as well as time preferences and the capital share in production. Note that if $\nu = \xi = 0$ and there is no idiosyncratic risk, then $\hat{s} = \beta\theta$. If furthermore $\delta = 1$, then $\hat{\delta} = 1$ and the model collapses to the standard representative agent stochastic neoclassical growth model (which with log-utility and full depreciation

–and only then– has a closed-form solution for the aggregate law of motion for capital. We will return to a comparison of our model with the neoclassical growth model and the Solow model in Section 7.

We now use the general results thus far to characterize analytically a stationary equilibrium, the transition path after an unexpected (transitory or permanent) productivity shock and the fully stochastic equilibrium, but under specific assumptions on the aggregate productivity process. This will allow us to show that Assumptions 2 and 3 can be restated as assumptions purely on fundamentals, and that the steady state distribution of assets satisfies Assumption 4.

4 Stationary Equilibrium

We derive the stationary equilibrium in this section. For the purpose of this section we maintain Assumption 1.1 and thus productivity is constant at A_0. In a stationary equilibrium the wage w, the gross interest rate R and the aggregate capital stock K are constants (over time and across aggregate states).

From Proposition 1 it immediately follows that optimal wage-deflated consumption and asset choices, as function of the wait time s, are given as

$$\frac{\mathbf{c}_0}{w} = c_0 = \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}\zeta \tag{43}$$

$$\frac{\mathbf{c}_s}{w} = c_s = (\beta R)^s c_0 \qquad \text{for } s = 1, 2, \cdots \tag{44}$$

$$\frac{\mathbf{a}_0}{w} = a_0 = 0 \tag{45}$$

$$\frac{\mathbf{a}_1}{w} = a_1 = \frac{\beta}{1-(1-\nu-\xi)\beta}\zeta \tag{46}$$

$$\frac{\mathbf{a}_s}{w} = a_s = (\beta R)a_{s-1} = (\beta R)^{s-1}a_1 \qquad \text{for } s = 2, 3, \cdots \tag{47}$$

High income agents consume a constant fraction of their labor income, and consumption of low income agents drifts down at a constant rate, βR, until they switch to a high income state and renew the contract. As long as $\beta R < 1$, consumption converges to zero in the long run, $\lim_{s\to\infty} c_s = 0$.

4.1 Aggregation

We can of course use the stationary version of the aggregate law of motion (40) in Proposition 2

$$K = \hat{s}(A_0)^{1-\theta}K^{\theta} + (1-\hat{\delta})K \tag{48}$$

to determine the aggregate capital stock and the associated stationary wage and interest rate, denoted by (K_0, R_0, w_0). This delivers

$$K_0 = A_0 \left(\frac{\hat{s}}{\hat{\delta}}\right)^{\frac{1}{1-\theta}} \tag{49}$$

$$R_0 = \theta A_0^{1-\theta} K_0^{\theta-1} + 1 - \delta = \theta \left(\frac{\hat{\delta}}{\hat{s}}\right) + 1 - \delta \tag{50}$$

$$w_0 = (1-\theta) A_0^{1-\theta} K_0^{\theta} = (1-\theta) A_0 \left(\frac{\hat{s}}{\hat{\delta}}\right)^{\frac{\theta}{1-\theta}} \tag{51}$$

where $(\hat{s}, \hat{\delta})$ are defined in (41) and(42).

However, it is instructive to carry out the explicit aggregation in the stationary equilibrium to obtain some intuition for the aggregate law of motion, and derive a graphical representation of the capital market clearing condition for our model akin to that in the Aiyagari (1994) model.

The capital market clearing condition in the steady state reads as

$$\begin{aligned} K &= \sum_{s=0}^{\infty} \phi_s a_s w = \sum_{s=1}^{\infty} \phi_s a_s w \\ &= \frac{\beta\zeta}{1-(1-\nu-\xi)\beta} \frac{\nu\xi}{\xi+\nu} w + \sum_{s=2}^{\infty} \phi_s a_s w \\ &= \frac{\beta\zeta}{1-(1-\nu-\xi)\beta} \frac{\nu\xi}{\xi+\nu} w + \beta R(1-\nu) \sum_{s=2}^{\infty} \phi_{s-1} a_{s-1} w \\ &= \frac{\beta\zeta}{1-(1-\nu-\xi)\beta} \frac{\nu\xi}{\xi+\nu} w + \beta R(1-\nu) K \end{aligned} \tag{52}$$

The first row is due to the fact that saving for the high idiosyncratic state $z = \zeta$ is zero, and thus, $a_0 = 0$. The second row splits the demand for assets (supply of capital) into the part coming from productive agents saving for the low income state (a_1) and the part stemming from unproductive agents rolling over parts of their assets (a_s) for $s > 1$. The third row exploits the optimal asset allocation in equation (47) and the form of the stationary wait time distribution ϕ_s in equation (32), and the last row uses the capital market clearing condition. Plugging in for (R, w) from the firm's optimality conditions and rearranging delivers back the stationary version of the aggregate law of motion in equation (48).

4.2 Existence, Uniqueness and Comparative Statics of Partial Insurance Stationary Equilibrium

Equation (52) can also be used to display the determination of the stationary equilibrium interest rate and capital stock graphically, derive its comparative statics properties and

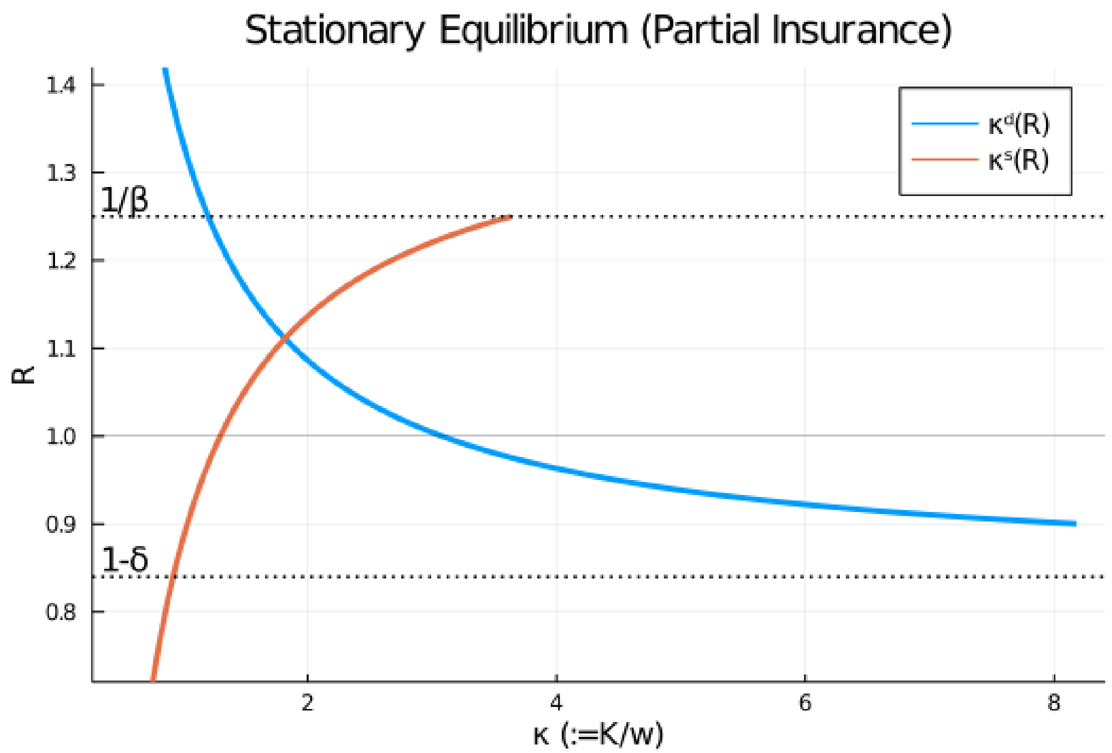

Figure 1: Determination of Market Clearing Interest Rate and Capital Stock

clarify the conditions needed for the existence of a stationary equilibrium with only partial consumption insurance. To do so, it is instructive to divide both sides of (52) by the wage w and use the firm's optimality condition with respect to capital (and the normalization of expected productivity to one, (6), to write both sides as a function of the gross interest rate R. This yields

$$K(R)/w(R) =: \kappa^d(R) = \frac{\theta}{(1-\theta)(R-1+\delta)} = \frac{\xi\beta}{[1-(1-\nu)\beta R]\,[1-(1-\nu-\xi)\beta]} := \kappa^s(R) \tag{53}$$

Figure 1 plots the (wage-normalized) demand for capital $\kappa^d(R)$ by the production firms (the relation between the return and the capital stock determined by the first order condition for capital). It has exactly the same form as in the original Aiyagari (1994) paper, sloping downward and the capital tock diverging to ∞ as the net interest rate approaches $-\delta$. The supply of capital κ^s from the household side is finite at $R = 1-\delta$, strictly increasing in the interest rate (something that is hard to prove in the original Aiyagari (1994) model), and also finite at $R = 1/\beta$. Equation (53) also allows to determine unambiguous comparative statics as the parameters of the model shift either the demand curve (in case of the production parameters (θ, δ)) or the supply curve (in case of the idiosyncratic risk parameters (ξ, ν) and the preference parameter β). Finally it also shows that a necessary and sufficient condition for a unique stationary partial insurance equilibrium with $R_0 < 1/\beta$ is that $\kappa^d(1/\beta) < \kappa^s(1/\beta)$. This leads to the following

Assumption 5.

$$\frac{\theta}{(1-\theta)\left(\frac{1}{\beta}-1+\delta\right)} < \frac{\xi}{\nu\left(\frac{1}{\beta}-1+\xi+\nu\right)}$$

This assumption is satisfied if the chance of productivity falling ξ and the risk of it not recovering quickly (as given by $1-\nu$) is sufficiently large.[7] The assumption insures that $\kappa^d(1/\beta) < \kappa^s(1/\beta)$. Equipped with this assumption, defined purely in terms of exogenous parameters of the model, we can state the following proposition, completely characterizing the stationary equilibrium of the model.

Proposition 3 (Stationary Equilibrium)**.** *Suppose Assumption 5 holds. Then, there exists a stationary partial insurance equilibrium, where the equilibrium interest rate R_0 is given in a closed form:*

$$R_0 = \frac{\xi(1-\theta)(1-\delta)+\theta\left(\xi+\nu+\frac{1}{\beta}-1\right)}{\xi(1-\theta)+\beta\theta(1-\nu)\left(\xi+\nu+\frac{1}{\beta}-1\right)} \tag{54}$$

The equilibrium capital and wage are given by:

$$K_0 = A_0\left[\frac{\xi(1-\theta)+\beta\theta(1-\nu)\left(\xi+\nu+\frac{1}{\beta}-1\right)}{\left[1-(1-\delta)\beta(1-\nu)\right]\left(\xi+\nu+\frac{1}{\beta}-1\right)}\right]^{\frac{1}{1-\theta}}, \tag{55}$$

$$w_0 = (1-\theta)A_0\left[\frac{\xi(1-\theta)+\beta\theta(1-\nu)\left(\xi+\nu+\frac{1}{\beta}-1\right)}{\left[1-(1-\delta)\beta(1-\nu)\right]\left(\xi+\nu+\frac{1}{\beta}-1\right)}\right]^{\frac{\theta}{1-\theta}}.$$

The equilibrium interest rate R_0 is strictly increasing in the capital share θ, strictly decreasing in the depreciation rate δ, the time discount factor β as well as the risk of productivity falling ξ and remaining low $1-\nu$ and is independent of the level of productivity A_0. The capital stock K_0 is strictly increasing in the time discount factor β as well as the risk of productivity falling ξ and remaining low $1-\nu$, strictly decreasing in the depreciation rate δ, and is proportional to the level of productivity A_0. The comparative statics of w_0 is the same as for K_0. The stationary equilibrium is unique[8] *in the sense that there is no other simple stationary equilibrium in which the stationary consumption and wealth allocation (and its associated cross-sectional distribution) is just a function of the wait time s.*

[7] Defining the time discount rate ρ by $\beta = \frac{1}{1+\rho}$, we can restate the assumption as

$$\frac{\theta}{(1-\theta)(\rho+\delta)} < \frac{\xi}{\nu(\rho+\xi+\nu)}.$$

which coincides with the assumption insuring the existence of a stationary equilibrium of the continuous-time model in Krueger and Uhlig (2022).

[8] We have not been able to rule out stationary equilibria in which allocations are more complex functions of idiosyncratic histories, although we conjecture such equilibria do not exist under Assumption 1.

Proof. See Appendix A.2 for the formal argument. Intuitively, it follows directly from Figure 1. That is, existence and uniqueness follows directly from the monotonicity of $\kappa^d(R)$ and $\kappa^s(R)$ as well Assumption 5. The comparative statics results with respect to R_0follow directly from the fact that the curve $\kappa^d(R)$ shifts to the right with an increase in θ and a decrease in δ whereas $\kappa^s(R)$ is independent of these parameters, and the fact that $\kappa^s(R)$ shifts to the right with an increase in $\xi, 1-\nu, \beta$ and $\kappa^d(R)$ is independent of these parameters. The comparative statics results with respect to K_0 and w_0 then follow from the firm optimality conditions, given the comparative statics with respect to R_0 □

Note that Assumption 5 is also a necessary condition for the existence of a simple partial insurance equilibrium.[9] We now characterize, again analytically, the transition path induced by an unexpected (transitory or permanent) change in productivity, starting from a partial insurance steady state (R_0, K_0) and associated asset distribution determined by the optimal asset allocation in equation (47) and the wait time distribution ϕ_s in equation (32).

5 Transitional Dynamics

5.1 The Thought Experiment

We now assume that starting from a stationary partial insurance equilibrium (i.e., starting with an initial wealth distribution determined by a partial insurance stationary equilibrium determined in the previous section with Assumption 5 in place), at the beginning of period $t = 1$ the economy experiences a completely unexpected, zero probability shock (a so-called MIT shock) that alters productivity from A_0 to a new deterministic sequence $\{A_t\}_{t=1}^{\infty}$. There are no further surprises about productivity (or any other parameters of the economy) thereafter; that is, aggregate productivity now satisfies Assumption 1, part 2. The optimal household allocations and the aggregate law of motion for capital are special cases of Propositions 1 and 2, respectively, and the latter now follows the deterministic first-order nonlinear difference equation:

$$\begin{aligned} K_{t+1} &= \left[\frac{\xi\beta(1-\theta)}{1-(1-\nu-\xi)\beta} + (1-\nu)\beta\theta\right] A_t^{1-\theta} K_t^{\theta} + (1-\nu)\beta(1-\delta)K_t \\ &= \hat{s}Y_t + (1-\hat{\delta})K_t \end{aligned} \tag{56}$$

[9] The violation of Assumption 5 does not exclude the possibility for the existence of a stationary equilibrium with $\beta R \geq 1$. The characterization of optimal consumption and asset allocations in Proposition 1 requires, in a stationary equilibrium, that $\beta R < 1$ and is no longer valid if $\beta R \geq 1$, and thus the ensuing aggregation analysis no longer applies.

where we recall that the constants $(\hat{s}, \hat{\delta})$ are given by

$$\hat{s} = \frac{\xi\beta(1-\theta)}{1-(1-\nu-\xi)\beta} + (1-\nu)\beta\theta \tag{57}$$

$$\hat{\delta} = 1-(1-\nu)\beta(1-\delta) \approx \nu+\rho+\delta. \tag{58}$$

This expression resembles the law of motion of capital in the classic Solow growth model, but with a depreciation rate $\hat{\delta}$ that is larger (by $\nu+\rho$) than the physical depreciation rate δ and a saving rate $\hat{s}$ that is an explicit function of the structural parameters of the model and depends negatively on $\nu+\rho$ and positively on the risk of income falling to zero ξ. We discuss the relation to the Solow model, and the literature more broadly, in Section 7.

Studying the special case of unexpected transitions is useful for three purposes. First, it will allow us, in Subsection 5.2, to derive a sufficient condition purely on productivity process such that the no-savings condition $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ is satisfied for all periods along the transition. Second, we show in Section 5.3 that (and explain why) the original steady state allocation (chosen under the assumption by households that wages and interest rates will never change) remains optimal in period 1, after the MIT shock has hit and wages and interest rates undergo the unexpected transition. Third, there we will also clarify why the aggregate transition induced by a change in productivity is independent from whether this change was unanticipated (as in the MIT shock thought experiment) or anticipated. This in turn suggests that the model with aggregate shocks in Section 6 remains analytically tractable and retains the same characteristics as the model with unexpected MIT transitions.

5.2 Sufficient Conditions for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ along the Transition Path

Thus far, we have derived the dynamics of the capital stock in equation (40) under the maintained assumption that the limited commitment constraint of households receiving high income is always binding along the transition path. Equivalently, phrased in terms of state-contingent asset accumulation, it was assumed that households have an implied asset position of zero when starting the period with high idiosyncratic productivity. We showed in Proposition 1 that such a contract satisfies the optimality conditions when $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all $t \geq 0$. Intuitively, when interest rates are low and/or wages are expected to be higher in the future than today, individuals have little incentive to save for the contingency of high idiosyncratic labor productivity tomorrow. In this subsection, we derive sufficient conditions that insure that $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all $t \geq 0$ after a positive and after a negative productivity shock, respectively. Broadly speaking, the shock to total factor productivity A cannot be too large in either direction. Furthermore, if depreciation is 100%, then no further assumptions besides those already made to ensure the existence of partial insurance steady state from which the transition starts are necessary, as we demonstrate next.

5.2.1 Full Depreciation

With full depreciation, $\delta = \hat{\delta} = 1$, the equilibrium law of motion for capital is given from equation (56), with $\delta = 1$ by

$$K_{t+1} = \left[\frac{\xi\beta(1-\theta)}{1-(1-\nu-\xi)\beta} + (1-\nu)\beta\theta \right] A_t^{1-\theta} K_t^{\theta} = \hat{s} A_t^{1-\theta} K_t^{\theta} \tag{59}$$

as long as, along the transition $\beta R_{t+1} < \frac{w_{t+1}}{w_t} < 1$. Lemma 10 in Appendix A.3.3 shows that with full depreciation $R_{t+1}\frac{w_t}{w_{t+1}} = R_0$, for all $t \geq 0$. But since under Assumption 5 there exists a stationary partial insurance equilibrium (with $\beta R_0 < 1$) and by assumption this is the starting point for the unexpected transition, along this transition $\beta R_{t+1} < \frac{w_{t+1}}{w_t} < 1$ is guaranteed by Assumption 5, independent of the sequence of productivity levels.

Also note that in the limit, as idiosyncratic risk vanishes (ν and ξ converge to zero), the saving rate $\hat{s}$ in our model approaches that of the standard representative agent model with log-utility and full depreciation $s = \beta\theta$. Finally, with full depreciation the nonlinear first order difference equation in (59) has a closed-form solution (since it implies that the log of the capital stock obeys a linear first order difference equation which can easily be solved in closed form). This immediately leads to the following proposition

Proposition 4. *Let Assumption 5 be satisfied and suppose the economy is originally in a partial insurance steady state* (K_0, R_0, w_0) *and* $\{(\mathbf{a}_{s,0}, \mathbf{c}_{s,0})\}_{s=0}^{\infty}$ *characterized in Proposition 3. Then the aggregate capital stock in period* t *of the transition induced by an unexpected change in productivity after period 0 to the sequence* $\{A_t\}_{t=1}^{\infty}$ *is determined as*

$$K_t = \exp\left[(1-\theta)\left[\sum_{\tau=1}^{t-1} \theta^{t-1-\tau} \log A_\tau \right] + \frac{1-\theta^{t-1}}{1-\theta} \log \hat{s} + \theta^{t-1} \log K_0 \right], \tag{60}$$

the factor prices (R_t, w_t) *are given by the firm's optimality conditions* (14) *and* (15)*, and individual household allocations* $\{(\mathbf{a}_{s,t}, \mathbf{c}_{s,t})\}$ *are as stated in Proposition 1, given the dynamics of the capital stock* K_t *in* (60).

Proof. Follows directly from taking logs on both sides of equation (59) and solving the linear first order difference equation for $\log(K_t)$,

$$\log K_{t+1} = \log \hat{s} + (1-\theta)\log A_t + \theta \log K_t$$

and then exponentiating. This delivers (60). □

With less than full depreciation, $\delta < 1$, Assumption 5 is not sufficient to insure that the no-savings condition $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ is satisfied. For arbitrary sequences of productivity it is difficult to establish general conditions purely in terms of the fundamentals of the economy, but in the case of fully *permanent* shocks this is possible since for this case we can establish that the capital stock evolves monotonically over time. To do so it is useful to distinguish positive technology shocks (which induce positive wage growth along the transition and temporarily elevated interest rates) from negative technology shocks (with negative wage growth and depressed interest rates along the transition), since the two cases differ in the restrictiveness of the assumptions needed to ensure that the no-savings condition is satisfied.

5.2.2 Permanent Shocks

We first establish the monotone convergence of the capital stock following a permanent shock to productivity.

Proposition 5 (Monotone Convergence of (K_t, R_t, w_t))**.** *Assume the economy is in a stationary equilibrium associated with aggregate productivity, A_0 and associated capital K_0 at time $t = 0$, and suppose at time $t = 1$, productivity unexpectedly and permanently changes to A_1 with $A_1 > A_0$. Furthermore, suppose $\beta R_t < \frac{w_{t+1}}{w_t}$ for all $t \geq 0$ (Assumption 3). Then, aggregate capital K_t and wages w_t monotonically increase and converge to their new stationary equilibrium values, and the interest rate jumps up on impact and then converges back monotonically to the old (and new) stationary equilibrium from above:*

$$K_0 = K_1 < K_2 < \cdots < K^* = \frac{A_1}{A_0} K_0, \tag{61}$$

$$w_0 < w_1 < w_2 < \cdots < w^* = \frac{A_1}{A_0} w_0, \tag{62}$$

$$R_0 < R_1 > R_2 > \cdots > R^* = R_0. \tag{63}$$

Symmetrically, following a permanent negative productivity shock $A_t = A_1 < A_0$ for all $t \geq 1$, the aggregate capital stock and wages monotonically decrease along the transition, and the interest rate falls on impact before converges back to the old (and new) stationary equilibrium from below:

$$K_0 = K_1 > K_2 > \cdots > K^* = \frac{A_1}{A_0} K_0, \tag{64}$$

$$w_0 > w_1 > w_2 > \cdots > w^* = \frac{A_1}{A_0} w_0, \tag{65}$$

$$R_0 > R_1 < R_2 < \cdots < R^* = R_0. \tag{66}$$

Proof. See Appendix A.3.1 □

Equipped with this result we can now give sufficient conditions, purely in terms of fundamentals, for the condition $\beta R_t < \frac{w_{t+1}}{w_t}$ to be satisfied for all t. We first consider the case of a positive productivity shock. In this case, the capital stock and thus wages $w_t = (1-\theta)A_t^{1-\theta}K_t^{\theta}$ are monotonically increasing over time, and so $\beta R_{t+1} < 1$ is a sufficient condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$. Furthermore, from the previous proposition the interest rate jumps up at $t = 1$ and then monotonically converges to the (old and new) stationary equilibrium interest rate. Therefore, $\beta R_1 < 1$ guarantees that $\beta R_{t+1} < 1$ and thus $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all $t \geq 0$. The following proposition provides a sufficient condition for this to be the case.

Proposition 6 (Sufficient Condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ $\forall t \geq 0$ after a Positive Shock)**.** *Let Assumption 5 be satisfied and let the economy be in a stationary equilibrium with* $\beta R_0 < 1$*. After a permanent positive MIT productivity shock at* $t = 1$ *(*$A_t = A_1 > A_0$ *for all* $t \geq 1$*),* $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ $\forall t \geq 0$ *is satisfied if the shock is not too large, that is, if* $A_1 \in [A_0, \bar{A}_1)$ *where the threshold* $\bar{A}_1$ *satisfies*

$$\frac{\bar{A}_1}{A_0} = \left[\frac{1-\beta(1-\delta)}{\theta}\,\frac{\xi(1-\theta)+(1-\nu)\theta\left[1-(1-\nu-\xi)\beta\right]}{\left[1-(1-\nu)\beta(1-\delta)\right]\left[1-(1-\nu-\xi)\beta\right]}\right]^{\frac{1}{1-\theta}} > 1. \tag{67}$$

Proof. See Appendix A.3.2. □

Intuitively, if $A_1/A_0 > 1$ is sufficiently small, the initial jump in the interest rate is not too large, and we can guarantee $\beta R_{t+1} < 1$ along the transition path. This, coupled with positive wage growth induced by the positive productivity shock insures that $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all $t \geq 0$ along the transition path, and high-productivity households have no incentive to save along the transition, confirming the existence of a simple, no-savings equilibrium .

The case of a negative technology shock is more challenging because wages are declining along the transition (see the previous proposition), and thus high-productivity individuals face stronger incentives to save in anticipation of lower labor income in the future. We can nevertheless give a sufficient condition on the size of the productivity decline that guarantees the no-savings condition be satisfied.

Proposition 7 (Sufficient Condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ $\forall t \geq 0$ after a Negative Shock)**.** *Let Assumption 5 be satisfied and let the economy be in a stationary equilibrium with* $\beta R_0 < 1$*. After a permanent negative MIT productivity shock at* $t = 1$ *(*$A_t = A_1 < A_0$ *for all* $t \geq 1$*),* $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ $\forall t \geq 0$ *is satisfied if* $A_1 \in (\underline{A}_1', A_0]$ *holds, where the threshold satisfies*

$$\underline{A}_1'/A_0 = \left[1-\nu+\nu\,\frac{1-(1-\delta)\beta(1-\nu)}{\beta\nu(1-\delta)}\,\frac{\xi(1-\theta)-\beta\theta\nu(\xi+\nu+\frac{1}{\beta}-1)}{\xi(1-\theta)+\beta\theta(1-\nu)\left(\xi+\nu+\frac{1}{\beta}-1\right)}\right]^{\frac{1}{\theta-1}} < 1. \tag{68}$$

Proof. See Appendix A.3.4. □

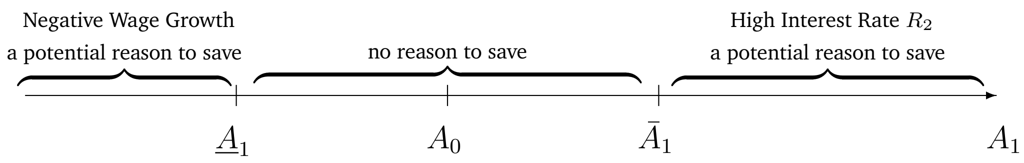

Figure 2: Thresholds for A_1

Figure 2 illustrates Propositions 6 and 7 graphically. Note that the conditions stated in these two propositions are sufficient but not necessary for the household limited commitment constraint to be binding in the high income state.[10] To summarize, Proposition 6 and 7 state that if the productivity shock is not too large, $A_1 \in (\underline{A}_1', \bar{A}_1)$, the condition on interest rate and wage growth, $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$, is satisfied for all $t \geq 0$.

In Appendix B.3.4 we generalize the results in this subsection to arbitrary monotone deterministic and convergent sequences $\{A_t\}_{t=1}^{\infty}$ with $A_0 < A_1 \leq A_2 \leq \cdots$ or with $A_0 > A_1 \geq A_2 \geq \cdots$ with $\lim_{t\to\infty} A_t = A^*$. The results are similar to the ones for permanent shocks (but require a first-order approximation of the capital stock dynamics), in that they require that the initial productivity shock A_1 cannot be too large or too small.

5.3 MIT Shocks, Anticipated Shocks and Consumption on Impact

In Proposition 1 we provided a general characterization of the optimal consumption allocation under the no-savings in the high-state condition. We now draw out one perhaps unexpected implication of this general characterization in the context of MIT shocks: despite the unexpected change in wages and interest rates starting in period $t = 1$, the optimal consumption allocation of low-productivity individuals satisfies the standard Euler equation even through the surprise, i.e., between period $t = 0$ and $t = 1$.

Corollary 2 (Consumption at the time of a shock)**.** *Consider an unexpected shock to productivity at $t = 1$ and assume that Assumptions 2, 3, and 4 hold. Then consumption of high-income agents ($\mathbf{c}_{h,t}$) is a constant fraction of their income, and consumption of low-income agents ($\mathbf{c}_{s,t}$ for $s \geq 1$) satisfies the Euler equation between periods $t = 0$ and $t = 1$:*

$$\mathbf{c}_{h,1} = \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta} z w_1$$
$$\mathbf{c}_{s,1} = \beta R_1 \mathbf{c}_{s-1,0} \qquad \textit{for } s \geq 1. \tag{69}$$

[10]Proposition 24 in Appendix B.3.1 shows a less tight condition on A_1 in case of negative shocks. See Appendix B.3.2 for numerical examples and Appendix B.3.3 for numerical comparative statics with respect to permanent productivity shocks.

Proof. Follows directly from the general characterization in Proposition 1. See Appendix A.3.5 for detail. □

The key to this result is that with log-utility, unconstrained households consume a constant fraction of their assets cum-interest, see (22)

$$\mathbf{c}_{s,1} := w_1 c_{s,1} = [1 - \beta(1-\nu)]\, R_1 \mathbf{a}_{s,1} \qquad \forall s \geq 1, \forall t \tag{70}$$

Two observations are crucial here. First, current consumption (and assets chosen for tomorrow) does not depend on *future* interest rates with log-utility.[11]. Second, the surprise change in current (period 1) TFP does impact the marginal product of capital and thus R_1 (even though the capital stock in $t = 1$ is predetermined), and therefore consumption $\mathbf{c}_{s,1}$ changes in period 1 relative to what the household had planned in the initial steady state ($\mathbf{c}_{s,0}$), as equation (70) indicates (and even though $\mathbf{a}_{s,1}$ is predetermined from the previous period). But since the impact of R_1 on consumption is proportional, it exactly cancels out with the direct change in R_1 in the Euler equation, and thus the adjusted consumption level $\mathbf{c}_{s,1}$ continues to satisfy the Euler equation between period $t = 0$ and $t = 1$.

In this section we have analyzed the transitional dynamics after an unanticipated productivity shock (MIT shock) at $t = 1$, starting from the initial steady state. Finally, note that the assumption that the TFP changes are completely unanticipated is irrelevant for the transition dynamics. As we saw from equation (70), low-income agents consume a constant fraction of their implied asset position regardless of future interest rates, and high-income agents also consume a constant fraction of their labor income. Therefore, aggregate consumption in the economy does not depend on future interest rates and wages, and the law of motion of aggregate capital does not depend on these future prices either.[12] Thus the dynamics of the economy unfolds the same, regardless of whether future productivity shocks are anticipated or unanticipated. It is important to note that these results do not hold in our model if households have a CRRA utility function with $\sigma \neq 1$; they also do not hold in the standard neoclassical growth model. We will discuss each case in the Appendix B.3.5.

[11]We discuss the relation to the literature also exploiting log-utility to obtain analytical tractability in heterogeneous-agent macro models in section 7

[12]As we see in equation (23), savings for the next period do not depend on future interest rates either:

$$\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}; z_{t+1} = 0) = \begin{cases} \frac{\beta}{1-(1-\nu-\xi)\beta} \zeta w_t(A^t) & \text{if } z_t = \zeta \\ \beta R_t(A^t) \mathbf{a}_t(a_0, z^t, A^t) & \text{if } z_t = 0 \end{cases}$$

With log utility, the effect of a higher return on assets on savings in a state with higher TFP (substitution effect) is completely offset by the lower marginal utility from consumption (income effect). Thus, households save the same amount regardless of future aggregate productivity.

Remark 1. *In the case of unexpected shocks, the payoffs of the contingent claims unexpectedly change. Is that a problem? Note that there are no negative Arrow securities positions, and for positive positions, the assets just pay surprisingly little or surprisingly much, fully in line with the surprise in the marginal product of capital. But this might need further thought, and will replace the discussion of intermediaries making losses or profits, since we do not have them explicitly anymore.*

6 Aggregate Shocks to Productivity

We now consider an economy with stochastic productivity growth, introducing aggregate risk into the economy. Assume that the productivity process $\{A_t\}$ is stochastic, following a probability distribution $\pi(A_{t+1}|A^t)$, which is independent of idiosyncratic shocks.[13] In Section 3, we have considered a general productivity process, where the probability of A_{t+1} may depend on the entire history of aggregate states $A^t \equiv \{A_0, A_1, \cdots, A_t\}$. Under the assumption of $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all t and for all possible states (A^t, A_{t+1}), we have derived the optimal allocation of households' consumption and savings and the law of motion of aggregate capital.

We now verify that under an assumption on fundamentals (exogenous parameters), $\beta R_{t+1}(A^{t+1}) < \frac{w_{t+1}(A^{t+1})}{w_t(A^t)}$ holds for all t and (A^t, A_{t+1}). In Section 6.1, we specify a productivity process with *iid* growth rates. Productivity growth can take two values, $\frac{A_{t+1}}{A_t} \in \{1-\epsilon, 1+\epsilon\}$, with equal probability (see equation (1) in Assumption 1). With this specification, we find a sufficient condition on ϵ (Assumption G) such that $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ holds for all t and all (A^t, A_{t+1}). Proposition 9 shows that under Assumption 5 (assumption for a partial insurance stationary equilibrium), there always exists $\bar{\epsilon} > 0$ such that Assumption G holds for all $0 \leq \epsilon < \bar{\epsilon}$.

6.1 A Sufficient Condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 0$ and A^t

We will derive a sufficient condition on ϵ for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all possible states. The idea is the following. First, we express the condition (21) in Assumption 3, $\beta R_{t+1}(A^{t+1}) < \frac{w_{t+1}(A^{t+1})}{w_t(A^t)}$, as a function of $\tilde{K}_t := \frac{K_t}{A_t}$ and $\frac{A_{t+1}}{A_t}$. Given a value of $\tilde{K}_t$, the condition (21) is tighter for a smaller value of $\frac{A_{t+1}}{A_t}$ (Lemma 2, see also Figure 3). Besides, $R_{t+1}(A^{t+1})$ is decreasing in $\tilde{K}_t$, and $\frac{w_{t+1}(A^{t+1})}{w_t(A^t)}$ is decreasing in $\tilde{K}_t$.[14] Therefore, we derive an upper

[13] A_t takes a positive real value, $A_t \in \mathbb{R}_+$, and the number of possible states in each period is finite.

[14] As seen in Figure 3, $\frac{\beta R_{t+1}}{w_{t+1}/w_t}$ can be increasing or decreasing in $\tilde{K}_t$. Because the argmax of $\frac{\beta R_{t+1}}{w_{t+1}/w_t}$ with respect to $\tilde{K}_t$ can be $\tilde{K}_t^{max}$ or $\tilde{K}_t^{min}$ (or somewhere in between), we focus on a sufficient condition

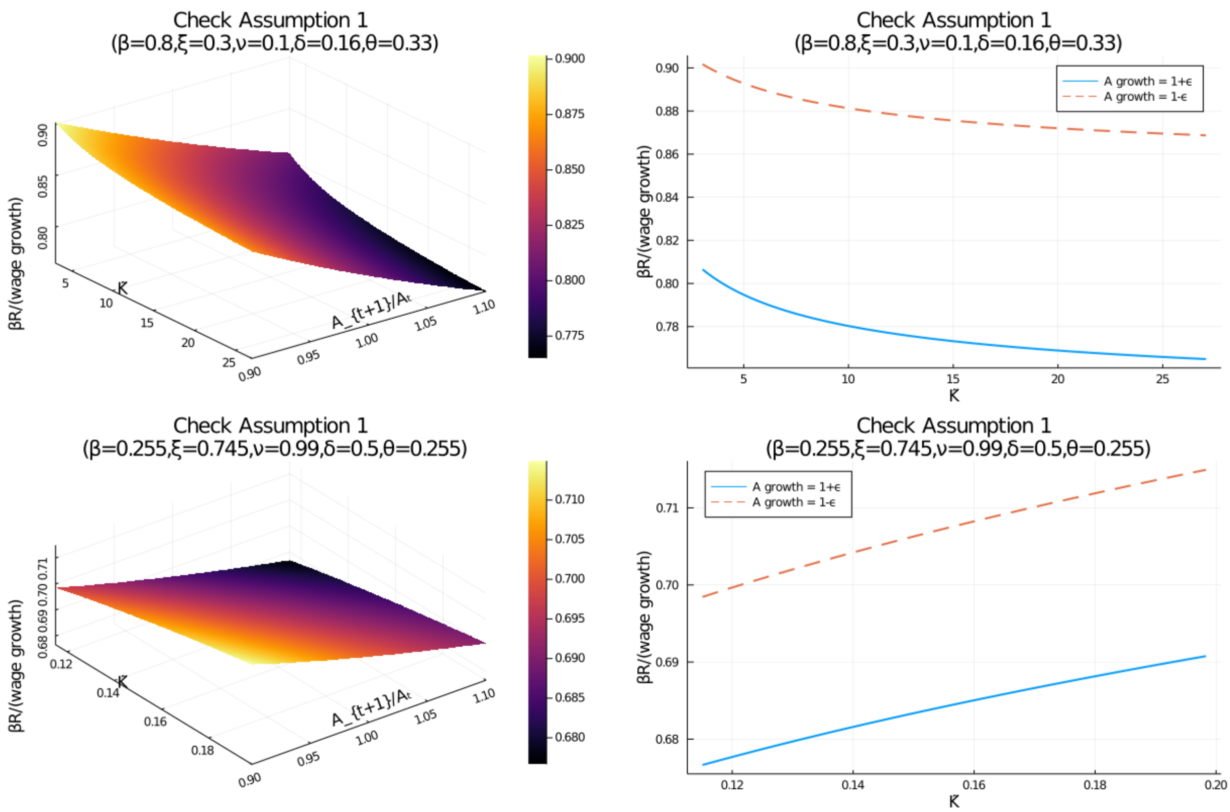

Figure 3: $\frac{\beta R_{t+1}}{w_{t+1}/w_t}$ given $\tilde{K}_t$ and $\frac{A_{t+1}}{A_t}$

bound on $\epsilon \geq 0$ such that

$$\beta R_{t+1}(A^{t+1})\Big|_{\tilde{K}_t=\tilde{K}_t^{min},\frac{A_{t+1}}{A_t}=1-\epsilon} < \frac{w_{t+1}(A^{t+1})}{w_t(A^t)}\Big|_{\tilde{K}_t=\tilde{K}_t^{max},\frac{A_{t+1}}{A_t}=1-\epsilon}. \tag{72}$$

Under such a condition on ϵ, $\beta R_{t+1}(A^{t+1}) < \frac{w_{t+1}(A^{t+1})}{w_t(A^t)}$ is satisfied for any sequence of $\{A_t\}_{t\geq 0}$ that follows a stochastic process (1). Note that although we derive the condition on ϵ under a specific stochastic process (1), the same logic goes through if the number of possible states of A_{t+1} is finite with a bound on A_{t+1} given by $A_{t+1} \in [(1-\epsilon)A_t, (1+\epsilon)A_t]$, instead of two values $A_{t+1} \in \{(1-\epsilon)A_t, (1+\epsilon)A_t\}$. Moreover, under Assumption G, $\beta R_{t+1}(A^{t+1}) < \frac{w_{t+1}(A^{t+1})}{w_t(A^t)}$ holds for any possible states (A^t, A_{t+1}) regardless of the probability of each state. This implies that we may not need an *iid* assumption on productivity growth rate.

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on ϵ derived from equation (72). Note that $\frac{\beta R_{t+1}}{w_{t+1}/w_t}$ is expressed as

$$\frac{\beta R_{t+1}}{w_{t+1}/w_t} = \frac{\beta\left[\theta\left\{\frac{A_t}{A_{t+1}}\left[\hat{s}(\tilde{K}_t)^\theta + (1-\hat{\delta})\tilde{K}_t\right]\right\}^{\theta-1} + 1 - \delta\right]}{\left(\frac{A_{t+1}}{A_t}\right)^{1-\theta}\left[\hat{s}(\tilde{K}_t)^{\theta-1} + 1 - \hat{\delta}\right]^\theta} \tag{71}$$

given the law of motion of capital (40).

6.1.1 Lemmas

We state Lemmas that are useful to find a sufficient condition on ϵ.

Lemma 1. *Define $\tilde{K}_t := \frac{K_t}{A_t}$. The law of motion of capital (40) is expressed as:*

$$\tilde{K}_{t+1} = \frac{A_t}{A_{t+1}} \left[\hat{s}\tilde{K}_t^{\theta} + (1-\hat{\delta})\tilde{K}_t \right]. \tag{73}$$

The maximum value and the minimum value of $\frac{K_t}{A_t}$ are given by:

$$\tilde{K}^{max} = \left(\frac{\hat{s}}{\hat{\delta} - \epsilon} \right)^{\frac{1}{1-\theta}} \text{ and } \tilde{K}^{min} = \left(\frac{\hat{s}}{\hat{\delta} + \epsilon} \right)^{\frac{1}{1-\theta}} \tag{74}$$

Proof. See Appendix A.4. □

Lemma 2.

1. *The constraint $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ is tighter at the time of negative shock ($\frac{A_{t+1}}{A_t} = 1 - \epsilon$).*
2. *If $\tilde{K}_t = \tilde{K}^{min}$, $\tilde{K}_{t+1} = \frac{A_t}{A_{t+1}}(1+\epsilon)\tilde{K}^{min}$. If $\tilde{K}_t = \tilde{K}^{max}$, $\tilde{K}_{t+1} = \frac{A_t}{A_{t+1}}(1-\epsilon)\tilde{K}^{max}$.*
3. *Suppose $\frac{A_{t+1}}{A_t} = 1 - \epsilon$ and $\tilde{K}_t \in [\tilde{K}^{min}, \tilde{K}^{max}]$. R_{t+1} is highest at $\tilde{K}_t = \tilde{K}^{min}$. $\frac{w_{t+1}}{w_t}$ is lowest at $\tilde{K}_t = \tilde{K}^{max}$*

Proof. See Appendix A.4. □

We prove that $\tilde{K}^{min} < \tilde{K}_t < \tilde{K}^{max}$ implies $\tilde{K}^{min} < \tilde{K}_{t+1} < \tilde{K}^{max}$. This implies that if the economy starts with aggregate capital satisfying $\tilde{K}^{min} < \tilde{K}_0 < \tilde{K}^{max}$, all future capital at $t \geq 1$ satisfies $\tilde{K}^{min} < \tilde{K}_t < \tilde{K}^{max}$.

Lemma 3. *$\tilde{K}^{min} \leq \tilde{K}_t \leq \tilde{K}^{max}$ implies $\tilde{K}^{min} \leq \tilde{K}_{t+1} \leq \tilde{K}^{max}$*

Proof. See Appendix A.4. □

6.1.2 Propositions

We now state a condition on ϵ to guarantee $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all $\tilde{K}^{min} < \tilde{K}_t < \tilde{K}^{max}$ and $\frac{A_{t+1}}{A_t}$.

Assumption G.

$$\beta \left[\theta \left(\frac{1-\epsilon}{1+\epsilon} \right)^{1-\theta} \frac{\hat{\delta} + \epsilon}{\hat{s}} + 1 - \delta \right] < 1 - \epsilon, \tag{75}$$

where $\hat{s} := \left[\frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta) + (1-\nu)\beta\theta \right]$ *and* $1 - \hat{\delta} := (1-\nu)\beta(1-\delta)$.

Proposition 8. *Suppose Assumption 2, 4, and 5 hold and the aggregate capital at $t = 0$ satisfies $\tilde{K}^{min} < \tilde{K}_0 < \tilde{K}^{max}$. Under Assumption G,*

$$\beta R_{t+1} < \frac{w_{t+1}}{w_t} \text{ for all } t \geq 0 \text{ with probability } 1.$$

Proof. See Appendix A.4. □

The next proposition shows that the set of ϵ that satisfies condition (75) is non-empty under Assumption 5.

Proposition 9. *Suppose Assumption 5 holds. There exists $\bar{\epsilon}$ such that Assumption G holds for all $0 \leq \epsilon < \bar{\epsilon}$. Given such $\bar{\epsilon}$, $0 \leq \epsilon < \bar{\epsilon}$ and $\tilde{K}^{min} < \tilde{K}_0 < \tilde{K}^{max}$ imply $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all $t \geq 0$ with probability 1.*

Proof. See Appendix A.4. □

The corollary below shows that if $\delta = 1$, we have $\bar{\epsilon}$ in a closed form. Note also that if $\delta = 1$, we don't need an assumption on ϵ as $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ does not depend on $\{A_t\}$. (See Section 5.2.1)

Corollary 3. *Suppose Assumption 5 holds and capital fully depreciates ($\delta = 1$). If $0 \leq \epsilon < \bar{\epsilon}$, where*

$$\bar{\epsilon} := \frac{\left(\frac{\hat{s}}{\beta\theta}\right)^{\frac{1}{\theta}} - 1}{\left(\frac{\hat{s}}{\beta\theta}\right)^{\frac{1}{\theta}} + 1} > 0, \tag{76}$$

Assumption G is satisfied.

Proof. See Appendix A.4. □

6.2 Quantitative Exploration

Corollary 3 shows that under full depreciation of capital ($\delta = 1$), there exists an upper bound on ϵ in closed form for Assumption G to hold. If $\delta < 1$, we need to solve a nonliner equation (75) to compute $\bar{\epsilon}$. Figure 4 illustrates the relationship between δ and $\bar{\epsilon}$. As δ increases, Assumption G can be satisfied for larger ϵ. Assumption G is a joint condition for parameters $(\beta, \xi, \nu, \delta, \theta, \epsilon)$. Since the left hand side of equation (75) is decreasing in ξ and increasing in ν (since $\hat{s}$ is increasing in ξ and decreasing in ν, $\hat{\delta}$ is increasing in ν), Assumption G becomes less tight as ξ increases and ν decreases. Intuitively, if households are more likely to switch to a zero income state (higher ξ) and stay unproductive for a long time (lower ν), they save more in a high-income state through the insurance contract. Larger capital supply leads to lower interest rate in the economy, so the condition $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ is likely to be satisfied.

In Figure 5, the economy starts from a steady state at $t = 0$ with $A_0 = 1$, and the productivity follows stochastic process in equation (1). Under the parameters chosen that satisfy Assumption G, βR_{t+1} is smaller than $\frac{w_{t+1}}{w_t}$ from $t = 0$ to $t = 10$, consistent with Proposition 8.

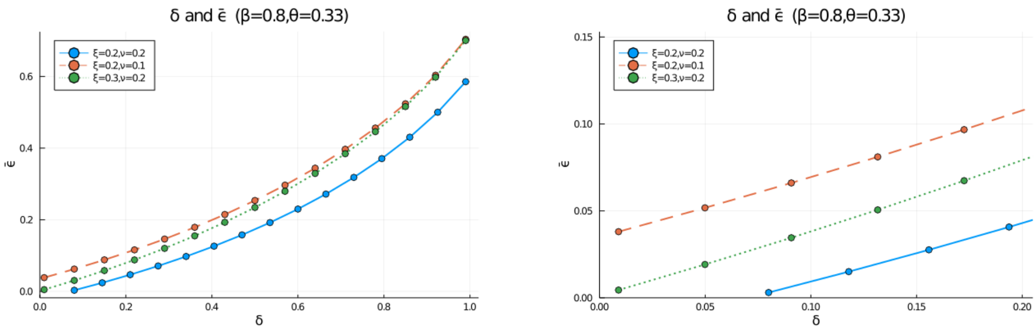


Figure 4: δ and $\bar{\epsilon}$

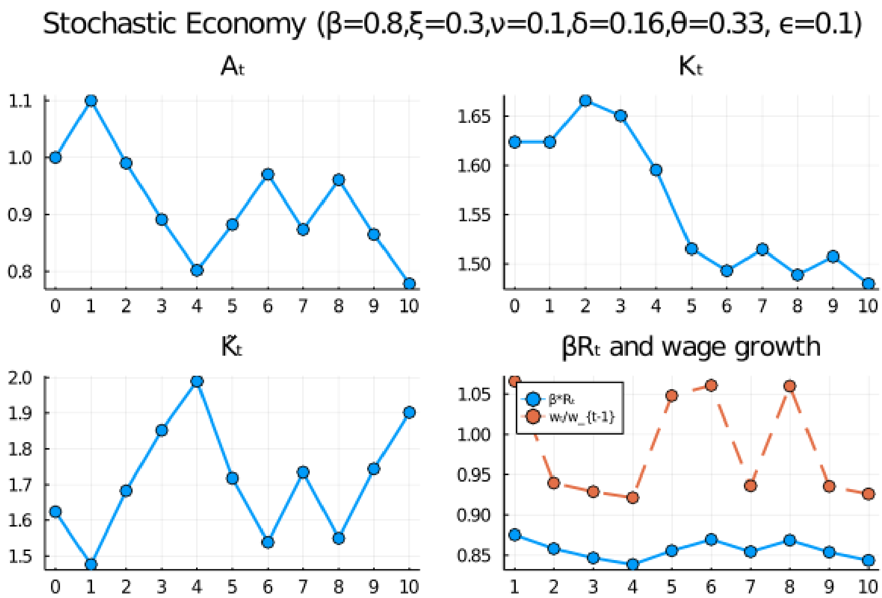


Figure 5: Path of a Stochastic Economy

7 Comparison to the Literature

We obtain a closed form for the transitional dynamics of aggregate capital. This is a surprising result given that a neoclassical growth model without a commitment constraint does not have a closed-form solution unless full depreciation is assumed. We discuss why it is the case.

We have four findings in this section. First, we consider a limited-commitment model in which the transition probabilities of income shocks approach zero: $\xi \to 0$ and $\nu \to 0$. With full depreciation of capital ($\delta = 1$), the law of motion of capital in this economy is the same as a standard neoclassical growth model. This is not the case with partial depreciation of capital ($0 < \delta < 1$). Second, we consider a modified neoclassical growth model in which workers earn labor income and capital owners earn capital income. Then, the law of motion of capital is the same between the modified neoclassical growth model and a limited-commitment model with $\xi \to 0$ and $\nu \to 0$. We relate this result with Moll (2014). Third, we derive the aggregate capital at arbitrary time t, K_t, in a closed form using a law of motion of capital for a limited commitment model with $\delta = 1$. We haven't found a closed form if capital partially depreciates ($0 < \delta < 1$), in contrast to a continuous time model with a constant saving rate discussed in Jones (2000). Finally, we show that a limited-commitment model converges to a steady state faster than a Solow growth model.

7.1 Comparison with a Standard Neoclassical Growth Model

The transitional dynamics of aggregate capital in an economy with limited commitment is characterized by equation (40):

$$K_{t+1} = \frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta)A_t^{1-\theta}K_t^{\theta} + (1-\nu)\beta\left[\theta A_t^{1-\theta}K_t^{\theta} + (1-\delta)K_t\right].$$

We consider a limit where idiosyncratic income states do not change over time, $\xi \to 0$ and $\nu \to 0$. We see that under full depreciation of capital ($\delta = 1$), the law of motion of capital is the same as a standard neoclassical growth model.

Proposition 10. *In a limited-commitment model, as $\xi \to 0$ and $\nu \to 0$, the transitional dynamics is described by two equations:*

$$K_{t+1} = \beta\left[\theta A_t^{1-\theta}K_t^{\theta} + (1-\delta)K_t\right], \tag{77}$$

$$K^* = A\left[\frac{\beta\theta}{1-\beta(1-\delta)}\right]^{\frac{1}{1-\theta}}. \tag{78}$$

Under full depreciation of capital ($\delta = 1$), the steady-state capital stock and the law of motion of capital is the same as a standard neoclassical growth model with full depreciation of capital and a log utility function.

Proof. See Appendix A. □

Note that a standard neoclassical growth model does not have a closed form for a law of motion of capital if $0 < \delta < 1$. Therefore, the law of motion is not the same between the two models if capital partially depreciates.

7.2 A Modified Neoclassical Growth Model

In general, the law of motion of capital in the standard neoclassical growth model does not have a closed-form solution. A reason is that a representative agent has two sources of income (wage and capital income). Thus, she wouldn't consume a constant fraction of capital income. In a limited-commitment model, however, high-income agents earn wage income, and low(zero)-income agents receive insurance payment as an annuity value of their asset. Under the maintained assumption (Assumption 5 and $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$), high-income agents consume a constant fraction of labor income, $\mathbf{c_h} = \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta} w_t z$, and low-income agents consume a constant fraction of capital income, $\mathbf{c_{s,t}} = [1-(1-\nu)\beta]\, R_t\, \mathbf{a_{s,t}}$. This property gives rise to a law of motion of capital in a closed form.

As an illustration, we describe a modified neoclassical growth economy which we can think of as a counterpart of the limited commitment model with $\xi \to 0$ and $\nu \to 0$. There are two types of agents: a worker and a capital owner. A worker earns labor income and consumes all of the income each period. A capital owner receives capital income and consumes a $1-\beta$ fraction of her income, following the property of a log utility function. Therefore, consumption functions are given by:

$$C^{worker} = w_t, \tag{79}$$

$$C^{owner} = (1-\beta) R_t K_t. \tag{80}$$

The market clearing condition implies that:

$$\begin{aligned} K_{t+1} &= A_t^{1-\theta} K_t^{\theta} + (1-\delta)K_t - C^{worker} - C^{owner} \\ &= A_t^{1-\theta} K_t^{\theta} + (1-\delta)K_t - (1-\theta)A_t^{1-\theta} K_t^{\theta} - (1-\beta)[\theta A_t^{1-\theta} K_t^{\theta} + (1-\delta)K_t] \\ &= \beta\theta A_t^{1-\theta} K_t^{\theta} + \beta(1-\delta)K_t \end{aligned} \tag{81}$$

This coincides with (77). Thus, a reason for obtaining a closed-form solution for the law of motion of capital in the limited commitment model is the separation between wage income and capital income.

7.2.1 Comparison with Moll (2014)

Online Appendix to Moll (2014) discusses why his almost standard neoclassical growth model has a constant saving rate as in a Solow model.[15] It is a consequence of three assumptions: (i) the separation of agents into workers and capitalists, (ii) that workers cannot save, (iii) that capitalist's utility is logarithmic. These assumptions imply that capitalists face individual constant returns and that capitalists save at a constant rate.

7.2.2 Constant Saving Rate in Our Model

Our model has a similar structure as in Moll (2014). High-income agents have zero assets at the beginning of the period if $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all t and consume a constant fraction of labor income. Low-income agents have zero labor income and face a constant return on asset (R_t). The optimal insurance contract implicitly solves the following problem on behalf of low-income agents:

$$\max \sum_{u=0}^{\infty} [(1-\nu)\beta]^u \log(\mathbf{c}_{s+u,t+u})$$

$$\text{s.t. } (1-\nu)\mathbf{a}_{s+u+1,t+u+1} = R_t \mathbf{a}_{s+u,t+u} - \mathbf{c}_{s+u,t+u}.$$

The low-income agents switch to a high-income state with probability ν and renew the contract. Hence, the effective discount factor is $(1-\nu)\beta$. If she switches to the high state, her asset in the next period is zero. Therefore, she needs to save $(1-\nu)\ \mathbf{a}_{s+1,t+1}$ today to have $\mathbf{a}_{s+1,t+1}$ tomorrow in a low-income state. Because of the log-utility function, the optimal decision rules takes:

$$(1-\nu)\mathbf{a}_{s+1,t+1} = (1-\nu)\beta R_t \mathbf{a}_{s,t}. \tag{82}$$

After aggregating the savings by high-income agents and by low-income agents, we have:

$$K_{t+1} = \frac{\xi\beta}{1-(1-\nu-\xi)\beta} w_t + (1-\nu)\beta R_t K_t,$$

or equivalently,

$$K_{t+1} = \hat{s} Y_t + (1-\hat{\delta})K_t,$$

$$\text{where } \hat{s} = \frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta) + (1-\nu)\beta\theta,$$

$$\hat{\delta} = 1-(1-\nu)\beta(1-\delta).$$

Therefore, the model with limited commitment has a constant saving rate as in a Solow model. Three properties play a role here: (i) the separation between high-income agents

[15] The law of motion of capital is given by $k_{t+1} = \alpha\beta A k_t^{\alpha} + \beta(1-\delta)k_t$. The note is available on his website: `https://benjaminmoll.com/wp-content/uploads/2019/07/capitalists-workers.pdf`

and low-income agents, (ii) that high-income agents have zero capital and consume a constant fraction of their labor income, (iii) that low-income agents have zero labor income and consume a constant fraction of their capital income.

7.3 Aggregate Capital in a Closed Form at time t

With full depreciation of capital, we derive aggregate capital at any time t along the transition in a closed form. By substituting $\delta = 1$ into (40), we have:

$$K_{t+1} = \hat{s} A_t^{1-\theta} K_t^{\theta},$$
$$\text{where } \hat{s} = \frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta) + (1-\nu)\beta\theta.$$

By taking a log and substituting $A_t = A_1$ for $t \geq 1$,

$$\log K_{t+1} = \theta \log K_t + \log(\hat{s} A_1^{1-\theta}) \text{ for } t \geq 1$$

This is a first-order difference equation and has a closed-form solution:

$$\log K_t = \theta^{t-1} \log K_1 + (1-\theta^{t-1}) \frac{\log(\hat{s} A_1^{1-\theta})}{1-\theta} \text{ for } t \geq 1$$

where $K_1 = K_0$ is predetermined in a stationary equilibrium. We haven't found a closed-form solution for general $\delta \neq 1$. This is in contrast to a continuous time model with a constant saving rate.[16]

7.4 Speeds of Convergence

As discussed in Section **??**, the speed of convergence is increasing in ν (i.e., the probability from low-income state to high-income state). In Figure 6, aggregate capital in a limited-commitment model converges to a new stationary equilibrium faster than the neoclassical growth model if $\nu = 0.3$ and $\xi = 0.7$ but slower than the neoclassical growth

[16] Jones (2000) presents the closed-form solution to a continuous-time Solow model. Under a Cobb-Douglas production function and a law of motion of capital:

$$y_t = k_t^{\alpha},$$
$$\dot{k}_t = s k_t^{\alpha} - \delta k_t,$$

the capital at time t is given by:

$$k_t = \left[\frac{s}{\delta}\left(1 - e^{-(1-\alpha)\delta t}\right) + k_0^{1-\alpha} e^{-(1-\alpha)\delta t}\right]^{\frac{1}{1-\alpha}}. \quad (83)$$

The note is available from his website: `https://web.stanford.edu/~chadj/closedform.pdf` model if $\nu = \xi = 0.1$.[17] The numerical result shows that ξ does not affect the speed of convergence.[18]

Compared to a Solow growth model with a save saving rate $\hat{s}$ and a depreciation rate δ, a limited-commitment model converges faster. This is because the effective depreciation rate in a limited-commitment model, $\hat{\delta}$, is higher (since $1-\hat{\delta} = (1-\nu)\beta(1-\delta) < 1-\delta$), and thus the capital persists less than a Solow model. In Figure 6, $\hat{s}$ is the same as the limited commitment-model with $\xi = \nu = 0.2$.

In Section C.5.2, we linearly approximate the transitional dynamics and derive the aggregate capital at time t. A limited-commitment model has a constant saving rate, $\hat{s}$, with an effective depreciation rate $\hat{\delta}$. The capital at t is approximated by:

$$K_t \approx \left[1-(1-\theta)\hat{\delta}\right]^{t-1}(K_1 - K^*) + K^*, \tag{396}$$

$$\text{where } \hat{\delta} = 1-(1-\nu)\beta(1-\delta) > \delta$$

This equation shows that the speed of convergence is given by $(1-\theta)\hat{\delta}$. That is, the aggregate capital converges to the new steady state, K^*, faster if $(1-\theta)\hat{\delta}$ is closer to 1. The speed of convergence depends on the share of capital in a Cobb-Douglas production function, θ, and the effective depreciation rate, $\hat{\delta}$, but not on the saving rate.[19] Since $\hat{\delta} > \delta$, a limited-commitment model converges faster than a Solow growth model. (See Barro and Sala-i-Martin (2003) for a discussion about speeds of convergence in a continuous-time model.)

8 Application 1: Inequality over the Cycle

We have shown that transitional dynamics after a productivity shock are described in a closed form and that it coincides with the dynamics of the economy with aggregate risk when the sample path of aggregate productivity displays a one-time positive (or negative) shock before returning back to its unconditional mean. [This discussion has to be made more precise]. We now assess the impact of such a shock on consumption- and wealth inequality, thereby analyzing how inequality evolves over the business cycle

[17] We normalize the initial capital, K_0, to one in order to compare the speed of convergence. The level of capital in a stationary equilibrium depends on ν and ξ.

[18] The figure is created by a Julia code "Neoclassical.jl".

[19] Equation (**??**) does not use a linear approximation, and the growth rate of capital depends on $\hat{\delta}$ and θ but not on the saving rate:

$$g_{t,t+1} := \frac{K_{t+1}-K_t}{K_t} = \left[\left(\frac{K^*}{K_t}\right)^{1-\theta} - 1\right]\left[1-(1-\nu)\beta(1-\delta)\right].$$

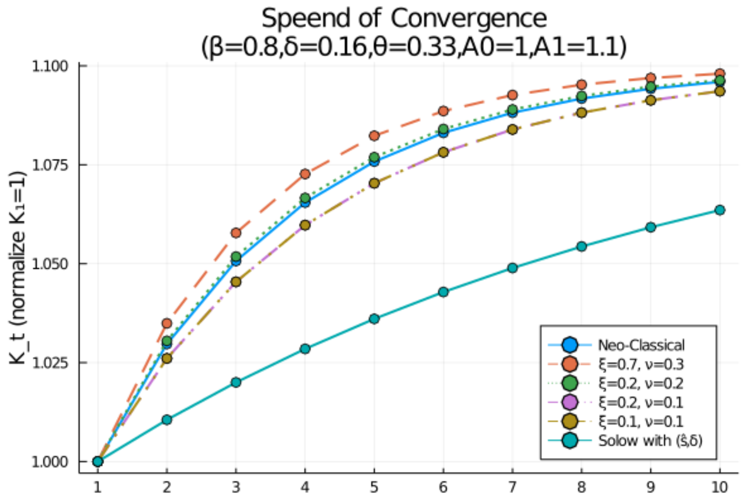


Figure 6: Speed of Convergence

in our model. [For the open question section: can we deal with aggregate TFP following a standard AR(1) process as in the standard RBC literature, or are iid growth shocks required?]

Five findings are discussed in this section: (i) A deflated consumption distribution in the long run does not depend on the level of aggregate productivity, A_t; (ii) With full depreciation of capital ($\delta = 1$), the deflated consumption distribution is time-invariant regardless of aggregate shocks; (iii) With partial depreciation ($\delta < 1$), the consumption inequality expands at the time of a positive productivity shock; (iv) In case of a permanent productivity shock at $t = 1$, the consumption gaps between high-income agents and low-income agents expand at $t = 1$, shrink from $t = 2$ onwards, and overshoot the original gaps along the transition for some parameter values. A sufficient condition for the overshooting phenomenon is also discussed; (v) We have a symmetric result for a negative productivity shock.

8.1 Consumption Distribution in the Long Run

We saw in Corollary 1 that a consumption distribution follows a simple structure (33), which gives the following deflated consumption, $c_{s,t} = \frac{\mathbf{c}_{s,t}}{w_t}$,

$$c_{s,t}(A^t) = \begin{cases} c_0 := \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}\zeta & \text{if } s = 0 \\ \beta R_t(A^t)\frac{w_{t-1}(A^{t-1})}{w_t(A^t)} c_{s-1,t-1}(A^{t-1}) & \text{if } s \geq 1 \end{cases} \tag{84}$$

Remember that s is the number of periods in a low-income state since the last high income state. In a stationary equilibrium, the consumption distribution is characterized by βR^*, which does not depend on productivity (A_t). This implies that after aggregate shocks have subsided, the consumption distribution will be back to the initial stationary distribution in the long run.

Proposition 11 (Consumption Distribution in the Long Run)**.** *Suppose that Assumptions 2, 3, and 4 hold and that an economy is in a steady state at $t = 0$ with productivity A_0. Suppose also that after a productivity shock, aggregate productivity settles down at A_1. Then, the deflated consumption distribution in the long run is the same as the initial distribution:*

$$c_s^* = (\beta R^*)^s c_0 \text{ for } s = 0, 1, 2, \cdots,$$

where the steady-state interest rate, R^*, does not depend on productivity A.

Proof. See Appendix A.6.

□

8.2 Consumption Distribution with $\delta = 1$

We consider a case with full depreciation of capital, $\delta = 1$. In this case, the deflated consumption distribution is constant over the cycle.

Proposition 12. *Suppose Assumptions 2, 3, and 4 hold. Suppose that an economy is in a stationary equilibrium at $t = 0$ with a deflated consumption distribution $\{c_s^*\}_{s \geq 0}$. With full depreciation of capital ($\delta = 1$), the deflated consumption distribution is time-invariant for any sequence of $\{A_t\}_{t \geq 0}$:*

$$c_{s,t}(A^t) = c_s^* \text{ for any } s \geq 0 \text{ at all} t \geq 0 \text{ and } A^t. \tag{85}$$

Proof. See Appendix A.6. □

8.3 Consumption Inequality after a Positive Shock with $\delta < 1$

We analyze the consumption inequality over the transitional dynamics after an MIT shock at $t = 1$. As discussed in Section **??**, it does not matter whether a productivity shock is anticipated or unanticipated. Therefore, the same result goes through in the case of a stochastic economy with a particular realization of productivity shocks $\{A_t\}_{t \geq 0} = (A_0, A_1, \cdots, A_1, \cdots)$.

8.3.1 Consumption Inequality on Impact (t=1)

The deflated consumption of low-income agents declines at the time of a positive shock ($t = 1$) if capital partially depreciates ($\delta < 1$), whereas the deflated consumption of high-income agents is constant over time. Therefore, the inequality expands when an economy faces a positive productivity shock.

Proposition 13. *Consider an economy in a stationary equilibrium at $t = 0$ and a positive productivity shock at $t = 1$ ($A_1 > A_0$), where $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all $t \geq 0$. Under $0 < \delta < 1$ and $0 < \theta < 1$, the degree of inequality rises at the time of a shock ($t = 1$) in the sense that the consumption of all low-income agents ($s = 1, 2, \cdots$) declines relative to high-income agents:*

$$c_{s,1} < c_{s,0}, \tag{86}$$

$$\text{while } c_{0,t} = c_0 := \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}\zeta \text{ for all } t \geq 0.$$

Proof. See Appendix A.6. □

8.3.2 Consumption Inequality after t=2

From time $t = 2$ onwards, aggregate capital adjusts, following the law of motion of capital (40). Given a sequence of $\{w_t, R_t\}_{t\geq 0}$, consumption gaps between high-income agents and low-income agents evolve over time.

Because low-income agents ($s \geq 1$) earn zero labor income and consume a fraction of savings made in the last high-income state, consumption gaps depend crucially on whether their last high income happened before or after the productivity shock. The low-income agents with $s \geq t$ haven't experienced high income after the shock, and their wages in the last high-income state are given by w_0. On the other hand, the low-income agents with $s < t$ have experienced high income after the shock, and wages in the last high-income state are w_{t-s} with $w_{t-s} > w_0$. With this intuition in mind, the consumption gap is characterized as follows.

Proposition 14. *Suppose an economy is in a stationary equilibrium at $t = 0$, and a productivity shock is realized at $t = 1$. Suppose Assumption 3 holds. The evolution of consumption gap between high-income and low-income agents relative to the stady state is described by:*

$$-\log\left(\frac{c_{s,t}}{c_{0,t}}\right) + \log\left(\frac{c_{s,0}}{c_{0,0}}\right) = \begin{cases} \log\left(\frac{w_1}{w_0}\frac{R_0}{R_1}\right) & \text{for all } s \geq 1 \text{ if } t = 1 \\ \log\left(\frac{w_1}{w_0}\frac{R_0}{R_1}\right) + \sum_{u=1}^{t-1}\log\left(\frac{w_{u+1}}{w_u}\frac{R_0}{R_{u+1}}\right) & \text{if } s \geq t \text{ and } t \geq 2 \\ \sum_{u=t-s}^{t-1}\log\left(\frac{w_{u+1}}{w_u}\frac{R_0}{R_{u+1}}\right) & \text{if } s < t \text{ and } t \geq 2 \end{cases} \tag{87}$$

Assume $0 < \delta < 1$. Then, the consumption gap expands at time 1, since $\log\left(\frac{w_1}{w_0}\frac{R_0}{R_1}\right) > 0$. From time 1 until time $s \geq 2$, the consumption gap continues to be higher than the stationary equilibrium if $\log\left(\frac{w_1}{w_0}\frac{R_0}{R_1}\right) + \sum_{u=1}^{t-1}\log\left(\frac{w_{u+1}}{w_u}\frac{R_0}{R_{u+1}}\right) > 0$. From time $s+1$ onwards, the consumption gap is smaller than the stationary equilibrium if $\sum_{u=t-s}^{t-1}\log\left(\frac{w_{u+1}}{w_u}\frac{R_0}{R_{u+1}}\right) < 0$.

Proof. See Appendix A.6. □

Figure 7 shows the evolution of consumption gaps. The gap is determined by three factors: (i) wage at time t compared to time 0 benefits high-income agents, which expands the consumption gap; (ii) wage at time $t-s$ compared to time 0 benefits s-th low-income agents; and (iii) higher interest rate from time $t-s+1$ to time t benefits s-th low-income agents because of the higher return on savings.

This decomposition facilitates Figure 8. Because wage increases monotonically after the positive productivity shock, the wage effect: $\log\left(\frac{w_t}{w_0}\right) - \log\left(\frac{w_{t-s}}{w_0}\right) = \log\left(\frac{w_t}{w_{t-s}}\right)$ expands the consumption gap. On the other hand, because interest rate is higher than the stationary equilibrium along the transition path, the interest rate effect: $\sum_{u=t-s+1}^{t} \log\left(\frac{R_u}{R_0}\right)$ shrinks the consumption gap. The overall impact depends on the relative magnitude of the two effects.

In Figure 7, the consumption gap starts to shrink at time $t=2$ and overshoots the original gap before converging to a new stationary equilibrium. In Proposition 19 in Appendix A.6, we derive a sufficient condition for the overshooting phenomenon in terms of $\{R_t, w_t\}$. The evolution of the key term, $\frac{w_{t+1}}{w_t}\frac{R_0}{R_{t+1}}$, in Proposition 19 is expressed in terms of capital in Proposition 20.

8.4 Consumption Inequality after a Negative Shock

The evolution of the deflated consumption distribution is symmetric for a permanent negative productivity shock. The symmetric results are stated in Corollaries 6–8 in Appendix A.6.

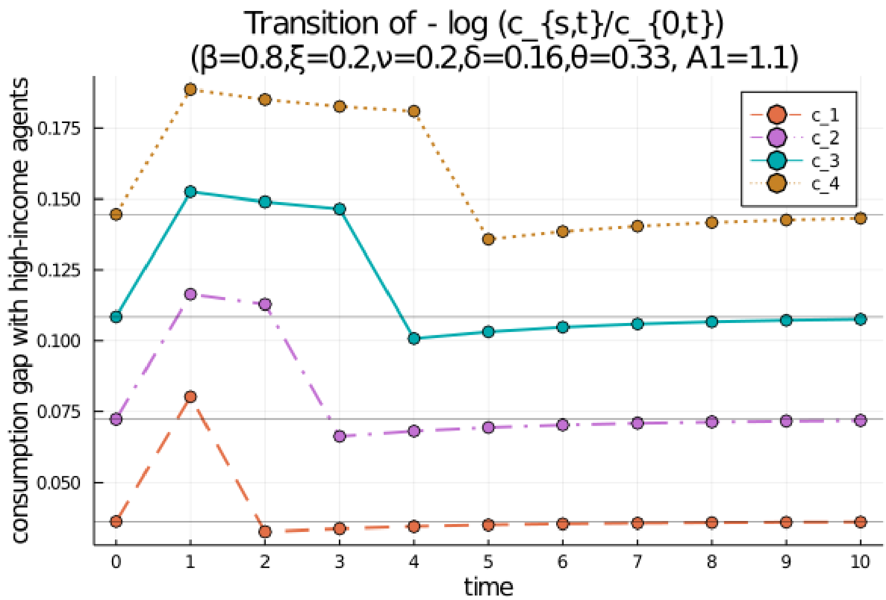
There are two differences between a positive shock and a negative shock. The first difference is in sufficient conditions for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all $t \geq 1$. After a positive shock, $\beta R_{t+1} < 1$ is sufficient since wages are increasing along the transition path. After a negative shock, agents may have an incentive to save even under $\beta R_{t+1} < 1$, as wages are declining over time. The sufficient condition (138) is derived in Appendix A.3.3.

Second, low-income agents may experience higher consumption at the time of a negative shock than high-income agents. This is because high-income agents unexpectedly suffer from lower wages due to the shock, and the decline in wages is larger in magnitude than the decline in interest rates at time $t=1$ if $0<\delta<1$.

9 Application 2: Asset Pricing

We examine asset pricing in the economy with aggregate shocks. The risk-free rate and the risk premium in the limited-commitment model and the representative agent model

Figure 7: Transition of the Consumption Distribution



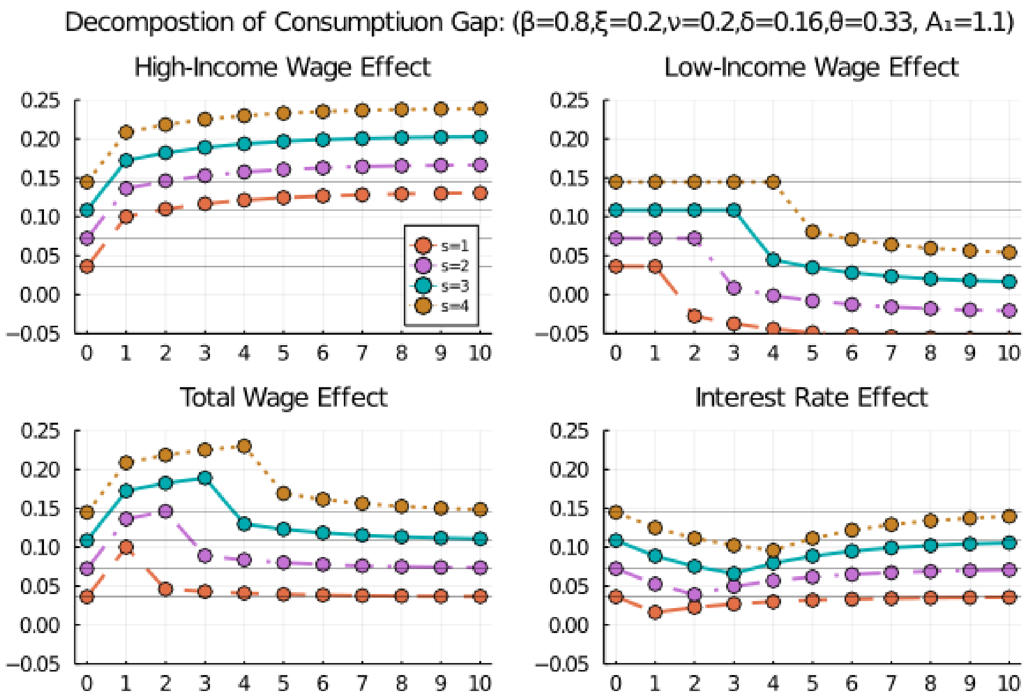


Figure 8: Decomposition of the Consumption Gap

are compared. We examine a conjecture that the presence of idiosyncratic shocks with a limited-commitment constraint lowers the return on risk-free assets but does not impact the risk premium. (cf. Krueger and Lustig, 2010)

In Section 9.1, we derive the risk-free rate and the risk premium as a function of interest rates and aggregate consumption in the two models. In Section 9.2, we show in an economy with full depreciation of capital ($\delta = 1$), the limited-commitment model has a lower risk-free rate, but the risk premium is the same as the representative-agent model. In Section 9.3, we discuss the intuition and why the same conclusion would not follow in an economy with $\delta \neq 1$, while providing a result in an endowment economy.

9.1 The Risk-Free rate and the Risk Premium in the Two Models

We derive the price of risk-free bonds $q^B(A^t)$ and the risk premium $1 + \lambda_t(A^t)$, defined as the ratio of the expected return on risky assets and the risk-free rate, in the limited-commitment model (Lemma 4) and in the representative-agent model (Lemma 5).

Lemma 4. *In the limited commitment model, the price of risk-free bonds and the risk premium at aggregate state A^t is given by:*

$$q_t^{B,LC}(A^t) = \mathbb{E}_t\left[\frac{1}{R_{t+1}(A^{t+1})}\right] \tag{88}$$

$$1 + \lambda_t^{LC}(A^t) := \frac{\mathbb{E}_t[R_{t+1}(A^{t+1})]}{\mathbb{E}_t[1/q^B(A^t)]} = \mathbb{E}_t[R_{t+1}(A^{t+1})]\ \mathbb{E}_t\left[\frac{1}{R_{t+1}(A^{t+1})}\right] > 1. \tag{89}$$

$\mathbb{E}_t[\cdot]$ *denotes the expectation conditional on* A^t*,* $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|A^t]$

Proof. See Appendix A.7.1. □

Lemma 5. *In the representative agent model, the price of risk-free bonds and the risk premium at aggregate state A^t is given by:*

$$q_t^{B,Rep}(A^t) = \mathbb{E}_t\left[\beta\frac{C_t(A^t)}{C_{t+1}(A^{t+1})}\right] \tag{90}$$

$$1 + \lambda_t^{Rep}(A^t) := \frac{\mathbb{E}_t[R_{t+1}(A^{t+1})]}{\mathbb{E}_t[1/q^B(A^t)]} = \mathbb{E}_t[R_{t+1}(A^{t+1})]\ \mathbb{E}_t\left[\beta\frac{C_t(A^t)}{C_{t+1}(A^{t+1})}\right]. \tag{91}$$

Proof. See Appendix A.7.1. □

9.2 Economy with Full Depreciation of Capital ($\delta = 1$)

We analyze the case with full depreciation of capital ($\delta = 1$). In this case, both the limited-commitment economy and the representative agent economy have a law of motion of capital in closed form, which allows us to derive the risk-free rate and the risk

Table 1: Asset Pricing in the Two Economies with $\delta = 1$

	LC		Rep
Low of Motion	$K_{t+1}^{LC} = \hat{s}^{LC} A_t^{1-\theta} K_t^{\theta}$	$>$	$K_{t+1}^{Rep} = \beta\theta A_t^{1-\theta} K_t^{\theta}$
Interest Rate	$R_{t+1}^{LC} = \theta \left(\frac{K_{t+1}^{LC}}{A_{t+1}}\right)^{\theta-1}$	$<$	$R_{t+1}^{Rep} = \theta \left(\frac{K_{t+1}^{Rep}}{A_{t+1}}\right)^{\theta-1}$
Risk-Free Rate $\left(\frac{1}{q^B(A^t)}\right)$	$\frac{1}{\mathbb{E}_t\left[1/R_{t+1}^{LC}(A^{t+1})\right]}$	$<$	$\frac{1}{\mathbb{E}_t\left[1/R_{t+1}^{Rep}(A^{t+1})\right]}$
Risk Premium $\left(\frac{\mathbb{E}_t[R_{t+1}]}{1/q^B(A^t)}\right)$	$\mathbb{E}_t\left[A_{t+1}^{1-\theta}\right]\mathbb{E}_t\left[\frac{1}{A_{t+1}^{1-\theta}}\right]$	$=$	$\mathbb{E}_t\left[A_{t+1}^{1-\theta}\right]\mathbb{E}_t\left[\frac{1}{A_{t+1}^{1-\theta}}\right]$

With $\delta = 1$, Assumption 5, $\frac{\theta}{(1-\theta)\left(\frac{1}{\beta}-1+\delta\right)} < \frac{\xi}{\nu\left(\frac{1}{\beta}-1+\xi+\nu\right)}$, is equivalent to $\beta\theta < \hat{s}^{LC} \left(:= \frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta) + (1-\nu)\beta\theta\right)$. Assumption 5 and $\delta = 1$ implies $\beta R^{*LC} < 1$ and $\beta R_{t+1}^{LC}(A^{t+1}) < \frac{w_{t+1}^{LC}(A^{t+1})}{w_t^{LC}(A^t)}$ for any (t, A^t, A_{t+1}). K_t is given by: $\log K_t = (1+\theta+\cdots+\theta^{t-2})\log s + (1-\theta)\left[\sum_{\tau=1}^{t-1}\theta^{\tau-1}\log A_{t-\tau}\right] + \theta^{t-1}\log K_0$, where $s \in \{\hat{s}^{LC}, s^{Rep} := \beta\theta\}$.

premium explicitly. We show that the limited-commitment model has a lower risk-free rate, but the risk premium is the same as in the representative agent model.

Proposition 15. *Consider an economy with full depreciation of capital $\delta = 1$. Given the same amount of aggregate capital K_t, the risk-free rate, which is the inverse of the price of risk-free bonds, is lower in the limited-commitment model than a representative-agent model:*

$$\frac{1}{q_t^{B,LC}(A^t)} < \frac{1}{q_t^{B,Rep}(A^t)} \textit{ for all } A^t. \tag{92}$$

The risk premium is the same in the two models and is given by:

$$1+\lambda_t = \mathbb{E}_t[A_{t+1}^{1-\theta}]\mathbb{E}_t\left[\frac{1}{A_{t+1}^{1-\theta}}\right] > 1. \tag{93}$$

If the productivity growth rate $\frac{A_{t+1}}{A_t}$ follows an iid process, the risk premium $1+\lambda_t$ is constant over time.

Proof. See Appendix A.7.2. □

Table 1 summarizes the results in this subsection. The limited-commitment model has a higher saving rate and hence a lower interest rate. However, since the interest rates move proportionally in the two models, the risk premium is the same. Appendix A.7.2 shows that the limited-commitment model accumulates more capital than the representative agent model.

9.3 Intuition: Different Risk Premia if $\delta \neq 1$

We have seen that the two models (Rep and LC) have the same risk premium in the economy with $\delta = 1$. We provide intuition why it is the case and why it will not go through in an economy with $\delta \neq 1$. Detailed discussions can be found in Appendix A.7.4.

9.3.1 Endowment Economy

To help understand intuition, we provide the result in an endowment economy. In this economy, the aggregate consumption $\{C_t(A^t)\}_{t,A^t}$ is exogenous, and the interest rate is proportional to aggregate shocks. Then, the risk premium is the same in the two models. The details are in Appendix A.7.3.

Proposition 16. *Consider an endowment economy with exogenous aggregate endowment $\{C_t(A^t)\}_{t,A^t}$. Lucas tree yields α fraction of aggregate endowment at all ($t+1$, A^{t+1}) and is priced at $q_t^{LT}(A^t)$ at state A^t. Assume that parameters satisfy:*

$$\frac{\alpha}{(1-\alpha)\left(\frac{1}{\beta}-1\right)} < \frac{\xi}{\nu\left(\frac{1}{\beta}-1+\xi+\nu\right)}, \tag{94}$$

so that Assumption 3 is satisfied. The risk-free rate, which is the inverse of the price of risk-free bonds, is lower in the limited-commitment model:

$$\frac{1}{q_t^{B,LC}(A^t)} < \frac{1}{q_t^{B,Rep}(A^t)} \text{ for all } A^t. \tag{95}$$

The risk premium is the same in the two models and is given by:

$$1+\lambda_t = \mathbb{E}_t[C_{t+1}(A^{t+1})]\mathbb{E}_t\left[\frac{1}{C_{t+1}(A^{t+1})}\right] > 1 \tag{96}$$

If the growth rate of exogenous consumption $\frac{C_{t+1}}{C_t}$ follows an iid process, the risk premium $1+\lambda_t$ is constant over time.

Proof. See Appendix A.7.3. □

9.3.2 Multiplicative Stochastic Discount Factors

In Krueger and Lustig (2010),[20] a key property is that the stochastic discount factors in the two models differ only by a non-random multiplicative term. In our case, the stochastic discount factor $\beta\frac{c_t}{c_{t+1}}$ in the two models is proportional to $\frac{C_t}{C_{t+1}}$ or $\frac{1}{A_{t+1}^{1-\theta}K_{t+1}^{\theta-1}}$ in the

[20] They show the same risk premium between a representative agent model and an Arrow model with uninsurable idiosyncratic risks in an endowment economy.

endowment economy or in the production economy with $\delta = 1$, respectively. However, this is not the case in a production economy with $\delta \neq 1$. In fact, there is no closed-form solution to aggregate consumption C_{t+1} in the representative agent economy.

Proposition 17. *Consider the representative agent model and the limited commitment model with aggregate shocks. Exogenous shocks, $\{A_t\}_{t\geq 0}$, follow a common stochastic process with probability of A_{t+1} given by $\pi(A_{t+1}|A^t)$ that potentially depends on the entire history $A^t = (A_0, A_1, \cdots, A_t)$. In an endowment economy, the total resources available in the economy, denoted by Υ_t, is exogenous and depends only on A_t. In a production economy, Υ_t is the sum of produced output, $K_t^\theta A_t^{1-\theta}$, and undepreciated capital, $(1-\delta)K_t$, where K_t depends on the history of aggregate shocks A^{t-1} and the initial capital K_0. Thus, we denote Υ_t as a function of A^t.*

$$\Upsilon_t(A^t) = \begin{cases} C_t(A_t) & \text{in an endowment economy} \\ K_t^\theta A_t^{1-\theta} + (1-\delta)K_t & \text{in a production economy} \end{cases} \tag{97}$$

A stochastic discount factor $m_{t,t+1}(A^{t+1})$ satisfies:

$$\mathbb{E}_t\left[m_{t,t+1}(A^{t+1})R^j_{t,1}(A^{t+1})\right] = 1 \tag{98}$$

for any asset j with one-period return $R^j_{t,1}(A^{t+1})$. In our setup with logarithmic utility, the stochastic discount factor is given by:

$$m_{t,t+1}(A^{t+1}) = \beta\frac{c_t}{c_{t+1}} \tag{99}$$

where c_t is the consumption of unconstrained agents (representative households in the Rep model or low-income households in the LC model). If unconstrained agents consume a non-random fraction (determined at t) of total resources, the stochastic discount factor is proportional to $\frac{\Upsilon_t}{\Upsilon_{t+1}}$ and satisfies:

$$m_{t,t+1}(A^{t+1}) := \beta\frac{c_t}{c_{t+1}} = \gamma_t\frac{\Upsilon_t(A^t)}{\Upsilon_{t+1}(A^{t+1})}, \tag{100}$$

where non-random variable γ_t is potentially time-varying but does not depend on A_{t+1}.

First, in the endowment economy and the production economy with $\delta = 1$, equation (100) is satisfied in both models.

Second, in the endowment economy and the production economy with $\delta = 1$, $\frac{\Upsilon_t}{\Upsilon_{t+1}}$ is proportional between the two models, meaning there is a non-random variable γ'_t satisfying:

$$\frac{\Upsilon_t^{LC}(A^t)}{\Upsilon_{t+1}^{LC}(A^{t+1})} = \gamma'_t\frac{\Upsilon_t^{Rep}(A^t)}{\Upsilon_{t+1}^{Rep}(A^{t+1})}, \tag{101}$$

where $\gamma'_t = 1$ in the endowment economy.

Then, (100) and (101) imply that $\frac{m^{LC}_{t,t+1}}{m^{Rep}_{t,t+1}}$ is non-random at t, meaning there exists another non-random variable γ''_t that does not depend on A_{t+1} and satisfies:

$$\frac{m^{LC}_{t,t+1}(A^{t+1})}{m^{Rep}_{t,t+1}(A^{t+1})} = \gamma''_t \left(:= \gamma'_t \frac{\gamma^{LC}_t}{\gamma^{Rep}_t} \right) \tag{102}$$

In this case, the two models have the same risk premium.

Proof. See Appendix A.7.4. □

If the two models do not have multiplicative stochastic discount factors, i.e., γ''_t in equation (102) depends on A_{t+1}, then the risk premia are generally different.[21]

In the production economy with $\delta \neq 1$, the risk premia are generally different between the two models. This is because: (a) consumption of unconstrained agents is not proportional to total resources in the economy due to a non-constant saving rate. Thus, (100) does not hold; (b) total resources are not proportional to aggregate shocks ($C_t(A^t)$ or $A_t^{1-\theta}$) due to undepreciated capital. Hence, (101) does not hold; (c) the interest rate is not proportional to aggregate shocks ($C_t(A^t)$ or $A_t^{1-\theta}$) due to undepreciated capital. We can say that the fundamental reason is (b): non-proportional total resources in the economy so that (a) agents do not have a constant saving rate and that (c) the interest rate depends on aggregate capital.

10 Conclusion

[21] To be more precise, γ''_t needs to be mean independent of A_{t+1}, i.e., $\mathbb{E}_{t+1}[\gamma''_t] = \mathbb{E}_t[\gamma''_t]$, to have the same risk premium. Denote $m^{LC}_{t,t+1} = \gamma''_t m^{Rep}_{t,t+1}$ and $R^{LC}_{t+1} = \tilde{\gamma}_t R^{Rep}_{t+1}$. In the endowment economy and the production economy with $\delta = 1$, both γ''_t and $\tilde{\gamma}_t$ are mean independent of A_{t+1}. Therefore, the risk premium satisfies:

$$1 + \lambda^{LC}_t = \frac{\mathbb{E}_t[R^{LC}_{t+1}]\mathbb{E}_t[m^{LC}_{t,t+1}]}{\mathbb{E}_t[R^{LC}_{t+1} m^{LC}_{t,t+1}]} = \frac{\mathbb{E}_t[\tilde{\gamma}_t]\mathbb{E}_t[R^{\text{Rep}}_{t+1}]\mathbb{E}_t[\gamma''_t]\mathbb{E}_t[m^{\text{Rep}}_{t,t+1}]}{\mathbb{E}_t[\tilde{\gamma}_t]\mathbb{E}_t[\gamma''_t]\mathbb{E}_t[R^{\text{Rep}}_{t+1} m^{\text{Rep}}_{t,t+1}]} = 1 + \lambda^{Rep}_t, \tag{103}$$

where the second equality utilizes the law of iterated expectations. (e.g., $\mathbb{E}_t[\gamma''_t m^{\text{Rep}}_{t,t+1}] = \mathbb{E}_t[\mathbb{E}_{t+1}[\gamma''_t m^{\text{Rep}}_{t,t+1}]] = \mathbb{E}_t[\mathbb{E}_{t+1}[\gamma''_t] m^{\text{Rep}}_{t,t+1}] = \mathbb{E}_t[\gamma''_t]\mathbb{E}_t[m^{\text{Rep}}_{t,t+1}]$)

11 Open Questions and Concerns

11.1 Open questions discussed by Yoshiki and Dirk

1. Uniqueness of the optimal contract

   - Given the sequence of prices $\{w_t(A^t), R_t(A^t)\}_{t\geq 0, A^t}$ that satisfies Assumption 3 and the Arrow security price with Assumption 2 (and under the initial condition 4), is the optimal choice of the households unique?

     We want to show uniqueness of the optimal choice by arguing that both the Kuhn-Tucker conditions and the TVC are necessary, and that the proposed allocation is the only allocation satisfying these conditions. See Appendix B.1.

   - Is the competitive equilibrium unique given the assumptions on parameters (Assumption 5 and Assumption G)?

2. In Section 8.2, we show that consumption distribution is constant over the cycle if $\delta = 1$. Does this hold in the Bewley model with $\delta = 1$?

11.2 Comments and Concerns from the presentation

1. When making assumption on $q(A^{t+1}|A^t) = \pi(A^{t+1}|A^t)$ need to give intuition immediately why this is true, perhaps quickly arguing that it is true in the representative agent model

   → The price of interest rate strip that pays $R_{t+1}(A^{t+1})$ at A_{t+1} (and zero in other aggregate states) is given by:

   $$q(A_{t+1}|A^t) = \beta \frac{c_t(A^t)}{c_{t+1}(A^{t+1})} \pi(A_{t+1}|A^t) R_{t+1}(A^{t+1}), \tag{104}$$

   where $c_t(A^t)$ is consumption of unconstrained agents. In the optimal allocation in Proposition 1, households trade assets only for a low-income state (the limited-commitment constraint binds in a high-income state). Consumption in a low-income state satisfies:

   $$\beta \frac{c_t(A^t)}{c_{t+1}(A^{t+1}; z_{t+1} = 0)} = \frac{1}{R_{t+1}(A^{t+1})} \tag{105}$$

   The two equations give:

   $$q(A_{t+1}|A^t) = \pi(A_{t+1}|A^t). \tag{106}$$

Therefore, the conjectured price of the interest rate strips will be supported in an equilibrium.

I misunderstood (I'm sorry) about the price in the complete-market/rep-agent model. The Euler equation in the rep-agent model implies:

$$1 = \mathbb{E}_t \left[\beta \frac{c_t(A^t)}{c_{t+1}(A^{t+1})} R_{t+1}(A^{t+1}) \right], \tag{107}$$

but (105) does not hold state-by-state unless $\delta = 1$.[22] Therefore, (106) is not true in the rep-agent model.

2. Related, so not call these assets Arrow securities, but immediately relate them to Arrow securities (I think we do so in the paper in an extensive footnote).

3. Clearer discussion on the relation to the Solow model. What is the right notion of the saving rate in that model, what is used in the literature? Saving out of net or gross (including nondepreciated) output.

   → The standard Solow model is:

$$K_{t+1} = sY_t + (1-\delta)K_t, \tag{108}$$

   meaning the constant saving rate out of the net output. On the other hand, the law of motion in the limited-commitment model is:

$$\begin{aligned} K_{t+1}(A^t) &= \left[\frac{\xi\beta(1-\theta)}{1-(1-\nu-\xi)\beta} + (1-\nu)\beta\theta \right] A_t^{1-\theta} K_t(A^{t-1})^\theta + (1-\nu)\beta(1-\delta)K_t(A^{t-1}) \\ &= \hat{s} A_t^{1-\theta} K_t(A^{t-1})^\theta + (1-\hat{\delta})K_t(A^{t-1}). \end{aligned}$$

   Here, low-income agents save a constant fraction of their asset income ($\mathbf{a}_{t+1} = \beta R_t \mathbf{a}_t$), where the return on assets includes the return from undepreciated capital ($R_t = \theta A_t^{1-\theta} K_t^{\theta-1} + 1 - \delta$).

4. Is the direction of the movement of the Lorenz curve (less inequality after a negative technology shock) a theoretical result that we can prove?

   → Proposition 13 states that consumption of all low-income agents declines relative to high-income agents at the time of a positive productivity shock (if

[22] If $\delta = 1$, $C_t^{Rep} = (1-\beta\theta)A_t^{1-\theta}K_t^\theta$, $K_{t+1}^{Rep} = \beta\theta A_t^{1-\theta}K_t^\theta$. This implies:

$$\begin{aligned} \beta \frac{c_t(A^t)}{c_{t+1}(A^{t+1})} &= \beta \frac{A_t^{1-\theta}K_t^\theta}{A_{t+1}^{1-\theta}K_{t+1}^\theta} \\ &= \frac{1}{\theta A_{t+1}^{1-\theta}K_{t+1}^{\theta-1}} = \frac{1}{R_{t+1}(A_{t+1})} \end{aligned}$$

$\delta < 1$). Consumption of all low-income agents move proportionally, since $\mathbf{c}_t = [1 - (1-\nu)\beta]R_t\mathbf{a}_t$ and $\mathbf{a}_t$ is pre-determined at $t-1$ (equation 23).

5. MIT shock versus full stochastic dynamics: can we use the model's analytical solution to show whether it makes a big difference here whether we have a sequence of MIT shocks versus a sequence of realizations of the aggregate shocks in the model with aggregate shocks

6. Do our results generalize to more general aggregate shock processes (i.e., what depends on the aggregate shock process being a two state iid growth process and what does not?)

   $\rightarrow$ I think this generalization is promising. Since we have checked the most extreme case (infinitely many negative shocks followed by a positive shock, and vice versa), we should be able to allow any magnitude of shocks bounded by $1-\epsilon$ and $1+\epsilon$: $\frac{A_{t+1}}{A_t} \in [1-\epsilon, 1+\epsilon]$ as well as any (non-*iid*) probability for each aggregate state $0 \leq \pi(A_{t+1}|A^t) \leq 1$ for all $A_{t+1} \in [A_t(1-\epsilon), A_t(1+\epsilon)]$.

7. More general results with respect to asset pricing away from the $\delta = 1$ case. For $\delta < 1$, is equity premium smaller than than in the representative agent model? Because then the asset pricers (the low productivity, high wealth agents) face income that is less risky than aggregate income because the depreciation bit is not hit by the aggregate productivity shock.

   $\rightarrow$ We need to be clever to prove a conjecture that the risk-free rate and the risk premium are lower in the limited commitment model, as we don't have a closed form of the law of motion in the rep-agent model. For example, there must be larger savings in the limited-commitment model than in the rep-agent model, but we don't have a closed form for the savings in the rep-agent model.

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A Proofs of Propositions

A.1 Proofs: Section 3 (Characterization of Equilibrium)

A.1.1 Proof of Proposition 1

We prove Proposition 1 in several steps. First, we propose a candidate optimal consumption and asset allocation. We then show in a sequence of steps that this proposed allocation is indeed an optimal choice of the household. To do so, Lemma 6 derives the Kuhn-Tucker conditions for the household optimization problem (12)–(13) for given a sequence of prices $\{R_t(A^t), w_t(A^t), q_t(A_{t+1}, z_{t+1}|A^t, z^t)\}_{t\geq 0, A^t, A_{t+1}, z^t, z_{t+1}}$. Then, Lemmas 7 and 8 show that the proposed allocation satisfies the household's budget constraint and the Kuhn-Tucker conditions. Finally, Proposition 1 shows that an allocation that satisfies the Kuhn-Tucker conditions and a transversality condition, it is an optimal choice of the maximization problem.

We conjecture that, under the maintained assumptions on prices stipulating sufficiently low interest rates/sufficiently high wage growth, individuals have no incentives to save for the high-income state tomorrow, and for the low-income state tomorrow consumption and asset choices are governed by a standard complete-markets Euler equation. That is, we conjecture that the optimal household consumption-asset choice is given by:

$$\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) = \begin{cases} 0 & \text{if } z_{t+1} = \zeta \\ \frac{\beta}{1-(1-\nu-\xi)\beta}\zeta w_t(A^t) & \text{if } z_t = \zeta \text{ and } z_{t+1} = 0 \\ \beta R_t(A^t)\mathbf{a}_t(a_0, z^t, A^t) & \text{if } z_t = 0 \text{ and } z_{t+1} = 0 \end{cases} \tag{23}$$

$$\mathbf{c}_t(a_0, z^t, A^t) = \begin{cases} w_t(A^t)c_0, \text{ where } c_0 := \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}\zeta, & \text{if } z_t = \zeta \\ [1-(1-\nu)\beta]R_t(A^t)\mathbf{a}_t(a_0, z^t, A^t) & \text{if } z_t = 0 \end{cases} \tag{22}$$

where $\mathbf{a}_0(a_0, z^0, A^0) = w_0(A^0)a_0$ are the initial asset holdings of the household (an exogenous initial condition).

It is straightforward to verify that (23) and (22) imply that under the proposed allocation for currently low-income individuals ($z_t = 0$) the standard complete markets Euler equation for consumption holds (a fact that will be useful below for some of the derivations):

$$\mathbf{c}_t(a_0, z^t, A^t) = \beta R_t(A^t)\mathbf{c}_{t-1}(a_0, z^{t-1}, A^{t-1}). \tag{24}$$

To see this consider first an individual with $z_{t-1} = 0$. Then

$$\begin{aligned} \mathbf{a}_t(a_0, z^t, A^t) &= \beta R_{t-1}(A^{t-1})\mathbf{a}_{t-1}(a_0, z^{t-1}, A^{t-1}) \\ &= \beta R_{t-1}(A^{t-1})\frac{\mathbf{c}_{t-1}(a_0, z^{t-1}, A^{t-1})}{[1-(1-\nu)\beta]R_{t-1}(A^{t-1})}, \end{aligned}$$

where the first line follows from equation (23), while the second line stems from equation (22). Therefore

$$\begin{aligned}\mathbf{c}_t(a_0, z^t, A^t) &= [1-(1-\nu)\beta]R_t(A^t)\mathbf{a}_t(a_0, z^t, A^t)\\ &= \beta R_t(A^t)\mathbf{c}_{t-1}(a_0, z^t, A^t).\end{aligned}$$

Now consider an individual with $z_{t-1} = \zeta$. Then

$$\begin{aligned}\mathbf{a}_t(a_0, z^t, A^t) &= \frac{\beta}{1-(1-\nu-\xi)\beta}\zeta w_{t-1}(A^{t-1})\\ &= \frac{\beta}{1-(1-\nu)\beta}\mathbf{c}_{t-1}(a_0, z^{t-1}, A^{t-1})\\ \therefore \mathbf{c}_t(a_0, z^t, A^t) &= [1-(1-\nu)\beta]R_t(A^t)\mathbf{a}_t(a_0, z^t, A^t)\\ &= \beta R_t(A^t)\mathbf{c}_{t-1}(a_0, z^t, A^t).\end{aligned}$$

We now derive the Kuhn-Tucker condition for the household maximization problem in the following Lemma.

Lemma 6. *Given a sequence of prices* $\{R_t(A^t), w_t(A^t), q_t(A_{t+1}, z_{t+1}|A^t, z^t)\}_{t\geq 0, A^t, A_{t+1}}$*, FOCs to the household's optimization problem (12)–(13) give the following Kuhn-Tucker condition:*

$$\frac{\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1})}{\mathbf{c}_t(a_0, z^t, A^t)} = \frac{\pi(A_{t+1}|A^t)\pi(z_{t+1}|z_t)}{q_t(A_{t+1}, z_{t+1}|A^t, z^t)}\left[\beta R_{t+1}(A^{t+1}) + \lambda(a_0, z^{t+1}, A^{t+1})\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1})\right] \tag{109}$$

with $\lambda(a_0, z^{t+1}, A^{t+1})\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) = 0,\ \lambda(a_0, z^{t+1}, A^{t+1}) \geq 0,\ \mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) \geq 0,$ (110)

where $\lambda(a_0, z^{t+1}, A^{t+1})$ *denotes a Lagrangian multiplier for a shortsale constraint at state* (z^{t+1}, A^{t+1}).

Proof. Households' Lagrangian problem is given by:

$$\begin{aligned}U(a_0, z_0) = &\max_{\{\mathbf{c}_t(a_0,z^t,A^t),\mathbf{a}_{t+1}(a_0,z^{t+1},A^{t+1})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\sum_{A^t}\sum_{z^t}\beta^t\pi(A^t)\pi(z^t)\log(\mathbf{c}_t(a_0, z^t, A^t))\\ &+ \sum_{t=0}^{\infty}\sum_{A^t}\sum_{z^t}\mu(z^t, A^t)\Bigg[w_t(A^t)z_t + R_t(A^t)\mathbf{a}_t(a_0, z^t, A^t)\\ &\qquad - \mathbf{c}_t(a_0, z^t, A^t) - \sum_{A_{t+1}}\sum_{z_{t+1}}q_t(A_{t+1}, z_{t+1}|A^t, z^t)\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1})\Bigg]\\ &+ \sum_{t=0}^{\infty}\sum_{A^{t+1}}\sum_{z^{t+1}}\beta^t\pi(A^{t+1})\pi(z^{t+1})\lambda(a_0, z^{t+1}, A^{t+1})\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}),\end{aligned} \tag{111}$$

where $\mu(z^t, A^t)$ and $\lambda(a_0, z^{t+1}, A^{t+1})$ are Lagrangian multipliers for budget constraints and shortsale constraints. FOCs with respect to $\mathbf{c}_t(a_0, z^t, A^t)$ and $\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1})$ are:

$$[\mathbf{c}_t(a_0, z^t, A^t)] : \ \beta^t \pi(A^t)\pi(z^t)\frac{1}{\mathbf{c}_t(a_0, z^t, A^t)} = \mu(z^t, A^t) \tag{112}$$

$$\begin{aligned}[\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1})] : \ & \mu(z^{t+1}, A^{t+1})R_{t+1}(A^{t+1}) + \beta^t \pi(A^{t+1})\pi(z^{t+1})\lambda(a_0, z^{t+1}, A^{t+1}) \\ & = \mu(z^t, A^t) q_t(A_{t+1}, z_{t+1}|A^t, z^t) \end{aligned} \tag{113}$$

By substituing $\mu(z^t, A^t)$, we obtain the following Kuhn-Tucker condition:

$$\frac{1}{\mathbf{c}_t(a_0, z^t, A^t)} = \frac{\pi(A_{t+1}|A^t)\pi(z_{t+1}|z_t)}{q_t(A_{t+1}, z_{t+1}|A^t, z^t)} \left[\beta R_{t+1}(A^{t+1})\frac{1}{\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1})} + \lambda(a_0, z^{t+1}, A^{t+1})\right] \tag{114}$$

where $\lambda(a_0, z^{t+1}, A^{t+1})\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) = 0, \ \lambda(a_0, z^{t+1}, A^{t+1}) \geq 0, \ \mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) \geq 0.$

Here we use the conditional probability: $\pi(A_{t+1}|A^t) = \frac{\pi(A^{t+1})}{\pi(A^t)}$, where $A^{t+1} = (A^t, A_{t+1})$, and $\pi(z_{t+1}|z^t) = \frac{\pi(z^{t+1})}{\pi(z^t)}$, where $z^{t+1} = (z^t, z_{t+1})$. Because the idiosyncratic shocks follow Markov, only the current state z_t matters for the probability of z_{t+1}. Hence, $\pi(z_{t+1}|z^t) = \pi(z_{t+1}|z_t)$. Since $\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1})$ takes non-zero value (otherwise the utility would be negative infinite), (114) can be expressed as:

$$\frac{\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1})}{\mathbf{c}_t(a_0, z^t, A^t)} = \frac{\pi(A_{t+1}|A^t)\pi(z_{t+1}|z_t)}{q_t(A_{t+1}, z_{t+1}|A^t, z^t)} \left[\beta R_{t+1}(A^{t+1}) + \lambda(a_0, z^{t+1}, A^{t+1})\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1})\right]$$

□

The next two lemmas show that the conjectured allocation satisfies the budget constraint and the Kuhn-Tucker condition for households' optimization problem.

Lemma 7. *Suppose Assumption 2 on contingent claim prices is satisfied. Then, the allocation defined in equations* (22) *and* (23) *satisfies the household's budget constraint (8) and the Euler equation between the current state and a future low-income state (and hence the Kuhn-Tucker condition 109):*

$$\frac{1}{\mathbf{c}_t(a_0, z^t, A^t)} = \beta R_{t+1}(A^{t+1})\frac{1}{\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1}; z_{t+1} = 0)} \text{ for all } A_{t+1}. \tag{115}$$

Proof. We first check the budget constraint. In a high-income state ($z_t = \zeta$), substituting the conjectured consumption and asset choice (22) and (23) into equation (8) gives:

$$w_t(A^t)c_0 + \sum_{A_{t+1}}\sum_{z_{t+1}} q_t(A_{t+1}, z_{t+1}|A^t, z^t)\frac{\beta}{1-(1-\nu-\xi)\beta}\zeta w_t(A^t) = w_t(A^t)\zeta$$

$$\therefore \ w_t(A^t)\left[\frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}\zeta + \sum_{A_{t+1}}\pi(A_{t+1}|A^t)\underbrace{\pi(z_{t+1}=0|z_t=\zeta)}_{=\xi}\frac{\beta}{1-(1-\nu-\xi)\beta}\zeta\right] = w_t(A^t)\zeta$$

where the second line uses Assumption 2: $q_t(A_{t+1}, z_{t+1}|A^t, z^t) = \pi(A_{t+1}|A^t)\pi(z_{t+1}|z_t)$. The equality holds since $\sum_{A_{t+1}} \pi(A_{t+1}|A^t) = 1$.

In a low-income state ($z_t = 0$), equation (8) becomes:

$$\begin{aligned}
&[1-(1-\nu)\beta]R_t(A^t)\mathbf{a}_t(z^t, A^t) \\
&\qquad + \sum_{A_{t+1}}\sum_{z_{t+1}} \pi(A_{t+1}|A^t)\underbrace{\pi(z_{t+1}=0|z_t=0)}_{=1-\nu}\beta R_t(A^t)\mathbf{a}_t(z^t, A^t) = R_t(A^t)\mathbf{a}_t(z^t, A^t) \\
\Leftrightarrow\ & R_t(A^t)\mathbf{a}_t(z^t, A^t) - (1-\nu)\beta\left[1 - \sum_{A_{t+1}} \pi(A_{t+1}|A^t)\right] R_t(A^t)\mathbf{a}_t(z^t, A^t) = R_t(A^t)\mathbf{a}_t(z^t, A^t)
\end{aligned}$$

This holds with equality since $\sum_{A_{t+1}} \pi(A_{t+1}|A^t) = 1$.

Second, we examine the Euler equation for low-income households.[23] The condition on Lagrange multipliers (110) implies $\lambda(a_0, z^{t+1}, A^{t+1}) = 0$, since $\lambda(a_0, z^{t+1}, A^{t+1})\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) = 0$ and $\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) > 0$ in the proposed allocation. Under $q_t(A_{t+1}, z_{t+1}|A^t, z^t) = \pi(A_{t+1}|A^t)\pi(z_{t+1}|z_t)$, the Kuhn-Tucker condition (109) is given by:

$$\frac{1}{\mathbf{c}_t(a_0, z^t, A^t)} = \beta R_{t+1}(A^{t+1})\frac{1}{\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1}; z_{t+1} = 0)}.$$

$\mathbf{c}_{t+1}$ at a low-income state given by (22) satisfies this Euler equation. □

The claim that households make no savings for high-income states follows the logic in Lemmas 19 and 20 (which is shown in a deterministic case).[24] We show below that under Assumption 3: $\beta R_{t+1}(A^{t+1}) < \frac{w_{t+1}(A^{t+1})}{w_t(A^t)}$ at all $t \geq 0$ and A^{t+1}, the Kuhn-Tucker conditions for high-income states are satisfied if households do not save for high-income states.

Lemma 8. *Suppose Assumption 2 on contingent claim prices is satisfied and suppose that the sequence of wages and interest rates $\{w_t(A^t), R_t(A^t)\}_{t=0}^{\infty}$ satisfies the no-savings Assumption 3 and that the initial wealth distribution satisfies Assumption 4. Then, the allocation defined in equations* (22) *and* (23) *satisfies the Kuhn-Tucker conditions for high-income states at any time $t \geq 0$. It also implies equation (26):*

$$\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1}) > \beta R_{t+1}(A^{t+1})\mathbf{c}_t(a_0, z^t, A^t) \textit{ if } z_{t+1} = \zeta.$$

[23] Since households always save for a low-income state, the Euler equation holds with equality. If households enter a low-income state with zero assets, their period utility would be negative infinite.

[24] At $t = 0$, given an initial state ($a_0 = 0, z_0 = \zeta$) or ($a_0 \leq \bar{a}_0 := \frac{\beta}{1-(1-\nu-\xi)\beta}\zeta, z_0 = 0$), households cannot achieve higher utility at $t = 1$ by saving for a high-income state at $t = 1$ if $\beta R_1 \frac{w_0}{w_1} < 1$. Since consumption choice $c_t(a_0, z^t; z_t = \zeta)$ cannot be larger than $c_0 := \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}\zeta$ at any $t \geq 0$, making positive savings for a high-income state at $t = 1$ wouldn't give higher utility at any time $t \geq 1$, under Assumption 3: $\beta R_{t+1}(A^{t+1}) < \frac{w_{t+1}(A^{t+1})}{w_t(A^t)}$ at all $t \geq 0$ and A^{t+1}. By induction, households at any time $t \geq 1$ do not save for a high-income state at $t+1$, since they enter a state at time $t \geq 1$ with ($a_t = 0, z_t = \zeta$) or ($a_t \leq \bar{a}_0, z_t = 0$).

Proof. We check the Kuhn-Tucker conditions, (109) and (110), for a high-income state ($z_{t+1} = \zeta$) under Assumption 2, $q_t(A_{t+1}, z_{t+1}|A^t, z^t) = \pi(A_{t+1}|A^t)\pi(z_{t+1}|z_t)$, and Assumption 4, $\beta R_{t+1}(A^{t+1}) < \frac{w_{t+1}(A^{t+1})}{w_t(A^t)}$. Substituting $\mathbf{c}_{t+1}(z^{t+1}, A^{t+1}; z_{t+1} = \zeta) = w_{t+1}(A^{t+1})c_0$ and $\mathbf{a}_{t+1}(z^{t+1}, A^{t+1}) = 0$ into equation (109) gives:

$$\frac{1}{w_t(A^t)c_t(a_0, z^t, A^t)} = \beta R_{t+1}(A^{t+1})\frac{1}{w_{t+1}(A^{t+1})c_{t+1}(a_0, z^{t+1}, A^{t+1}; z_{t+1} = \zeta)} + \lambda(a_0, z^{t+1}, A^{t+1})$$

$$\Leftrightarrow \underbrace{\beta R_{t+1}(A^{t+1})\frac{w_t(A^t)}{w_{t+1}(A^{t+1})}}_{<1}\frac{c_t(a_0, z^t, A^t)}{c_0} + \lambda(a_0, z^{t+1}, A^{t+1})w_t(A^t)c_t(a_0, z^t, A^t) = 1$$

As long as $c_t(a_0, z^t, A^t) \leq c_0$ for all (a_0, z^t, A^t), which we will show below, the Lagrangian multiplier $\lambda(a_0, z^{t+1}, A^{t+1})$ that solves equation (109) satisfies $\lambda(a_0, z^{t+1}, A^{t+1}) > 0$. Then, the Kuhn-Tucker condition for a high-income state is satisfied. Specifically, the Kuhn-Tucker condition for a high-income state with $\lambda(a_0, z^{t+1}, A^{t+1}) > 0$ implies:

$$\frac{1}{\mathbf{c}_t(z^t, A^t)} > \beta R_{t+1}(A^{t+1})\frac{1}{\mathbf{c}_{t+1}(z^{t+1}, A^{t+1}; z_{t+1} = \zeta)}. \tag{116}$$

This gives equation (26). Because the marginal utility of consumption at time t is higher than the discounted marginal utility of consumption at state (z^{t+1}, A^{t+1}) with $z_{t+1} = \zeta$, households do not have incentives to save for a high-income state.

Under $\beta R_{t+1}(A^{t+1}) < \frac{w_{t+1}(A^{t+1})}{w_t(A^t)}$ for all (t, A^t, A_{t+1}), we will show that $c_t(a_0, z^t, A^t) \leq c_0$ for all (a_0, z^t, A^t). We prove by induction. At $t = 0$, the initial wealth distribution satisfies Assumption 4. Given the consumption rule (22) and $\beta R_0 < 1$, the initial consumption satisfies $c_0(a_0, z_0, A_0) \leq c_0$ for all (a_0, z_0), as we see the following:

$$\begin{aligned} w_0c_0(a_0, z_0 = 0, A_0) =: \mathbf{c}_0(a_0, z_0 = 0, A_0) &= [1 - (1-\nu)\beta]R_0(A_0)\mathbf{a}_0(a_0, z_0, A_0) \\ &< [1 - (1-\nu)\beta]R_0(A_0)w_0\frac{\beta}{1 - (1-\nu-\xi)\beta}\zeta \\ &< \beta R_0c_0w_0 \\ &< c_0w_0 \end{aligned}$$

Suppose $c_t(a_0, z^t, A^t) \leq c_0$ for all (a_0, z^t, A^t) at time $t \geq 0$. $c_{t+1}(a_0, z^{t+1}, A^{t+1})$ is given by equation (22):

$$c_{t+1}(a_0, z^{t+1}, A^{t+1}) = \begin{cases} c_0 & \text{if } z_{t+1} = \zeta \\ \beta R_{t+1}(A^{t+1})\frac{w_t(A^t)}{w_{t+1}(A^{t+1})}c_t(a_0, z^t, A^t) & \text{if } z_{t+1} = 0 \end{cases}$$

Since $\beta R_{t+1}(A^{t+1})\frac{w_t(A^t)}{w_{t+1}(A^{t+1})} < 1$ and $c_t(a_0, z^t, A^t) \leq c_0$, we have $c_{t+1}(a_0, z^{t+1}, A^{t+1}) \leq c_0$ for all (c_0, z^{t+1}, A^{t+1}). □

Finally, given that the conjectured allocation satisfies the Kuhn-Tucker conditions, we prove that the conjectured allocation is indeed optimal.

Proposition 1 (Optimal Household Consumption and Asset Allocation). *Suppose Assumption 2 on contingent claims prices is satisfied and suppose that the sequence of wages and interest rates* $\{w_t(A^t), R_t(A^t)\}_{t=0}^{\infty}$ *satisfies the no-savings Assumption 3 and that the initial wealth distribution satisfies Assumption 4. Then the optimal consumption and asset allocation of individual household is given by*

$$\mathbf{c}_t(a_0, z^t, A^t) = \begin{cases} w_t(A^t)c_0, \text{ where } c_0 := \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}\zeta, & \text{if } z_t = \zeta \\ [1-(1-\nu)\beta]R_t(A^t)\mathbf{a}_t(a_0, z^t, A^t) & \text{if } z_t = 0 \end{cases} \tag{22}$$

$$\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) = \begin{cases} 0 & \text{if } z_{t+1} = \zeta \\ \frac{\beta}{1-(1-\nu-\xi)\beta}\zeta w_t(A^t) & \text{if } z_t = \zeta \text{ and } z_{t+1} = 0 \\ \beta R_t(A^t)\mathbf{a}_t(a_0, z^t, A^t) & \text{if } z_t = 0 \text{ and } z_{t+1} = 0 \end{cases} \tag{23}$$

where $\mathbf{a}_0(a_0, z^0, A^0) = w_0(A^0)a_0$.

Proof. In Lemmas 7 and 8, we have shown that under Assumptions 3 and 4, the conjectured allocation (22)–(23) satisfies the budget constraints and the Kuhn-Tucker conditions. The shortsale constraints are also satisfied. Now we want to show that the conjectured allocation maximizes the objective (12) under the constraints (8) and (13). We apply the standard proof (e.g., Sims (2002) and Krusell (2014)) to our setup. The upshot is that since the utility function is concave and the constraint set is convex (the constraint $\mathbf{a}_{t+1}(z^{t+1}, A^{t+1}) \geq 0$ is linear in $\mathbf{a}_{t+1}(z^{t+1}, A^{t+1})$), the Kuhn-Tucker conditions and a transversality condition (130) are jointly sufficient for optimality.

We will show that the conjectured allocation (22)–(23), denoted by $(\{\mathbf{c}_t^*(z^t, A^t), \mathbf{a}_{t+1}^*(z^{t+1}, A^{t+1})\})$, gives (weakly) higher expected utility than any other feasible allocations:

$$\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \mathbb{E}\left[\log(\mathbf{c}_t^*(z^t, A^t))\right] \geq \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \mathbb{E}\left[\log(\mathbf{c}_t(z^t, A^t))\right], \tag{117}$$

where feasible allocations $(\{\mathbf{c}_t(z^t, A^t), \mathbf{a}_{t+1}(z^{t+1}, A^{t+1})\})$ satisfy the budget constraints and the shortsale constraints. Since $\mathbf{c}_t(z^t, A^t)$ is uniquely determined by the budget constraint given $(\mathbf{a}_t(z^t, A^t), \{\mathbf{a}_{t+1}(z^{t+1}, A^{t+1})\}_{z^{t+1}, A^{t+1}})$, denote:

$$\begin{aligned} u_t(\mathbf{a}_t, \{\mathbf{a}_{t+1}\}) :=& \log(\mathbf{c}_t(z^t, A^t)) \\ =& \log\Big(w_t(A^t)z_t + R_t(A^t)\mathbf{a}_t(z^t, A^t) \\ & - \sum_{A_{t+1}} \sum_{z_{t+1}} \pi(A_{t+1}|A^t)\pi(z_{t+1}|z_t)\mathbf{a}_{t+1}(z^{t+1}, A^{t+1})\Big). \end{aligned}$$

With this notation, the Kuhn-Tucker condition implies the following:[25]

$$D_2 u_t(\mathbf{a}_t, \{\mathbf{a}_{t+1}\}) + \beta \mathbb{E}_t \left[D_1 u_{t+1}(\mathbf{a}_{t+1}, \{\mathbf{a}_{t+2}\}) \right] + \mathbb{E}_t \left[\lambda(z^{t+1}, A^{t+1}) \right] = 0 \tag{123}$$

Define the difference in the sum of expected utility up to time T:

$$\tilde{V}_T(\mathbf{a}) := \sum_{t=0}^{T} \beta^t \mathbb{E} \left[u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) - u_t(\mathbf{a}_t, \{\mathbf{a}_{t+1}\}) \right] \tag{124}$$

[25]Lagrangian is given by:

$$\begin{aligned}
\mathcal{L} &= \sum_{t=0}^{\infty} \beta^t \sum_{A^t, z^t} \pi(z^t)\pi(A^t) u_t \left(\mathbf{a}_t(z^t, A^t), \{\mathbf{a}_{t+1}(z^{t+1}, A^{t+1})\}_{z^{t+1}, A^{t+1}|z^t, A^t} \right) \\
&+ \sum_{t=0}^{\infty} \beta^t \sum_{A^{t+1}, z^{t+1}} \pi(z^{t+1})\pi(A^{t+1}) \lambda_{t+1}(z^{t+1}, A^{t+1}) \\
&=: \sum_{t=0}^{\infty} \beta^t \mathbb{E} \left[u_t(\mathbf{a}_t, \{\mathbf{a}_{t+1}\}) \right] + \sum_{t=0}^{\infty} \beta^t \mathbb{E} \left[\lambda_{t+1} \right]
\end{aligned}$$

From Lemma 8, we know that the Kuhn-Tucker condition is satisfied for any (z^{t+1}, A^{t+1}):

$$\frac{1}{\mathbf{c}_t(z^t, A^t)} = \beta R_{t+1}(A^{t+1}) \frac{1}{\mathbf{c}_{t+1}(z^{t+1}, A^{t+1})} + \lambda(z^{t+1}, A^{t+1}) \tag{118}$$

This implies, since $\sum_{z^{t+1}, A^{t+1}|z^t, A^t} \pi(z^{t+1}|z^t)\pi(A^{t+1}|A^t) = 1$,

$$\frac{1}{\mathbf{c}_t(z^t, A^t)} = \sum_{z^{t+1}, A^{t+1}|z^t, A^t} \pi(z^{t+1}|z^t)\pi(A^{t+1}|A^t) \left[\beta R_{t+1}(A^{t+1}) \frac{1}{\mathbf{c}_{t+1}(z^{t+1}, A^{t+1})} + \lambda(z^{t+1}, A^{t+1}) \right] \tag{119}$$

$$=: \mathbb{E}_t \left[\beta R_{t+1}(A^{t+1}) \frac{1}{\mathbf{c}_{t+1}(z^{t+1}, A^{t+1})} \right] + \mathbb{E}_t \left[\lambda(z^{t+1}, A^{t+1}) \right] \tag{120}$$

We define the derivative of the flow utility as:

$$\begin{aligned}
D_1 u_t(\mathbf{a}_t, \{\mathbf{a}_{t+1}\}) &:= \frac{\partial}{\partial \mathbf{a}_t} u_t(\mathbf{a}_t, \{\mathbf{a}_{t+1}\}) \\
&= R_t(A^t) \frac{1}{\mathbf{c}_t(z^t, A^t)} \qquad (121) \\
D_2 u_t(\mathbf{a}_t, \{\mathbf{a}_{t+1}\}) &:= \sum_{z^{t+1}, A^{t+1}} \frac{\partial}{\partial \mathbf{a}_{t+1}(z^{t+1}, A^{t+1})} u_t(\mathbf{a}_t(z^t, A^t), \{\mathbf{a}_{t+1}(z^{t+1}, A^{t+1})\}) \\
&= - \sum_{z^{t+1}, A^{t+1}} \pi(z^{t+1}|z^t)\pi(A^{t+1}|A^t) \frac{1}{\mathbf{c}_t(z^t, A^t)} \\
&= - \frac{1}{\mathbf{c}_t(z^t, A^t)} \qquad (122)
\end{aligned}$$

Therefore, by substituting them into (120), we obtain:

$$D_2 u_t(\mathbf{a}_t, \{\mathbf{a}_{t+1}\}) + \beta \mathbb{E}_t \left[D_1 u_{t+1}(\mathbf{a}_{t+1}, \{\mathbf{a}_{t+2}\}) \right] + \mathbb{E}_t \left[\lambda(z^{t+1}, A^{t+1}) \right] = 0.$$

We will show that:

$$\lim_{T\to\infty} \tilde{V}_T(\mathbf{a}) \geq 0 \tag{125}$$

Since $\log(\cdot)$ is a concave function, we have:[26]

$$\mathbb{E}\left[\sum_{t=0}^{T} \beta^t u_t(\mathbf{a}_t, \{\mathbf{a}_{t+1}\})\right] \geq \mathbb{E}\Big[\sum_{t=0}^{T} \beta^t \Big\{u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) + D_1 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) \cdot (\mathbf{a}_t - \mathbf{a}_t^*) \\ + D_2 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) \cdot (\mathbf{a}_{t+1} - \mathbf{a}_{t+1}^*)\Big\}\Big] \tag{126}$$

Using this, we obtain:

$$\begin{aligned}
\tilde{V}_T(\mathbf{a}) \geq{}& \mathbb{E}\Big[\sum_{t=0}^{T} \beta^t \Big\{D_1 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) \cdot (\mathbf{a}_t^* - \mathbf{a}_t) + D_2 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) \cdot (\mathbf{a}_{t+1}^* - \mathbf{a}_{t+1})\Big\}\Big] \\
={}& D_1 u_0(\mathbf{a}_0^*, \{\mathbf{a}_1^*\}) \cdot (\mathbf{a}_0^* - \mathbf{a}_0) \\
&+ \mathbb{E}\Big[\sum_{t=0}^{T-1} \beta^t \big[D_2 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) + \beta \mathbb{E}_t[D_1 u_{t+1}(\mathbf{a}_{t+1}^*, \{\mathbf{a}_{t+2}^*\})]\big] \cdot (\mathbf{a}_{t+1}^* - \mathbf{a}_{t+1})\Big] \\
&+ \beta^T \mathbb{E}\left[D_2 u_T(\mathbf{a}_T^*, \{\mathbf{a}_{T+1}^*\}) \cdot (\mathbf{a}_{T+1}^* - \mathbf{a}_{T+1})\right]
\end{aligned} \tag{127}$$

where the first term is zero given the same initial condition $\mathbf{a}_0^* = \mathbf{a}_0$. The second term is non-negative.[27] This is because the Kuhn-Tucker condition implies (123):

$$D_2 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) + \mathbb{E}_t[\lambda_{t+1}^*] + \beta \mathbb{E}_t[D_1 u_{t+1}(\mathbf{a}_{t+1}^*, \{\mathbf{a}_{t+2}^*\})] = 0$$

and thus, the second term is larger or equal to zero:[28]

$$\mathbb{E}\Big[\sum_{t=0}^{T-1} \beta^t \big(-\mathbb{E}_t[\lambda_{t+1}^*]\big) \cdot (\mathbf{a}_{t+1}^* - \mathbf{a}_{t+1})\Big] = \mathbb{E}\Big[\sum_{t=0}^{T-1} \beta^t \underbrace{\mathbb{E}_t[\lambda_{t+1}^*]\mathbf{a}_{t+1}}_{\geq 0}\Big] - \mathbb{E}\Big[\sum_{t=0}^{T-1} \beta^t \underbrace{\mathbb{E}_t[\lambda_{t+1}^*]\mathbf{a}_{t+1}^*}_{=0}\Big] \geq 0$$

---

[26]Here we denote:

$$D_2 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) \cdot (\mathbf{a}_{t+1} - \mathbf{a}_{t+1}^*) := \\ \sum_{z^{t+1}, A^{t+1}} \frac{\partial}{\partial \mathbf{a}_{t+1}(z^{t+1}, A^{t+1})} u_t(\mathbf{a}_t(z^t, A^t), \{\mathbf{a}_{t+1}(z^{t+1}, A^{t+1})\}) \cdot (\mathbf{a}_{t+1}(z^{t+1}, A^{t+1}) - \mathbf{a}_{t+1}^*(z^{t+1}, A^{t+1}))$$

[27]Here we use the law of iterated expectations:

$$\begin{aligned}
&\mathbb{E}\Big[\big[D_2 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) + \beta D_1 u_{t+1}(\mathbf{a}_{t+1}^*, \{\mathbf{a}_{t+2}^*\})\big] \cdot (\mathbf{a}_{t+1}^* - \mathbf{a}_{t+1})\Big] \\
&= \mathbb{E}\Big[\mathbb{E}_t\big[\big\{D_2 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) + \beta D_1 u_{t+1}(\mathbf{a}_{t+1}^*, \{\mathbf{a}_{t+2}^*\})\big\} \cdot (\mathbf{a}_{t+1}^* - \mathbf{a}_{t+1})\big]\Big] \\
&= \mathbb{E}\Big[\big[D_2 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) + \beta \mathbb{E}_t[D_1 u_{t+1}(\mathbf{a}_{t+1}^*, \{\mathbf{a}_{t+2}^*\})]\big] \cdot (\mathbf{a}_{t+1}^* - \mathbf{a}_{t+1})\Big]
\end{aligned}$$

[28]Here we denote:

$$\mathbb{E}_t[\lambda_{t+1}^*]\mathbf{a}_{t+1} := \sum_{z^{t+1}, A^{t+1}} \pi(z^{t+1}|z^t)\pi(A^{t+1}|A^t)\lambda^*(z^{t+1}, A^{t+1})\mathbf{a}_{t+1}(z^{t+1}, A^{t+1})$$

Therefore, (125) is satisfied if the third term is non-negative:

$$\lim_{T\to\infty} \beta^T \mathbb{E}\left[D_2 u_T(\mathbf{a}_T^*, \{\mathbf{a}_{T+1}^*\}) \cdot (\mathbf{a}_{T+1}^* - \mathbf{a}_{T+1})\right] \geq 0. \tag{128}$$

Using the Kuhn-Tucker condition again, this is equivalent to:

$$\lim_{T\to\infty} \beta^T \mathbb{E}\left[\left\{\mathbb{E}_T[\lambda_{T+1}^*] + \beta \mathbb{E}_T[D_1 u_{T+1}(\mathbf{a}_{T+1}^*, \{\mathbf{a}_{T+2}^*\})]\right\} \cdot (\mathbf{a}_{T+1} - \mathbf{a}_{T+1}^*)\right] \geq 0. \tag{129}$$

Since $\lambda_{T+1}^*(z^{T+1}, A^{T+1})\mathbf{a}_{T+1}(z^{T+1}, A^{T+1}) \geq 0$, $\lambda_{T+1}^*(z^{T+1}, A^{T+1})\mathbf{a}_{T+1}^*(z^{T+1}, A^{T+1}) = 0$ for all (z^{T+1}, A^{T+1}), and $\beta D_1 u_{T+1}(\mathbf{a}_{T+1}^*, \{\mathbf{a}_{T+2}^*\})\mathbf{a}_{T+1} \geq 0$, the following is sufficient for (125):

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[D_1 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\})\mathbf{a}_t^*\right] = 0, \tag{130}$$

$$\text{where } D_1 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) = R_t(A^t)\frac{1}{\mathbf{c}_t^*(z^t, A^t)}.$$

Equation (130) is a transversality condition. Our conjectured allocation satisfies:

$$R_t(A^t)\frac{\mathbf{a}_t^*(z^t, A^t)}{\mathbf{c}_t^*(z^t, A^t)} = \begin{cases} 0 & \text{if } z_t = \zeta \\ \frac{1}{1-(1-\nu)\beta} & \text{if } z_t = 0 \end{cases} \tag{131}$$

Hence, (130) is satisfied in the conjectured allocation. Therefore, the conjectured allocation gives (weakly) higher expected utility than any other feasible allocations and maximizes the objective. □

A.1.2 Proof of Proposition 2

Proposition 2 (A Law of Motion of Aggregate Capital)**.** *Under the assumptions maintained in Proposition 1 and thus the household consumption and saving allocations are given by* (22) *and* (23)*, the law of motion for the aggregate capital stock is given by:*

$$\begin{aligned} K_{t+1}(A^t) &= \left[\frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta) + (1-\nu)\beta\theta\right] A_t^{1-\theta} K_t(A^{t-1})^\theta + (1-\nu)\beta(1-\delta)K_t(A^{t-1}) \\ &= \hat{s} A_t^{1-\theta} K_t(A^{t-1})^\theta + (1-\hat{\delta})K_t(A^{t-1}) \end{aligned} \tag{40}$$

where

$$\hat{s} = \frac{\xi\beta(1-\theta)}{1-(1-\nu-\xi)\beta} + (1-\nu)\beta\theta \tag{41}$$

$$\hat{\delta} = 1 - (1-\nu)\beta(1-\delta). \tag{42}$$

Proof. Aggregate saving is the sum of individual savings.

$$
\begin{aligned}
K_{t+1} &= \int \sum_{z^{t+1}} \hat{\mathbf{a}}_{t+1}(a_0, z^{t+1}, A^{t+1}) \pi(z^{t+1}) d\Phi(a_0, z_0) \\
&= \underbrace{\pi(z^{t+1}; z_t = \zeta, z_{t+1} = 0)}_{=\xi} \underbrace{\pi(z^t; z_t = \zeta)}_{=\frac{\nu}{\xi+\nu}} \hat{\mathbf{a}}_{t+1}(a_0, z^{t+1}, A^{t+1}; z_t = \zeta, z_{t+1} = 0) \\
&\quad + \int \sum_{z^{t+1}} \underbrace{\pi(z^{t+1}; z_t = 0, z_{t+1} = 0)}_{=1-\nu} \hat{\mathbf{a}}_{t+1}(a_0, z^{t+1}, A^{t+1}; z_t = 0, z_{t+1} = 0) d\Phi(a_0, z_0) \\
&= \xi \frac{\beta}{1 - (1 - \nu - \xi)\beta} w_t(A^t) + (1 - \nu)\beta R_t(A^t) \underbrace{\int \sum_{z^t} \hat{\mathbf{a}}_t(a_0, z^t, A^t; z_t = 0) d\Phi(a_0, z_0)}_{=K_t},
\end{aligned}
$$

The second line decomposes the summation into four groups by current and next states: $(z_t = \zeta, z_{t+1} = 0)$, $(z_t = 0, z_{t+1} = 0)$, $(z_t = \zeta, z_{t+1} = \zeta)$, and $(z_t = 0, z_{t+1} = \zeta)$. It sums up only the first two groups, since households do not save for a high-income state. The third line follows the households' saving rule (23). To obtain $K_t = \int \sum_{z^t} \hat{\mathbf{a}}_t(a_0, z^t, A^t; z_t = 0) d\Phi(a_0, z_0)$, note that high-income households have zero savings at the initial period ($t = 0$) and that households do not save for a high-income state at $t \geq 1$. Hence, aggregate savings at any time $t \geq 0$ is the sum of savings by low-income households ($z_t = 0$).

By substituting, $w_t(A^t) = (1 - \theta)(A_t)^{1-\theta} K_t^\theta$ and $R_t(A^t) = \theta(A_t)^{1-\theta} K_t^{\theta-1} + 1 - \delta$, we obtain:

$$
K_{t+1} = \frac{\xi\beta(1 - \theta)}{1 - (1 - \nu - \xi)\beta} (A_t)^{1-\theta} K_t^\theta + (1 - \nu)\beta \left[\theta(A_t)^{1-\theta} K_t^{\theta-1} + 1 - \delta\right] K_t. \tag{132}
$$

□

A.2 Proof: Section 4 (Stationary Equilibrium)

Proposition 3 (Stationary Equilibrium)**.** *Suppose Assumption 5 holds. Then, there exists a stationary partial insurance equilibrium, where the equilibrium interest rate R_0 is given in a closed form:*

$$
R_0 = \frac{\xi(1 - \theta)(1 - \delta) + \theta \left(\xi + \nu + \frac{1}{\beta} - 1\right)}{\xi(1 - \theta) + \beta\theta(1 - \nu) \left(\xi + \nu + \frac{1}{\beta} - 1\right)} \tag{54}
$$

The equilibrium capital and wage are given by:

$$K_0 = A_0 \left[\frac{\xi(1-\theta) + \beta\theta(1-\nu)\left(\xi + \nu + \frac{1}{\beta} - 1\right)}{\left[1-(1-\delta)\beta(1-\nu)\right]\left(\xi+\nu+\frac{1}{\beta}-1\right)} \right]^{\frac{1}{1-\theta}}, \tag{55}$$

$$w_0 = (1-\theta)A_0 \left[\frac{\xi(1-\theta) + \beta\theta(1-\nu)\left(\xi + \nu + \frac{1}{\beta} - 1\right)}{\left[1-(1-\delta)\beta(1-\nu)\right]\left(\xi+\nu+\frac{1}{\beta}-1\right)} \right]^{\frac{\theta}{1-\theta}}.$$

The equilibrium interest rate R_0 is strictly increasing in the capital share θ, strictly decreasing in the depreciation rate δ, the time discount factor β as well as the risk of productivity falling ξ and remaining low $1-\nu$ and is independent of the level of productivity A_0. The capital stock K_0 is strictly increasing in the time discount factor β as well as the risk of productivity falling ξ and remaining low $1-\nu$, strictly decreasing in the depreciation rate δ, and is proportional to the level of productivity A_0. The comparative statics of w_0 is the same as for K_0. The stationary equilibrium is unique in the sense that there is no other simple stationary equilibrium in which the stationary consumption and wealth allocation (and its associated cross-sectional distribution) is just a function of the wait time s.

Proof. We first show that under Assumption 5, we can always find a stationary equilibrium with $\beta R_0 < 1$, i.e., the existence of a partial insurance equilibrium. Since we consider a stationary equilibrium, aggregate productivity A_0 is constant over time and across aggregate states.

Suppose an interest rate satisfies $\beta R_0 < 1$. We will later verify that the equilibrium interest rate indeed satisfies $\beta R_0 < 1$ under Assumption 5. Under $\beta R_0 < 1$, households' consumption and asset choices are given by equations (43)–(47). Hence, aggregate capital supply is given by equation (52). In order for the capital market to clear, the interest rate satisfies the equation (53):

$$K(R)/w(R) =: \kappa^d(R) = \frac{\theta}{(1-\theta)(R-1+\delta)} = \frac{\xi\beta}{\left[1-(1-\nu)\beta R\right]\left[1-(1-\nu-\xi)\beta\right]} := \kappa^s(R).$$

If the equilibrium interest rate R_0 that solves (53) satisfies $\beta R_0 < 1$, then we show the existence of a partial insurance equilibrium with $\beta R_0 < 1$.

Note that the wage-normalized capital demand is positive infinite in the limit $R \to 1-\delta$:

$$\lim_{R\to 1-\delta} \kappa^d(R) \left(:= \frac{\theta}{(1-\theta)(R-1+\delta)} \right) = +\infty,$$

and $\kappa^d(R)$ is strictly decreasing in $R \in (1-\delta, \frac{1}{\beta})$. Also, the wage-normalized capital supply is finite at $R = 1-\delta$:

$$\lim_{R\to 1-\delta} \kappa^s(R) \left(:= \frac{\xi\beta}{\left[1-(1-\nu)\beta R\right]\left[1-(1-\nu-\xi)\beta\right]} \right) < \infty,$$

since $0 < 1-(1-\nu)\beta(1-\delta) < 1$ and $0 < 1-(1-\nu-\xi)\beta < 1$. $\kappa^s(R)$ is strictly increasing in $R \in (1-\delta, \frac{1}{\beta})$. This means that the excess demand for capital, $\kappa^d(R) - \kappa^s(R)$, is positive infinite at the limit $R \to 1-\delta$ and (strictly) monotonically decreasing in $R \in (1-\delta, \frac{1}{\beta})$. Therefore, an equilibrium interest rate with $\beta R_0 < 1$ exists if:

$$\kappa^d(R) - \kappa^s(R) < 0 \text{ at } R = \frac{1}{\beta}.$$

This condition is equivalent to Assumption 5. Hence, under Assumption 5, there exists a partial insurance equilibrium with $\beta R_0 < 1$.

From the FOCs for a representative firm (equations 14 and 15), we have:

$$K_0 = A_0 \left(\frac{\theta}{R_0 - 1 + \delta} \right)^{\frac{1}{1-\theta}},$$
$$w_0 = (1-\theta)(A_0)^{1-\theta}(K_0)^{\theta}.$$

By substituting the equilibrium interest rate, we obtain the equilibrium capital and the wage.

Comparative statics are straightforward. As discussed in the main text, $\kappa^d(R)$ is strictly increasing in θ and strictly decreasing in δ, while $\kappa^s(R)$ does not depend on θ or δ. Given that $\kappa^d(R) - \kappa^s(R)$ is strictly decreasing in R, higher θ implies higher R_0, and higher δ implies lower R_0. On the other hand, since $\kappa^s(R)$ is strictly increasing in β, ξ, and $1-\nu$,[29] while $\kappa^d(R)$ does not depend on β, ξ, or $1-\nu$. Thus, higher ξ and $1-\nu$ implies lower R_0. Since K_0 and w_0 are negatively related with R_0, we have the opposite comparative statics with respect to $(\beta, \xi, 1-\nu)$. To show that K_0 is decreasing in δ, we show that $R_0 - 1 + \delta$ is increasing in δ:

$$\begin{aligned} R_0 - (1-\delta) &= \frac{\xi(1-\theta)(1-\delta) + \theta\left(\xi + \nu + \frac{1}{\beta} - 1\right)}{\xi(1-\theta) + \beta\theta(1-\nu)\left(\xi + \nu + \frac{1}{\beta} - 1\right)} - (1-\delta) \\ &= \frac{\theta\left(\xi + \nu + \frac{1}{\beta} - 1\right)}{\xi(1-\theta) + \beta\theta(1-\nu)\left(\xi + \nu + \frac{1}{\beta} - 1\right)} - \frac{\beta\theta(1-\nu)\left(\xi + \nu + \frac{1}{\beta} - 1\right)}{\xi(1-\theta) + \beta\theta(1-\nu)\left(\xi + \nu + \frac{1}{\beta} - 1\right)}(1-\delta) \end{aligned}$$

From the FOCs for representative firms, we see that K_0 (and hence w_0) is decreasing in $R_0 - 1 + \delta$ and hence increasing in δ.

Finally, given the aggregate capital supply function derived from the optimal households' consumption and asset allocation that yields a simple equilibrium, since $\kappa^d(R) - \kappa^s(R)$ is strictly monotonically decreasing in $R \in (1-\delta, \frac{1}{\beta})$, the solution to an equation (53) is unique if it exists. □

[29] $\kappa^s(R)$ is strictly increasing in ξ because the derivative with respect to ξ is strictly positive:

$$\frac{\partial}{\partial \xi}\kappa^s(R) = \frac{\beta}{[1-(1-\nu)\beta R][1-(1-\nu-\xi)\beta]} \left[1 - \frac{\xi}{\frac{1}{\beta} - 1 + \nu + \xi} \right] > 0$$

A.3 Proofs: Section 5 (Transitional Dynamics)

A.3.1 Proof of Monotone Convergence

Proposition 5 (Monotone Convergence of (K_t, R_t, w_t)). *Assume the economy is in a stationary equilibrium associated with aggregate productivity, A_0 and associated capital K_0 at time $t = 0$, and suppose at time $t = 1$, productivity unexpectedly and permanently changes to A_1 with $A_1 > A_0$. Furthermore, suppose $\beta R_t < \frac{w_{t+1}}{w_t}$ for all $t \geq 0$ (Assumption 3). Then, aggregate capital K_t and wages w_t monotonically increase and converge to their new stationary equilibrium values, and the interest rate jumps up on impact and then converges monotonically to the new stationary equilibrium from above:*

$$\begin{aligned} K_0 = K_1 < K_2 < \cdots < K^* &= \frac{A_1}{A_0} K_0, \\ w_0 < w_1 < w_2 < \cdots < w^* &= \frac{A_1}{A_0} w_0, \\ R_0 < R_1 > R_2 > \cdots > R^* &= R_0. \end{aligned}$$

Symmetrically, following a permanent negative productivity shock $A_t = A_1 < A_0$ for all $t \geq 1$, the aggregate capital monotonically decreases along the transition:

$$\begin{aligned} K_0 = K_1 > K_2 > \cdots > K^* &= \frac{A_1}{A_0} K_0, \\ w_0 > w_1 > w_2 > \cdots > w^* &= \frac{A_1}{A_0} w_0, \\ R_0 > R_1 < R_2 < \cdots < R^* &= R_0. \end{aligned}$$

Proof. Consider a positive permanent shock, $A_t = A_1 \ \forall t \geq 1$ with $A_1 > A_0$. Denote K^* as the new stationary equilibrium capital associated with A_1. We know from equation (49) that $K_0 < K^*$ since $A_0 < A_1$. We want to show that given $K_t < K^*$, capital in the next period satisfies $K_t < K_{t+1} < K^*$ at any $t \geq 1$, implying a monotone convergence of capital to the new stationary equilibrium capital.

The law of motion of capital is given by equation (40):

$$K_{t+1} = \hat{s} A_t^{1-\theta} K_t^{\theta} + (1 - \hat{\delta}) K_t$$

where $\hat{s} = \dfrac{\xi \beta (1-\theta)}{1 - (1 - \nu - \xi)\beta} + (1-\nu)\beta\theta$ and $\hat{\delta} = 1 - (1-\nu)\beta(1-\delta)$.

First, we show $K_t < K_{t+1}$ at any $t \geq 1$, by using $\hat{s} = \hat{\delta} \left(\frac{K^*}{A_1}\right)^{1-\theta}$,

$$\begin{aligned} K_{t+1} - K_t &= \hat{s} A_1^{1-\theta} K_t^{\theta} + (1-\hat{\delta}) K_t - K_t \\ &= \hat{\delta} \underbrace{\left[\left(\frac{K^*}{K_t} \right)^{1-\theta} - 1 \right]}_{>0 \text{ given } K_t < K^*} K_t > 0. \end{aligned}$$

Because $\hat{\delta} > 0$, the increment in capital $(K_{t+1} - K_t)$ is strictly positive until K_t converges to K^*. Second, we show $K_{t+1} < K^*$ at any $t \geq 1$:

$$\begin{aligned} K_{t+1} - K^* &= \hat{s}A_1^{1-\theta}K_t^{\theta} + (1-\hat{\delta})K_t - K^* \\ &= \hat{\delta}\left(\frac{K_t}{K^*}\right)^{\theta} K^* + (1-\hat{\delta})K_t - K^* \\ &= \hat{\delta}\underbrace{\left[\left(\frac{K_t}{K^*}\right)^{\theta} - 1\right]}_{<0 \text{ if } K_t<K^*} K^* + (1-\hat{\delta})\underbrace{(K_t - K^*)}_{<0 \text{ if } K_t<K^*} < 0 \end{aligned}$$

We have shown that if $K_t < K^*$, $K_t < K_{t+1} < K^*$. This holds for all $t = 1, 2, ...$ as we start from $K_1 = K_0 < K^*$. Therefore, we have $K_1 < K_2 < \cdots < K_t < \cdots < K^*$.

The wage and interest rate follows the FOCs:

$$\begin{aligned} w_t &= (1-\theta)A_t^{1-\theta}K_t^{\theta}, \\ R_t &= \theta A_t^{1-\theta}K_t^{\theta-1} + 1 - \delta \text{ for all } t \geq 1. \end{aligned}$$

Given $K_1 = K_0$, both w_t and R_t jump up at $t = 1$. From $t = 2$ onwards, since $0 < \theta < 1$, $K_t < K_{t+1}$ implies $w_t < w_{t+1}$ and $R_t > R_{t+1}$. Therefore, the monotone convergence of capital implies a monotone convergence of wages and interest rates.

In case of a negative shock, the inequality holds in the opposite direction.

□

A.3.2 Sufficient Condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ $\forall t \geq 0$ after a Positive Shock

Proposition 6 (Sufficient Condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ after a Positive Shock)**.** *Let Assumption 5 be satisfied and let the economy be in a stationary equilibrium with $\beta R_0 < 1$. After a positive and permanent productivity shock at $t = 1$ ($A_t = A_1 > A_0$ for all $t \geq 1$), $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ $\forall t \geq 0$ is satisfied if $A_1 \in [A_0, \bar{A}_1)$ holds, where the threshold satisfies*

$$\frac{\bar{A}_1}{A_0} = \left[\frac{1-\beta(1-\delta)}{\theta}\, \frac{\xi(1-\theta) + (1-\nu)\theta\left[1-(1-\nu-\xi)\beta\right]}{\left[1-(1-\nu)\beta(1-\delta)\right]\left[1-(1-\nu-\xi)\beta\right]}\right]^{\frac{1}{1-\theta}} > 1. \tag{67}$$

Proof. Since wages, $w_t = (1-\theta)A_t^{1-\theta}K_t^{\theta}$, are monotonically increasing after a positive productivity shock, $\beta R_{t+1} < 1$ is a sufficient condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$. After a positive productivity shock at $t = 1$, the interest rate jumps and monotonically converges to the one in a stationary equilibrium, see Corollary **??** in Section **??**. Therefore, $\beta R_1 < 1$ guarantees that $\beta R_{t+1} < 1$ for all $t \geq 0$.

A condtion for $\beta R_1 < 1$ follows directly from the expression for R_1 and the fact that $K_1 = K_0 = A_0\left(\frac{\theta}{R_0 - 1 + \delta}\right)^{\frac{1}{1-\theta}}$ is predetermined from the initial stationary equilibrium

$$R_1 = \theta A_1^{1-\theta}K_0^{\theta-1} + 1 - \delta, \tag{133}$$

and therefore $\beta R_1 < 1$ if

$$R_1 = \left(\frac{A_1}{A_0}\right)^{1-\theta}(R_0 - 1 + \delta) + 1 - \delta < \frac{1}{\beta} \tag{134}$$

$$\frac{A_1}{A_0} < \left(\frac{\frac{1}{\beta} - 1 + \delta}{R_0 - 1 + \delta}\right)^{\frac{1}{1-\theta}} \tag{135}$$

$$\frac{A_1}{A_0} < \left[\frac{1 - \beta(1-\delta)}{\theta}\,\frac{\xi(1-\theta) + (1-\nu)\theta\left[1 - (1-\nu-\xi)\beta\right]}{\left[1 - (1-\nu)\beta(1-\delta)\right]\left[1 - (1-\nu-\xi)\beta\right]}\right]^{\frac{1}{1-\theta}}. \tag{136}$$

This gives the threshold stated in the proposition. Since $R_0 < 1/\beta$, equation (135) implies that $\bar{A}_1 > A_0$. □

A.3.3 Sufficient Condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ $\forall t \geq 0$ after a Negative Shock

In this subsection, we derive a sufficient condition on the magnitude of a negative productivity shock ($A_1 < A_0$ and $A_t = A_1$ for all $t \geq 1$) such that $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ is satisfied for all $t \geq 0$.

First note that in a stationary equilibrium, i.e., $\frac{K_t}{A_t} = \frac{K_{t+1}}{A_{t+1}} = \frac{K_0}{A_0}$, this condition is equivalent to Assumption 5. This must be true since $\beta R_{t+1}\frac{w_t}{w_{t+1}}$ converges to βR^* as $R_t \to R^*$ and $w_t \to w^*$. We show that $\beta R_{t+1}\frac{w_t}{w_{t+1}} < 1$ is always satisfied under $\delta = 1$ and Assumption 5 (Lemma 10).

Second, we derive a sufficient condition for $\beta R_{t+1}\frac{w_t}{w_{t+1}} < 1$ $\forall t \geq 1$. After a negative shock, the aggregate capital monotonically declines and converges to a new stationary equilibrium. Using this property, we derive a lower bound on A_1 that guarantees $\beta R_{t+1}\frac{w_t}{w_{t+1}} < 1$ $\forall t \geq 1$ (Proposition 18).

Third, the condition in Proposition 18 ($A_1 \in (\underline{A_1}, A_0]$) does not guarantee $\beta R_1 < \frac{w_1}{w_0}$. If $\beta R_1 > \frac{w_1}{w_0}$, low-income households may consume more than high-income households at $t = 1$. This gives rise to a possibility that low-income households at $t = 1$ have an inventive to save for a high-income state at $t = 2$. Hence, we derive a condition for $\beta R_1 < \frac{w_1}{w_0}$, which is sufficient to prevent this possibility (Proposition 7).

In Appendix B.3.1, we claim that the fact that low-income households consume more than high-income households is per se not a problem. Instead, we derive a sufficient condition for low-income households at $t = 1$ not to save for the next high-income state (Proposition 24). Numerical examples in Appendix B.3.2 show that this condition is less tight than the condition in Proposition 7.

Condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ $\forall t \geq 1$ We state three useful lemmas and use them to derive a proposition. Note that using the FOCs for production firms and the law of motion of capital, $\beta R_{t+1}\frac{w_t}{w_{t+1}}$ can be written as:

$$\begin{aligned}
\beta R_{t+1}\frac{w_t}{w_{t+1}} &= \beta\left[\theta A_{t+1}^{1-\theta}K_{t+1}^{\theta-1}+1-\delta\right]\frac{(1-\theta)A_t^{1-\theta}K_t^{\theta}}{(1-\theta)A_{t+1}^{1-\theta}K_{t+1}^{\theta}} \\
&= \beta\left[\theta+(1-\delta)\left(\frac{K_{t+1}}{A_{t+1}}\right)^{1-\theta}\right]\frac{A_t^{1-\theta}K_t^{\theta}}{K_{t+1}}, \\
&= \beta\left[\frac{\theta+(1-\delta)\left(\frac{K_{t+1}}{A_{t+1}}\right)^{1-\theta}}{\hat{s}+(1-\hat{\delta})\left(\frac{K_t}{A_t}\right)^{1-\theta}}\right] \quad \text{for all } t\geq 0 \qquad (137)
\end{aligned}$$

where $K_{t+1}=\hat{s}A_t^{1-\theta}K_t^{\theta}+(1-\hat{\delta})K_t$,

$$\hat{s}=\frac{\xi\beta(1-\theta)}{1-(1-\nu-\xi)\beta}+(1-\nu)\beta\theta, \text{ and } 1-\hat{\delta}=(1-\nu)\beta(1-\delta).$$

After a negative shock, the aggregate capital monotonically decreases and converges to a new stationary equilibrium. Therefore, $\left(\frac{K_{t+1}}{A_1}\right)^{1-\theta}<\left(\frac{K_t}{A_1}\right)^{1-\theta}$ for all $t\geq 1$. A sufficient condition for $\beta R_{t+1}<\frac{w_{t+1}}{w_t}$ $\forall t\geq 1$ is then written as:

$$\left[\left(\frac{K_t}{A_1}\right)^{1-\theta}-(1-\nu)\left(\frac{K_t}{A_1}\right)^{1-\theta}\right]=\nu\left(\frac{K_t}{A_1}\right)^{1-\theta}<\frac{1}{1-\delta}\left[\frac{\xi(1-\theta)}{1-(1-\nu-\xi)\beta}-\theta\nu\right] \text{ for all } t\geq 1.$$

As $\frac{K_1}{A_1}>\frac{K_2}{A_1}>\cdots>\frac{K^*}{A_1}$, it is sufficient to satisfy the condition at time $t=1$. By solving this inequality, we have:

$$\left(\frac{A_0}{A_1}\right)^{1-\theta}<\left[\frac{1-(1-\delta)\beta(1-\nu)}{\beta\nu(1-\delta)}\right]\left[\frac{\xi(1-\theta)-\beta\theta\nu\left[\xi+\nu+\frac{1}{\beta}-1\right]}{\xi(1-\theta)+\beta\theta(1-\nu)(\xi+\nu+\frac{1}{\beta}-1)}\right]. \qquad (138)$$

Lemma 9. *Assumption 5 implies $\xi(1-\theta)>\beta\theta\nu(\xi+\nu+\frac{1}{\beta}-1)$. Hence, the right hand side of inequality (138) is strictly positive.*

Proof. We restate Assumption 5:

$$\frac{\theta}{(1-\theta)\left[\frac{1}{\beta}-1+\delta\right]}<\frac{\xi}{\nu\left[\frac{1}{\beta}-1+\xi+\nu\right]}.$$

As all terms are positive under $0<\beta<1$, it is equivalent to:

$$\xi(1-\theta)[1-\beta(1-\delta)]>\beta\theta\nu\left[\frac{1}{\beta}-1+\xi+\nu\right].$$

Since $0<\beta(1-\delta)<1$, we have:

$$\xi(1-\theta)>\xi(1-\theta)[1-\beta(1-\delta)]>\beta\theta\nu(\xi+\nu+\frac{1}{\beta}-1). \qquad (139)$$

Therefore, all terms in the right hand side of inequality (138) are strictly positive. In the limit $\delta\to 1$, it is positive infinity: $\lim_{\delta\to 1}\left[\frac{1-(1-\delta)\beta(1-\nu)}{\beta\nu(1-\delta)}\right]\left[\frac{\xi(1-\theta)-\beta\theta\nu\left[\xi+\nu+\frac{1}{\beta}-1\right]}{\xi(1-\theta)+\beta\theta(1-\nu)(\xi+\nu+\frac{1}{\beta}-1)}\right]=+\infty$. □

Lemma 10. *Assume Assumption 5 and consider transitional dynamics after a productivity shock. With full depreciation of capital ($\delta = 1$), $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ is satisfied for all $t \geq 1$. Furthermore, $R_{t+1}\frac{w_t}{w_{t+1}} = R_0$ for all $t \geq 1$ if $\delta = 1$.*

Proof. Substituting $\delta = 1$ into the condition (137) gives:

$$\beta R_{t+1}\frac{w_t}{w_{t+1}} = \beta\left[\frac{\theta}{\frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta)+(1-\nu)\beta\theta}\right] \tag{140}$$

Then, $\beta R_{t+1}\frac{w_t}{w_{t+1}} < 1$ is equivalent to:

$$\theta < \frac{\xi}{1-(1-\nu-\xi)\beta}(1-\theta)+(1-\nu)\theta$$
$$\Leftrightarrow \nu\theta\left[1-(1-\nu-\xi)\beta\right] < \xi(1-\theta).$$

This inequality is satisfied under Assumption 5 as we saw in the previous Lemma.

We derive $R_{t+1}\frac{w_t}{w_{t+1}} = R_0$ under full depreciation ($\delta = 1$) using equation (54):

$$\begin{aligned} R_0\Big|_{\delta=1} &= \frac{\xi(1-\theta)(1-\delta)+\theta\left(\xi+\nu+\frac{1}{\beta}-1\right)}{\xi(1-\theta)+\beta\theta(1-\nu)\left(\xi+\nu+\frac{1}{\beta}-1\right)}\Bigg|_{\delta=1} \\ &= \frac{\theta}{\frac{\xi(1-\theta)}{\xi+\nu+\frac{1}{\beta}-1}+\beta\theta(1-\nu)} = R_{t+1}\frac{w_t}{w_{t+1}}\Bigg|_{\delta=1} \end{aligned} \tag{141}$$

□

Lemma 11. *Under Assumption 5, the right hand side of inequality (138) is strictly larger than 1. Hence, there exists a negative productivity shock A_1 with $A_1 < A_0$ such that (138) holds. Define $\underline{A_1}$ such that (138) holds with equality. Then $\underline{A_1} < A_0$.*

Proof. Consider $0 < \delta < 1$. The first and second terms in the right hand side of (138) are expressed as:

$$\left[\frac{1-(1-\delta)\beta(1-\nu)}{\beta\nu(1-\delta)}\right] = 1+\frac{\frac{1}{\beta(1-\delta)}-1}{\nu} \tag{142}$$

$$\left[\frac{\xi(1-\theta)-\beta\theta\nu\left[\xi+\nu+\frac{1}{\beta}-1\right]}{\xi(1-\theta)+\beta\theta(1-\nu)(\xi+\nu+\frac{1}{\beta}-1)}\right] = \frac{1}{1+\frac{\beta\theta\left(\xi+\nu+\frac{1}{\beta}-1\right)}{\xi(1-\theta)-\beta\theta\nu\left[\xi+\nu+\frac{1}{\beta}-1\right]}}. \tag{143}$$

Therefore, the product is larger than 1 if:

$$\frac{\frac{1}{\beta(1-\delta)}-1}{\nu} > \frac{\beta\theta\left(\xi+\nu+\frac{1}{\beta}-1\right)}{\xi(1-\theta)-\beta\theta\nu\left[\xi+\nu+\frac{1}{\beta}-1\right]} \tag{144}$$

Since all terms are positive (remember Lemma 9), this is equivalent to:

$$\begin{aligned} &\left(\frac{1}{\beta(1-\delta)}-1\right)\xi(1-\theta)-\frac{\theta\nu}{(1-\delta)}\left(\xi+\nu+\frac{1}{\beta}-1\right)+\beta\theta\nu\left(\xi+\nu+\frac{1}{\beta}-1\right)>\beta\theta\nu\left(\xi+\nu+\frac{1}{\beta}-1\right)\\ \Leftrightarrow\ &\left(\frac{1-\beta(1-\delta)}{\beta(1-\delta)}\right)\xi(1-\theta)>\frac{\theta\nu}{(1-\delta)}\left(\xi+\nu+\frac{1}{\beta}-1\right)\\ \Leftrightarrow\ &\frac{\xi}{\nu\left(\xi+\nu+\frac{1}{\beta}-1\right)}>\frac{\theta}{(1-\theta)(\frac{1}{\beta}-1+\delta)}\end{aligned}$$

The last condition is equivalent to Assumption 5. Therefore, the right hand side of inequality (138) is strictly greater than 1 for any $0<\delta<1$If $\delta=1$, the right hand side of (138) goes to positive infinity. Put together, this means that for any $0<\delta\leq 1$, there exists $A_1<A_0$ such that the condition (138) holds.

We can solve for A_1 such that (138) holds with equality:

$$\underline{A_1}=A_0\left[\frac{\beta\nu(1-\delta)}{1-(1-\delta)\beta(1-\nu)}\frac{\xi(1-\theta)+\beta\theta(1-\nu)(\xi+\nu+\frac{1}{\beta}-1)}{\xi(1-\theta)-\beta\theta\nu\left[\xi+\nu+\frac{1}{\beta}-1\right]}\right]^{\frac{1}{1-\theta}}.$$

Because $\left[\frac{\beta\nu(1-\delta)}{1-(1-\delta)\beta(1-\nu)}\frac{\xi(1-\theta)+\beta\theta(1-\nu)(\xi+\nu+\frac{1}{\beta}-1)}{\xi(1-\theta)-\beta\theta\nu\left[\xi+\nu+\frac{1}{\beta}-1\right]}\right]<1$, $\underline{A_1}<A_0$. □

We use these lemmas to prove a proposition.

Proposition 18 (Sufficient Condition for $\beta R_{t+1}<\frac{w_{t+1}}{w_t}$ for $t\geq 1$ after a Negative Shock)**.** *Let Assumption 5 be satisfied and let the economy be in a stationary equilibrium with $\beta R_0<1$. After a negative and permanent productivity shock at $t=1$ ($A_t=A_1<A_0$ for all $t\geq 1$), $\beta R_{t+1}<\frac{w_{t+1}}{w_t}$ $\forall t\geq 1$ is satisfied if $A_1\in(\underline{A_1},A_0]$ holds, where the threshold satisfies*

$$\underline{A_1}/A_0=\left[\frac{\beta\nu(1-\delta)}{1-(1-\delta)\beta(1-\nu)}\frac{\xi(1-\theta)+\beta\theta(1-\nu)(\xi+\nu+\frac{1}{\beta}-1)}{\xi(1-\theta)-\beta\theta\nu\left[\xi+\nu+\frac{1}{\beta}-1\right]}\right]^{\frac{1}{1-\theta}}<1. \quad (145)$$

Proof. We want to derive a sufficient condition for $\beta R_{t+1}\frac{w_t}{w_{t+1}}<1$ $\forall t\geq 1$. We focus on the case of $0<\delta<1$, because with full depreciation of capital ($\delta=1$), $\beta R_{t+1}\frac{w_t}{w_{t+1}}=\beta R_0<1$ $\forall t\geq 1$. Using equation (137) and monotonicity of capital, $K_{t+1}<K_t$, we derive the following.

$$\begin{aligned}\beta R_{t+1}\frac{w_t}{w_{t+1}}&=\beta\left[\frac{\theta+(1-\delta)\left(\frac{K_{t+1}}{A_1}\right)^{1-\theta}}{\frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta)+(1-\nu)\beta\theta+(1-\nu)\beta(1-\delta)\left(\frac{K_t}{A_1}\right)^{1-\theta}}\right]\\ &<\beta\left[\frac{\theta+(1-\delta)\left(\frac{K_t}{A_1}\right)^{1-\theta}}{\frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta)+(1-\nu)\beta\theta+(1-\nu)\beta(1-\delta)\left(\frac{K_t}{A_1}\right)^{1-\theta}}\right]\quad\text{for } t\geq 1\end{aligned}$$

The last line is less than 1 if

$$\beta\theta + \beta(1-\delta)\left(\frac{K_t}{A_1}\right)^{1-\theta} < \frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta) + (1-\nu)\beta\theta + (1-\nu)\beta(1-\delta)\left(\frac{K_t}{A_1}\right)^{1-\theta}$$
$$\Leftrightarrow \nu\beta(1-\delta)\left(\frac{K_t}{A_1}\right)^{1-\theta} < \frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta) - \nu\beta\theta$$

Because $K_1 > K_2 > \cdots$, this condition is satisfied for all $t \geq 1$ if it is satisfied at time $t = 1$:

$$\left(\frac{K_1}{A_1}\right)^{1-\theta} < \frac{1}{\nu(1-\delta)}\left[\frac{\xi(1-\theta)}{1-(1-\nu-\xi)\beta} - \nu\theta\right]. \tag{146}$$

Aggregate capital at time $t = 1$ is predetermined at $t = 0$:

$$K_1 = K_0 = A_0\left[\frac{\xi(1-\theta) + \beta\theta(1-\nu)\left(\xi+\nu+\frac{1}{\beta}-1\right)}{[1-(1-\delta)\beta(1-\nu)]\,(\xi+\nu+\frac{1}{\beta}-1)}\right]^{\frac{1}{1-\theta}}.$$

Substituting K_1 into equation (146) yields:

$$\left(\frac{A_0}{A_1}\right)^{1-\theta} < \left[\frac{1-(1-\delta)\beta(1-\nu)}{\beta\nu(1-\delta)}\right]\left[\frac{\xi(1-\theta) - \beta\theta\nu\left[\xi+\nu+\frac{1}{\beta}-1\right]}{\xi(1-\theta) + \beta\theta(1-\nu)(\xi+\nu+\frac{1}{\beta}-1)}\right]. \tag{147}$$

We obtain equation (145) by solving for A_1. When $A_1 = \underline{A_1}$, the equation holds with equality. Lemma 11 shows that $\underline{A_1} < A_0$ under Assumption 5. □

A.3.4 Condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at $t = 0$

Proposition 18 shows that after a negative and permanent productivity shock at $t = 1$, $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ $\forall t \geq 1$ is satisfied if $A_1 \in (\underline{A_1}, A_0]$ holds. However, this condition does not guarantee $\beta R_1 < \frac{w_1}{w_0}$. If $\beta R_1 > \frac{w_1}{w_0}$, the argument in Lemma 8 breaks down, i.e., $c_1(a_0, z^1, A^1) \leq c_0$ may not hold for some a_0. Then, low-income households at $t = 1$ may have the incentive to save for the next high-income state. In such a case, the contract stipulated in Proposition 1 may not be optimal. Hence, we derive a sufficient condition for $\beta R_1 < \frac{w_1}{w_0}$. It turns out that the sufficient condition for $\beta R_1 < \frac{w_1}{w_0}$, given by $A_1 \in (\underline{A}'_1, A_0]$, implies the sufficient condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ $\forall t \geq 1$, i.e., $A_1 \in (\underline{A_1}, A_0]$, derived in Proposition 18. Therefore, $A_1 \in (\underline{A}'_1, A_0]$ is a sufficient condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ $\forall t \geq 0$.

Proposition 7 (Sufficient Condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ $\forall t \geq 0$ after a Negative Shock)**.** *Let Assumption 5 be satisfied and let the economy be in a stationary equilibrium with* $\beta R_0 < 1$*. After a negative and permanent productivity shock at* $t = 1$ *(*$A_t = A_1 < A_0$ *for all* $t \geq 1$*),*

$\beta R_{t+1} < \frac{w_{t+1}}{w_t}\ \forall t \geq 0$ *is satisfied if* $A_1 \in (\underline{A}_1', A_0]$ *holds, where the threshold satisfies*

$$\underline{A}_1'/A_0 = \left[1-\nu+\nu\ \frac{1-(1-\delta)\beta(1-\nu)}{\beta\nu(1-\delta)}\ \frac{\xi(1-\theta)-\beta\theta\nu(\xi+\nu+\frac{1}{\beta}-1)}{\xi(1-\theta)+\beta\theta(1-\nu)\left(\xi+\nu+\frac{1}{\beta}-1\right)}\right]^{\frac{1}{\theta-1}} < 1. \tag{68}$$

Proof. $\beta R_{t+1}\frac{w_t}{w_{t+1}}$ can be written as:

$$\beta R_{t+1}\frac{w_t}{w_{t+1}} = \beta\left[\frac{\theta+(1-\delta)\left(\frac{K_{t+1}}{A_{t+1}}\right)^{1-\theta}}{\hat{s}+(1-\hat{\delta})\left(\frac{K_t}{A_t}\right)^{1-\theta}}\right]$$

At $t=0$, $\beta R_{t+1}\frac{w_t}{w_{t+1}} < 1$ is equivalent to:

$$\beta\theta+\beta(1-\delta)\left(\frac{K_1}{A_1}\right)^{1-\theta} < \hat{s}+(1-\hat{\delta})\left(\frac{K_0}{A_0}\right)^{1-\theta},$$

$$\text{where } \hat{s} = \frac{\xi\beta(1-\theta)}{1-(1-\nu-\xi)\beta}+(1-\nu)\beta\theta \text{ and } 1-\hat{\delta} = (1-\nu)\beta(1-\delta).$$

The condition can be written as:

$$(1-\delta)\left(\frac{K_0}{A_0}\right)^{1-\theta}\left[\left(\frac{A_0}{A_1}\right)^{1-\theta}-(1-\nu)\right] < \frac{\xi(1-\theta)}{1-(1-\nu-\xi)\beta}-\nu\theta$$

$$\Leftrightarrow \left(\frac{A_0}{A_1}\right)^{1-\theta} < 1-\nu+\frac{1}{1-\delta}\left(\frac{K_0}{A_0}\right)^{\theta-1}\left[\frac{\xi(1-\theta)}{1-(1-\nu-\xi)\beta}-\nu\theta\right], \tag{148}$$

$$\text{where } \frac{K_0}{A_0} = \left[\frac{\xi(1-\theta)+\beta\theta(1-\nu)\left(\xi+\nu+\frac{1}{\beta}-1\right)}{[1-(1-\delta)\beta(1-\nu)]\,(\xi+\nu+\frac{1}{\beta}-1)}\right]^{\frac{1}{1-\theta}}.$$

Using the following derivations:

$$\begin{aligned}
&\frac{1}{1-\delta}\left(\frac{K_0}{A_0}\right)^{\theta-1}\left[\frac{\xi(1-\theta)}{1-(1-\nu-\xi)\beta}-\nu\theta\right]\\
&=\frac{1}{1-\delta}\left[\frac{[1-(1-\delta)\beta(1-\nu)]\,(\xi+\nu+\frac{1}{\beta}-1)}{\xi(1-\theta)+\beta\theta(1-\nu)\left(\xi+\nu+\frac{1}{\beta}-1\right)}\right]\left[\frac{\xi(1-\theta)-\nu\theta\beta(\xi+\nu+\frac{1}{\beta}-1)}{\beta(\xi+\nu+\frac{1}{\beta}-1)}\right]\\
&=\nu\ \frac{1-(1-\delta)\beta(1-\nu)}{\beta\nu(1-\delta)}\ \frac{\xi(1-\theta)-\nu\theta\beta(\xi+\nu+\frac{1}{\beta}-1)}{\xi(1-\theta)+\beta\theta(1-\nu)\left(\xi+\nu+\frac{1}{\beta}-1\right)}\\
&=\nu\left(\frac{\underline{A}_1}{A_0}\right)^{\theta-1},
\end{aligned}$$

the condition for $\beta R_1 < \frac{w_1}{w_0}$ is given by:

$$\left(\frac{A_0}{A_1}\right)^{1-\theta} < 1-\nu+\nu\left(\frac{A_1}{A_0}\right)^{\theta-1}$$

$$\therefore\ \frac{A_1}{A_0} > \left[1-\nu+\nu\left(\frac{A_1}{A_0}\right)^{\theta-1}\right]^{\frac{1}{\theta-1}} =: \frac{A_1'}{A_0}, \tag{149}$$

$$\text{where } \frac{\underline{A}_1}{A_0} < \frac{A_1'}{A_0} < 1. \tag{150}$$

We obtain the last inequality (150), since $0 < \nu < 1$ and $0 < \frac{\underline{A}_1}{A_0} < 1$. Hence, $A_1 \in (\underline{A}_1', A_0]$ implies $A_1 \in (\underline{A}_1, A_0]$. By substituting $\frac{\underline{A}_1}{A_0}$ using equation (145), we obtain equation (68) in the statement. □

A.3.5 Consumption on Impact

Corollary 2 (Consumption at the time of a shock)**.** *Consider an unexpected shock to productivity at $t = 1$ and assume that Assumptions 2, 3, and 4 hold. Then consumption of high-income agents ($\mathbf{c}_{h,t}$) is a constant fraction of their income, and consumption of low-income agents ($\mathbf{c}_{s,t}$ for $s \geq 1$) satisfies the standard Euler equation between periods $t = 0$ and $t = 1$:*

$$\mathbf{c}_{h,1} = \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta} z w_1$$

$$\mathbf{c}_{s,1} = \beta R_1 \mathbf{c}_{s-1,0} \qquad \textit{for } s \geq 1.$$

Proof. At $t = 1$, the asset distribution is predetermined and given by $\{\mathbf{a}_{s,t}\}_{s=0,t=1}^{\infty}$ that satisfies Assumption 4. In particular, we consider the case in which $\{\mathbf{a}_{s,t}\}_{s=0,t=1}^{\infty}$ follows a stationary distribution given by equations (45)–(47), where w_0 follows (51). Note that households purchased contingent assets $\{\mathbf{a}_{s,t}(A^t)\}_{s=0,t=1}^{\infty}$ at $t = 0$, expecting the steady-state productivity for all future periods with probability 1, $A_t = A^*$ for all $t \geq 1$, but an unanticipated aggregate state is realized at $t = 1$. We assume that households hold assets at $t = 1$ as if the anticipated aggregate state (A^*) is realized.

At time $t = 1$, a deterministic sequence of productivity $\{A_t\}_{t=1}^{\infty}$ is unexpectedly realized. The corresponding sequence of prices $\{q_t(A_{t+1}, z_{t+1}|A^t, z^t), w_t(A^t), R_t(A^t)\}_{A^t,z^t,t\geq 1}$ satisfies Assumptions 2 and 3. Then, by Proposition 1, the optimal consumption at $t = 1$ is given by (22). Equation (24) implies that the consumption satisfies the Euler equation between $t = 0$ and $t = 1$. □

A.4 Proofs: Section 6 (Aggregate Shocks)

Lemma 1. *Define $\tilde{K}_t := \frac{K_t}{A_t}$. The law of motion of capital (40) is expressed as:*

$$\tilde{K}_{t+1} = \frac{A_t}{A_{t+1}} \left[\hat{s}\tilde{K}_t^{\theta} + (1-\hat{\delta})\tilde{K}_t \right]. \tag{151}$$

The maximum value and the minimum value of $\frac{K_t}{A_t}$ is given by:

$$\tilde{K}^{max} = \left(\frac{\hat{s}}{\hat{\delta} - \epsilon} \right)^{\frac{1}{1-\theta}} \text{ and } \tilde{K}^{min} = \left(\frac{\hat{s}}{\hat{\delta} + \epsilon} \right)^{\frac{1}{1-\theta}}$$

Proof. We deflate the law of motion of capital by A_{t+1}. Define $\tilde{K}_t := \frac{K_t}{A_t}$. Equation (40) is given by:

$$\begin{aligned} \tilde{K}_{t+1} &= \frac{A_t}{A_{t+1}} \left[\hat{s}\tilde{K}_t^{\theta} + (1-\hat{\delta})\tilde{K}_t \right] \\ \text{where } \hat{s} &:= \left[\frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta) + (1-\nu)\beta\theta \right] \\ \text{and } 1-\hat{\delta} &:= (1-\nu)\beta(1-\delta). \end{aligned}$$

$\tilde{K}_t := \frac{K_t}{A_t}$ takes the maximum and minimum after the economy experiences negative ($\frac{A_{t+1}}{A_t} = 1-\epsilon$) and positive ($\frac{A_{t+1}}{A_t} = 1+\epsilon$) shocks infinitely many times, respectively. This is because $\tilde{K}_t$ is decreasing in $\frac{A_{t+1}}{A_t}$ in equation (73). Also, since $\tilde{K}_{t+1}$ is increasing in $\tilde{K}_t$, $\tilde{K}_t' > \tilde{K}_t$ implies $\tilde{K}_{t+1}' > \tilde{K}_{t+1}$. Then, $\tilde{K}^{min}$ solves:

$$\begin{aligned} \tilde{K}^{min} &= \frac{1}{1+\epsilon} \left[\hat{s}\left(\tilde{K}^{min}\right)^{\theta} + (1-\hat{\delta})\tilde{K}^{min} \right]. \\ \therefore\ \tilde{K}^{min} &= \left(\frac{\hat{s}}{\hat{\delta}+\epsilon} \right)^{\frac{1}{1-\theta}}. \end{aligned}$$

Similarly, $\tilde{K}^{max}$ solves:

$$\begin{aligned} \tilde{K}^{max} &= \frac{1}{1-\epsilon} \left[\hat{s}\left(\tilde{K}^{max}\right)^{\theta} + (1-\hat{\delta})\tilde{K}^{max} \right]. \\ \therefore\ \tilde{K}^{max} &= \left(\frac{\hat{s}}{\hat{\delta}-\epsilon} \right)^{\frac{1}{1-\theta}}. \end{aligned}$$

□

Lemma 2.

1. *The constraint $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ is tighter at the time of negative shock ($\frac{A_{t+1}}{A_t} = 1-\epsilon$).*
2. *If $\tilde{K}_t = \tilde{K}^{min}$, $\tilde{K}_{t+1} = \frac{A_t}{A_{t+1}}(1+\epsilon)\tilde{K}^{min}$. If $\tilde{K}_t = \tilde{K}^{max}$, $\tilde{K}_{t+1} = \frac{A_t}{A_{t+1}}(1-\epsilon)\tilde{K}^{max}$.*
3. *Suppose $\frac{A_{t+1}}{A_t} = 1-\epsilon$ and $\tilde{K}_t \in [\tilde{K}^{min}, \tilde{K}^{max}]$. R_{t+1} is highest at $\tilde{K}_t = \tilde{K}^{min}$. $\frac{w_{t+1}}{w_t}$ is lowest at $\tilde{K}_t = \tilde{K}^{max}$*

Proof.

1. $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ is given by:

$$\begin{aligned}
\beta \left[\theta \tilde{K}_{t+1}^{\theta-1} + 1 - \delta\right] &< \frac{A_{t+1}}{A_t}\left(\frac{\tilde{K}_{t+1}}{\tilde{K}_t}\right)^{\theta} \\
\Leftrightarrow \ \beta \left[\theta \left(\frac{A_{t+1}}{A_t}\right)^{1-\theta}\left(\frac{K_{t+1}}{A_t}\right)^{\theta-1} + 1 - \delta\right] &< \left(\frac{A_{t+1}}{A_t}\right)^{1-\theta}\left(\frac{K_{t+1}}{K_t}\right)^{\theta} \\
\Leftrightarrow \ \beta \left[\theta \left(\frac{K_{t+1}}{A_t}\right)^{\theta-1} + (1-\delta)\left(\frac{A_t}{A_{t+1}}\right)^{1-\theta}\right] &< \left(\frac{K_{t+1}}{K_t}\right)^{\theta}
\end{aligned}$$

The left hand side is stricly decreasing in $\frac{A_{t+1}}{A_t}$ unless $\delta = 1$. Therefore, the condition is tigher if $\frac{A_{t+1}}{A_t} = 1 - \epsilon$. Since the condition does not depend on $\frac{A_{t+1}}{A_t}$ under $\delta = 1$, it is still sufficient to check the condition at the time of negative shock ($\frac{A_{t+1}}{A_t} = 1 - \epsilon$) for any $0 < \delta \leq 1$.

2. $\tilde{K}_{t+1}$ is given by (73):

$$\tilde{K}_{t+1} = \frac{A_t}{A_{t+1}}\left[\hat{s}\tilde{K}_t^{\theta} + (1-\hat{\delta})\tilde{K}_t\right].$$

If $\tilde{K}_t = \tilde{K}^{min}$ and $\frac{A_{t+1}}{A_t} = 1 + \epsilon$, we know:

$$\begin{aligned}
\tilde{K}_{t+1} &= \frac{1}{1+\epsilon}\left[\hat{s}(\tilde{K}^{min})^{\theta} + (1-\hat{\delta})\tilde{K}^{min}\right] = \tilde{K}^{min} \\
\therefore \ \left[\hat{s}(\tilde{K}^{min})^{\theta} + (1-\hat{\delta})\tilde{K}^{min}\right] &= (1+\epsilon)\tilde{K}^{min}
\end{aligned}$$

Hence, if $\tilde{K}_t = \tilde{K}^{min}$, we derive:

$$\tilde{K}_{t+1} = \frac{A_t}{A_{t+1}}(1+\epsilon)\tilde{K}^{min} \tag{152}$$

Similarly, if $\tilde{K}_t = \tilde{K}^{max}$, we have

$$\tilde{K}_{t+1} = \frac{A_t}{A_{t+1}}(1-\epsilon)\tilde{K}^{max}. \tag{153}$$

3. R_{t+1} is decreasing in $\tilde{K}_{t+1}$, since R_{t+1} is given by:

$$R_{t+1} = \theta\tilde{K}_{t+1}^{\theta-1} + 1 - \delta.$$

$\tilde{K}_{t+1}$ is determined by (73):

$$\tilde{K}_{t+1} = \frac{A_t}{A_{t+1}}\left[\hat{s}\tilde{K}_t^{\theta} + (1-\delta)\tilde{K}_t\right].$$

Given $\frac{A_t}{A_{t+1}} = \frac{1}{1-\epsilon}$, $\tilde{K}_{t+1}$ is increasing in $\tilde{K}_t$. Therefore, R_{t+1} is decreasing in $\tilde{K}_t$ and takes the maximum value at $\tilde{K}_t = \tilde{K}^{min}$.

$\frac{w_{t+1}}{w_t}$ is given by:

$$\frac{w_{t+1}}{w_t} = \frac{A_{t+1}}{A_t}\left(\frac{\tilde{K}_{t+1}}{\tilde{K}_t}\right)^{\theta}$$
$$= \left(\frac{A_{t+1}}{A_t}\right)^{1-\theta}\left[\hat{s}\tilde{K}_t^{\theta-1} + 1 - \hat{\delta}\right]^{\theta},$$

which is decreasing in $\tilde{K}_t$. Therefore, $\frac{w_{t+1}}{w_t}$ takes the minimum value at $\tilde{K}_t = \tilde{K}^{max}$. □

Lemma 3. *$\tilde{K}^{min} \leq \tilde{K}_t \leq \tilde{K}^{max}$ implies $\tilde{K}^{min} \leq \tilde{K}_{t+1} \leq \tilde{K}^{max}$*

Proof.

1. $\tilde{K}_t \geq \tilde{K}^{min}$ implies $\tilde{K}_{t+1} \geq \tilde{K}^{min}$.

Consider a positive shock. $\tilde{K}_{t+1}$ is given by:

$$\tilde{K}_{t+1} = \frac{1}{1+\epsilon}\left[\hat{s}\tilde{K}_t^{\theta} + (1-\hat{\delta})\tilde{K}_t\right].$$

If $\tilde{K}_t = \tilde{K}^{min}$, $\tilde{K}_{t+1} = \tilde{K}^{min}$.
If $\tilde{K}_t > \tilde{K}^{min}$, since the right hand side is strictly increasing in $\tilde{K}_t$, $\tilde{K}_{t+1} > \tilde{K}^{min}$. Therefore, $\tilde{K}_t \geq \tilde{K}^{min}$ implies $\tilde{K}_{t+1} \geq \tilde{K}^{min}$ after a positive shock. Since

$$\frac{1}{1+\epsilon}\left[\hat{s}\tilde{K}_t^{\theta} + (1-\hat{\delta})\tilde{K}_t\right] < \frac{1}{1-\epsilon}\left[\hat{s}\tilde{K}_t^{\theta} + (1-\hat{\delta})\tilde{K}_t\right],$$

$\tilde{K}_t \geq \tilde{K}^{min}$ implies $\tilde{K}_{t+1} \geq \tilde{K}^{min}$ after a negative shock as well.

2. $\tilde{K}_t \leq \tilde{K}^{max}$ implies $\tilde{K}_{t+1} \leq \tilde{K}^{max}$.

Consider a negative shock. $\tilde{K}_{t+1}$ is given by:

$$\tilde{K}_{t+1} = \frac{1}{1-\epsilon}\left[\hat{s}\tilde{K}_t^{\theta} + (1-\hat{\delta})\tilde{K}_t\right].$$

If $\tilde{K}_t = \tilde{K}^{max}$, $\tilde{K}_{t+1} = \tilde{K}^{max}$.
If $\tilde{K}_t < \tilde{K}^{max}$, since the right hand side is strictly increasing in $\tilde{K}_t$, $\tilde{K}_{t+1} < \tilde{K}^{max}$. Therefore, $\tilde{K}_t \leq \tilde{K}^{max}$ implies $\tilde{K}_{t+1} \leq \tilde{K}^{max}$ after a negative shock. Since

$$\frac{1}{1-\epsilon}\left[\hat{s}\tilde{K}_t^{\theta} + (1-\hat{\delta})\tilde{K}_t\right] > \frac{1}{1+\epsilon}\left[\hat{s}\tilde{K}_t^{\theta} + (1-\hat{\delta})\tilde{K}_t\right],$$

$\tilde{K}_t \leq \tilde{K}^{max}$ implies $\tilde{K}_{t+1} \leq \tilde{K}^{max}$ after a positive shock as well. □

Proposition 8 . *Suppose Assumption 2, 4, and 5 hold and the aggregate capital at $t = 0$ satisfies $\tilde{K}^{min} < \tilde{K}_0 < \tilde{K}^{max}$. Under Assumption G,*

$$\beta R_{t+1} < \frac{w_{t+1}}{w_t} \textit{ for all } t \geq 0 \textit{ with probability } 1.$$

Proof. By Lemma 3, $\tilde{K}^{min} < \tilde{K}_0 < \tilde{K}^{max}$ implies $\tilde{K}^{min} < \tilde{K}_t < \tilde{K}^{max}$ for all $t \geq 1$. We saw in Lemma 2 that the condition $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ is tighter at the time of negative shock. Given a negative shock, R_{t+1} achieves the maximum if $\tilde{K}_t = \tilde{K}^{min}$, and $\frac{w_{t+1}}{w_t}$ takes the minimum if $\tilde{K}_t = \tilde{K}^{max}$. If the maximum of βR_{t+1} is smaller than the minimum of $\frac{w_{t+1}}{w_t}$ under $\frac{A_{t+1}}{A_t} = 1-\epsilon$, the condition is satisfied for all $\tilde{K}_t$ and $\frac{A_{t+1}}{A_t}$. By imposing these, we have a sufficient condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$, where

$$\begin{aligned} &\beta R_{t+1} < \frac{w_{t+1}}{w_t} \\ \Leftrightarrow \ &\beta \left[\theta \tilde{K}_{t+1}^{\theta-1} + 1 - \delta\right] < \left(\frac{A_{t+1}}{A_t}\right)^{1-\theta} \left[\hat{s}\tilde{K}_t^{\theta-1} + 1 - \hat{\delta}\right]^{\theta}. \end{aligned}$$

$\tilde{K}_{t+1}$ in the left hand side takes the minimum at $\tilde{K}_{t+1} = \frac{A_t}{A_{t+1}}(1+\epsilon)\tilde{K}^{min}$. $\tilde{K}_t$ in the right hand side takes the maximum at $\tilde{K}_t = \tilde{K}^{max}$. $\frac{A_{t+1}}{A_t}$ is given by $1-\epsilon$. Threfore, a sufficient condition is given by:

$$\beta \left[\theta \left[\frac{A_t}{A_{t+1}}(1+\epsilon)\tilde{K}^{min}\right]^{\theta-1} + 1 - \delta\right] < \left(\frac{A_{t+1}}{A_t}\right)^{1-\theta} \left[\hat{s}(\tilde{K}^{max})^{\theta-1} + 1 - \hat{\delta}\right]^{\theta} \tag{154}$$

By subsituting $\tilde{K}^{max}$ and $\tilde{K}^{min}$, we obtain Assumption G. Under this condition, $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all $t \geq 0$ and all states $\frac{A_{t+1}}{A_t} \in \{1-\epsilon, 1+\epsilon\}$. □

Proposition 9. *Suppose Assumption 5 holds. There exists $\bar{\epsilon}$ such that Assumption G holds for all $0 \leq \epsilon < \bar{\epsilon}$. Given such $\bar{\epsilon}$, $0 \leq \epsilon < \bar{\epsilon}$ and $\tilde{K}^{min} < \tilde{K}_0 < \tilde{K}^{max}$ imply $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all $t \geq 0$ with probability 1.*

Proof. If $\epsilon = 0$ in Assumption G, we have:

$$\beta \left[\theta \frac{\hat{\delta}}{\hat{s}} + 1 - \delta\right] < 1. \tag{155}$$

In the steady state, $\tilde{K}$ satisfies:

$$\tilde{K}^* = \left(\frac{\hat{\delta}}{\hat{s}}\right)^{\frac{-1}{1-\theta}}. \tag{156}$$

Therefore, equation (155) is equivalent to:

$$\begin{aligned} &\beta \left[\theta(\tilde{K}^*)^{\theta-1} + 1 - \delta\right] < 1 \\ \Leftrightarrow \ &\beta R^* < 1. \end{aligned}$$

This is equivalent to Assumption 5. This means that Assumption G is satisfied in an open neighborhood of $\epsilon = 0$, since Assumption G is continuous in ϵ.

We show that Assumption G becomes monotonically tighter as ϵ increases. Assumption G is equivalent to:

$$\left[\theta\left(\frac{1+\epsilon}{1-\epsilon}\right)^{\theta}\left(\frac{\hat{\delta}+\epsilon}{1+\epsilon}\right)\frac{1}{\hat{s}}+\frac{1-\delta}{1-\epsilon}\right]<1.$$

$\left(\frac{1+\epsilon}{1-\epsilon}\right)$ and $\frac{1-\delta}{1-\epsilon}$ are strictly increasing in ϵ. $\frac{\hat{\delta}+\epsilon}{1+\epsilon}$ is weakly increasing in ϵ for all $0<\hat{\delta}\leq 1$. Threfore, the left hand side is strictly increasing in ϵ. This means that the condition becomes tighter as ϵ increases. Hence, Assumption G is satisfied for all $0<\epsilon<\bar{\epsilon}$. □

Corollary 3. *Suppose Assumption 5 holds and capital fully depreciates ($\delta=1$). If $0\leq\epsilon<\bar{\epsilon}$, where*

$$\bar{\epsilon}:=\frac{\left(\frac{\hat{s}}{\beta\theta}\right)^{\frac{1}{\theta}}-1}{\left(\frac{\hat{s}}{\beta\theta}\right)^{\frac{1}{\theta}}+1}>0, \tag{157}$$

Assumption G is satisfied.

Proof. If $\delta=1$ (which implies $\hat{\delta}=1$), Assumption G becomes:

$$\frac{\beta\theta}{\hat{s}}\left(\frac{1+\epsilon}{1-\epsilon}\right)^{\theta}<1. \tag{158}$$

This holds with equality if

$$\frac{1+\bar{\epsilon}}{1-\bar{\epsilon}}=\left(\frac{\hat{s}}{\beta\theta}\right)^{\frac{1}{\theta}},$$

which gives $\bar{\epsilon}$ in equation (157). Since the left hand side of equation (158) is strictly increasing in ϵ, the inequality holds for all ϵ with $0<\epsilon<\bar{\epsilon}$. If $\frac{\hat{s}}{\beta\theta}>1$, we will have $0<\bar{\epsilon}<1$. $\frac{\hat{s}}{\beta\theta}>1$ is equivalent to:

$$\frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta)+(1-\nu)\beta\theta>\beta\theta.$$

We know from Lemma 9 that Assumption 5 implies:

$$\xi(1-\theta)>\beta\theta\nu\left[\frac{1}{\beta}-1+\nu+\epsilon\right].$$

Therefore, $\bar{\epsilon}$ satisfies $0<\bar{\epsilon}<1$. □

A.5 Proof: Section 7 (Literature)

Proposition 10. *In a limited-commitment model, as $\xi\to 0$ and $\nu\to 0$, the transitional dynamics is described by two equations:*

$$K_{t+1}=\beta\left[\theta A_t^{1-\theta}K_t^{\theta}+(1-\delta)K_t\right], \tag{77}$$

$$K^{*}=A\left[\frac{\beta\theta}{1-\beta(1-\delta)}\right]^{\frac{1}{1-\theta}}. \tag{78}$$

Under full depreciation of capital ($\delta = 1$), the steady-state capital stock and the law of motion of capital is the same as a standard neoclassical growth model with full depreciation of capital and a log utility function.

Proof. The transitional dynamics of aggregate capital in an economy with limited commitment is characterized by equation (40):

$$K_{t+1} = \frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta)A_t^{1-\theta}K_t^\theta + (1-\nu)\beta\left[\theta A_t^{1-\theta}K_t^\theta + (1-\delta)K_t\right],$$

$$\text{where } K^* = A\left[\frac{\xi\beta(1-\theta)+(1-\nu)\beta\theta\left[1-(1-\nu-\xi)\beta\right]}{\left[1-(1-\nu)\beta(1-\delta)\right]\left[1-(1-\nu-\xi)\beta\right]}\right]^{\frac{1}{1-\theta}}.$$

$$R^* = \frac{\xi(1-\theta)(1-\delta)+\theta\left(\xi+\nu+\frac{1}{\beta}-1\right)}{\xi(1-\theta)+\beta\theta(1-\nu)\left(\xi+\nu+\frac{1}{\beta}-1\right)}.$$

We consider a limit where idiosyncratic income states do not change over time, $\xi \to 0$ and $\nu \to 0$, and see the relation to a standard neoclassical growth model. As $\xi \to 0$ and $\nu \to 0$, the equations above converge to:

$$K_{t+1} = \beta\left[\theta A_t^{1-\theta}K_t^\theta + (1-\delta)K_t\right], \tag{77}$$

$$K^* = A\left[\frac{\beta\theta}{1-\beta(1-\delta)}\right]^{\frac{1}{1-\theta}}, \tag{78}$$

$$R^* = \frac{1}{\beta}. \tag{159}$$

We will see that this corresponds to a neoclassical growth model if $\delta = 1$.

We describe a standard neoclassical growth model with a log utility function, $u(C) = \log(C)$.

$$F(K,L) = K^\theta(AL)^{1-\theta} \text{ with } L = 1,$$

$$\rightarrow \begin{cases} R_t &= \theta A_t^{1-\theta}K_t^{\theta-1} + 1 - \delta, \\ w_t &= (1-\theta)A_t^{1-\theta}K_t^\theta. \end{cases}$$

The equilibrium is characterized by the Euler equation and the resource constraint:

$$\frac{1}{C_t} = \beta R_{t+1}\frac{1}{C_{t+1}}, \tag{160}$$

$$C_t + K_{t+1} = A_t^{1-\theta}K_t^\theta + (1-\delta)K_t. \tag{161}$$

In the steady state, $C_t = C_{t+1}$ gives $\beta R_t = 1$. Therefore, we have:

$$R^* := \theta A^{1-\theta}K^{\theta-1} + 1 - \delta = \frac{1}{\beta} \tag{162}$$

$$\Leftrightarrow K^* = A\left(\frac{\theta}{\frac{1}{\beta}-1+\delta}\right)^{\frac{1}{1-\theta}}. \tag{163}$$

We see that the steady-state level of capital coincides with our model (78).
The law of motion of capital is given by:

$$\frac{1}{A_t^{1-\theta}K_t^\theta + (1-\delta)K_t - K_{t+1}} = \beta \frac{\theta A_{t+1}^{1-\theta}K_{t+1}^{\theta-1} + 1 - \delta}{A_{t+1}^{1-\theta}K_{t+1}^\theta + (1-\delta)K_{t+1} - K_{t+2}}$$
$$\Leftrightarrow K_{t+2} = A_{t+1}^{1-\theta}K_{t+1}^\theta + (1-\delta)K_{t+1} - \beta \underbrace{\left(\theta A_{t+1}^{1-\theta}K_{t+1}^{\theta-1} + 1 - \delta\right)}_{=R_{t+1}} \underbrace{\left[A_t^{1-\theta}K_t^\theta + (1-\delta)K_t - K_{t+1}\right]}_{C_t}. \tag{164}$$

Generally, the equation (353) doesn't have a closed-form solution. Consider a special case with $\delta = 1$:

$$K_{t+2} = A_{t+1}^{1-\theta}K_{t+1}^\theta - \beta\theta A_{t+1}^{1-\theta}K_{t+1}^{\theta-1}\left[A_t^{1-\theta}K_t^\theta - K_{t+1}\right]. \tag{165}$$

We have a well-known closed-form solution in this case. Guess that

$$K_{t+1} = \beta\theta A_t^{1-\theta}K_t^\theta. \tag{166}$$

Then, (165) becomes:

$$\begin{aligned} K_{t+2} &= A_{t+1}^{1-\theta}K_{t+1}^\theta - \beta\theta A_{t+1}^{1-\theta}K_{t+1}^{\theta-1}\left[(1-\beta\theta)A_t^{1-\theta}K_t^\theta\right] \\ &= A_{t+1}^{1-\theta}K_{t+1}^\theta - A_{t+1}^{1-\theta}K_{t+1}^{\theta-1}\left[(1-\beta\theta)K_{t+1}\right] \\ &= \beta\theta A_{t+1}^{1-\theta}K_{t+1}^\theta. \end{aligned}$$

This is consistent with the guess. Under $\delta = 1$, this coincides with our model (77) with $\delta = 1$:

$$K_{t+1} = \beta\theta A_t^{1-\theta}K_t^\theta$$

Therefore, our model with $\xi \to 0$, $\nu \to 0$, and $\delta = 1$ coincides with the neoclassical growth model with $\delta = 1$. □

A.6 Proofs: Section 8 (Consumption Inequality)

Proposition 11 (Consumption Distribution in the Long Run)**.** *Suppose that Assumptions 2, 3, and 4 hold and that an economy is in a steady state at $t = 0$ with productivity A_0. Suppose also that after a productivity shock, aggregate productivity settles down at A_1. Then, the deflated consumption distribution in the long run is the same as the initial distribution:*

$$c_s^* = (\beta R^*)^s c_0 \text{ for } s = 0, 1, 2, \cdots,$$

where the steady-state interest rate, R^*, does not depend on productivity A.

Proof. As we see in Proposition 5, in the transitional dynamics, aggregate capital will monotonically converge to a new steady state. Then, the interest rate also converges to a steady-state interest rate (Corollary **??**).

In a stationary equilibrium, the deflated consumption of low-income agents is determined by:

$$c_s = \beta R^* c_{s-1}.$$

Hence, the consumption distribution is characterized by:

$$c_s = (\beta R^*)^s c_0 \ \text{ for } s = 0, 1, 2, \cdots,$$

where the mass of each agent is given by equation (373):

$$\phi_s = \begin{cases} \frac{\nu}{\xi+\nu} & \text{if } \ s = 0 \\ \frac{\nu\xi}{\xi+\nu}(1-\nu)^{s-1} & \text{if } \ s \geq 1. \end{cases}$$

Because the equilibrium interest rate does not depend on productivity A, shown in equation (54):

$$R^* = \frac{\xi(1-\theta)(1-\delta) + \theta\left(\xi + \nu + \frac{1}{\beta} - 1\right)}{\xi(1-\theta) + \beta\theta(1-\nu)\left(\xi + \nu + \frac{1}{\beta} - 1\right)},$$

the deflated consumption distribution is the same across stationary equilibia with different A.

□

Lemma 12. *Suppose Assumptions 2, 3, and 4 hold. With full depreciation of capital (*$\delta = 1$*), the following equation holds for any sequence of aggregate shocks* $\{A_t\}_{t\geq 0}$*:*

$$\beta R_{t+1}(A^{t+1}) \frac{w_t(A^t)}{w_{t+1}(A^{t+1})} = \frac{\beta\theta}{\hat{s}} \tag{167}$$

Proof. With full depreciation of capital ($\delta = 1$), the law of motion of capital (40) is given by:

$$K_{t+1} = \underbrace{\left[\frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta) + (1-\nu)\beta\theta\right]}_{=:\hat{s}} A_t^{1-\theta} K_t^{\theta} \tag{168}$$

The interest rate and wage are:

$$R_{t+1}(A^{t+1}) = \theta\left(\frac{K_{t+1}}{A_{t+1}}\right)^{\theta-1}$$
$$w_t(A^t) = (1-\theta)A_t^{1-\theta}K_t^{\theta}$$

Hence, given K_t, the following equation holds for any (A^t, A_{t+1}):

$$\begin{aligned}\beta R_{t+1}(A^{t+1})\frac{w_t(A^t)}{w_{t+1}(A^{t+1})} &= \beta\theta A_{t+1}^{1-\theta}K_{t+1}^{\theta-1}\frac{(1-\theta)A_t^{1-\theta}K_t^{\theta}}{(1-\theta)A_{t+1}^{1-\theta}K_{t+1}^{\theta}} \\ &= \beta\theta\frac{A_t^{1-\theta}K_t^{\theta}}{K_{t+1}} \\ &= \frac{\beta\theta}{\hat{s}}. \end{aligned} \tag{169}$$

□

Proposition 12. *Suppose Assumptions 2, 3, and 4 hold. Suppose that an economy is in a stationary equilibrium at* $t = 0$ *with deflated consumption distribution* $\{c_s^*\}_{s\geq 0}$*. With full depreciation of capital (*$\delta = 1$*), the deflated consumption distribution is time-invariant for any sequence of* $\{A_t\}_{t\geq 0}$*:*

$$c_{s,t}(A^t) = c_s^* \text{ for any } t \geq 0 \text{ and } A^t. \tag{85}$$

Proof. The stationary distribution at $t = 0$ is given by:

$$c_s^* = (\beta R^*)^s c_0, \text{ where } c_0 := \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}\zeta.$$

Note that with $\delta = 1$, the interest rate in the statioanry equilibrium (54) is given by:

$$\begin{aligned} R^* &= \frac{\theta\left(\xi+\nu+\frac{1}{\beta}-1\right)}{\xi(1-\theta)+\beta\theta(1-\nu)\left(\xi+\nu+\frac{1}{\beta}-1\right)}. \\ &= \frac{\theta}{\hat{s}} \end{aligned} \tag{170}$$

$$\text{where } \hat{s} := \frac{\xi(1-\theta)+(1-\nu)\beta\theta(\xi+\nu+\frac{1}{\beta}-1)}{\xi+\nu+\frac{1}{\beta}-1}.$$

From $t = 1$ onwards, the deflated consumption distribution evolves acccording to equation (84):

$$c_{s,t}(A^t) = \begin{cases} c_0 & \text{if } s = 0 \\ \beta R_t(A^t)\frac{w_{t-1}(A^{t-1})}{w_t(A^t)}c_{s-1,t-1}(A^{t-1}) & \text{if } s \geq 1 \end{cases}$$

Lemma 12 shows that:

$$\beta R_t(A^t)\frac{w_{t-1}(A^{t-1})}{w_t(A^t)} = \frac{\beta\theta}{\hat{s}}$$

for any $t \geq 1$ and (A^{t-1}, A_t). Combined with equation (170), this means that:

$$c_{s,t}(A^t) = \begin{cases} c_0 & \text{if } s = 0 \\ \beta R^* c_{s-1,t-1}(A^{t-1}) & \text{if } s \geq 1 \end{cases}$$

Hence, starting from the stationary distribution at $t = 0$, the deflated consumption distribution is time-invariant:

$$c_{s,t}(A^t) = c_s^* \text{ for all } t \geq 0 \text{ and } A^t.$$

$\square$

Proposition 13. *Consider an economy in a stationary equilibrium at $t = 0$ and a positive productivity shock at $t = 1$ ($A_1 > A_0$), where $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all $t \geq 0$. Under $0 < \delta < 1$ and $0 < \theta < 1$, the degree of inequality rises at the time of a shock ($t = 1$) in the sense that the consumption of all low-income agents ($s = 1, 2, \cdots$) declines relative to high-income agents:*

$$c_{s,1} = \frac{\theta A_0^{1-\theta} K_0^{\theta-1} + (1-\delta)\left(\frac{A_0}{A_1}\right)^{1-\theta}}{\theta A_1^{1-\theta} K_0^{\theta-1} + 1 - \delta} c_{s,0} < c_{s,0}, \tag{86}$$

$$\text{while } c_{h,t} = \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta} z \text{ for all } t \geq 0.$$

Proof. We derive that $c_{s,1} < c_{s,0}$ for all $s \geq 1$ if $A_1 > A_0$ and $\delta < 1$, meaning that the deflated consumption of low-income agents at time $t = 1$ is lower than the deflated consumption in a stationary equilibrium ($t = 0$) for all $s \geq 1$. By equation (**??**),

$$\begin{aligned}
c_{s,1} &= \frac{w_0}{w_1}\frac{R_1}{R_0} c_{s,0}, \\
\text{where } R_t &= \theta A_t^{1-\theta} K_t^{\theta-1} + 1 - \delta, \\
w_t &= (1-\theta) A_t^{1-\theta} K_t^{\theta}. \\
\therefore\ c_{s,1} &= \left(\frac{A_0^{1-\theta}}{A_1^{1-\theta}}\right) \frac{\theta A_1^{1-\theta} K_0^{\theta-1} + 1 - \delta}{\theta A_0^{1-\theta} K_0^{\theta-1} + 1 - \delta} c_{s,0} \\
&= \frac{\theta A_0^{1-\theta} K_0^{\theta-1} + (1-\delta)\left(\frac{A_0}{A_1}\right)^{1-\theta}}{\theta A_0^{1-\theta} K_0^{\theta-1} + 1 - \delta} c_{s,0}
\end{aligned}$$

We use the fact that $K_1 = K_0$ as the aggregate capital at $t = 1$, K_1, is predetermined at time $t = 0$. Given that $0 < \delta < 1$, $0 < \theta < 1$, and $A_1 > A_0$, we have $(1-\delta)\left(\frac{A_0}{A_1}\right)^{1-\theta} < 1-\delta$. Therefore, $c_{s,1} < c_{s,0}$ for all $s \geq 1$. $\square$

In order to obtain the expression of consumption gap in Proposition 14, we first derive the consumption gap $\frac{c_{s,t}}{c_{s,0}}$ in the following Lemma.

Lemma 13. *Suppose an economy is in a stationary equilibrium at $t = 0$, and a productivity shock is realized at $t = 1$. Assume $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ holds for all $t \geq 0$. The evolution of deflated consumption for low-income agents is characterized as:*

$$\frac{c_{s,t}}{c_{s,0}} = \begin{cases} \left(\prod_{u=1}^{t} \frac{R_u}{R_0}\right) \frac{w_0}{w_t} & \text{if } s \geq t \\ \left(\prod_{u=t-s+1}^{t} \frac{R_u}{R_0}\right) \frac{w_{t-s}}{w_t} & \text{if } 1 \leq s < t \end{cases} \tag{171}$$

Notivce that because $R_u = R_0$ for $u \leq 0$ and $w_{t-s} = w_0$ for $s \geq t$, the latter expression includes the former as a special case.

Proof. The ratio of deflated consumption between time t and time 0 for agents s with $s \geq t$, $\frac{c_{s,t}}{c_{s,0}}$, is derived using the Euler equation:

$$\begin{aligned} w_t c_{s,t} &= \beta R_t w_{t-1} c_{s-1,t-1} \\ &= (\beta R_t)(\beta R_{t-1}) \cdots (\beta R_1) w_0 c_{s-t,0} \\ c_{s,0} &= (\beta R_0)^t c_{s-t,0} \\ \therefore \ \frac{c_{s,t}}{c_{s,0}} &= \left(\frac{R_1}{R_0}\right) \cdots \left(\frac{R_t}{R_0}\right) \frac{w_0}{w_t} \end{aligned}$$

This equation can be expressed in a sequential way:

$$\begin{aligned} \frac{c_{s,t}}{c_{s,0}} &= \frac{c_{s,t-1}}{c_{s,0}} \left(\frac{R_t}{R_0}\right) \left(\frac{w_{t-1}}{w_t}\right) \\ &= \prod_{u=1}^{t} \left(\frac{R_u}{R_0}\right) \left(\frac{w_{u-1}}{w_u}\right) = \left(\prod_{u=1}^{t} \frac{R_u}{R_0}\right) \frac{w_0}{w_t}. \end{aligned}$$

If $s < t$, the low-income agents have experienced high income after the shock. Hence, wage at the time of last high income is w_{t-s} with $w_{t-s} > w_0$. The ratio of deflated consumption between time t and time 0 is given by:

$$\begin{aligned} w_t c_{s,t} &= \beta R_t w_{t-1} c_{s-1,t-1} \\ &= (\beta R_t) \cdots (\beta R_{t-s+1}) w_{t-s} c_{h,t-s} \\ c_{s,0} &= (\beta R_0)^s c_{h,0} \\ \therefore \ \frac{c_{s,t}}{c_{s,0}} &= \left(\frac{R_{t-s+1}}{R_0}\right) \cdots \left(\frac{R_t}{R_0}\right) \frac{w_{t-s}}{w_t} = \left(\prod_{u=t-s+1}^{t} \frac{R_u}{R_0}\right) \frac{w_{t-s}}{w_t} \end{aligned}$$

□

Proposition 14. *Suppose an economy is in a stationary equilibrium at $t = 0$, and a productivity shock is realized at $t = 1$. Suppose Assumption 3 holds. The evolution of consumption gap between high-income and low-income agents relative to the stady state is described by:*

$$-\log\left(\frac{c_{s,t}}{c_{0,t}}\right) + \log\left(\frac{c_{s,0}}{c_{0,0}}\right) = \begin{cases} \log\left(\frac{w_1}{w_0}\frac{R_0}{R_1}\right) & \text{for all } s \geq 1 \text{ if } t = 1 \\ \log\left(\frac{w_1}{w_0}\frac{R_0}{R_1}\right) + \sum_{u=1}^{t-1} \log\left(\frac{w_{u+1}}{w_u}\frac{R_0}{R_{u+1}}\right) & \text{if } s \geq t \text{ and } t \geq 2 \\ \sum_{u=t-s}^{t-1} \log\left(\frac{w_{u+1}}{w_u}\frac{R_0}{R_{u+1}}\right) & \text{if } s < t \text{ and } t \geq 2 \end{cases} \tag{87}$$

Assume $0 < \delta < 1$. Then, the consumption gap expands at time 1, since $\log\left(\frac{w_1}{w_0}\frac{R_0}{R_1}\right) > 0$. From time 1 until time $s \geq 2$, the consumption gap continues to be higher than the *stationary equilibrium if* $\log\left(\frac{w_1}{w_0}\frac{R_0}{R_1}\right)+\sum_{u=1}^{t-1}\log\left(\frac{w_{u+1}}{w_u}\frac{R_0}{R_{u+1}}\right)>0$. *From time* $s+1$ *onwards, the consumption gap is smaller than the stationary equilibrium if* $\sum_{u=t-s}^{t-1}\log\left(\frac{w_{u+1}}{w_u}\frac{R_0}{R_{u+1}}\right)<0$.

Proof. From equation (171), we know:

$$\frac{c_{s,t}}{c_{s,0}}=\left(\frac{R_{t-s+1}}{R_0}\right)\cdots\left(\frac{R_t}{R_0}\right)\frac{w_{t-s}}{w_t},$$
$$\frac{c_{s,t-1}}{c_{s,0}}=\left(\frac{R_{t-s}}{R_0}\right)\cdots\left(\frac{R_{t-1}}{R_0}\right)\frac{w_{t-1-s}}{w_{t-1}}.$$

Dividing the former equation by the latter gives:

$$\frac{c_{s,t}}{c_{s,t-1}}=\left(\frac{R_t}{R_{t-s}}\right)\left(\frac{w_{t-s}}{w_t}\right)\left(\frac{w_{t-1}}{w_{t-1-s}}\right).$$

This allows us to express $\frac{c_{s,t}}{c_{0,t}}$ in a sequential way, where we use $c_{0,t}=c_{0,t-1}=c_h$:

$$\begin{aligned}\frac{c_{s,t}}{c_{0,t}}&=\left(\frac{R_t}{R_{t-s}}\right)\left(\frac{w_{t-s}}{w_t}\right)\left(\frac{w_{t-1}}{w_{t-1-s}}\right)\frac{c_{s,t-1}}{c_{0,t-1}}\\&=\left[\prod_{u=1}^{t}\frac{R_u}{R_{u-s}}\frac{w_{u-s}}{w_u}\frac{w_{u-1}}{w_{u-1-s}}\right]\frac{c_{s,0}}{c_{0,0}}\text{ for }s\geq 1.\end{aligned}\tag{172}$$

By taking a log:

$$\begin{aligned}-\log\left(\frac{c_{s,t}}{c_{0,t}}\right)&=\sum_{u=1}^{t}\left[\underbrace{\log\left(\frac{w_u}{w_{u-1}}\right)}_{\text{wage increase for high income}}-\underbrace{\log\left(\frac{w_{u-s}}{w_{u-s-1}}\right)}_{\text{wage increase for }s\text{ agent}}-\underbrace{\log\left(\frac{R_u}{R_{u-s}}\right)}_{\text{higher interest rate}}\right]\underbrace{-\log\left(\frac{c_{s,0}}{c_{0,0}}\right)}_{\text{initial gap}}\\&=\log\left(\frac{w_t}{w_0}\right)-\log\left(\frac{w_{t-s}}{w_0}\right)-\sum_{u=t-s+1}^{t}\log\left(\frac{R_u}{R_0}\right)-\log\left(\frac{c_{s,0}}{c_{0,0}}\right).\end{aligned}\tag{173}$$

This equation holds for any $s\geq 1$. If $s\geq t$, we use the facts that:

$$\log\left(\frac{w_{t-s}}{w_0}\right)=\log\left(\frac{w_0}{w_0}\right)=0$$
$$\sum_{u=t-s+1}^{0}\log\left(\frac{R_u}{R_0}\right)=\sum_{u=t-s+1}^{0}\log\left(\frac{R_0}{R_0}\right)=0\text{ for }s>t.$$

Then, (173) is written as follows:

$$\begin{aligned}-\log\left(\frac{c_{s,t}}{c_{0,t}}\right)+\log\left(\frac{c_{s,0}}{c_{0,0}}\right)&=\log\left(\frac{w_t}{w_0}\right)-\sum_{u=1}^{t}\log\left(\frac{R_u}{R_0}\right)\text{ for }s\geq t\\&=\sum_{u=1}^{t}\log\left(\frac{w_u}{w_{u-1}}\frac{R_0}{R_u}\right)\\&=\begin{cases}\log\left(\frac{w_1}{w_0}\frac{R_0}{R_1}\right)+\sum_{u=1}^{t-1}\log\left(\frac{w_{u+1}}{w_u}\frac{R_0}{R_{u+1}}\right)&\text{if }t\geq 2\\\log\left(\frac{w_1}{w_0}\frac{R_0}{R_1}\right)&\text{if }t=1\end{cases}\end{aligned}\tag{174}$$

Combined with the case of $s < t$, we have:

$$-\log\left(\frac{c_{s,t}}{c_{0,t}}\right) + \log\left(\frac{c_{s,0}}{c_{0,0}}\right) = \begin{cases} \log\left(\frac{w_1}{w_0}\frac{R_0}{R_1}\right) & \text{for all } s \geq 1 \text{ if } t = 1 \\ \log\left(\frac{w_1}{w_0}\frac{R_0}{R_1}\right) + \sum_{u=1}^{t-1} \log\left(\frac{w_{u+1}}{w_u}\frac{R_0}{R_{u+1}}\right) & \text{if } s \geq t \text{ and } t \geq 2 \\ \sum_{u=t-s}^{t-1} \log\left(\frac{w_{u+1}}{w_u}\frac{R_0}{R_{u+1}}\right) & \text{if } s < t \text{ and } t \geq 2 \end{cases}$$

As we saw before,

$$\frac{w_1}{w_0}\frac{R_0}{R_1} = \left(\frac{A_1}{A_0}\right)^{1-\theta} \frac{\theta A_0^{1-\theta} K_0^{\theta-1} + 1 - \delta}{\theta A_1^{1-\theta} K_0^{\theta-1} + 1 - \delta}$$
$$= \frac{\theta A_1^{1-\theta} K_0^{\theta-1} + (1-\delta)\left(\frac{A_1}{A_0}\right)^{1-\theta}}{\theta A_1^{1-\theta} K_0^{\theta-1} + 1 - \delta} > 1 \text{ if } \delta < 1. \tag{175}$$

This confirms that consumption gap between high-income agents and s-th low-income agents expands at time 1 if $\delta < 1$. This equation illustrates that if $0 < \delta < 1$, wage growth at time 1, given by $\left(\frac{A_1}{A_0}\right)^{1-\theta}$, is higher than interest rate growth. Therefore, the productivity shock benefits high-income agents more than low-income agents at time 1. □

Proposition 19 (Sufficient Condition for Overshooting). *Consider an economy in a stationary equilibrium at $t = 0$ and a positive productivity shock at $t = 1$ ($A_t = A_1 > A_0$ for all $t \geq 1$), where $\beta R_t < \frac{w_{t+1}}{w_t}$ for all $t \geq 0$. The consumption gap between s-th low-income agents and high-income agents at time $s+1$, $-\log\left(\frac{c_{s,s+1}}{c_{0,s+1}}\right)$, is smaller than the initial gap in the stationary equilibrium, $-\log\left(\frac{c_{s,0}}{c_{0,0}}\right)$, if the following condition is satisfied:*

$$\frac{w_{t+1}}{w_t}\frac{R_0}{R_{t+1}} < 1 \textit{ for all } t = 1, \cdots, s. \tag{176}$$

Proof. The overshooting phenomenon happens at time $s+1$ for s-th low-income agents if $-\log\left(\frac{c_{s,t}}{c_{0,t}}\right) + \log\left(\frac{c_{s,0}}{c_{0,0}}\right) < 0$ at $t = s+1$. We rewrite this condition:

$$-\log\left(\frac{c_{s,s+1}}{c_{0,s+1}}\right) + \log\left(\frac{c_{s,0}}{c_{0,0}}\right) = \log\left(\frac{w_{s+1}}{w_1}\right) - \sum_{u=2}^{s+1}\log\left(\frac{R_u}{R_0}\right)$$
$$= \sum_{u=1}^{s}\left[\log\left(\frac{w_{u+1}}{w_u}\right) - \log\left(\frac{R_{u+1}}{R_0}\right)\right]$$
$$= \sum_{u=1}^{s}\log\left(\frac{w_{u+1}}{w_u}\frac{R_0}{R_{u+1}}\right)$$

This equation shows that the overshooting phenomenon happens for agent s if $\frac{w_{t+1}}{w_t}\frac{R_0}{R_{t+1}} < 1$ for all $t = 1, ..., s$. □

A.6.1 Evaluation of $\frac{w_{t+1}}{w_t}\frac{R_0}{R_{t+1}}$ at time $t \geq 1$

We saw in the last proposition that the overshooting of consumption gap happens if $\frac{w_{t+1}}{w_t}\frac{R_0}{R_{t+1}} < 1$ for all $t = 1, \cdots, s$. We characterize this term in terms of capital (K_0, K_t, K_{t+1}).

Proposition 20. *Suppose an economy is in a stationary equilibrium at $t = 0$, and a productivity shock is realized at $t = 1$. Assume $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ holds for all $t \geq 0$. $\frac{w_{t+1}}{w_t}\frac{R_0}{R_{t+1}}$ for $t \geq 1$ is expressed as:*

$$\frac{w_{t+1}}{w_t}\frac{R_0}{R_{t+1}} = \left[\hat{\delta} + (1-\hat{\delta})\left(\frac{K_t/A_1}{K_0/A_0}\right)^{1-\theta}\right]\left[\frac{\theta + (1-\delta)\left(\frac{K_0}{A_0}\right)^{1-\theta}}{\theta + (1-\delta)\left(\frac{K_{t+1}}{A_1}\right)^{1-\theta}}\right]. \tag{177}$$

Proof. We use the following:

$$\begin{aligned}
R_t &= \theta A_t^{1-\theta} K_t^{\theta-1} + 1 - \delta, \\
w_t &= (1-\theta) A_t^{1-\theta} K_t^{\theta}, \\
K_{t+1} &= \hat{s} A_t^{1-\theta} K_t^{\theta} + (1-\hat{\delta}) K_t, \\
\text{where } \hat{s} &= \frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta) + (1-\nu)\beta\theta = \hat{\delta}\left(\frac{K_0}{A_0}\right)^{1-\theta}, \\
\hat{\delta} &= 1 - (1-\nu)\beta(1-\delta).
\end{aligned}$$

Then, $\frac{w_{t+1}}{w_t}\frac{R_0}{R_{t+1}}$ is expressed by:

$$\begin{aligned}
\frac{w_{t+1}}{w_t}\frac{R_0}{R_{t+1}} &= \left[\frac{(1-\theta)A_{t+1}^{1-\theta}K_{t+1}^{\theta}}{(1-\theta)A_t^{1-\theta}K_t^{\theta}}\right]\left[\frac{\theta A_0^{1-\theta}K_0^{\theta-1} + 1 - \delta}{\theta A_{t+1}^{1-\theta}K_{t+1}^{\theta-1} + 1 - \delta}\right] \text{ for } t = 1, \cdots, s \qquad (178) \\
&= \left[\frac{K_{t+1}K_{t+1}^{\theta-1}}{K_t^{\theta}}\right]\left[\frac{\theta A_0^{1-\theta}K_0^{\theta-1} + 1 - \delta}{\theta A_1^{1-\theta}K_{t+1}^{\theta-1} + 1 - \delta}\right] \\
&= \left[\frac{\hat{s}A_t^{1-\theta}K_t^{\theta} + (1-\hat{\delta})K_t}{K_t^{\theta}}\right]\left[\frac{\theta A_0^{1-\theta}K_0^{\theta-1} + 1 - \delta}{\theta A_1^{1-\theta} + (1-\delta)K_{t+1}^{1-\theta}}\right] \\
&= \left[\hat{\delta}\left(\frac{K_0}{A_0}\right)^{1-\theta} A_1^{1-\theta} + (1-\hat{\delta})K_t^{1-\theta}\right]\left[\frac{\theta A_0^{1-\theta} + (1-\delta)K_0^{1-\theta}}{\theta A_1^{1-\theta} + (1-\delta)K_{t+1}^{1-\theta}}\right] K_0^{\theta-1} \\
&= \left[\hat{\delta}\left(\frac{A_1}{A_0}\right)^{1-\theta} + (1-\hat{\delta})\left(\frac{K_t}{K_0}\right)^{1-\theta}\right]\left[\frac{\theta A_0^{1-\theta} + (1-\delta)K_0^{1-\theta}}{\theta A_1^{1-\theta} + (1-\delta)K_{t+1}^{1-\theta}}\right] \\
&= \underbrace{\left[\hat{\delta} + (1-\hat{\delta})\left(\frac{K_t/A_1}{K_0/A_0}\right)^{1-\theta}\right]}_{<1 \text{ if } \delta<1}\underbrace{\left[\frac{\theta + (1-\delta)\left(\frac{K_0}{A_0}\right)^{1-\theta}}{\theta + (1-\delta)\left(\frac{K_{t+1}}{A_1}\right)^{1-\theta}}\right]}_{>1 \text{ if } \delta<1}
\end{aligned}$$

□

Note that $\frac{K_t}{A_1} < \frac{K_{t+1}}{A_1} < \cdots < \frac{K^*}{A_1} = \frac{K_0}{A_0}$ on the transition path after a positive productivity shock. Equation (177) is equal to 1 if $\delta = 1$, as $\hat{\delta} = 1 - (1-\nu)\beta(1-\delta) = 1$ under $\delta = 1$.

Corollary 4. *Consider a transition path after a productivity shock.*

$$\text{If } \delta = 1, \ \frac{w_{t+1}}{w_t}\frac{R_0}{R_{t+1}} = 1 \text{ for all } t \geq 0.$$

This is consistent with the constant deflated consumption distribution.

Proof. Plug in $\delta = \hat{\delta} = 1$ for the equations (175) and (177). □

In the long run, $\frac{K_t}{A_t}$ and $\frac{K_{t+1}}{A_1}$ converge to $\frac{K_0}{A_0}$. Therefore, the term, $\frac{w_{t+1}}{w_t}\frac{R_0}{R_{t+1}}$, converges to 1, meaning the consumption gap between high-income and low-income agents goes back to the original level.

Corollary 5. *Consider a transition path after a productivity shock. The consumption gap converges to the initial level for sufficiently large $t \geq s+1$.*

Proof. $\lim_{t\to\infty} \frac{w_{t+1}}{w_t}\frac{R_0}{R_{t+1}} = 1$ imply that $\lim_{t\to\infty} \sum_{u=t-s}^{t-1} \log\left(\frac{w_{u+1}}{w_u}\frac{R_0}{R_{u+1}}\right) = 0$ □

A.6.2 Consumption Distribution after a Negative Productivity Shock

This subsection summarizes symmetric results for an unexpected negative productivity shock at time 1. Figure 9 illustrates that the transition path of consumption gaps after a negative productivity shock is symmetric with a transition path after a positive shock.

Corollary 6 (Negative Productivity Shock with Full Depreciation)**.** *Consider a transition path after a negative productivity shock.*

$$\text{If } \delta = 1, \ \frac{w_{t+1}}{w_t}\frac{R_0}{R_{t+1}} = 1 \text{ for all } t \geq 0.$$

This implies that with full depreciation of capital ($\delta = 1$), the deflated consumption distribution is constant along the transition.

Corollary 7 (Transition of Consumption Gap after a Negative Shock)**.** *Consider a transition path after a negative productivity shock at time 1. The evolution of the consumption gap between high-income agents and s-th low-income agents is given by (87):*

$$-\log\left(\frac{c_{s,t}}{c_{0,t}}\right) + \log\left(\frac{c_{s,0}}{c_{0,0}}\right) = \begin{cases} \log\left(\frac{w_1}{w_0}\frac{R_0}{R_1}\right) & \text{if } t = 1 \\ \log\left(\frac{w_1}{w_0}\frac{R_0}{R_1}\right) + \sum_{u=1}^{t-1} \log\left(\frac{w_{u+1}}{w_u}\frac{R_0}{R_{u+1}}\right) & \text{if } 2 \leq t \leq s \text{ and } s \geq 2 \\ \sum_{u=t-s}^{t-1} \log\left(\frac{w_{u+1}}{w_u}\frac{R_0}{R_{u+1}}\right) & \text{if } t \geq s+1 \end{cases}$$

Figure 9: Transition of the Consumption Distribution (Positive & Negative Productivity Shock)

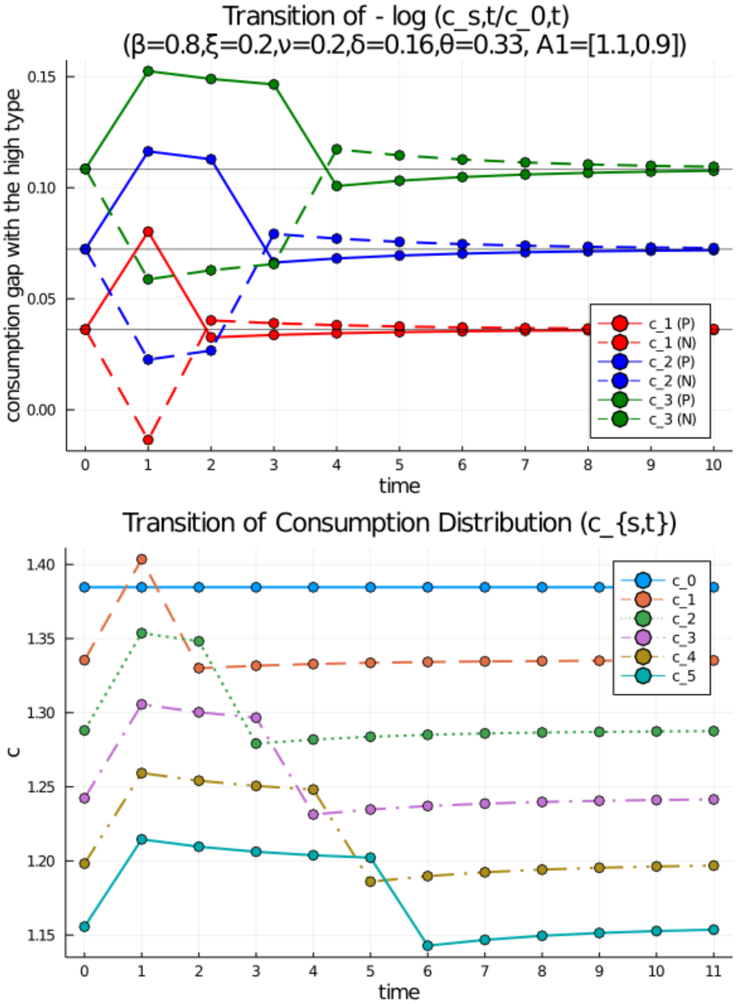


Figure 10: Transition of the Deflated Consumption Distribution (Negative Productivity Shock)

Assume $0 < \delta < 1$. *Then, the consumption gap shrinks at time 1, since* $\log\left(\frac{w_1}{w_0}\frac{R_0}{R_1}\right) < 0$. *From time 1 until time* $s \geq 2$*, the consumption gap continues to be lower than the stationary equilibrium if* $\log\left(\frac{w_1}{w_0}\frac{R_0}{R_1}\right) + \sum_{u=1}^{t-1}\log\left(\frac{w_{u+1}}{w_u}\frac{R_0}{R_{u+1}}\right) < 0$. *From time* $s+1$ *onwards, the consumption gap is higher than the stationary equilibrium if* $\sum_{u=t-s}^{t-1}\log\left(\frac{w_{u+1}}{w_u}\frac{R_0}{R_{u+1}}\right) > 0$. *Therefore, a sufficient condition for having overshooting of consumption gaps is:*

$$\frac{w_{t+1}}{w_t}\frac{R_0}{R_{t+1}} > 1 \textit{ for all } t = 1, \cdots, s.$$

Corollary 8 (Long-Run Consumption Gap after a Negative Shock)**.** *Consider a transition path after a negative productivity shock at time 1. For sufficiently large* $t \geq s+1$*, the consumption gap converges to the initial level for all* $s \geq 1$*. This means that the deflated consumption distribution in the new stationary equilibrium is the same as the initial stationary equilibrium.*

A.7 Proofs: Section 9 (Asset Pricing)

A.7.1 The Risk-Free Rate and the Risk Premium

Lemma 4 . *In the limited commitment model, the price of risk-free bonds and the risk premium at aggregate state* A^t *is given by:*

$$q_t^{B,LC}(A^t) = \mathbb{E}_t\left[\frac{1}{R_{t+1}(A^{t+1})}\right]$$
$$1 + \lambda_t^{LC}(A^t) := \frac{\mathbb{E}_t[R_{t+1}(A^{t+1})]}{\mathbb{E}_t[1/q^B(A^t)]} = \mathbb{E}_t[R_{t+1}(A^{t+1})]\,\mathbb{E}_t\left[\frac{1}{R_{t+1}(A^{t+1})}\right] > 1.$$

$\mathbb{E}_t[\cdot]$ *denotes the expectation conditional on* A^t, $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|A^t]$.

Proof. We first derive a pricing kernel that allows us to compute the price of any securities, including the price of risk-free bonds $q^B(A^t)$ and the price of risky capital $q^K(A^t)$. A pricing kernel is defined as the price of one unit of non-deflated consumption goods at time $t+1$ in a state A^{t+1} conditional on the state at t being A^t:[30]

$$Q(A^{t+1}|A^t) = \beta\frac{u'(\mathbf{c}_{t+1}(A^{t+1}))}{u'(\mathbf{c}_t(A^t))}\pi(A^{t+1}|A^t). \tag{179}$$

Since we assume a logarithmic utility function, the marginal utility of consumption is given by:

$$u'(\mathbf{c}_t) = \frac{1}{\mathbf{c}_t}.$$

[30] See Ljungqvist and Sargent (2018) p.270

In addition, in the limited commitment model, consumption of asset holders (i.e., households in a low-income state at $t+1$) follows the Euler equation:

$$\mathbf{c}_{t+1}(A^{t+1}; z_{t+1}=0) = \beta R_{t+1}(A^{t+1})\mathbf{c}_t(A^t).$$

Therefore, the pricing kernel in the limited-commitment model is given by:

$$Q(A^{t+1}|A^t) = \frac{1}{R_{t+1}(A^{t+1})}\pi(A^{t+1}|A^t). \tag{180}$$

Given this pricing kernel, we can derive the price of any securities. The price of risk-free bonds that yield one unit of consumption at $t+1$ regardless of the aggregate state A^{t+1} is given by:

$$\begin{aligned} q^B(A^t) &= \sum_{A^{t+1}|A^t} Q(A^{t+1}|A^t)\cdot 1 \\ &= \sum_{A^{t+1}|A^t} \frac{1}{R_{t+1}(A^{t+1})}\pi(A^{t+1}|A^t) \\ &= \mathbb{E}_t\left[\frac{1}{R_{t+1}(A^{t+1})}\right]. \end{aligned} \tag{181}$$

The price of risky assets that yields $R_{t+1}(A^{t+1})$ depending on the aggregate state A^{t+1} is given by:

$$\begin{aligned} q^K(A^t) &= \sum_{A^{t+1}|A^t} Q(A^{t+1}|A^t)R_{t+1}(A^{t+1}) \\ &= \sum_{A^{t+1}|A^t} \frac{1}{R_{t+1}(A^{t+1})}\pi(A^{t+1}|A^t)R_{t+1}(A^{t+1}) = 1. \end{aligned} \tag{182}$$

Now we compute the expected return on these two assets. The expected rate of return on risk-free bonds is given by:

$$\mathbb{E}_t\left[\frac{1}{q^B(A^t)}\right] = \frac{1}{\sum_{A^{t+1}|A^t}\frac{1}{R_{t+1}(A^{t+1})}\pi(A^{t+1}|A^t)} =: \frac{1}{\mathbb{E}_t[1/R_{t+1}(A^{t+1})]}. \tag{183}$$

The expected rate of return on risky assets is given by:

$$\mathbb{E}_t\left[\frac{R_{t+1}(A^{t+1})}{1}\right] = \mathbb{E}_t\left[R_{t+1}(A^{t+1})\right]. \tag{184}$$

$$\mathbb{E}_t\left[\frac{1}{R_{t+1}(A^{t+1})}\right] > \frac{1}{\mathbb{E}_t\left[R_{t+1}(A^{t+1})\right]}. \tag{185}$$

Hence, the risk premium is given by:

$$1+\lambda_t^{LC} := \frac{\mathbb{E}_t[R_{t+1}(A^{t+1})]}{\mathbb{E}_t[1/q^B(A^t)]} = \mathbb{E}_t[R_{t+1}(A^{t+1})]\,\mathbb{E}_t\left[\frac{1}{R_{t+1}(A^{t+1})}\right] > 1. \tag{186}$$

The risk premium is strictly larger than 1 because $R_{t+1}(A^{t+1})$ is a non-trivial random variable and Jensen's inequality holds with strict inequality.[31] □

Lemma 5 . *In the representative agent model, the price of risk-free bonds and the risk premium at aggregate state A^t is given by:*

$$q_t^{B,Rep}(A^t) = \mathbb{E}_t \left[\beta \frac{C_t(A^t)}{C_{t+1}(A^{t+1})} \right]$$
$$1 + \lambda_t^{Rep}(A^t) := \frac{\mathbb{E}_t[R_{t+1}(A^{t+1})]}{\mathbb{E}_t[1/q^B(A^t)]} = \mathbb{E}_t[R_{t+1}(A^{t+1})]\ \mathbb{E}_t \left[\beta \frac{C_t(A^t)}{C_{t+1}(A^{t+1})} \right].$$

Proof. As before, the pricing kernel is given by:

$$\begin{aligned} Q(A^{t+1}|A^t) &= \beta \frac{u'(\mathbf{c}_{t+1}(A^{t+1}))}{u'(\mathbf{c}_t(A^t))} \pi(A^{t+1}|A^t) \\ &= \beta \frac{C_t(A^t)}{C_{t+1}(A^{t+1})} \pi(A^{t+1}|A^t). \end{aligned} \tag{187}$$

The second line holds since consumption by a unit measure of representative households is the same as the aggregate consumption.

Then, the price of bonds is given by:

$$\begin{aligned} q^B(A^t) &= \sum_{A^{t+1}|A^t} Q(A^{t+1}|A^t) \cdot 1 \\ &= \mathbb{E}_t \left[\beta \frac{C_t(A^t)}{C_{t+1}(A^{t+1})} \right] \end{aligned} \tag{188}$$

The risk premium is given by:

$$1 + \lambda_t^{Rep} := \frac{\mathbb{E}_t[R_{t+1}(A^{t+1})]}{\mathbb{E}_t[1/q^B(A^t)]} = \mathbb{E}_t[R_{t+1}(A^{t+1})]\ \mathbb{E}_t \left[\beta \frac{C_t(A^t)}{C_{t+1}(A^{t+1})} \right] \tag{189}$$

A sequence of aggregate consumption C_t follows the Euler equation:

$$\frac{1}{C_t(A^t)} = \beta \mathbb{E}_t \left[R_{t+1}(A^{t+1}) \frac{1}{C_{t+1}(A^{t+1})} \right] \tag{190}$$

This gives:

$$\begin{aligned} 1 &= \mathbb{E}_t \left[\beta R_{t+1}(A^{t+1}) \frac{C_t(A^t)}{C_{t+1}(A^{t+1})} \right] \\ &= \mathbb{E}_t \left[R_{t+1}(A^{t+1}) \right] \mathbb{E}_t \left[\beta \frac{C_t(A^t)}{C_{t+1}(A^{t+1})} \right] + cov_t \left(R_{t+1}(A^{t+1}), \beta \frac{C_t(A^t)}{C_{t+1}(A^{t+1})} \right). \end{aligned} \tag{191}$$

If the covariance term is negative, the risk premium in the representative agent model is positive. □

[31] If $g(\cdot)$ is a convex function, $\mathbb{E}[g(X)] \geq g(\mathbb{E}[X])$, where X is a random variable. Equality holds only if $P(g(X) = a + bX) = 1$, where $a + bX$ is tangent to $g(\cdot)$ at $\mathbb{E}[X]$.

A.7.2 Economy with $\delta = 1$

Proposition 15. *Consider an economy with full depreciation of capital $\delta = 1$. Given the same amount of aggregate capital K_t, the risk-free rate, which is the inverse of the price of risk-free bonds, is lower in the limited-commitment model than a representative-agent model:*

$$\frac{1}{q_t^{B,LC}(A^t)} < \frac{1}{q_t^{B,Rep}(A^t)} \text{ for all } A^t. \tag{92}$$

The risk premium is the same in the two models and is given by:

$$1 + \lambda_t = \mathbb{E}_t[A_{t+1}^{1-\theta}]\mathbb{E}_t\left[\frac{1}{A_{t+1}^{1-\theta}}\right] > 1. \tag{93}$$

If the productivity growth rate $\frac{A_{t+1}}{A_t}$ follows an iid process, the risk premium $1+\lambda_t$ is constant over time.

Proof. Under full depreciation of capital, the interest rate is given by:

$$R_{t+1}(A^{t+1}) = \theta\left(\frac{K_{t+1}}{A_{t+1}}\right)^{\theta-1}, \tag{192}$$

where the law of motion of capital follows:

$$K_{t+1}^{LC}(A^t) = \hat{s}A_t^{1-\theta}K_t^{\theta} \tag{193}$$

$$K_{t+1}^{rep}(A^t) = \beta\theta A_t^{1-\theta}K_t^{\theta} \tag{194}$$

with $\beta\theta < \hat{s}$ under Assumption 5. This expression shows that given the same amount of capital at time t, the limited-commitment economy accumulates more capital than the representative-agent economy, implying that the interest rate is lower in the limited-commitment economy. Since the interest rate is lower in the limited-commitment economy given K_t, it implies that the risk-free rate is lower in the limited-commitment model:

$$\frac{1}{q^B(A^t)} = \frac{1}{\mathbb{E}_t\left[1/R_{t+1}(A^{t+1})\right]} \tag{195}$$

$$\text{where } R_{t+1}^{LC}(A^{t+1}) < R_{t+1}^{Rep}(A^{t+1}) \text{ given } K_t \tag{196}$$

As we derived in Section 9.1, the risk premium in the two economies is given by:

$$1 + \lambda_t^{LC} = \mathbb{E}_t[R_{t+1}^{LC}(A^{t+1})]\ \mathbb{E}_t\left[\frac{1}{R_{t+1}^{LC}(A^{t+1})}\right]$$

$$1 + \lambda_t^{Rep} = \mathbb{E}_t[R_{t+1}^{Rep}(A^{t+1})]\ \mathbb{E}_t\left[\beta\frac{C_t^{Rep}(A^t)}{C_{t+1}^{Rep}(A^{t+1})}\right] \tag{197}$$

The goods market clearing condition and the law of motion of capital pin down the aggregate consumption at time t:

$$C_t + K_{t+1} = K_t^{\theta} A_t^{1-\theta} + (1-\delta)K_t \text{ with } \delta = 1 \tag{198}$$

$$\begin{aligned} \therefore C_t^{LC} &= K_t^{\theta} A_t^{1-\theta} - K_{t+1}^{LC} \\ &= (1-\hat{s}) A_t^{1-\theta} K_t^{\theta} \end{aligned} \tag{199}$$

$$\begin{aligned} C_t^{Rep} &= K_t^{\theta} A_t^{1-\theta} - K_{t+1}^{Rep} \\ &= (1-\beta\theta) A_t^{1-\theta} K_t^{\theta} \end{aligned} \tag{200}$$

Note that the closed form of aggregate consumption in the representative-agent model (200) simplifies the second term in equation (197):

$$\begin{aligned} \mathbb{E}_t\left[\beta \frac{C_t^{Rep}(A^t)}{C_{t+1}^{Rep}(A^{t+1})}\right] &= \mathbb{E}_t\left[\beta \frac{(1-\beta\theta)A_t^{1-\theta}K_t^{\theta}}{(1-\beta\theta)A_{t+1}^{1-\theta}K_{t+1}^{\theta}}\right] \\ &= \mathbb{E}_t\left[\frac{\beta A_t^{1-\theta}K_t^{\theta}}{K_{t+1}A_{t+1}^{1-\theta}K_{t+1}^{\theta-1}}\right] \text{ where } K_{t+1} = \beta\theta A_t^{1-\theta}K_t^{\theta} \\ &= \mathbb{E}_t\left[\frac{1}{\theta A_{t+1}^{1-\theta}K_{t+1}^{\theta-1}}\right] = \mathbb{E}_t\left[\frac{1}{R_{t+1}^{Rep}(A^{t+1})}\right] \end{aligned} \tag{201}$$

This means that the second term, which is the inverse of risk-free rate, has the same expression in the two economies. The crucial property is the constant saving rate in the representative agent economy, which implies that consumption C_t is proportional to aggregate output $A_t^{1-\theta}K_t^{\theta}$ and that capital in the next period K_{t+1} is also proportional to aggregate output. Without the assumption of $\delta = 1$, this is not generally the case. Hence, the derivation below to show that the two economies have the same risk premium would not hold.

We explicitly derive the risk premium for each economy. In the limited-commitment economy, the risk premium is:

$$\begin{aligned} 1 + \lambda_t^{LC} &= \mathbb{E}_t\left[R_{t+1}^{LC}(A^{t+1})\right] \mathbb{E}_t\left[\frac{1}{R_{t+1}^{LC}(A^{t+1})}\right] \\ &= \mathbb{E}_t\left[\theta(K_{t+1}^{LC})^{\theta-1}(A_{t+1})^{1-\theta}\right] \mathbb{E}_t\left[\frac{1}{\theta(K_{t+1}^{LC})^{\theta-1}(A_{t+1})^{1-\theta}}\right] \\ &= \mathbb{E}_t\left[\theta(\hat{s}A_t^{1-\theta}K_t^{\theta})^{\theta-1}(A_{t+1})^{1-\theta}\right] \mathbb{E}_t\left[\frac{1}{\theta(\hat{s}A_t^{1-\theta}K_t^{\theta})^{\theta-1}(A_{t+1})^{1-\theta}}\right] \\ &= \mathbb{E}_t\left[A_{t+1}^{1-\theta}\right] \mathbb{E}_t\left[\frac{1}{A_{t+1}^{1-\theta}}\right] \end{aligned} \tag{202}$$

In the representative agent economy,

$$
\begin{aligned}
1+\lambda_t^{Rep} &= \mathbb{E}_t\left[R_{t+1}^{Rep}(A^{t+1})\right]\mathbb{E}_t\left[\beta\frac{C_t^{Rep}(A^t)}{C_{t+1}^{Rep}(A^{t+1})}\right] \\
&= \mathbb{E}_t\left[\theta(K_{t+1}^{Rep})^{\theta-1}(A_{t+1})^{1-\theta}\right]\mathbb{E}_t\left[\beta\frac{(1-\beta\theta)A_t^{1-\theta}K_t^{\theta}}{(1-\beta\theta)A_{t+1}^{1-\theta}(K_{t+1}^{Rep})^{\theta}}\right] \\
&= \mathbb{E}_t\left[A_{t+1}^{1-\theta}\right]\mathbb{E}_t\left[\frac{1}{A_{t+1}^{1-\theta}}\right] \qquad (203)
\end{aligned}
$$

This shows that the risk premium is the same between the two economies under full depreciation of capital. □

Saving Rate in the two models We can rewrite $\hat{s}^{LC}$ as:

$$
\begin{aligned}
\hat{s}^{LC} &= \beta\theta + \frac{\xi\beta(1-\theta)}{1-(1-\nu-\xi)\beta} - \nu\beta\theta \\
&= \beta\theta + (1-\theta)\nu\left[\frac{\xi}{\nu\left(\frac{1}{\beta}-1+\xi+\nu\right)} - \frac{\theta}{(1-\theta)\frac{1}{\beta}}\right] \qquad (204)
\end{aligned}
$$

Under $\delta = 1$, the second term is strictly positive if and only if Assumption 5 holds.

With $\delta = 1$, we also derive K_t in closed form. Since the economy has a constant saving rate,

$$
\begin{aligned}
K_{t+1} &= sA_t^{1-\theta}K_t^{\theta}, \text{ where } s \in \{\hat{s}^{LC}, s^{Rep} := \beta\theta\} \\
\log K_{t+1} &= \log s + (1-\theta)\log A_t + \theta\log K_t
\end{aligned}
$$

This implies:

$$
\begin{aligned}
\log K_2 &= \log s + (1-\theta)\log A_1 + \theta\log K_1, \text{ where } K_1 = K_0 \\
\log K_t &= (1+\theta+\cdots+\theta^{t-2})\log s + (1-\theta)\left[\sum_{\tau=1}^{t-1}\theta^{\tau-1}\log A_{t-\tau}\right] + \theta^{t-1}\log K_0 \qquad (205) \\
&= (1-\theta)\left[\sum_{\tau=1}^{t-1}\theta^{t-1-\tau}A_\tau\right] + \frac{1-\theta^{t-1}}{1-\theta}\log\hat{s} + \theta^{t-1}\log K_0 \qquad (206)
\end{aligned}
$$

Comparison given $(K_0, \{A_t\}_{t=0}^{\infty})$ The expression of K_{t+1} above implies that for any given K_t and A_t, the limited-commitment model always accumlates more capital:

$$
\log K_{t+1}^{LC} - \log K_{t+1}^{Rep} = \underbrace{\log\hat{s}^{LC} - \log s^{Rep}}_{>0 \text{ under Assumption 5}} + \theta\left(\log K_t^{LC} - \log K_t^{Rep}\right). \qquad (207)
$$

Starting from the same initial capital K_0, the difference in K_t is expressed as:

$$\log K_t^{LC} - \log K_t^{Rep} = (1 + \theta + \theta^2 + \cdots + \theta^t)(\log \hat{s}^{LC} - \log s^{Rep}), \tag{208}$$

where $(\log \hat{s}^{LC} - \log s^{Rep}) = \log \left[\frac{\xi}{1-(1-\nu-\xi)\beta} \frac{1-\theta}{\theta} + 1 - \nu \right] > 0$ under Assumption 5.

In the long-run, this will converge to:

$$\lim_{t\to\infty} (\log K_t^{LC} - \log K_t^{Rep}) = \frac{1}{1-\theta} \log \left[\frac{\hat{s}}{\beta\theta} \right], \tag{209}$$

which is consistent with the capital ratio in the steady state:

$$\frac{K^{*LC}}{K^{*Rep}} = \left(\frac{\hat{s}}{\beta\theta} \right)^{\frac{1}{1-\theta}}. \tag{210}$$

We make three remarks here: (i) Given $(K_0, \{A_t\}_{t=0}^{\infty})$, K_t^{LC} is always larger than K_t^{Rep}, (ii) For any sequence of $\{A_t\}_{t\geq 0}$, $\frac{K_t^{LC}}{K_t^{Rep}}$ monotonically converges to the ratio in the steady state $\frac{K^{*LC}}{K^{*Rep}}$, (iii) As ξ and ν approach zero ($\xi \to 0$ and $\nu \to 0$), the law of motion of capital in the limited-commitment model also approches to the one in the representative-agent model:

$$\lim_{\xi\to 0, \nu\to 0} K_t^{LC} = K_t^{Rep} \text{ for any } t \geq 1, \text{ given } K_0. \tag{211}$$

A.7.3 Endowment Economy

In an endowment economy, aggregate consumption $\{C_t(A^t)\}_{t,A^t}$ is exogenous. Households face idiosyncratic income shocks as they draw $z_t \in \{0, \zeta\}$ each period that follows a Markov transition probability as before. Households trade a state contingent Lucas tree $\sigma_{t+1}(a_0, z^{t+1}, A^{t+1})$ that delivers dividends depending on aggregate states A^{t+1} at time $t+1$. In the limited commitment model, households are subject to a tight borrowing constraint. In a standard complete market model, the borrowing constraint never binds. Since we obtain the same stochastic discount factor in the standard complete market model and the representative agent model, we call such an economy a representative agent model.

Lemma 14 shows that when aggregate consumption $C_t(A^t)$ is exogneous and follows a common stochastic process in the two models, the limited commitment model has a higher bond price:

$$q_t^{B,LC}(A^t) > q_t^{B,Rep}(A^t) \text{ for all } A^t, \tag{212}$$

and hence a lower risk-free rate.

To explicitly compute the price of bonds and the risk premium, we derive the interest rate in the two models. The interest rate $R_{t+1}(A^{t+1})$ is determined to clear goods and security (Lucas tree) markets. Then, we confirm that the price of risk-free bonds is higher in the limited-commitment model.

Lemma 14. *If the aggregate consumption is the same between the limited-commitment model and the representative agent model, the limited-commitment has a higher price of risk-free bonds and a lower-risk free rate.*

Proof. The price of risk-free bonds in the two models is given by:

$$q^{B,LC}(A^t) = \mathbb{E}_t\left[\frac{1}{R^{LC}_{t+1}(A^{t+1})}\right] = \sum_{A^{t+1}|A^t} \frac{1}{R^{LC}_{t+1}(A^{t+1})}\pi(A^{t+1}|A^t) \tag{213}$$

$$q^{B,Rep}(A^t) = \mathbb{E}_t\left[\beta\frac{C^{Rep}_t(A^t)}{C^{Rep}_{t+1}(A^{t+1})}\right] = \sum_{A^{t+1}|A^t} \beta\frac{C^{Rep}_t(A^t)}{C^{Rep}_{t+1}(A^{t+1})}\pi(A^{t+1}|A^t) \tag{214}$$

In the limited commitment model,

$$\frac{1}{R^{LC}_{t+1}(A^{t+1})} = \beta\frac{\mathbf{c}_t(A^t)}{\mathbf{c}_{t+1}(A^{t+1}; z_{t+1}=0)} > \beta\frac{C^{LC}_t(A^t)}{C^{LC}_{t+1}(A^{t+1})}, \tag{215}$$

since $\mathbf{c}_{t+1}(A^{t+1}; z_{t+1}=\zeta) > \mathbf{c}_{t+1}(A^{t+1}; z_{t+1}=0)$.

Put differently, we can show that $C^{LC}_{t+1} > \beta R^{LC}_{t+1} C^{LC}_t$:

$$\begin{aligned}
C_{t+1} &:= \sum_{s=0}^{\infty}\phi_s\mathbf{c}_{s,t+1} \\
&= \phi_0 \underbrace{\mathbf{c}_{0,t+1}}_{>\beta R_{t+1}\mathbf{c}_{0,t}} + \sum_{s=1}^{\infty}\phi_s \underbrace{\mathbf{c}_{s,t+1}}_{=\beta R_{t+1}\mathbf{c}_{s-1,t}} \\
&> \beta R_{t+1}\left[\phi_0\mathbf{c}_{0,t} + \sum_{s=1}^{\infty}\phi_s \underbrace{\mathbf{c}_{s-1,t}}_{>\mathbf{c}_{s,t}\text{under Assumption1}}\right] \\
&> \beta R_{t+1}\left[\phi_0\mathbf{c}_{0,t} + \sum_{s=1}^{\infty}\phi_s\mathbf{c}_{s,t}\right] = \beta R_{t+1}C_t
\end{aligned}$$

Therefore, if the aggregate consumption is the same between the two models for all (t, A^t), the limited-commitment has a higher price of risk-free bonds and a lower risk-free rate.

□

Lemma 15. *Consider the limited-commitment model with exogenous aggregate endowment $\{C_t(A^t)\}_{t,A^t}$. Lucas tree yields α fraction of aggregate endowment at all t and A^t and is priced at $q^{LT}_t(A^t)$ at state A^t. Assume that parameters satisfy:*

$$\frac{\alpha}{(1-\alpha)\left(\frac{1}{\beta}-1\right)} < \frac{\xi}{\nu\left(\frac{1}{\beta}-1+\xi+\nu\right)}. \tag{94}$$

The interest rate, defined as the return on Lucas tree, is given by:

$$\begin{aligned} R_{t+1}(A^{t+1}) &:= \frac{\alpha C_{t+1}(A^{t+1}) + q_{t+1}^{LT}(A^{t+1})}{q_t^{LT}(A^t)} \\ &= \frac{\alpha+\bar{q}}{\bar{q}} \frac{C_{t+1}(A^{t+1})}{C_t(A^t)} \\ \textit{where } \bar{q} &:= \frac{\xi(1-\alpha)+\beta(1-\nu)\alpha(\xi+\nu+\frac{1}{\beta}-1)}{[1-\beta(1-\nu)](\xi+\nu+\frac{1}{\beta}-1)}. \end{aligned} \tag{216}$$

Under Assumption (94), Assumption 3 is satisfied for all (t, A^t, A_{t+1})*:*

$$\beta R_{t+1}(A^{t+1}) < \frac{w_{t+1}(A^{t+1})}{w_t(A^t)} \textit{ for all } t, A^t, A_{t+1}. \tag{217}$$

Proof. In the limited-commitment model, the price of Lucas tree is determined to clear the goods market and the asset market:

$$\sum_{z^t} \int \pi(z^t)\mathbf{c}_t(a_0, z^t, A^t) d\Phi(a_0, z_0) = C_t(A^t) \tag{218}$$

$$\sum_{z^t} \int \pi(z^t)\sigma_t(a_0, z^t, A^t) d\Phi(a_0, z_0) = 1, \tag{219}$$

where $\sigma_t(a_0, z^t, A^t)$ represents the share of Lucas tree held by households with state (a_0, z^t). In an equilibrium, high-income households receive labor income $[1-\alpha]C_t(A^T)\zeta$ but no dividend, and low-income households receive dividends from a Lucas tree $\alpha C_t(A^t)\sigma_t(a_0, z^t, A^t)$. We assume that a labor share, $1-\alpha \in (0,1)$, is constant over time and across states.[32]

We derive $\{q_t^{LT}(A^t), R_t(A^t)\}$, where $q_t^{LT}(A^t)$ is the price of Lucas tree and $R_t(A^t)$ is the interest rate, that are consistent with the equilibrium allocation in the production economy. In the endowment economy, households purchase state-contingent shares of Lucas tree. The budget constraint is given by:

$$\begin{aligned} &\mathbf{c}_t(a_0, z^t, A^t) + \sum_{A_{t+1}} \sum_{z_{t+1}} q_t^{LT}(A^t)\sigma_{t+1}(a_0, z^{t+1}, A^{t+1}) \\ &= \underbrace{[1-\alpha]C_t(A^t)z_t}_{\text{labor income}} + [\underbrace{\alpha C_t(A^t)}_{\text{dividends}} + q_t^{LT}(A^t)]\sigma_t(a_0, z^t, A^t) \end{aligned} \tag{220}$$

[32] The share of labor income in the total resources available in the economy is also constant in a production economy with $\delta = 1$. However, in a production economy with $\delta < 1$, the share:

$$\frac{w_t L}{A_t^{1-\theta}K_t^\theta + (1-\delta)K_t} = \frac{(1-\theta)A_t^{1-\theta}K_t^\theta}{A_t^{1-\theta}K_t^\theta + (1-\delta)K_t}$$

is not constant. Since one of the key assumptions in Krueger and Lustig (2009) is violated, the conjecture (the same risk premium) may not be true in such an economy. We will come back to this point later.

As a counterpart of allocation in the production economy with aggregate shocks (22)–(24), the conjectured allocation is expressed as:

$$\begin{aligned}
\mathbf{c}_{0,t} &= \left(\frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}\right)[1-\alpha]C_t(A^t)\zeta \\
\mathbf{c}_{s,t} &= [1-(1-\nu)\beta]\,[\alpha C_t(A^t)+q_t^{LT}(A^t)]\sigma_{s,t} \qquad \text{for } s=1,2,\cdots \\
\mathbf{c}_{s+1,t+1} &= \beta R_{t+1}\mathbf{c}_{s,t} \qquad \text{for } s=0,1,\cdots \\
\sigma_{0,t} &= 0 \\
q_t^{LT}(A^t)\sigma_{1,t+1} &= \left(\frac{\beta}{1-(1-\nu-\xi)\beta}\right)[1-\alpha]C_t(A^t)\zeta \qquad (221)\\
q_t^{LT}(A^t)\sigma_{s+1,t+1} &= \beta[\alpha C_t(A^t)+q_t^{LT}(A^t)]\sigma_{s,t} \qquad \text{for } s=1,2,\cdots \qquad (222)
\end{aligned}$$

For this allocation to be optimal, we need to verify that equation (217) (Assumption 3) is satisfied:

$$\beta R_{t+1}(A^{t+1}) < \frac{w_{t+1}(A^{t+1})}{w_t(A^t)} \text{ for all } t, A^t, A_{t+1}$$

As we will see, the equilibrium interest rate is given by $R_{t+1}(A^{t+1}) = \frac{\alpha+\bar{q}}{\bar{q}}\frac{C_{t+1}}{C_t}$, and the growth rate in labor income is given by $\frac{w_{t+1}}{w_t} = \frac{(1-\alpha)C_{t+1}}{(1-\alpha)C_t}$. Under the assumption on parameters ($\beta\frac{\alpha+\bar{q}}{\bar{q}} < 1$), which is equivalent to (94), the condition (217) is satisfied.

We now derive the equilibrium price of Lucas tree and the equilibrium interest rate in the limited-commitment economy. Consider that C_t is realized at time t. A market clearing condition for the share of Lucas tree at $t+1$ is:

$$\begin{aligned}
1 &= \sum_{s=1}^{\infty}\phi_s\sigma_{s,t+1} \\
&= \frac{\nu\xi}{\xi+\nu}\sigma_{1,t+1} + \sum_{s=2}^{\infty}\frac{\nu\xi}{\xi+\nu}(1-\nu)^{s-1}\sigma_{s,t+1} \qquad (223)
\end{aligned}$$

From equations (221)–(222), we know:

$$\begin{aligned}
\sigma_{1,t+1} &= \left(\frac{\beta}{1-(1-\nu-\xi)\beta}\right)\frac{[1-\alpha]C_t(A^t)}{q_t^{LT}(A^t)}\zeta \qquad (224)\\
\sigma_{s,t+1} &= \beta\frac{\alpha C_t(A^t)+q_t^{LT}(A^t)}{q_t^{LT}(A^t)}\sigma_{s-1,t} \qquad \text{for } s\geq 2 \qquad (225)
\end{aligned}$$

By substituing into (223), we have:

$$\begin{aligned}
1 &= \frac{\nu\xi}{\xi+\nu}\left(\frac{\beta}{1-(1-\nu-\xi)\beta}\right)\frac{[1-\alpha]C_t(A^t)}{q_t^{LT}(A^t)}\zeta \\
&\quad + \beta\left[\frac{\alpha C_t(A^t)+q_t^{LT}(A^t)}{q_t^{LT}(A^t)}\right]\underbrace{\sum_{s=2}^{\infty}\frac{\nu\xi}{\xi+\nu}(1-\nu)^{s-1}\sigma_{s-1,t}}_{=(1-\nu)\sum_{s=1}^{\infty}\frac{\nu\xi}{\xi+\nu}(1-\nu)^{s-1}\sigma_{s,t}=1-\nu} \qquad (226)
\end{aligned}$$

By soliving this, we obtain:

$$q_t^{LT} = \underbrace{\frac{\xi(1-\alpha) + \beta(1-\nu)\alpha(\xi + \nu + \frac{1}{\beta} - 1)}{[1-\beta(1-\nu)](\xi + \nu + \frac{1}{\beta} - 1)}}_{\text{denote as } \bar{q}} C_t \tag{227}$$

Given $q_t^{LT}(A^t)$, $R_{t+1}(A^{t+1})$ is defined as:

$$\begin{aligned} R_{t+1}(A^{t+1}) &:= \frac{\alpha C_{t+1} + q_{t+1}^{LT}}{q_t^{LT}} \\ &= \frac{\alpha + \bar{q}}{\bar{q}} \frac{C_{t+1}}{C_t} \\ \text{where } \bar{q} &:= \frac{\xi(1-\alpha) + \beta(1-\nu)\alpha(\xi + \nu + \frac{1}{\beta} - 1)}{[1-\beta(1-\nu)](\xi + \nu + \frac{1}{\beta} - 1)} \end{aligned}$$

□

Stationary Equilibrium in the LC model We verify that $R_t(A^t) = R^*$ if $C_t = C_{t-1}$, where R^* is the interest rate in a stationary equilibrium that we derive in the next proposition. We assume parameter restirictions (94) that correspond to Assumption 5 in the production economy. Under this assumption, we obtain a partial insurance equilibirum in a stationary equilibrium, in which households do not save for a high-income state and the Euler equation between the current state and a future high-income state does not hold. We verify that the stationary equilibrium interest rate satisfies $\beta R^* < 1$.

Proposition 21. *Consider the limited-commitment model with exogenous aggregate endowment $\{C_t(A^t)\}_{t,A^t}$. Assume that parameters satisfy (94):*

$$\frac{\alpha}{(1-\alpha)\left(\frac{1}{\beta} - 1\right)} < \frac{\xi}{\nu\left(\frac{1}{\beta} - 1 + \xi + \nu\right)}.$$

In the stationary equilibrium ($C_t = C$ for all t), the interest rate is given by:

$$R^* = \frac{\xi(1-\alpha) + \alpha(\xi + \nu + \frac{1}{\beta} - 1)}{\xi(1-\alpha) + \beta\alpha(1-\nu)(\xi + \nu + \frac{1}{\beta} - 1)}, \tag{228}$$

which is equal to $\frac{\alpha+\bar{q}}{\bar{q}}$. Under Assumption (94),

$$\beta R^* < 1.$$

Hence, the economy is in a partial insurance equilibirum in a stationary equilibirum.

Proof. In the stationary equilibrium ($C_t(A^t) = C$ for all t and A^t), the conjectured allocation becomes:

$$
\begin{aligned}
\mathbf{c}_0 &= \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}[1-\alpha]C\zeta \\
\mathbf{c}_{s,t} &= [1-(1-\nu)\beta]\,[\alpha C + q^{LT}]\sigma_s \qquad \text{for } s = 1, 2, \cdots \\
\mathbf{c}_{s+1} &= \beta R \mathbf{c}_s \qquad\qquad\qquad\qquad\qquad\qquad \text{for } s = 0, 1, \cdots \\
\sigma_0 &= 0 \\
\sigma_1 &= \left(\frac{\beta}{1-(1-\nu-\xi)\beta}\right)\frac{[1-\alpha]C\zeta}{q^{LT}} \\
\sigma_{s+1} &= \beta R \sigma_s \qquad\qquad\qquad\qquad\qquad\qquad \text{for } s = 1, 2, \cdots
\end{aligned}
$$

The market clearing condition for Lucas tree is:

$$
\begin{aligned}
\sum_{s=1}^{\infty}\phi_s\sigma_s := \sum_{s=1}^{\infty}\frac{\nu\xi}{\xi+\nu}(1-\nu)^{s-1}(\beta R)^{s-1}\sigma_1 = 1 \qquad (229)\\
\Leftrightarrow\ \frac{\nu\xi}{\xi+\nu}\,\frac{1}{1-(1-\nu)\beta R}\left(\frac{\beta}{1-(1-\nu-\xi)\beta}\right)\frac{[1-\alpha]C\zeta}{q^{LT}} = 1
\end{aligned}
$$

By solving this equation, while using:

$$
\begin{aligned}
R = \frac{\alpha C + q^{LT}}{q^{LT}} \Leftrightarrow\ q^{LT} = \frac{\alpha C}{R-1}, \qquad (230)\\
\text{and } \frac{\nu}{\xi+\nu}\zeta \equiv 1,
\end{aligned}
$$

we obtain:

$$
R^* = \frac{\xi(1-\alpha) + \alpha(\xi+\nu+\frac{1}{\beta}-1)}{\xi(1-\alpha) + \beta\alpha(1-\nu)(\xi+\nu+\frac{1}{\beta}-1)}, \qquad (231)
$$

which is strictly larger than 1 since $0 < \beta(1-\nu) < 1$. $\beta R^* < 1$ is equivalent to (94):

$$
\frac{\alpha}{(1-\alpha)\left(\frac{1}{\beta}-1\right)} < \frac{\xi}{\nu\left(\frac{1}{\beta}-1+\xi+\nu\right)},
$$

which we assume in the endowment economy.

□

The Interest Rate in the Representative Agent Economy In the representative agent model, the interest rate, defined as the return on Lucas tree as before, is given by:

$$
\begin{aligned}
R^{Rep}_{t+1}(A^{t+1}|A^t) &:= \frac{\alpha C_{t+1}(A^{t+1}) + q^{\mathrm{LT}}_{t+1}(A^{t+1})}{q^{\mathrm{LT}}_t(A^t)} \\
&= \frac{1}{\beta}\frac{C_{t+1}(A^{t+1})}{C_t(A^t)}. \qquad (232)
\end{aligned}
$$

The derivation is standard as in Ljungqvist and Sargent (2018). The price of Lucas tree, which yields α fraction of aggregate endowment, follows:

$$q_t^{\text{Lucas}}(A^t) = \sum_{A^{t+1}} Q(A^{t+1}|A^t)\left[\alpha C_{t+1}(A^{t+1}) + q_{t+1}^{\text{Lucas}}(A^{t+1})\right], \tag{233}$$

$$\text{where } Q(A^{t+1}|A^t) = \beta \frac{C_t(A^t)}{C_{t+1}(A^{t+1})}\pi(A^{t+1}|A^t)$$

Using recursion of the equation, the price of Lucas tree is expressed as:

$$q_t^{\text{Lucas}}(A^t) = \frac{1}{u'(C_t(A^t))}\left[\mathbb{E}_t \sum_{j=1}^{\infty} \beta^j u'(C_{t+j}(A^{t+j}))\alpha C_{t+j}(A^{t+j}) + \mathbb{E}_t \lim_{k\to\infty} \beta^k u'(C_{t+k}(A^{t+k}))q_{t+k}(A^{t+k})\right]. \tag{234}$$

The last term must be zero to clear the market. Under a logarithmic utility function, which gives $u'(C_{t+j})C_{t+j} = 1$, the price of Lucas tree is proportional to the current aggregate endowment:

$$q_t^{\text{Lucas}}(A^t) = \frac{\beta}{1-\beta}\alpha C_t(A^t). \tag{235}$$

Then, the interest rate is given by:

$$R_{t+1}^{Rep}(A^{t+1}|A^t) = \frac{1}{\beta}\frac{C_{t+1}(A^{t+1})}{C_t(A^t)} \tag{236}$$

Proposition 16. *Consider an endowment economy with exogenous aggregate endowment $\{C_t(A^t)\}_{t,A^t}$. Lucas tree yields α fraction of aggregate endowment at all ($t+1$, A^{t+1}) and is priced at $q_t^{LT}(A^t)$ at state A^t. Assume that parameters satisfy:*

$$\frac{\alpha}{(1-\alpha)\left(\frac{1}{\beta}-1\right)} < \frac{\xi}{\nu\left(\frac{1}{\beta}-1+\xi+\nu\right)}, \tag{94}$$

so that Assumption 3 is satisfied. The risk-free rate, which is the inverse of the price of risk-free bonds, is lower in the limited-commitment model:

$$\frac{1}{q_t^{B,LC}(A^t)} < \frac{1}{q_t^{B,Rep}(A^t)} \textit{ for all } A^t. \tag{95}$$

The risk premium is the same in the two models and is given by:

$$1+\lambda_t = \mathbb{E}_t[C_{t+1}(A^{t+1})]\mathbb{E}_t\left[\frac{1}{C_{t+1}(A^{t+1})}\right] > 1 \tag{96}$$

If the growth rate of exogenous consumption $\frac{C_{t+1}}{C_t}$ follows an iid process, the risk premium $1+\lambda_t$ is constant over time.

Proof. In the representative agent model (complete market model), we have seen that under logarithmic utility, the interest rate is given by (232):

$$R_{t+1}^{Rep}(A^{t+1}) = \frac{1}{\beta}\frac{C_{t+1}(A^{t+1})}{C_t(A^t)}$$

In the limited commitment model, the interest rate is given by (216):

$$R_{t+1}^{LC}(A^{t+1}) = \frac{\alpha+\bar{q}}{\bar{q}}\frac{C_{t+1}(A^{t+1})}{C_t(A^t)}. \tag{237}$$

The interest rates are proportional to consumption growth $\frac{C_{t+1}(A^{t+1})}{C_t(A^t)}$ in the two economies. In a partial insurance equilibrium, we assume that parameters $(\xi, \nu, \beta, \alpha)$ deliver $\beta R^* < 1$ in the stationary equilibrium, which implies:

$$\beta R^* < 1 \Leftrightarrow \beta\frac{\alpha+\bar{q}}{\bar{q}} < 1 \tag{238}$$

$$\therefore R_{t+1}^{LC}(A^{t+1}) < R_{t+1}^{Rep}(A^{t+1}). \tag{239}$$

This confirms that the limited commitment model has a lower interest rate. The price of risk-free bond is given by:

$$q_t^{LC}(A^t) = \mathbb{E}_t\left[\frac{1}{R_{t+1}^{LC}(A^{t+1})}\right] = \mathbb{E}_t\left[\frac{\bar{q}}{\alpha+\bar{q}}\frac{C_t(A^t)}{C_{t+1}(A^{t+1})}\right] \tag{240}$$

$$q_t^{Rep}(A^t) = \mathbb{E}_t\left[\beta\frac{C_t(A^t)}{C_{t+1}(A^{t+1})}\right] \tag{241}$$

Since $\beta < \frac{\bar{q}}{\alpha+\bar{q}}$, the limited-commitment model has a higher price of risk-free bonds and a lower risk-free rate.

Remember that the risk premium in the two economies is given by:

$$1+\lambda_t^{Rep} = \mathbb{E}_t[R_{t+1}^{Rep}(A^{t+1})]\ \mathbb{E}_t\left[\beta\frac{C_t(A^t)}{C_{t+1}(A^{t+1})}\right]$$
$$1+\lambda_t^{LC} = \mathbb{E}_t[R_{t+1}^{LC}(A^{t+1})]\ \mathbb{E}_t\left[\frac{1}{R_{t+1}^{LC}(A^{t+1})}\right].$$

Given the equilibrium interest rates, the risk premium in the two economies is given by:

$$1+\lambda_t^{Rep} = \mathbb{E}_t[C_{t+1}(A^{t+1})]\mathbb{E}_t\left[\frac{1}{C_{t+1}(A^{t+1})}\right] > 1 \tag{242}$$

$$1+\lambda_t^{LC} = \mathbb{E}_t[C_{t+1}(A^{t+1})]\mathbb{E}_t\left[\frac{1}{C_{t+1}(A^{t+1})}\right] > 1 \tag{243}$$

Therefore, we see that the two economies have the same risk premium. If the growth rate of exogenous consumption $\frac{C_{t+1}}{C_t}$ follows an *iid* process, the risk premium $1+\lambda_t$ is constant over time. □

A.7.4 Intuition

In the endowment economy and the production economy with $\delta = 1$, the two properties hold: (i) Income from capital (αC_t in the endowment economy and $\theta K_t^{\theta} A_t^{1-\theta}$ in the production economy with $\delta = 1$) is proportional to aggregate consumption C_t. Because of the constant saving rate, consumption of unconstrained agents are proportional to total resources. (ii) Future aggregate shocks ($\{A_{t+2}, A_{t+3}, \cdots\}$) do not impact the return on risky assets (a Lucas tree or capital) due to a logarithmic utility function and the fact that capital fully depreciates by time $t+2$. Since dividend at $t+1$ is proportional to exogenous shocks (C_{t+1} or $A_{t+1}^{1-\theta}$), R_{t+1} is also proportional to exogenous shocks:

$$R_{t+1} \propto \begin{cases} \frac{C_{t+1}}{C_t} & \text{in an endowment economy} \\ \left(\frac{K_{t+1}}{A_{t+1}}\right)^{\theta-1} & \text{in a production economy with } \delta = 1 \end{cases} \tag{244}$$

These two properties lead to the same risk premium. The Euler equation holds for agents who are unconstrained in their trade of securities (representative agents or low-income households in the LC model):

$$\frac{1}{c_t} = \beta \mathbb{E}_t \left[R_{t+1} \frac{1}{c_{t+1}} \right]. \tag{245}$$

Generally, this Euler equation holds in expectation but not state-by-state. However, in the endowment economy or in the production economy with $\delta = 1$, since income from capital is always proportional to total income of unconstraint agents, their income effect and substitution effect cancel out, i.e., high aggregate productivity increases the return on risky assets but decreases the marginal utility of consumption by the same rate. Hence, $\frac{R_{t+1}}{C_{t+1}}$ is constant across states, and

$$\frac{1}{c_t} = \beta R_{t+1} \frac{1}{c_{t+1}} \tag{246}$$

holds state-by-state. Then, we can rewrite the risk premium in both models as:[33]

$$1 + \lambda_t = \mathbb{E}_t \left[\frac{c_{t+1}}{\beta c_t} \right] \mathbb{E}_t \left[\frac{\beta c_t}{c_{t+1}} \right], \tag{247}$$

where c_t is consumption of unconstrained agents. Since c_t is proportional to aggregate shocks (C_t or $A_t^{1-\theta}$), the risk premium is the same in the two economies.

[33]Remember that the risk premium is given by:

$$1 + \lambda_t^{LC} = \mathbb{E}_t[R_{t+1}^{LC}(A^{t+1})] \, \mathbb{E}_t \left[\frac{1}{R_{t+1}^{LC}(A^{t+1})} \right]$$
$$1 + \lambda_t^{Rep} = \mathbb{E}_t[R_{t+1}^{Rep}(A^{t+1})] \, \mathbb{E}_t \left[\beta \frac{C_t^{Rep}(A^t)}{C_{t+1}^{Rep}(A^{t+1})} \right]$$

If $\delta \neq 1$, this argument breaks down. First, income from capital, $R_{t+1}K_{t+1} = \theta K_{t+1}^{\theta} A_{t+1}^{1-\theta} + (1-\delta)K_{t+1}$, is not proportional to total resources in the economy, $K_{t+1}^{\theta}A_{t+1}^{1-\theta} + (1-\delta)K_{t+1}$, and the representative agent model does not have a constant saving rate. Second, since the interest rate, $R_{t+1} = \theta K_{t+1}^{\theta-1}A_{t+1}^{1-\theta} + 1 - \delta$ is not proportional to the stochastic variable $A_{t+1}^{1-\theta}$, the risk premium is impacted by K_{t+1}, which follows a different law of motion of capital in the two models.

The following proposition clarifies the logic that consumption of unconstrained agents being proportional to total resources in the economy and proportional return on risky assets in the two models imply multiplicative stochastic discount factors and the same risk premium. The representative-agent production economy with $\delta \neq 1$ does not have a constant saving rate. Besides, the return on capital depends on aggregate capital at $t+1$ due to the undepreciated part of the capital. Since the two models have a different law of motion of capital, the return on capital will not be proportional between the two models.

Proposition 17. *Consider the representative agent model and the limited commitment model with aggregate shocks. Exogenous shocks, $\{A_t\}_{t\geq 0}$, follow a common stochastic process with probablity of A_{t+1} given by $\pi(A_{t+1}|A^t)$ that potentially depends on the entire history $A^t = (A_0, A_1, \cdots, A_t)$. In an endowment economy, the total resources available in the economy, denoted by Υ_t, is exogenous and depends only on A_t. In a production economy, Υ_t is the sum of produced output, $K_t^{\theta}A_t^{1-\theta}$, and undepreciated capital, $(1-\delta)K_t$, where K_t depends on the history of aggregate shocks A^{t-1} and the initial capital K_0. Thus, we denote Υ_t as a function of A^t.*

$$\Upsilon_t(A^t) = \begin{cases} C_t(A_t) & \text{in an endowment economy} \\ K_t^{\theta}A_t^{1-\theta} + (1-\delta)K_t & \text{in a production economy} \end{cases} \tag{248}$$

A stochastic discount factor $m_{t,t+1}(A^{t+1})$ satisfies:

$$\mathbb{E}_t \left[m_{t,t+1}(A^{t+1}) R_{t,1}^{j}(A^{t+1}) \right] = 1 \tag{249}$$

for any asset j with one-period return $R_{t,1}^{j}(A^{t+1})$. In our setup with logarithmic utility, the stochastic discount factor is given by:

$$m_{t,t+1}(A^{t+1}) = \beta \frac{c_t}{c_{t+1}} \tag{250}$$

where c_t is consumption of unconstrained agents (representative households in the Rep model or low-income households in the LC model). If unconstrained agents consume a non-random fraction (determined at t) of total resources, the stochastic discount factor is proportional to $\frac{\Upsilon_t}{\Upsilon_{t+1}}$ and satisfies:

$$m_{t,t+1}(A^{t+1}) := \beta \frac{c_t}{c_{t+1}} = \gamma_t \frac{\Upsilon_t(A^t)}{\Upsilon_{t+1}(A^{t+1})}, \tag{100}$$

where non-random variable γ_t is potentially time-varying but does not depend on A_{t+1}.

First, in the endowment economy and the production economy with $\delta = 1$, equation (100) is satisfied in both models.

Second, in the endowment economy and the production economy with $\delta = 1$, $\frac{\Upsilon_t}{\Upsilon_{t+1}}$ is proportional between the two models, meaning there is a non-random variable γ'_t satisfying:

$$\frac{\Upsilon_t^{LC}(A^t)}{\Upsilon_{t+1}^{LC}(A^{t+1})} = \gamma'_t \frac{\Upsilon_t^{Rep}(A^t)}{\Upsilon_{t+1}^{Rep}(A^{t+1})}, \tag{101}$$

where $\gamma'_t = 1$ in the endowment economy.

Then, (100) and (101) imply that $\frac{m^{LC}_{t,t+1}}{m^{Rep}_{t,t+1}}$ is non-random at t, meaning there exists another non-random variable γ''_t that does not depend on A_{t+1} and satisfies:

$$\frac{m^{LC}_{t,t+1}(A^{t+1})}{m^{Rep}_{t,t+1}(A^{t+1})} = \gamma''_t \left(:= \gamma'_t \frac{\gamma_t^{LC}}{\gamma_t^{Rep}} \right) \tag{102}$$

In this case, the two models have the same risk premium.

Proof. First, we show that equation (100) holds in both models. Consider the representative agent model. In the endowment economy, we have:

$$c_t = C_t = \Upsilon_t, \tag{251}$$

meaning consumption of unconstrained agents (c_t), aggregate consumption (C_t), and total resources available in the economy (Υ_t) are all identical. In the production economy with $\delta = 1$, the constant saving rate implies that:

$$\begin{aligned} C_t^{Rep} &= (1-\beta\theta)A_t^{1-\theta}K_t^{\theta} \\ &= (1-\beta\theta)\Upsilon_t. \end{aligned} \tag{252}$$

In these two cases, equation (100) holds.

In the limited commitment model, low-income agents at $t+1$ are unconstrained and consume a constant fraction, $1-(1-\nu)\beta$, of return from savings. In the endowment economy, low-income agents with $\sigma_{s,t}$ share of Lucas tree consume:

$$\mathbf{c}_{s,t}(\sigma_{s,t}) = [1-(1-\nu)\beta][\alpha C_t(A_t) + q_t^{LT}(A_t)]\sigma_{s,t}, \tag{253}$$

where $q_t^{LT}(A_t) = \bar{q}C_t(A_t)$.

In the production economy, low-income agents with assets $\mathbf{a}_t(z^t, A^t)$ consume:

$$\mathbf{c}_{s,t}(z^t, A^t) = [1-(1-\nu)\beta]R_t(A^t)\mathbf{a}_t(z^t, A^t) \tag{254}$$

where $R_t(A^t) = \theta K_t^{\theta-1}A_t^{1-\theta} + 1 - \delta$.

We see that consumption of low-income agents are proportional to C_t in the endowment economy and $A_t^{1-\theta}$ in the production economy if $\delta = 1$.

Put differently, from the Euler equation of unconstrained agents, we have:

$$m_{t,t+1}^{LC} := \beta \frac{c_t}{c_{t+1}(z_{t+1} = 0)} = \frac{1}{R_{t+1}^{LC}} \tag{255}$$

We have seen in equation (216) that in the endowment economy,

$$R_{t+1} = \frac{\alpha + \bar{q}}{\bar{q}} \frac{C_{t+1}}{C_t} \tag{256}$$

holds. In the production economy with $\delta = 1$,

$$\begin{aligned} R_{t+1}^{LC} &= \theta K_{t+1}^{\theta-1} A_{t+1}^{1-\theta} \\ &= \theta \frac{K_{t+1}^{\theta} A_{t+1}^{1-\theta}}{\hat{s}^{LC} K_t^{\theta} A_t^{1-\theta}} \\ &= \frac{\theta}{\hat{s}^{LC}} \frac{\Upsilon_{t+1}}{\Upsilon_t} \end{aligned} \tag{257}$$

Hence, equation (100) holds in both cases in the limited commitment model.

Now we proceed to the second half of the proposition. In the endowment economy, the proof follows Theorem 4.2 in Krueger and Lustig (2010). The risk premium is defined as:

$$1 + \lambda_t := \frac{\mathbb{E}_t \left[R_{t,1}[\{e_{t+k}\}] \right]}{R_{t,1}[1]}, \tag{258}$$

where $R_{t,1}[\{e_{t+k}\}]$ is the one-period return of holding a claim $\{e_{t+k}\}_{k\geq 1}$ from time t to $t+1$. The Lucas tree yields α fraction of endowment in the economy, so $e_{t+k} = \alpha \Upsilon_{t+k}$ for all $k \geq 1$ in both models. Their derivation shows that the risk premium can be expressed as a weighted sum of risk premia on strips:

$$1 + \lambda_t = \sum_{k=1}^{\infty} \frac{\omega_k}{1/\mathbb{E}_t[m_{t,t+1}]} \mathbb{E}_t \left[R_{t,1}[e_{t+k}] \right], \tag{259}$$

$$\text{where } \omega_k = \frac{\mathbb{E}_t[m_{t,t+k} e_{t+k}]}{\sum_{j=1}^{\infty} \mathbb{E}_t[m_{t,t+j} e_{t+j}]}$$

If $m_{t,t+1}^{LC} = \gamma_t'' m_{t,t+1}^{Rep}$, where γ_t'' is a non-random multiplicative term,

$$\begin{aligned} 1 + \lambda_t^{LC} &:= \frac{\mathbb{E}_t \left[R_{t,1}^{LC}[e_{t+k}] \right]}{1/\mathbb{E}_t[m_{t,t+1}^{LC}]} = \frac{\mathbb{E}_t \left[\frac{E_{t+1}[m_{t+1,t+k}^{LC} e_{t+k}]}{E_t[m_{t,t+k}^{LC} e_{t+k}]} \right]}{1/\mathbb{E}_t[m_{t,t+1}^{LC}]} \\ &= \frac{\mathbb{E}_t \left[\frac{E_{t+1}[\gamma_{t+1}'' \cdots \gamma_{t+k-1}'' m_{t+1,t+k}^{Rep} e_{t+k}]}{E_t[\gamma_t'' \gamma_{t+1}'' \cdots \gamma_{t+k-1}'' m_{t,t+k}^{Rep} e_{t+k}]} \right]}{1/\mathbb{E}_t[\gamma_t m_{t,t+1}^{Rep}]} = \frac{\mathbb{E}_t \left[R_{t,1}^{Rep}[e_{t+k}] \right]}{1/\mathbb{E}_t[m_{t,t+1}^{Rep}]} =: 1 + \lambda_t^{Rep} \end{aligned} \tag{260}$$

holds for all $k \geq 1$, where $m_{t,t+k} = m_{t,t+1}m_{t+1,t+2}\cdots m_{t+k-1,t+k}$. Since ω_k is also the same between the two models, the two models have the same risk premium.

In the production economy, one unit of capital purchased at t yields $\theta K_{t+1}^{\theta-1}A_{t+1}^{1-\theta}+1-\delta$ at time $t+1$. By substituting $e_{t+1} = \theta K_{t+1}^{\theta-1}A_{t+1}^{1-\theta} + 1 - \delta$ and $e_{t+k} = 0$ for all $k \geq 2$, the risk premium is given by:

$$
\begin{aligned}
1+\lambda_t &:= \frac{\mathbb{E}_t\left[R_{t,1}[\{e_{t+k}\}]\right]}{R_{t,1}[1]} \\
&= \frac{\mathbb{E}_t\left[\frac{\mathbb{E}_{t+1}\left[\theta K_{t+1}^{\theta-1}A_{t+1}^{1-\theta}+1-\delta\right]}{\mathbb{E}_t\left[m_{t,t+1}(\theta K_{t+1}^{\theta-1}A_{t+1}^{1-\theta}+1-\delta)\right]}\right]}{1/\mathbb{E}_t\left[m_{t,t+1}\right]} \\
&= \frac{\mathbb{E}_t[\theta K_{t+1}^{\theta-1}A_{t+1}^{1-\theta}+1-\delta]\mathbb{E}_t[m_{t,t+1}]}{\mathbb{E}_t[(\theta K_{t+1}^{\theta-1}A_{t+1}^{1-\theta}+1-\delta)m_{t,t+1}]}
\end{aligned} \tag{261}
$$

If $\delta = 1$, this is simplified to:

$$
1+\lambda_t = \frac{\mathbb{E}_t[A_{t+1}^{1-\theta}]\mathbb{E}_t[m_{t,t+1}]}{\mathbb{E}_t[A_{t+1}^{1-\theta}m_{t,t+1}]}, \tag{262}
$$

where we use the fact that K_{t+1} is determined at t and is canceled out in the previous equation. If $m_{t,t+1}^{LC} = \gamma_t'' m_{t,t+1}^{Rep}$ holds, the representative agent model and the limited commitment model have the same risk premium.

Note that if $\delta \neq 1$, the risk premium depends on K_{t+1}, which follows a different law of motion in the two models. Hence, even multiplicative stochastic discount factors would not imply the same risk premium under $\delta \neq 1$.

In equations (260) and (262), we have seen that $m_{t,t+1}^{LC} = \gamma_t'' m_{t,t+1}^{Rep}$ imply $1+\lambda_t^{LC} = 1+\lambda_t^{Rep}$. If γ_t'' depends on A_{t+1}, then the term γ_t'' does not cancel out, so the risk premia are generally different across the two models. □

B Additional Discussions

B.1 Uniqueness of the Households' Optimal Choice

B.1.1 Necessity of the Kuhn-Tucker Condition

In the environment in this paper with finitely many event histories at any finite period t, it should be straightforward to extend the theorem on page 249 in Luenberger (1969). It states a generalized Kuhn-Tucker Theorem for a real-valued functional $f : X \to \mathbb{R}$ and an inequality constraint $G(x) \leq \theta$, where X is a vector space and G is a mapping from X to a normed space Z. This theorem is stated for a deterministic case, but our environment has a finite number of states at any given time t. Another way is to apply Blot (2009). This paper establishes a Pontryagin principle for a stochastic infinite-horizon discrete-time problem.

B.1.2 Necessity of the Transversality Condition

The proposed allocation satisfies both the sufficient TVC in Proposition 1:

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[\frac{R_t(A^t)}{\mathbf{c}_t^*(z^t, A^t)}\mathbf{a}_t^*(z^t, A^t)\right] = 0 \tag{263}$$

and the necessary TVC from Proposition 4.2 in Kamihigashi (2005):[34][35]

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[\frac{1}{\mathbf{c}_t^*(z^t, A^t)}\mathbf{a}_{t+1}^*(z^{t+1}, A^{t+1})\right] = 0 \tag{266}$$

[34]Note that:

$$\begin{aligned}
\mathbb{E}_t[\mathbf{a}_{t+1}^*(z^{t+1}, A^{t+1})|z_t = 0] &= (1-\nu)\beta R_t(A^t)\mathbf{a}_t^*(z^t, A^t)\\
\mathbb{E}_t[\mathbf{a}_{t+1}^*(z^{t+1}, A^{t+1})|z_t = \zeta] &= \frac{\xi\beta}{1-(1-\nu-\xi)\beta}\zeta w_t(A^t)
\end{aligned}$$

If $z_t = 0$, since $0 < (1-\nu)\beta < 1$, (263) implies (266), and vice versa. However, if $z_t = \zeta$, (263) and (266) are not equivalent.

[35]It states that under Assumptions 3.1–3.8 and logarithmic utility $u(\cdot) = \log(\cdot)$, the optimal path $\{x_t^*\}$ satisfies:

$$\lim_{t\uparrow\infty} \beta^t \mathbb{E} u'(g_t(x_t^*, x_{t+1}^*))g_{t,2}(x_t^*, x_{t+1}^*; -x_{t+1}^*) = 0, \tag{264}$$

$$\text{where } g_{t,2}(x_t^*, x_{t+1}^*; d) = \lim_{\epsilon\downarrow 0} \frac{g_t(x_t^*, x_{t+1}^* + \epsilon d) - g_t(x_t^*, x_{t+1}^*)}{\epsilon}. \tag{265}$$

In our environment,

$$g_t(\mathbf{a}_t, \{\mathbf{a}_{t+1}\}) := w_t(A^t)z_t + R_t(A^t)\mathbf{a}_t(z^t, A^t) - \sum_{A_{t+1}}\sum_{z_{t+1}} \pi(A_{t+1}|A^t)\pi(z_{t+1}|z^t)\mathbf{a}_{t+1}(z^{t+1}, A^{t+1}).$$

Given $q(A_{t+1}|A^t) = \pi(A_{t+1}|A^t)$, this gives:

$$\begin{aligned}
g_{t,2}(\mathbf{a}_t, \{\mathbf{a}_{t+1}\}; -\{\mathbf{a}_{t+1}\}) &= \sum_{A_{t+1}}\sum_{z_{t+1}} \pi(A_{t+1}|A^t)\pi(z_{t+1}|z^t)\mathbf{a}_{t+1}(z^{t+1}, A^{t+1})\\
&=: \mathbb{E}_t\left[\mathbf{a}_{t+1}(z^{t+1}, A^{t+1})\right].
\end{aligned}$$

Hence, the transversality condition is given by:

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[\frac{1}{\mathbf{c}_t^*(z^t, A^t)}\mathbf{a}_{t+1}^*(z^{t+1}, A^{t+1})\right] = 0$$

Assumptions 3.1–3.8 are satisfied in this environment. Let $(\Omega, \mathcal{F}, P)$ be a probability space and F be the set of all functions from Ω to $\mathbb{R}^n$. 3.1: $\mathbf{a}_0 \in F$ and $X_t := \{(\mathbf{a}_t(z^t, A^t), \{\mathbf{a}_{t+1}(z^{t+1}, A^{t+1})\}); \mathbf{a}_{t+1}(z^{t+1}, A^{t+1}) \geq 0, \mathbf{c}_t(z^t, A^t) \geq 0\} \subset F \times F$ for all $t \geq 0$. 3.2: $\beta \in (0,1)$, and $\log(\cdot)$ is C^1 on $\mathbb{R}_{++}$, concave, and strictly increasing. 3.3: For all $(\mathbf{a}_t, \{\mathbf{a}_{t+1}\}) \in X_t$, $g_t(\mathbf{a}_t, \{\mathbf{a}_{t+1}\}) \geq 0$, $g_t : \Omega \to \mathbb{R}_+$ is measurable, and $\mathbb{E}[\log(g_t(\mathbf{a}_t, \{\mathbf{a}_{t+1}\}))]$ exists in $[-\infty, \infty)$. 3.4: An optimal path $\{\mathbf{a}_t^*\}$ exists. 3.5: $g_{t,2}(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}; -\{\mathbf{a}_{t+1}^*\})$ is measurable. 3.6: For all $t \geq 0$, $\exists \underline{\lambda}_t \in [0,1), \forall \lambda \in [\lambda_t, 1), (\mathbf{a}_t^*, \lambda\{\mathbf{a}_{t+1}^*\}) \in X_t$ and $\forall \tau \geq t+1, (\lambda \mathbf{a}_\tau^*, \lambda\{\mathbf{a}_{\tau+1}^*\}) \in X_\tau$. 3.7: $g_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) > 0$ and $g_t(\lambda\mathbf{a}_t^*, \lambda\{\mathbf{a}_{t+1}^*\})$ is concave in $\lambda \in [\underline{\lambda}_0, 1]$. 3.8: $g_t(\mathbf{a}_t^*, \lambda\{\mathbf{a}_{t+1}^*\})$ is nonincreasing and continuous in $\lambda \in (\underline{\lambda}_t, 1]$. Finally, the logarithmic utility is assumed.

Proof. In the proposed allocation, if $z_t = \zeta$,

$$\mathbf{c}_t^*(z^t, A^t) = w_t(A^t)\frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}\zeta \text{ and } \mathbf{a}_t^*(z^t, A^t) = 0$$

$$\therefore \frac{R_t(A^t)}{\mathbf{c}_t^*(z^t, A^t)}\mathbf{a}_t^*(z^t, A^t) = 0 \text{ if } z_t = \zeta \tag{267}$$

If $z_t = 0$,

$$\mathbf{c}_t^*(z^t, A^t) = [1-(1-\nu)\beta]R_t(A^t)\mathbf{a}_t^*(z^t, A^t)$$

$$\therefore \frac{R_t(A^t)}{\mathbf{c}_t^*(z^t, A^t)}\mathbf{a}_t^*(z^t, A^t) = \frac{1}{1-(1-\nu)\beta} \text{ if } z_t = 0 \tag{268}$$

Hence, the proposed allocation satisfies the sufficient TVC:

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[\frac{R_t(A^t)}{\mathbf{c}_t^*(z^t, A^t)}\mathbf{a}_t^*(z^t, A^t)\right] \leq \lim_{t\to\infty} \beta^t \mathbb{E}\left[\frac{1}{1-(1-\nu)\beta}\right] = 0 \tag{269}$$

On the other hand, if $z_{t+1} = 0$, $\mathbf{a}_{t+1}^*(z^{t+1}, A^{t+1}) = 0$. If $z_t = \zeta$ and $z_{t+1} = 0$,

$$\mathbf{c}_t^*(z^t, A^t) = w_t(A^t)\frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}\zeta \text{ and } \mathbf{a}_{t+1}^*(z^{t+1}, A^{t+1}) = \frac{\beta}{1-(1-\nu-\xi)\beta}\zeta w_t(A^t)$$

$$\therefore \frac{\mathbf{a}_{t+1}^*(z^{t+1}, A^{t+1})}{\mathbf{c}_t^*(z^t, A^t)} = \frac{\beta}{1-(1-\nu)\beta} \tag{270}$$

If $z_t = 0$ and $z_{t+1} = 0$,

$$\mathbf{c}_t^*(z^t, A^t) = [1-(1-\nu)\beta]R_t(A^t)\mathbf{a}_t^*(z^t, A^t)$$

$$\mathbf{a}_{t+1}^*(z^{t+1}, A^{t+1}) = \beta R_t(A^t)\mathbf{a}_t^*(z^t, A^t)$$

$$\therefore \frac{\mathbf{a}_{t+1}^*(z^{t+1}, A^{t+1})}{\mathbf{c}_t^*(z^t, A^t)} = \frac{\beta}{1-(1-\nu)\beta} \tag{271}$$

Hence, the proposed allocation satisfies the necessary TVC:

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[\frac{1}{\mathbf{c}_t^*(z^t, A^t)}\mathbf{a}_{t+1}^*(z^{t+1}, A^{t+1})\right] \leq \lim_{t\to\infty} \beta^t \mathbb{E}\left[\frac{\beta}{1-(1-\nu)\beta}\right] = 0 \tag{272}$$

□

B.1.3 Uniqueness of the Allocation that Satisfies both the KTC and the TVC

We hope to prove that the proposed allocation is the only feasible allocation that satisfies both the Kuhn-Tucker condition and the (necessary) transversality condition. Here we show that an allocation with positive savings in a high-income state eventually violates the transversality condition. Proposition 22 considers households in a high-income state ($z_t = \zeta$) with zero assets, and Proposition 23 considers households in a low-income state ($z_t = 0$) with assets less than a certain amount. It is straightforward to show that the Euler equation is satisfied between the current state and the next low-income state (Lemma 16 below).

Proposition 22. *Suppose Assumption 2 on contigent claim prices is satisfied and suppose that the sequence of wages and interest rates* $\{w_t(A^t), R_t(A^t)\}_{t=0}^{\infty}$ *satisfies the no-savings Assumption 3. Consider households at a high-income state* $z_t = \zeta$ *with zero assets* $\mathbf{a}_t(a_0, z^t, A^t; z_t = \zeta) = 0$*. Suppose the households make positive savings* $\epsilon > 0$ *for the next high-income state:*

$$\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}; z_t = \zeta, z_{t+1} = \zeta) = \epsilon > 0.$$

Given that the households' consumption follows the Kuhn-Tucker condition, such a consumption allocation violates the transversality condition (266):

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[\frac{1}{\mathbf{c}_t(a_0, z^t, A^t)}\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1})\right] = 0$$

Proof. Any consumption allocation must satisfy the budget constraint:

$$\mathbf{c}_t(a_0, z^t, A^t) + \sum_{A_{t+1}}\sum_{z_{t+1}} \pi(A_{t+1}|A^t)\pi(z_{t+1}|z_t)\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) = w_t(A^t)z_t + R_t(A^t)\mathbf{a}_t(a_0, z^t, A^t),$$

where the contigent claim price $q_t(A^{t+1}, z^{t+1}|A^t, z^t) = \pi(A_{t+1}|A^t)\pi(z_{t+1}|z_t)$ is already imposed. Consider a consumption and savings rule at time t with positive savings $\epsilon > 0$ for the next high-income state:

$$\mathbf{c}_t(a_0, z^t, A^t; z_t = \zeta) = w_t(A^t)c_0 - (1-\xi)\epsilon$$
$$\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) = \begin{cases} \epsilon & \text{if } z_{t+1} = \zeta \\ \frac{\beta}{1-(1-\nu-\xi)\beta}\zeta w_t(A^t) & \text{if } z_{t+1} = 0 \end{cases}$$

Remember that $\pi(z_{t+1} = \zeta | z_t = \zeta) = 1 - \xi$ and $c_0 := \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}\zeta$. Note that households save the same amount across all aggregate states A_{t+1}. In the next low-income state ($z_{t+1} = 0$), the consumption is determined by the Euler equation. Such an allocation is feasible, and we don't further analyze this case.[36] In the next high-income state, since the savings are positive ($\epsilon > 0$), the consumption must follow the Euler equation. Then, $\mathbf{c}_{t+1}$ is given by:

$$\begin{aligned}\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1}; z_{t+1} = \zeta) &= \beta R_{t+1}(A^{t+1})\mathbf{c}_t(a_0, z^t, A^t; z_t = \zeta) \\ &= w_{t+1}(A^{t+1})c_0 - \left(1 - \beta R_{t+1}(A^{t+1})\frac{w_t(A^t)}{w_{t+1}(A^{t+1})}\right) w_{t+1}(A^{t+1})c_0 \\ &\quad - \beta R_{t+1}(A^{t+1})(1-\xi)\epsilon \qquad (273)\end{aligned}$$

[36] Since $\mathbf{c}_t(a_0, z^t, A^t; z_t = \zeta)$ is slightly smaller than the optimal consumption, $\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1}; z_{t+1} = 0)$ is also slightly smaller than the optimal consumption due to the Euler equation. Hence, the households accumulate small savings over time in the following low-income states.

Assuming that the consumption and saving rule goes back to the optimal rule from time $t+2$ onwards (meaning zero savings for the next high-income state),[37] $\mathbf{a}_{t+2}$ is given by:

$$\mathbf{a}_{t+2}(a_0, z^{t+2}, A^{t+2}) = \begin{cases} 0 & \text{if } z_{t+2} = \zeta \\ \frac{\beta}{1-(1-\nu-\xi)\beta}\zeta w_{t+1}(A^{t+1}) + \frac{1}{\xi}\Bigg[\left(1 - \beta R_{t+1}(A^{t+1})\frac{w_t(A^t)}{w_{t+1}(A^{t+1})}\right) & \\ \quad w_{t+1}(A^{t+1})c_0 + \beta R_{t+1}(A^{t+1})(1-\xi)\epsilon + R_{t+1}(A^{t+1})\epsilon\Bigg] & \text{if } z_{t+2} = 0 \end{cases}$$

Following the Euler equation and the budget constraint, $\mathbf{c}_{t+2}$ and $\mathbf{a}_{t+3}$ are given by:

$$\begin{aligned} \mathbf{c}_{t+2}(a_0, z^{t+2}, A^{t+2}; z_{t+2} = 0) &= \beta R_{t+2}(A^{t+2})\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1}; z_{t+1} = \zeta) \\ &= \beta R_{t+2}(A^{t+2}) w_{t+1}(A^{t+1})c_0 \\ &\quad - \beta R_{t+2}(A^{t+2})\left(1 - \beta R_{t+1}(A^{t+1})\frac{w_t(A^t)}{w_{t+1}(A^{t+1})}\right) w_{t+1}(A^{t+1})c_0 \\ &\quad - \beta R_{t+2}(A^{t+2})\beta R_{t+1}(A^{t+1})(1-\xi)\epsilon \end{aligned}$$

$$\begin{aligned} (1-\nu)\mathbf{a}_{t+3}(a_0, z^{t+3}, A^{t+3}; z_{t+3} = 0) &= R_{t+2}(A^{t+2})\mathbf{a}_{t+2}(a_0, z^{t+2}, A^{t+2}; z_{t+2} = 0) \\ &\quad - \mathbf{c}_{t+2}(a_0, z^{t+2}, A^{t+2}; z_{t+2} = 0) \\ &= \beta R_{t+2}(A^{t+2})\frac{(1-\nu)\beta}{1-(1-\nu-\xi)\beta} w_{t+1}(A^{t+1})\zeta \\ &\quad + \left(\frac{1}{\xi} + \beta\right) R_{t+2}(A^{t+2})(1 - \beta R_{t+1}\frac{w_t}{w_{t+1}}) w_{t+1}c_0 \\ &\quad + \left(\frac{1}{\xi} + \beta\right) R_{t+2}(A^{t+2})\beta R_{t+1}(1-\xi)\epsilon + R_{t+2}R_{t+1}\frac{1}{\xi}\epsilon \end{aligned}$$

Since we will take a limit $\epsilon \to 0$, so that the argument holds for any $\epsilon > 0$, we omit terms with ϵ for $\mathbf{c}_{t+3}$ and $\mathbf{a}_{t+4}$:[38]

$$\begin{aligned} \mathbf{c}_{t+3}(a_0, z^{t+3}, A^{t+3}; z_{t+3} = 0) &= \beta R_{t+3}\mathbf{c}_{t+2}(a_0, z^{t+2}, A^{t+2}; z_{t+2} = 0) \\ &< (\beta R_{t+3})(\beta R_{t+2}) w_{t+1}c_0 - (\beta R_{t+3})(\beta R_{t+2})\left(1 - \beta R_{t+1}\frac{w_t}{w_{t+1}}\right) w_{t+1}c_0 \\ \mathbf{a}_{t+4}(a_0, z^{t+4}, A^{t+4}; z_{t+4} = 0) &= \frac{1}{1-\nu}\left[R_{t+3}\mathbf{a}_{t+3}(a_0, z^{t+3}, A^{t+3}; z_{t+3} = 0) - \mathbf{c}_{t+3}(a_0, z^{t+3}, A^{t+3}; z_{t+3} = 0)\right] \\ &> (\beta R_{t+3})(\beta R_{t+2}) w_{t+1}\bar{a}_0 \\ &\quad + \left[\frac{1}{\xi(1-\nu)^2} + \frac{\beta}{(1-\nu)^2} + \frac{\beta^2}{1-\nu}\right] R_{t+3}R_{t+2}\left(1 - \beta R_{t+1}\frac{w_t}{w_{t+1}}\right) w_{t+1}c_0 \end{aligned}$$

[37]If the households save for a future high-income state again, the households accumulate even more savings in the following states. If the consumption rule under consideration violates the transversality condition, such an allocation will also violate the transversality condition.

[38]Since $\epsilon > 0$, we obtain an inequality. We also omit an argument A^t for wage and interest rate functions to simplify the notation.

where $\bar{a}_0 = \frac{\beta}{1-(1-\nu-\xi)\beta}\zeta$. By solving this sequentially, we obtain:

$$\begin{aligned}
&\lim_{\tau\to\infty} \mathbf{c}_{t+\tau}(a_0, z^{t+\tau}, A^{t+\tau}; z_t = z_{t+1} = \zeta, z_{t+2} = \cdots = z_{t+\tau} = 0) \\
&\quad < \left(\prod_{u=1}^{\tau-1} \beta R_{t+u+1}(A^{t+u+1})\right) w_{t+1}(A^{t+1}) c_0 \qquad (274) \\
&\lim_{\tau\to\infty} \mathbf{a}_{t+\tau+1}(a_0, z^{t+\tau+1}, A^{t+\tau+1}; z_t = z_{t+1} = \zeta, z_{t+2} = \cdots = z_{t+\tau+1} = 0) \\
&\quad > \left(\prod_{u=1}^{\tau-1} \beta R_{t+u+1}(A^{t+u+1})\right) w_{t+1}(A^{t+1}) \bar{a}_0 \\
&\qquad + \frac{1}{\xi(1-\nu)^{\tau-1}} \left(\prod_{u=1}^{\tau-1} R_{t+u+1}(A^{t+u+1})\right) \left(1 - \beta R_{t+1}(A^{t+1}) \frac{w_t(A^t)}{w_{t+1}(A^{t+1})}\right) w_{t+1}(A^{t+1}) c_0 \qquad (275)
\end{aligned}$$

We now evaluate the transversality condition:

$$\begin{aligned}
&\lim_{\tau\to\infty} \beta^{t+\tau} \mathbb{E}\left[\frac{1}{\mathbf{c}_{t+\tau}(a_0, z^{t+\tau}, A^{t+\tau})} \mathbf{a}_{t+\tau+1}(a_0, z^{t+\tau+1}, A^{t+\tau+1})\right] \\
&\quad \geq \lim_{\tau\to\infty} \beta^{t+\tau} \pi(z_t = z_{t+1} = \zeta, z_{t+2} = \cdots = z_{t+\tau} = 0) \\
&\qquad \frac{\mathbf{a}_{t+\tau+1}(a_0, z^{t+\tau+1}, A^{t+\tau+1}; z_t = z_{t+1} = \zeta, z_{t+2} = \cdots = z_{t+\tau+1} = 0)}{\mathbf{c}_{t+\tau}(a_0, z^{t+\tau}, A^{t+\tau}; z_t = z_{t+1} = \zeta, z_{t+2} = \cdots = z_{t+\tau} = 0)} \\
&\quad > \beta^t \pi(z^t) \lim_{t\to\infty} \beta^\tau \xi (1-\nu)^{\tau-1} \frac{\left(\prod_{u=1}^{\tau-1} R_{t+u+1}(A^{t+u+1})\right) \left(1 - \beta R_{t+1}(A^{t+1}) \frac{w_t(A^t)}{w_{t+1}(A^{t+1})}\right) w_{t+1}(A^{t+1}) c_0}{\xi(1-\nu)^{\tau-1} \left(\prod_{u=1}^{\tau-1} \beta R_{t+u+1}(A^{t+u+1})\right) w_{t+1}(A^{t+1}) c_0} \\
&\quad = \beta^{t+1} \pi(z^t) \left(1 - \beta R_{t+1}(A^{t+1}) \frac{w_t(A^t)}{w_{t+1}(A^{t+1})}\right) > 0 \qquad (276)
\end{aligned}$$

The last line is strictly positive since $\beta R_{t+1}(A^{t+1}) < \frac{w_{t+1}(A^{t+1})}{w_t(A^t)}$ by Assumption 3. We have seen that if the households in a high-income state $z_t = \zeta$ with zero assets at any finite period $t \geq 0$ make positive savings for the next high-income state ($z_{t+1} = \zeta$), the allocation that follows the Kuhn-Tucker condition violates the transversality condition.

□

Proposition 23. *Suppose Assumptions 2, 3, and 4 hold. Consider households in a low-income state $z_t = 0$ with assets less than the following amount:*

$$\mathbf{a}_t(a_0, z^t, A^t; z_t = 0) \leq \begin{cases} w_0 \bar{a}_0 & \text{if } t = 0 \\ w_{t-1}(A^{t-1}) \bar{a}_0 & \text{if } t \geq 1 \end{cases} \qquad (277)$$

where $\bar{a}_0 := \frac{\beta}{1-(1-\nu-\xi)\beta}\zeta$, and suppose they make positive savings $\epsilon > 0$ for the next high-income state:

$$\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}; z_t = 0, z_{t+1} = \zeta) = \epsilon > 0.$$

Given that the households' consumption follows the Kuhn-Tucker condition, such a consumption allocation violates the transversality condition (266):

$$\lim_{t\to\infty}\beta^t\mathbb{E}\left[\frac{1}{\mathbf{c}_t(a_0,z^t,A^t)}\mathbf{a}_{t+1}(a_0,z^{t+1},A^{t+1})\right]=0.$$

Proof. Consider a consumption and saving rule in the state $z_t=0$ with positive savings $\epsilon>0$ for the next high-income state:

$$\mathbf{c}_t(a_0,z^t,A^t;z_t=0)=[1-(1-\nu)\beta]R_t(A^t)\mathbf{a}_t(a_0,z^t,A^t;z_t=0)-\nu\epsilon \tag{278}$$

$$\mathbf{a}_{t+1}(a_0,z^{t+1},A^{t+1})=\begin{cases}\epsilon & \text{if } z_{t+1}=\zeta\\ \beta R_t(A^t)\mathbf{a}_t(a_0,z^t,A^t;z_t=0) & \text{if } z_{t+1}=0\end{cases} \tag{279}$$

Note that because of the restriction on $\mathbf{a}_t(a_0,z^t,A^t)$ given by (277), $\mathbf{c}_t(a_0,z^t,A^t)$ satisfies the following:

$$\mathbf{c}_t(a_0,z^t,A^t)\le\begin{cases}\beta R_0 w_0\frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}\zeta-\nu\epsilon & <w_0c_0-\nu\epsilon & \text{if } t=0\\ \beta R_t(A^t)w_{t-1}(A^{t-1})\frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}\zeta-\nu\epsilon & <w_t(A^t)c_0-\nu\epsilon & \text{if } t\ge1\end{cases} \tag{280}$$

The inequality holds since $\beta R_0<1$ and $\beta R_t(A^t)\frac{w_{t-1}(A^{t-1})}{w_t(A^t)}<1$ for $t\ge1$ by Assumption 3. Remember $\bar{a}_0:=\frac{\beta}{1-(1-\nu-\xi)\beta}\zeta$ and $c_0:=\frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}\zeta$. Then, $\mathbf{c}_{t+1}(a_0,z^{t+1},A^{t+1};z_t=0,z_{t+1}=\zeta)$ for any $t\ge0$ is bounded from above as follows:

$$\begin{aligned}\mathbf{c}_{t+1}(a_0,z^{t+1},A^{t+1};z_t=0,z_{t+1}=\zeta)&=\beta R_{t+1}(A^{t+1})\mathbf{c}_t(a_0,z^t,A^t;z_t=0)\\&<w_{t+1}(A^{t+1})c_0-\left(1-\beta R_{t+1}(A^{t+1})\frac{w_t(A^t)}{w_{t+1}(A^{t+1})}\right)w_{t+1}(A^{t+1})c_0\\&\quad-\beta R_{t+1}(A^{t+1})\nu\epsilon\end{aligned} \tag{281}$$

and $\mathbf{a}_{t+2}(a_0,z^{t+2},A^{t+2};z_{t+2}=0)$ satisfies:

$$\begin{aligned}\mathbf{a}_{t+2}(a_0,z^{t+2},A^{t+2};z_{t+2}=0)>\bar{a}_0w_{t+1}(A^{t+1})+\frac{1}{\xi}\Bigg[&\left(1-\beta R_{t+1}(A^{t+1})\frac{w_t(A^t)}{w_{t+1}(A^{t+1})}\right)\\&w_{t+1}(A^{t+1})c_0+\beta R_{t+1}(A^{t+1})\nu\epsilon+R_{t+1}(A^{t+1})\epsilon\Bigg]\end{aligned}$$

The rest of the derivations follows exactly the same as in Proposition 22. Since $\mathbf{c}_{t+\tau}(a_0,z^{t+\tau},A^{t+\tau};z_t=0,z_{t+1}=\zeta,z_{t+2}=\cdots=z_{t+\tau}=0)$ is bounded from above as in inequality (274), and $\mathbf{a}_{t+\tau+1}(a_0,z^{t+\tau+1},A^{t+\tau+1};z_t=0,z_{t+1}=\zeta,z_{t+2}=\cdots=z_{t+\tau+1}=0)$ is bounded from below as in inequality (275), the transversality condition is violated as in (276). Hence, given the restriction on the amount of assets in a low-income state (277), the consumption and saving rule with positive savings for the next high-income state will violate the transversality condition. □

Lemma 16. *Suppose Assumption 2 on contingent claims prices is satisfied. Then, the Kuhn-Tucker condition given by (109) and (110) implies that the Euler equation between the current state* (z^t, A^t) *and the next low-income state* (z^{t+1}, A^{t+1}) *with* $z_{t+1} = 0$ *is satisfied:*

$$\frac{\mathbf{c}(a_0, z^{t+1}, A^{t+1}; z_{t+1} = 0)}{\mathbf{c}(a_0, z^t, A^t)} = \beta R_{t+1}(A^{t+1}) \text{ for any } A_{t+1}. \tag{282}$$

Proof. The Kuhn-Tucker condition is given by (109) and (110):

$$\frac{\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1})}{\mathbf{c}_t(a_0, z^t, A^t)} = \frac{\pi(A_{t+1}|A^t)\pi(z_{t+1}|z_t)}{q_t(A^{t+1}, z^{t+1}|A^t, z^t)} \left[\beta R_{t+1}(A^{t+1}) + \lambda(a_0, z^{t+1}, A^{t+1})\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1})\right]$$

with $\lambda(a_0, z^{t+1}, A^{t+1})\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) = 0, \ \lambda(a_0, z^{t+1}, A^{t+1}) \geq 0, \ \mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) \geq 0$

Note that by Assumption 2, $q_t(A^{t+1}, z^{t+1}|A^t, z^t) = \pi(A_{t+1}|A^t)\pi(z_{t+1}|z_t)$. Consider a state $z_{t+1} = 0$. The Kuhn-Tucker condition requires that if $\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) > 0$, the Euler equation holds:

$$\frac{\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1}; z_{t+1} = 0)}{\mathbf{c}_t(a_0, z^t, A^t)} = \beta R_{t+1}(A^{t+1}).$$

Suppose $\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}; z_{t+1} = 0) = 0$. The budget constraint in state (z^{t+1}, A^{t+1}) is given by:

$$\begin{aligned} &\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1}) + \sum_{z_{t+2}} \sum_{A_{t+2}} \pi(z_{t+2}|z_{t+1})\pi(A_{t+2}|A_{t+1})\mathbf{a}_{t+2}(a_0, z^{t+2}, A^{t+2}) \\ &= w_{t+1}(A^{t+1}) \underbrace{z_{t+1}}_{=0} + R_{t+1}(A^{t+1}) \underbrace{\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1})}_{=0}, \end{aligned}$$

where the tight borrowing constraint requires $\mathbf{a}_{t+2}(a_0, z^{t+2}, A^{t+2}) \geq 0$. Hence, $\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1}) = 0$. Given the logarithmic utility function, this leads to negative infinite utility. Since there is a feasible allocation $\{\mathbf{c}_t(a_0, z^t, A^t), \mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1})\}_{t=0}^{\infty}$ such that $\mathbf{c}_t(a_0, z^t, A^t) > 0$ for all t and (z^t, A^t), an allocation with $\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) = 0$ and $z_{t+1} = 0$ is sub-optimal. □

B.2 The Optimal Consumption Contract in the Transitional Dynamics

We made a separate attempt to show the optimal contract in the transitional dynamics, so this part may be useful to show the uniqueness of the optimal contract.

We want to show that under Assumption 3, the following three properties hold in the household's optimization problem:

1. The shortsale constraint binds if $z_{t+1} = \zeta$ and thus $\mathbf{a}_{0,t+1} = a_{0,t+1} = 0$.

2. The standard complete markets Euler equation holds if $z_{t+1} = 0$ and thus for all $s, t \geq 0$,

$$\frac{w_{t+1}c_{s+1,t+1}}{w_t c_{s,t}} = \beta R_{t+1}$$

3. The transversality condition holds

$$\lim_{j\to\infty} \frac{(1-\nu)^j}{\prod_{\tau=0}^{j-1} R_{t+\tau}} \mathbf{a}_{s+j,t+j} = 0$$

Given $\{w_t, r_t\}_{t=0}^{\infty}$, the household consumption and asset allocation $\{\hat{\mathbf{c}}_t(a_0, z^t), \hat{\mathbf{a}}_{t+1}(a_0, z^{t+1})\}_{t=0}^{\infty}$ solves, for all (a_0, z_0),

$$\begin{aligned}
& \max_{\{\mathbf{c}_t(a_0,z^t),\mathbf{a}_{t+1}(a_0,z^{t+1})\}} \sum_{t=0}^{\infty} \sum_{z^t} \beta^t \pi(z^t) \log(\mathbf{c}_t(a_0, z^t)) && (283) \\
& s.t. \\
\mathbf{c}_t(a_0, z^t) &+ \sum_{z_{t+1}} \pi(z_{t+1}|z_t)\mathbf{a}_{t+1}(a_0, z^{t+1}) = w_t z_t + (1 + r_t)\mathbf{a}_t(a_0, z^t) && (284) \\
\mathbf{a}_{t+1}(a_0, z^{t+1}) &\geq 0 && (285)
\end{aligned}$$

This gives a Lagrangian problem:

$$\begin{aligned}
U(a_0, z_0) = & \max_{\{\mathbf{c}_t(a_0,z^t),\mathbf{a}_{t+1}(a_0,z^{t+1})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{z^t} \beta^t \pi(z^t) \log(\mathbf{c}_t(a_0, z^t)) \\
& + \sum_{t=0}^{\infty} \sum_{z^t} \mu(z^t) \left[w_t z_t + (1 + r_t)\mathbf{a}_t(a_0, z^t) - \mathbf{c}_t(a_0, z^t) - \sum_{z_{t+1}} \pi(z_{t+1}|z_t)\mathbf{a}_{t+1}(a_0, z^{t+1}) \right] \\
& + \sum_{t=0}^{\infty} \sum_{z^{t+1}} \beta^t \pi(z^{t+1})\lambda(z^{t+1})\mathbf{a}_{t+1}(a_0, z^{t+1}), \qquad (286)
\end{aligned}$$

where $\beta^t\pi(z^{t+1})$ in front of $\lambda(z^{t+1})$ will simplify expressions later. FOCs are:

$$\begin{aligned}
[\mathbf{c}_t(a_0, z^t)] : \quad & \beta^t \pi(z^t) \frac{1}{\mathbf{c}_t(a_0, z^t)} && = \mu(z^t) && (287) \\
[\mathbf{a}_{t+1}(a_0, z^{t+1})] : \quad & \mu(z^{t+1})(1 + r_{t+1}) + \beta^t \pi(z^{t+1})\lambda(z^{t+1}) && = \mu(z^t)\pi(z_{t+1}|z_t) && (288)
\end{aligned}$$

By substituing $\mu(z^t)$, we obtain the following Kuhn-Tucker condition:

$$\frac{1}{\mathbf{c}_t(a_0, z^t)} = \beta(1 + r_{t+1}) \frac{1}{\mathbf{c}_{t+1}(a_0, z^{t+1})} + \lambda(z^{t+1}), \qquad (289)$$

$$\text{where } \lambda(z^{t+1})\mathbf{a}_{t+1}(a_0, z^{t+1}) = 0, \ \lambda(z^{t+1}) \geq 0, \ \mathbf{a}_{t+1}(a_0, z^{t+1}) \geq 0. \qquad (290)$$

This condition means either of the following holds:

$$\mathbf{a}_{t+1}(a_0, z^{t+1}) > 0 \text{ and } \frac{1}{\mathbf{c}_t(a_0, z^t)} = \beta(1 + r_{t+1})\frac{1}{\mathbf{c}_{t+1}(a_0, z^{t+1})} \tag{291}$$

$$\mathbf{a}_{t+1}(a_0, z^{t+1}) = 0 \text{ and } \frac{1}{\mathbf{c}_t(a_0, z^t)} \geq \beta(1 + r_{t+1})\frac{1}{\mathbf{c}_{t+1}(a_0, z^{t+1})} \tag{292}$$

If the household saves for the state z^{t+1}, the Euler equation between states z^t and z^{t+1} holds with equality. If the Euler equation does not hold between z^t and z^{t+1}, savings for z^{t+1} should be equal to zero ($\mathbf{a}_{t+1}(a_0, z^{t+1}) = 0$).

Assumption 3. *The sequence of equilibrium interest rates and wages satisfy*

$$\beta(1 + r_0) < 1 \tag{293}$$

$$\beta(1 + r_{t+1}) < \frac{w_{t+1}}{w_t} \text{ for all } t \geq 0 \tag{294}$$

Lemma 17. *The standard complete markets Euler equation holds if $z_{t+1} = 0$ and thus for all $s, t \geq 0$,*

$$\frac{w_{t+1}c_{s+1,t+1}}{w_t c_{s,t}} = \beta R_{t+1}$$

Proof. Consider a state $z_{t+1} = 0$. The Kuhn-Tucker condition requires that if $\mathbf{a}_{t+1}(a_0, z^{t+1}) > 0$, the Euler equation holds:

$$\frac{\mathbf{c}_{t+1}(a_0, z^{t+1}; z_{t+1} = 0)}{\mathbf{c}_t(a_0, z^t)} = \beta R_{t+1}.$$

Suppose $\mathbf{a}_{t+1}(a_0, z^{t+1}) = 0$. The budget constraint (284) in state z^{t+1} is given by:

$$\mathbf{c}_{t+1}(a_0, z^{t+1}) + \sum_{z_{t+2}} \pi(z_{t+2}|z_{t+1})\mathbf{a}_{t+2}(a_0, z^{t+2}) = w_{t+1} \underbrace{z_{t+1}}_{=0} + R_{t+1} \underbrace{\mathbf{a}_{t+1}(a_0, z^{t+1})}_{=0},$$

where the borrowing constraint requires $\mathbf{a}_{t+2}(a_0, z^{t+2}) \geq 0$. Hence, $\mathbf{c}_{t+1}(a_0, z^{t+1}) = 0$. Given the logarithmic utility function, this leads to negative infinite utility. Given that there is a feasible allocation $\{\mathbf{c}_t(a_0, z^t), \mathbf{a}_{t+1}(a_0, z^{t+1})\}_{t=0}^{\infty}$ such that $\mathbf{c}_t(a_0, z^t) > 0$ for all t and z^t,[39] the allocation with $\mathbf{a}_{t+1}(a_0, z^{t+1}) = 0$ and $z_{t+1} = 0$ for some z^{t+1} is not optimal.

By denoting $\mathbf{c}_{t+1}(a_0, z^{t+1}; z_{t+1} = 0) = w_{t+1}c_{s+1,t+1}$ and $\mathbf{c}_t(a_0, z^t) = w_t c_{s,t}$, where $s \geq 0$ is the number of periods in a low-income state since the last high income, we obtain the Euler equation in the statement. □

[39] If the household saves positive assets for all low-income states, she can avoid zero consumption in all possible states. The optimal consumption that we establish:

$$\mathbf{c}_t(a_0, z^t; z_t = \zeta) = \frac{1 - (1-\nu)\beta}{1 - (1 - \nu - \xi)\beta} w_t \zeta$$

$$\mathbf{c}_t(a_0, z^t; z_t = 0) = [1 - (1-\nu)\beta] R_t \mathbf{a}_t(a_0, z^t; z_t = 0)$$

is one of feasible allocations.

Lemma 18. *In an initial state $(a_0, z_0) = (0, \zeta)$, any consumption choice with $c_0(a_0, z^0) > c_0^* := \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}\zeta$ cannot be optimal. In an initial state (a_0, z_0) with $z_0 = 0$, any consumption choice with $c_0(a_0, z^0) > [1-(1-\nu)\beta]R_0 a_0$ cannot be optimal. This applies for any time $t \geq 0$:*

$$c_t(a_0, z^t; z_t = \zeta) \leq c_0^* \quad \text{if} \quad a_t(a_0, z^t; z_t = \zeta) = 0 \tag{295}$$

$$c_t(a_0, z^t; z_t = 0) \leq [1-(1-\nu)\beta]R_t a_t(a_0, z^t; z_t = 0) \tag{296}$$

Proof. Lemma 17 shows that households always make positive savings for a low-income state. As households need to secure consumption in future low-income states, which is determined by the Euler equation, they face an upper bound on their consumption in the initial period.

Consider an initial state $(a_0, z_0) = (0, \zeta)$. The budget constraint in the state is given by:

$$\begin{aligned} w_0\zeta &= w_0 c_0(a_0, z^0) + \sum_{z_1} \pi(z_1|z_0 = \zeta)\mathbf{a}_1(a_0, z^1) \\ &\geq w_0 c_0(a_0, z^0) + \underbrace{\pi(z_1 = 0|z_0 = \zeta)}_{=\xi}\mathbf{a}_1(a_0, z^1; z_1 = 0), \end{aligned} \tag{297}$$

since $\mathbf{a}_1(a_0, z^1; z_1 = \zeta) \geq 0$. A budget constraint in a future low-income state, z^t with $z_t = 0$, is:

$$\begin{aligned} R_t\mathbf{a}_t(a_0, z^t) &= \mathbf{c}_t(a_0, z^t) + \sum_{z_{t+1}} \pi(z_{t+1}|z_t)\mathbf{a}_{t+1}(a_0, z^{t+1}) \\ &\geq \mathbf{c}_t(a_0, z^t) + \underbrace{\pi(z_{t+1} = 0|z_t = 0)}_{=1-\nu}\mathbf{a}_{t+1}(a_0, z^{t+1}; z_{t+1} = 0) \end{aligned}$$

$$\therefore\ \mathbf{a}_t(a_0, z^t; z_t = 0) \geq \frac{\mathbf{c}_t(a_0, z^t)}{R_t} + (1-\nu)\frac{1}{R_t}\mathbf{a}_{t+1}(a_0, z^{t+1}; z_{t+1} = 0) \text{ for all } t \geq 1. \tag{298}$$

Sequentially substituting (298) into (297) gives:

$$w_0\zeta \geq w_0 c_0 + \xi\left[\frac{\mathbf{c}_t(a_0, z^1; z_1 = 0)}{R_1} + (1-\nu)\frac{\mathbf{a}_2(a_0, z^2; z_1 = 0, z_2 = 0)}{R_1}\right] \tag{299}$$

$$\geq \mathbf{c}_0 + \xi\frac{\mathbf{c}_1(a_0, z^1; z_1 = 0)}{R_1} + \xi(1-\nu)\frac{\mathbf{c}_2(a_0, z^2; z_1 = 0; z_2 = 0)}{R_1 R_2} + \cdots + \lim_{t\to\infty}\xi(1-\nu)^{t-1}\frac{\mathbf{c}_t(a_0, z^t; z_1 = \cdots = z_t = 0)}{\Pi_{u=1}^t R_u} \tag{300}$$

The Euler equation in Lemma 17 states that

$$\begin{aligned} \mathbf{c}_{t+1}(a_0, z^{t+1}; z_{t+1} = 0) &= \beta R_{t+1}\mathbf{c}_t(a_0, z^t) \\ \therefore\ \mathbf{c}_t(a_0, z^t; z_1 = 0, \cdots, z_t = 0) &= \beta^t\left[\Pi_{u=1}^t R_u\right]\mathbf{c}_0(a_0, z^0). \end{aligned}$$

Hence, equation (300) is written as:

$$\begin{aligned} w_0\zeta &\geq \mathbf{c}_0 + \xi\beta\mathbf{c}_0 + \xi(1-\nu)\beta^2\mathbf{c}_0 + \cdots + \lim_{t\to\infty}\xi(1-\nu)^{t-1}\beta^t\mathbf{c}_0 \\ &= \left[1 + \frac{\xi\beta}{1-(1-\nu)\beta}\right]\mathbf{c}_0 \end{aligned} \tag{301}$$

Therefore, the Euler equation and the budget constraint in future low-income states impose the upper bound on consumption in the high-income state:

$$c_0(a_0 = 0, z^0 = \zeta) \leq \frac{1 - (1-\nu)\beta}{1 - (1 - \nu - \xi)\beta}\zeta.$$

A similar argument applies to an initial state (a_0, z_0) with $z_0 = 0$. The budget constraint in the state is given by:

$$\begin{aligned} R_0 w_0 a_0 &= w_0 c_0(a_0, z^0) + \sum_{z_1} \pi(z_1 | z_0 = 0) \mathbf{a}_1(a_0, z^1) \\ &\geq \mathbf{c}_0(a_0, z^0) + \underbrace{\pi(z_1 = 0 | z_0 = 0)}_{=1-\nu} \mathbf{a}_1(a_0, z^1; z_1 = 0), \end{aligned} \tag{302}$$

since $\mathbf{a}_1(a_0, z^1; z_1 = \zeta) \geq 0$. As we saw in equation (298), a budget constraint in a future low-income state gives:

$$\mathbf{a}_t(a_0, z^t; z_t = 0) \geq \frac{\mathbf{c}_t(a_0, z^t)}{R_t} + (1-\nu)\frac{1}{R_t}\mathbf{a}_{t+1}(a_0, z^{t+1}; z_{t+1} = 0) \text{ for all } t \geq 1.$$

Sequentially substituting (298) into (302) gives:

$$\begin{aligned} &R_0 w_0 a_0 \\ &\geq \mathbf{c}_0 + (1-\nu)\frac{\mathbf{c}_1(a_0, z^1; z_1 = 0)}{R_1} + (1-\nu)^2 \frac{\mathbf{c}_2(a_0, z^2; z_1 = 0; z_2 = 0)}{R_1 R_2} + \cdots + \lim_{t \to \infty} (1-\nu)^t \frac{\mathbf{c}_t(a_0, z^t; z_1 = \cdots = z_t = 0)}{\Pi_{u=1}^t R_u} \end{aligned} \tag{303}$$

By using the implication of the Euler equation:

$$\mathbf{c}_t(a_0, z^t; z_1 = 0, \cdots, z_t = 0) = \beta^t \left[\Pi_{u=1}^t R_u\right] \mathbf{c}_0(a_0, z^0),$$

equation (303) is written as:

$$\begin{aligned} R_0 w_0 a_0 &\geq \mathbf{c}_0 + (1-\nu)\beta\mathbf{c}_0 + (1-\nu)^2\beta^2\mathbf{c}_0 + \cdots + \lim_{t \to \infty}(1-\nu)^t\beta^t\mathbf{c}_0 \\ &= \frac{1}{1 - (1-\nu)\beta}\mathbf{c}_0 \end{aligned} \tag{304}$$

Therefore, consumption in the low-income state is subject to an upper bound:

$$c_0(a_0, z^0 = 0) \leq [1 - (1-\nu)\beta] R_0 a_0.$$

Because idiocyncratic productivity (z_t) follows Markov and does not depend on the past history of states, a household at time t with $a_t(a_0, z^t)$ and z_t faces the same problem as in a state $(\tilde{a}_0, \tilde{z}_0)$ if $a_t(a_0, z^t) = \tilde{a}_0$ and $z_t = \tilde{z}_0$. Hence, the same argument to obtain equations (301) and (304) applies for any periods $t \geq 0$. This results in equations (295) and (296).

□

Lemma 19. *Suppose Assumption 3 holds. In an initial state with ($a_0 = 0, z_0 = \zeta$), it is not optimal to save for a high-income state in the next period, i.e.,* $\mathbf{a}_1(a_0 = 0, z_0 = z_1 = \zeta) = 0$. *In an initial state with* $a_0 \leq \bar{a}_0 := \frac{\beta}{1-(1-\nu-\xi)\beta}\zeta$ *and* $z_0 = 0$*, it is not optimal to save for a high-income state, i.e.,* $\mathbf{a}_1(a_0, z_0 = 0, z_1 = \zeta) = 0 \ \forall a_0 \leq \bar{a}_0$.

Proof. Consider an initial state with $(a_0 = 0, z_0 = \zeta)$. In Lemma 18, we saw that consuming $c_0(a_0, z_0) = c_0^*$ in an initial state with ($a_0 = 0, z_0 = \zeta$) and following the Euler equation in all future low-income states is feasible, if savings for a high-income state are always zero. This applies to all future high-income states with zero assets as well. If the household enters a high-income state with $\mathbf{a}_t(a_0, z^t) = 0$, she faces the same problem as in an initial state with ($a_0 = 0, z = \zeta$). Hence, consumption rule:

$$\mathbf{c}_t(a_0, z^t; z_t = \zeta) = \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta} w_t \zeta \qquad \forall t \geq 0 \tag{305}$$

$$\mathbf{c}_t(a_0, z^t; z_t = 0) = \beta R_t \mathbf{c}_{t-1}(a_0, z^{t-1}) \qquad \forall t \geq 1 \tag{306}$$

$$\mathbf{a}_{t+1}(a_0, z^{t+1}; z_{t+1} = \zeta) = 0 \qquad \forall t \geq 0 \tag{307}$$

is feasible. We will show that deviating from this consumption rule and making positive savings for a high-income state cannot achieve higher utility.

If the household saves for a high-income state ($\mathbf{a}_1(a_0, z^1; z_1 = \zeta) > 0$), the K-T condition requires that the Euler equation holds between states z^0 and z^1:

$$\mathbf{c}_1(a_0, z^1; z_1 = \zeta) = \beta R_1 \mathbf{c}_0(a_0, z^0)$$

Since $\beta R_1 \frac{w_0}{w_1} < 1$, $c_1(a_0, z^1; z_1 = \zeta)$ is strictly smaller than c_0^*.[40] This means that even though the household saves for the high-income state, consumption at the state is strictly less than c_0^*, since the borrowing constraint does not bind and the optimal consumption should follow the Euler equation.

Consider an initial state with $(a_0 \leq \bar{a}_0, z_0 = 0)$. If the consumption and asset choice is given by:

$$\mathbf{c}_t(a_0, z^t; z_t = 0) = [1-(1-\nu)\beta] R_t w_t a_t \qquad \forall t \geq 0$$

$$\mathbf{a}_{t+1}(a_0, z^{t+1}; z_{t+1} = \zeta) = 0 \qquad \forall t \geq 0$$

$$\mathbf{a}_{t+1}(a_0, z^{t+1}; z_t = 0, z_{t+1} = 0) = \beta R_t w_t a_t \qquad \forall t \geq 0$$

and equation (305), it satisfies the budget constraint and the Euler equation in all future low-income states.[41] Given Lemma 18, the household cannot consume more than

[40] Remember that consumption in an initial high-income state, $c_0(a_0 = 0, z^0 = \zeta)$, is less than or equal to c_0^* by Lemma 18. If $\mathbf{c}_1(a_0, z^1; z_1 = \zeta)$ is given by $\beta R_1 \frac{w_0}{w_1} c_0(a_0, z^0)$, it is strictly smaller than c_0^*.

[41] The budget constraint in a low-income state is given by:

$$R_t \mathbf{a}_t(a_0, z^t) = \mathbf{c}_t(a_0, z^t) + \sum_{z_{t+1}} \pi(z_{t+1}|z_t) \mathbf{a}_{t+1}(a_0, z^{t+1})$$
$$= [1-(1-\nu)\beta] R_t \mathbf{a}_t + (1-\nu)\beta R_t \mathbf{a}_t = R_t \mathbf{a}_t.$$

$[1-(1-\nu)\beta]\,R_0w_0a_0$ in the initial state. Note that given $a_0 \leq \bar{a}_0 := \frac{\beta}{1-(1-\nu-\xi)\beta}\zeta$,

$$\begin{aligned} c_0 &\leq [1-(1-\nu)\beta]\,R_0\bar{a}_0 \\ &= \beta R_0 c_0^* < c_0^* \qquad \text{since } \beta R_0 < 1. \end{aligned}$$

Suppose she makes positive savings for the next high-income state (z^1 with $z_1 = \zeta$). This is possible only when she consumes less than $[1-(1-\nu)\beta]\,R_0w_0a_0$ in the initial period. We show that this will achieve stricly less consumption not only this period but also the high-income state in the next period. Since the savings are positive, consumption follows the Euler equation:

$$\begin{aligned} \mathbf{c}_1(a_0, z^1; z_0 = 0, z_1 = \zeta) &= \beta R_1 \mathbf{c}_0(a_0, z^0; z_0 = 0) \\ &\leq (\beta R_0)(\beta R_1) w_0 c_0^* \end{aligned}$$

Since $\beta R_0 < 1$ and $\beta R_1 \frac{w_0}{w_1} < 1$ under Assumption 3, we have $c_1(a_0, z^1; z_0 = 0, z_1 = \zeta) < c_0^*$. Hence, she consumes strictly less than what she would consume when she enters the high-income state with zero assets.

So far, we have shown that if the household saves for the high-income state (z^1 with $z_1 = \zeta$), consumption in the state is strictly smaller than $w_1 c_0^*$. We examine if the household could use the positive savings in later periods and achieve higher utility. Note that as far as the Euler equation is satisfied, consumption $c_{t+1}(a_0, z^{t+1})$ drifts down at a rate, $\beta R_{t+1}\frac{w_1}{w_{t+1}} < 1$. If consumption jumps up and the Euler equation is not satisfied, the K-T condition requires that the saving in such a future contingent state should be zero. However, we saw in Lemma 18 that consuming $c_t(a_0, z^t) > c_0^*$ when she has zero assets in a high-income state is not optimal. Entering a low-income state with zero assets is not optimal, either. Therefore, she cannot achieve a strictly higher utility than $\log(w_t c_0^*)$ at any future state if she saves for a high-income state and follows the optimality conditions in all future periods. Given that saving zero assets for a high-income state and consuming $c = c_0^*$ in all future high-income states is feasible, saving for a high-income state is not optimal. □

Lemma 20. *Suppose Assumption 3 holds. If initial states of households are given by either* $(a_0 = 0, z_0 = \zeta)$ *or* $(a_0 \leq \bar{a}_0, z_0 = 0)$*, where* $a_0 \leq \bar{a}_0 := \frac{\beta}{1-(1-\nu-\xi)\beta}\zeta$*, households never save for a high-income state:*

$$a_{t+1}(a_0, z^{t+1}; z_{t+1} = \zeta) = 0 \quad \textit{for all } t \geq 0. \tag{308}$$

Consumption at z^{t+1} with $z_{t+1} = 0$ is given by:

$$\begin{aligned} \mathbf{c}_{t+1}(a_0, z^{t+1}; z_{t+1} = 0) &= [1-(1-\nu)\beta]\,R_{t+1}\mathbf{a}_{t+1}(a_0, z^{t+1}; z_{t+1} = 0) \\ &= \beta R_{t+1}\mathbf{c}_t(a_0, z^t; z_t = 0), \end{aligned}$$

since $\mathbf{a}_{t+1}(a_0, z^{t+1}; z_{t+1} = 0) = \beta R_t \mathbf{a}_t(a_0, z^t) = \beta \frac{\mathbf{c}_t(a_0, z^t)}{1-(1-\nu)\beta}$. Hence, the Euler equation between states z^t and z^{t+1} is satisfied.

Proof. We saw in Lemma 19 that households at time $t = 0$ with high income and zero assets do not save for the next high-income state: $\mathbf{a}_1(a_0 = 0, z_0 = z_1 = \zeta) = 0$. Since idiosyncratic productivity is Markov, the argument applies for any time $t \geq 1$ as far as households enter a high-income state with zero assets:

$$a_{t+1}(a_0, z^{t+1}; z_t = z_{t+1} = \zeta) = 0 \quad \text{if } a_t(a_0, z^t; z_t = \zeta) = 0.$$

Therefore, if we show that low-income households in all possible states that can be reached from the initial states $(a_0 = 0, z_0 = \zeta)$ or $(a_0 \leq \bar{a}_0, z_0 = 0)$ do not save for a next high-income state, we obtain the statement.

We prove by induction. Lemma 19 shows that households with low income in the initial period do not save for the next high-income state. We now show that households in a low-income state at $t = 1$ do not save for the next high-income state:

$$a_2(a_0, z^2; z_1 = 0, z_2 = \zeta) = 0.$$

All households with low income at $t = 1$ enter the period with positive assets (Lemma 17). Therefore, their consumption at $t = 1$ is determined by the Euler equation:

$$c_1(a_0, z^1; z_1 = 0) = \beta R_1 \frac{w_0}{w_1} c_0(a_0, z^0).$$

Since we know $c_0(a_0, z^0) \leq c_0^*$ given the assumed initial states, $c_1(a_0, z^1; z_1 = 0) < c_0^*$ holds under Assumption 3. Suppose the household saves for the next high-income state, z^2 with $z_2 = \zeta$. Then, consumption at the state z^2 is determined by the Euler equation and drifts down at a rate $\beta R_2 \frac{w_1}{w_2} < 1$. Hence, it is strictly smaller than what she would consume when she enters the high-income state with zero assets, i.e., $c_2(a_0, z^2; z_2 = \zeta) < c_0^*$ if $a_2(a_0, z^2; z_2 = \zeta) > 0$. Following the same logic as in Lemma 19, she cannot achieve strictly higher utility in future periods by saving positive assets for the high-income state. Therefore, saving for the state, z^2, is zero, i.e., $a_2(a_0, z^2; z_2 = \zeta) = 0$ if $(a_0 = 0, z_0 = \zeta)$ or $(a_0 \leq \bar{a}_0, z_0 = 0)$. Since $c_1(a_0, z^1; z_1 = 0) < c_0^*$ and $c_1(a_0, z^1; z_1 = \zeta) \leq c_0^*$, consumption at a low-income state in the next period is strictly smaller than c_0^*:

$$c_2(a_0, z^2; z_2 = 0) = \beta R_2 \frac{w_1}{w_2} c_1(a_0, z^1) < c_0^*. \tag{309}$$

Suppose at time $t \geq 1$,

$$c_t(a_0, z^t; z_t = 0) < c_0^*$$

holds. Since we assume $\beta R_{t+1} \frac{w_t}{w_{t+1}} < 1$ (Assumption 3), making positive savings for the next high-income state makes consumption strictly smaller than c_0^*:

$$\begin{aligned} c_{t+1}(a_0, z^{t+1}; z_t = 0, z_{t+1} = \zeta) &= \beta R_{t+1} \frac{w_t}{w_{t+1}} c_t(a_0, z^t; z_t = 0) \text{ if } a_{t+1}(a_0, z^{t+1}; z_t = 0, z_{t+1} = \zeta) > 0 \\ &< c_0^* \end{aligned}$$

As long as she makes positive savings for future periods, her consumption goes down at rate $\beta R_{t+1}\frac{w_t}{w_{t+1}} < 1$. If she makes zero savings for a high-income state, Lemma 18 states consumption at the contingent state is less than or equal to c_0^*. Hence, she cannot achieve higher utility in the future periods by saving positive assets for state z^{t+1}:

$$c_{t+\tau}(a_0, z^{t+\tau}; z_{t+\tau} = \zeta) \leq c_0^* \text{ for all } \tau \geq 2$$
$$\text{if } a_{t+1}(a_0, z^{t+1}; z_t = 0, z_{t+1} = \zeta) > 0.$$

We obtain that it is not optimal to save for z^{t+1} with $z_{t+1} = \zeta$:

$$a_{t+1}(a_0, z^{t+1}; z_t = 0, z_{t+1} = \zeta) = 0. \tag{310}$$

Since $c_{t+1}(a_0, z^{t+1}; z_t = \zeta, z_{t+1} = 0) \leq \beta R_{t+1}\frac{w_t}{w_{t+1}} c_0^* < c_0^*$ by Lemma 18 and Assumption 3 and $c_{t+1}(a_0, z^{t+1}; z_t = 0, z_{t+1} = 0) = \beta R_{t+1}\frac{w_t}{w_{t+1}} c_t(a_0, z^t; z_t = 0) < c_0^*$ by Assumption 3, we also obtain:

$$c_{t+1}(a_0, z^{t+1}; z_{t+1} = 0) < c_0^*.$$

By induction, equation (310) holds for all $t \geq 1$. □

Lemma 21 (Necessity of the Transversality Condition). *Suppose Assumption 3 holds. Consider an optimization problem of a low-income agent with assets $\mathbf{a}_{s,t}$ at time t, where $\mathbf{a}_{s,t} = w_t a_{s,t}$ and $a_{s,t} \leq \bar{a}_0 := \frac{\beta}{1-(1-\nu-\xi)\beta}\zeta$. The transversality condition (**??**):*

$$\lim_{j\to\infty} \frac{(1-\nu)^j}{\prod_{\tau=0}^{j-1} R_{t+\tau}} \mathbf{a}_{s+j,t+j} = 0.$$

is a necessary condition for an optimal consumption and saving profile $\{\mathbf{c}_{s+j,t+j}, \mathbf{a}_{s+j+1,t+j+1}\}_{j=0}^{\infty}$.

Proof. A maximization problem of a low-income household with assets $\mathbf{a}_{s,t}$ at time t is given by:

$$\max_{\{\mathbf{a}_{t+j+1}(a_0, z^{t+j+1})\}_{j=0}^{\infty}} \sum_{j=0}^{\infty} \beta^j \pi(z^{t+j}) \log \left[w_{t+j} z_{t+j} + R_{t+j}\mathbf{a}_{t+j}(a_0, z^{t+j}) - \sum_{z_{t+j+1}} \pi(z_{t+j+1}|z_{t+j}) \mathbf{a}_{t+j+1}(a_0, z^{t+j+1}) \right]$$
$$\mathbf{a}_{t+j+1}(a_0, z^{t+j+1}) \geq 0 \qquad \forall j \geq 0$$
$$\mathbf{a}_t(a_0, z^t) = \mathbf{a}_{s,t}$$

where $\mathbf{c}_{t+j}(a_0, z^{t+j}) = w_{t+j} z_{t+j} + R_{t+j}\mathbf{a}_{t+j}(a_0, z^{t+j}) - \sum_{z_{t+j+1}} \pi(z_{t+j+1}|z_{t+j}) \mathbf{a}_{t+j+1}(a_0, z^{t+j+1}) \geq 0$ is substituted. We apply Proposition 4.2 in Kamihigashi (2005).[42] A transversality con-

[42] Assumptions 3.1–3.8 are satisfied in this environment. Let $(\Omega, \mathcal{F}, P)$ be a probability space and F be the set of all functions from Ω to $\mathbb{R}^n$. 3.1: $\mathbf{a}_{s,t} \in F$ and $X_j := \{(\mathbf{a}_{t+j}(a_0, z^{t+j}), \mathbf{a}_{t+j+1}(a_0, z^{t+j+1})); \mathbf{a}_{t+j+1}(a_0, z^{t+j+1}) \geq 0, \mathbf{c}_{t+j}(a_0, z^{t+j}) \geq 0\} \subset F \times F$ for all $j \geq 0$.

dition is given by:[43]

$$\lim_{j\to\infty}\beta^j\sum_{z^{t+j}}\pi(z^{t+j})\frac{R_{t+j}}{\mathbf{c}_{t+j}(a_0,z^{t+j})}\mathbf{a}_{t+j}(a_0,z^{t+j})=0. \tag{311}$$

Since $\pi(z^{t+j})\frac{R_{t+j}}{\mathbf{c}_{t+j}(a_0,z^{t+j})}\mathbf{a}_{t+j}(a_0,z^{t+j})\geq 0$ in all possible states z^{t+j}, a following condition holds in the state z^{t+j} with $z_t=z_{t+1}=\cdots=z_{t+j}=0$:

$$\lim_{j\to\infty}\beta^j(1-\nu)^j\frac{R_{t+j}}{\mathbf{c}_{t+j}(a_0,z^{t+j};z_t=\cdots=z_{t+j}=0)}\mathbf{a}_{t+j}(a_0,z^{t+j};z_t=\cdots=z_{t+j}=0)=0. \tag{312}$$

By substituting $\mathbf{c}_{t+j}(a_0,z^{t+j};z_t=\cdots=z_{t+j}=0)=\left[\prod_{\tau=1}^{j}\beta R_{t+\tau}\right]\mathbf{c}_t(a_0,z^t)$ and $\mathbf{a}_{t+j}(a_0,z^{t+j};z_t=\cdots=z_{t+j}=0)=\mathbf{a}_{s+j,t+j}$, we obtain:

$$\frac{R_t}{\mathbf{c}_{s,t}}\lim_{j\to\infty}\frac{(1-\nu)^j}{\prod_{\tau=0}^{j-1}R_{t+\tau}}\mathbf{a}_{s+j,t+j}=0 \tag{313}$$

Given that $\frac{R_t}{\mathbf{c}_{s,t}}$ is finite, as $\mathbf{c}_{s,t}=0$ gives negative infinite utility and cannot be optimal, we establish the transversality condition given by (**??**). □

B.3 Additional Discussion about Transitional Dynamics

B.3.1 A Condition for $\left(\beta R_1\frac{w_0}{w_1}\right)\left(\beta R_2\frac{w_1}{w_2}\right)<1$

Sufficiency of $\left(\beta R_1\frac{w_0}{w_1}\right)\left(\beta R_2\frac{w_1}{w_2}\right)<1$ We derived a condition for $\beta R_{t+1}<\frac{w_{t+1}}{w_t}\ \forall t\geq 0$ in Appendix A.3.4, which is sufficient for households not to save for a high-income state at any $t\geq 0$. However, a condition $\beta R_1<\frac{w_1}{w_0}$ is not necessary, as the household's saving decision between $t=0$ and $t=1$ is not impacted by the negative MIT shock at $t=1$. Nonetherless, the negative shock at $t=1$ lowers wage, w_1, and increases the amount of deflated assets held by low-income households:

$$a_1(a_0,z^1;z_1=0)=\frac{\mathbf{a}_1(a_0,z^1;z_1=0)}{w_1}, \tag{314}$$

3.2: $\beta\in(0,1)$, and $\log(\cdot)$ is C^1 on $\mathbb{R}_{++}$, concave, and strictly increasing. 3.3: Define $g_{t+j}(\mathbf{a}_{t+j},\mathbf{a}_{t+j+1}):=w_{t+j}z_{t+j}+R_{t+j}\mathbf{a}_{t+j}(a_0,z^{t+j})-\sum_{z_{t+j+1}}\pi(z_{t+j+1}|z_{t+j})\mathbf{a}_{t+j+1}(a_0,z^{t+j+1})$. For all $(\mathbf{a}_{t+j},\mathbf{a}_{t+j+1})\in X_j$, $g_{t+j}(\mathbf{a}_{t+j},\mathbf{a}_{t+j+1})\geq 0$, $g_{t+j}:\Omega\to\mathbb{R}_+$ is measurable, and $\mathbb{E}[\log(\mathbf{c}_{t+j}(a_{s,t},z^{t+j}))]$ exists in $[-\infty,\infty)$. 3.4: An optimal path $\{\mathbf{a}^*_{t+j}\}$ exists. 3.5: $g_{t+j,2}(\mathbf{a}^*_{t+j},\mathbf{a}^*_{t+j+1};-\mathbf{a}^*_{t+j+1})$ is measurable. 3.6: For all $j\geq 0$, $\exists\underline{\lambda}_{t+j}\in[0,1),\forall\lambda\in[\lambda_{t+j},1),(\mathbf{a}^*_{t+j},\lambda\mathbf{a}^*_{t+j+1})\in X_j$ and $\forall\tau\geq t+j+1,(\lambda\mathbf{a}^*_\tau,\lambda\mathbf{a}^*_{\tau+1})\in X_\tau$. 3.7: $g_{t+j}(\mathbf{a}^*_{t+j},\mathbf{a}^*_{t+j+1})>0$ and $g_{t+j}(\lambda\mathbf{a}^*_{t+j},\lambda\mathbf{a}^*_{t+j+1})$ is concave in $\lambda\in[\underline{\lambda}_0,1]$. 3.8: $g_{t+j}(\mathbf{a}^*_{t+j},\lambda\mathbf{a}^*_{t+j+1})$ is nonincreasing and continuous in $\lambda\in[\underline{\lambda}_j,1]$. Finally, the logarithmic utility is assumed.

[43] We use a familar form: $\lim_{j\to\infty}\beta^j\mathbb{E}[v_1(\mathbf{a}^*_{t+j},\mathbf{a}^*_{t+j+1})\mathbf{a}^*_{t+j}]=0$, which holds given the Kuhn-Tucker condition.

where $\mathbf{a}_1(a_0, z^1; z_1 = 0)$ is pre-determined at $t = 0$. We need a condition for low-income households at $t = 1$ not to save for a high-income state at $t = 2$.

We derive the condition following the same logic as before. If a low-income household at $t = 1$ cannot achieve higher utility in a high-income state at $t = 2$ (z^2 with $z_2 = \zeta$), then she cannot achieve higher utility in future periods as well by saving for the state (z^2). Hence, we need a condition for:

$$\beta R_2 \mathbf{c}_1(a_1, z^1) < w_2 c_0^*, \tag{315}$$

where the left hand side is consumption at $t = 2$ if the low-income household at $t = 1$ saves for the next high-income state. We will use an upper bound on $\mathbf{c}_1(a_1, z^1)$ given a_1 and an upper bound on a_1 given initial states. In the end, a sufficient condition for (315) is given by:

$$\left(\beta R_1 \frac{w_0}{w_1}\right) \left(\beta R_2 \frac{w_1}{w_2}\right) < 1 \tag{316}$$

First, we set up a maximization problem of a low-income household at $t = 1$:

$$\max_{\{\mathbf{c}_t(a_1,z^t),\mathbf{a}_{t+1}(a_1,z^{t+1})\}_{t=1}^{\infty}} \quad \sum_{t=1}^{\infty} \sum_{z^t} \beta^t \pi(z^t) \log(\mathbf{c}_t(a_1, z^t)) \tag{317}$$

$$s.t. \quad \mathbf{c}_t(a_1, z^t) + \sum_{z_{t+1}} \pi(z_{t+1}|z_t) \mathbf{a}_{t+1}(a_1, z^{t+1}) = w_t z_t + R_t \mathbf{a}_t(a_1, z^t) \tag{318}$$

$$\mathbf{a}_{t+1}(a_1, z^{t+1}) \geq 0 \qquad \forall t \geq 1$$

If the saving for the next high-income state is positive, $\mathbf{a}_2(a_1, z^2; z_2 = \zeta) = \epsilon > 0$, the budget constraint at $t = 1$ is given by:

$$\mathbf{c}_1(a_1, z^1) + \nu \underbrace{\mathbf{a}_2(a_1, z^2; z_2 = \zeta)}_{=\epsilon} + (1 - \nu) \mathbf{a}_2(a_1, z^2; z_2 = 0) = R_1 \mathbf{a}_1(a_1, z^1). \tag{319}$$

Since households need to secure consumption in future low-income states, sequentially imposing equation (298) gives:

$$\begin{aligned}
\mathbf{a}_2(a_0, z^2; z_2 = 0) &\geq \frac{\mathbf{c}_2(a_1, z^2; z_2 = 0)}{R_2} + (1 - \nu) \frac{1}{R_2} \mathbf{a}_3(a_0, z^3; z_3 = 0) \\
&\geq \frac{\mathbf{c}_2(a_1, z^2; z_2 = 0)}{R_2} \left[1 + \beta(1 - \nu) + \beta^2(1 - \nu)^2 + \cdots\right] \\
&= \frac{\beta}{1 - \beta(1 - \nu)} \mathbf{c}_1(a_1, z^1).
\end{aligned}$$

Hence, equation (319) becomes:

$$\mathbf{c}_1(a_1, z^1) \leq [1 - (1 - \nu)\beta] \left[R_1 \mathbf{a}_1(a_1, z^1) - \nu\epsilon\right]. \tag{320}$$

We use this upper bound on $\mathbf{c}_1(a_1, z^1)$ to derive a condition for (315).[44] Using (320) and $c_0^* := \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}\zeta$, (315) holds if

$$\begin{aligned} \beta R_2[1-(1-\nu)\beta]R_1\mathbf{a}_1(a_1, z^1) &< w_2 c_0^* \\ \Leftrightarrow \ \mathbf{a}_1(a_1, z^1) &< \frac{w_2}{\beta R_2}\frac{1}{\beta R_1}\frac{\beta\zeta}{1-(1-\nu-\xi)\beta} \end{aligned} \tag{321}$$

Since the maximum saving of low-income households at $t = 1$ is:[45]

$$\mathbf{a}_1(a_0 = 0, z_0 = \zeta, z_1 = 0) = \frac{\beta\zeta}{1-(1-\nu-\xi)\beta}w_0,$$

the condition (321) is given by:

$$\begin{aligned} w_0 &< \frac{w_2}{\beta R_2}\frac{1}{\beta R_1} \\ \Leftrightarrow \left(\beta R_1 \frac{w_0}{w_1}\right) \left(\beta R_2 \frac{w_1}{w_2}\right) &< 1 \end{aligned}$$

A Condition for $\left(\beta R_1 \frac{w_0}{w_1}\right) \left(\beta R_2 \frac{w_1}{w_2}\right) < 1$ It is possible for low-income households to consume more than high-income households at the time of MIT shock ($t = 1$) but still not want to save for high-income state at $t = 2$. The most relevant case is a household with ($z_0 = \zeta, z_1 = 0, z_2 = \zeta$). If she saves for $t = 2$, her consumption is given by:

$$\begin{aligned} &(\beta R_2 \frac{w_1}{w_2})\ c_1(a_0, z^1; z_1 = 0) \\ &= (\beta R_2 \frac{w_1}{w_2})(\beta R_1 \frac{w_0}{w_1}) c_0^* \end{aligned}$$

If she does not save for high-income state at $t = 2$, her consumption is given by c_0^*. If $\beta R_1 \frac{w_0}{w_1} > 1$, $c_1(a_0, z^1; z_1 = 0) > c_0^*$, i.e., her consumption at $t = 1$ in a low-income state is higher than c_0^*. However, if $(\beta R_2 \frac{w_1}{w_2})(\beta R_1 \frac{w_0}{w_1}) < 1$, she is not better off by saving for $t = 2$.

[44] If the saving for the next high-income state is positive, $\mathbf{a}_2(a_1, z^2; z_2 = \zeta) = \epsilon > 0$, $\mathbf{c}_2(a_1, z^2; z_2 = \zeta)$ is determined by the Euler equation. This gives lower utility than saving zero assets for the state if

$$\beta R_2 \mathbf{c}_1(a_1, z^1) < w_2 c_0^*.$$

[45] The maximum saving of low-income agents at $t = 1$ is given in a state either ($z_0 = \zeta, z_1 = 0$) or ($z_0 = 0, a_0 = \bar{a}_0$). For each case,

$$\begin{aligned} \mathbf{a}_1(a_0 = 0, z_0 = \zeta, z_1 = 0) &= \frac{\beta\zeta}{1-(1-\nu-\xi)\beta}w_0 \\ \mathbf{a}_1(a_0 = \bar{a}_0, z_0 = 0, z_1 = 0) &= \beta R_1 \frac{\beta\zeta}{1-(1-\nu-\xi)\beta}w_0. \end{aligned}$$

Since $\beta R_1 < \beta R_0 < 1$, given a negative productivity shock and Assumption 3, $\mathbf{a}_1(a_0 = 0, z_0 = \zeta, z_1 = 0) > \mathbf{a}_1(a_0 = \bar{a}_0, z_0 = 0, z_1 = 0)$. Threfore, we focus on households with ($z_0 = \zeta, z_1 = 0$).

A sufficient condition for $\beta R_2 \frac{w_1}{w_2} < 1$ is given by Prop. 8 ($\frac{A_1}{A_0} > \frac{A_1}{A_0}$), and the condition for $\beta R_1 \frac{w_0}{w_1} < 1$ is given by Prop. 8' ($\frac{A_1}{A_0} > \frac{A_1'}{A_0}$), where $\frac{A_1}{A_0} < \frac{A_1'}{A_0} < 1$. So, we could find $\frac{A_1''}{A_0}$ with $\frac{A_1}{A_0} < \frac{A_1''}{A_0} < \frac{A_1'}{A_0} < 1$ such that

$$(\beta R_2 \frac{w_1}{w_2})(\beta R_1 \frac{w_0}{w_1}) < 1 \text{ for all } A_1 \text{ with } \frac{A_1''}{A_0} < \frac{A_1}{A_0} < 1,$$
$$\text{where } (\beta R_2 \frac{w_1}{w_2})(\beta R_1 \frac{w_0}{w_1}) = \beta \left[\frac{\theta + (1-\delta)\left(\frac{K_2}{A_2}\right)^{1-\theta}}{\hat{s} + (1-\hat{\delta})\left(\frac{K_1}{A_1}\right)^{1-\theta}} \right] \beta \left[\frac{\theta + (1-\delta)\left(\frac{K_1}{A_1}\right)^{1-\theta}}{\hat{s} + (1-\hat{\delta})\left(\frac{K_0}{A_0}\right)^{1-\theta}} \right]$$

To derive a sufficient condition on A_1 for $(\beta R_2 \frac{w_1}{w_2})(\beta R_1 \frac{w_0}{w_1}) < 1$ in case of a negative MIT shock at $t = 1$, we use a property that $\frac{K_2}{A_2} = \frac{K_2}{A_1} < \frac{K_1}{A_1}$. Hence, it is sufficient to have:

$$\beta \left[\frac{\theta + (1-\delta)\left(\frac{K_1}{A_1}\right)^{1-\theta}}{\hat{s} + (1-\hat{\delta})\left(\frac{K_1}{A_1}\right)^{1-\theta}} \right] \beta \left[\frac{\theta + (1-\delta)\left(\frac{K_1}{A_1}\right)^{1-\theta}}{\hat{s} + (1-\hat{\delta})\left(\frac{K_0}{A_0}\right)^{1-\theta}} \right] < 1, \tag{322}$$

where $\left(\frac{K_1}{A_1}\right)^{1-\theta} = \left(\frac{A_0}{A_1}\right)^{1-\theta} \left(\frac{K_0}{A_0}\right)^{1-\theta}$. Hence (322) is written as:

$$\beta \left[\frac{\theta + (1-\delta)X\left(\frac{K_0}{A_0}\right)^{1-\theta}}{\hat{s} + (1-\hat{\delta})X\left(\frac{K_0}{A_0}\right)^{1-\theta}} \right] \beta \left[\frac{\theta + (1-\delta)X\left(\frac{K_0}{A_0}\right)^{1-\theta}}{\hat{s} + (1-\hat{\delta})\left(\frac{K_0}{A_0}\right)^{1-\theta}} \right] < 1, \tag{323}$$

where we denote $X := \left(\frac{A_0}{A_1}\right)^{1-\theta}$. This is equivalent to:

$$\beta^2 \left[\theta + (1-\delta)X\left(\frac{K_0}{A_0}\right)^{1-\theta} \right]^2 < \left[\hat{s} + (1-\hat{\delta})X\left(\frac{K_0}{A_0}\right)^{1-\theta} \right] \left[\hat{s} + (1-\hat{\delta})\left(\frac{K_0}{A_0}\right)^{1-\theta} \right],$$
$$\text{where } \left(\frac{K_0}{A_0}\right)^{1-\theta} = \frac{\xi(1-\theta) + \beta\theta(1-\nu)\left(\xi + \nu + \frac{1}{\beta} - 1\right)}{[1-(1-\delta)\beta(1-\nu)]\left(\xi + \nu + \frac{1}{\beta} - 1\right)}.$$

A threshold value of X solves a quadratic equation:

$$\beta^2(1-\delta)^2\left(\frac{K_0}{A_0}\right)^{2-2\theta} X^2 + \left[2\beta^2\theta(1-\delta)\left(\frac{K_0}{A_0}\right)^{1-\theta} - (1-\hat{\delta})\left(\frac{K_0}{A_0}\right)^{1-\theta} \left[\hat{s} + (1-\hat{\delta})\left(\frac{K_0}{A_0}\right)^{1-\theta} \right] \right] X$$
$$+ \beta^2\theta^2 - \hat{s}\left[\hat{s} + (1-\hat{\delta})\left(\frac{K_0}{A_0}\right)^{1-\theta} \right] < 0 \tag{324}$$

Proposition 24. *Let the economy be in a stationary equilibrium with $\beta R_0 < 1$. After a negative MIT shock at $t = 1$, low-income households at $t = 1$ do not save for the next high-income state, i.e., $\mathbf{a}_2(a_0, z^2; z_1 = 0, z_2 = \zeta) = 0$, if $A_1 \in (\underline{A_1''}, A_0]$ holds. A threshold value $\underline{A}_1''$ is given by $\frac{A_1''}{A_0} = (\bar{X})^{\frac{1}{\theta-1}}$, where $\bar{X}$ solves a quadratic equation (324):*

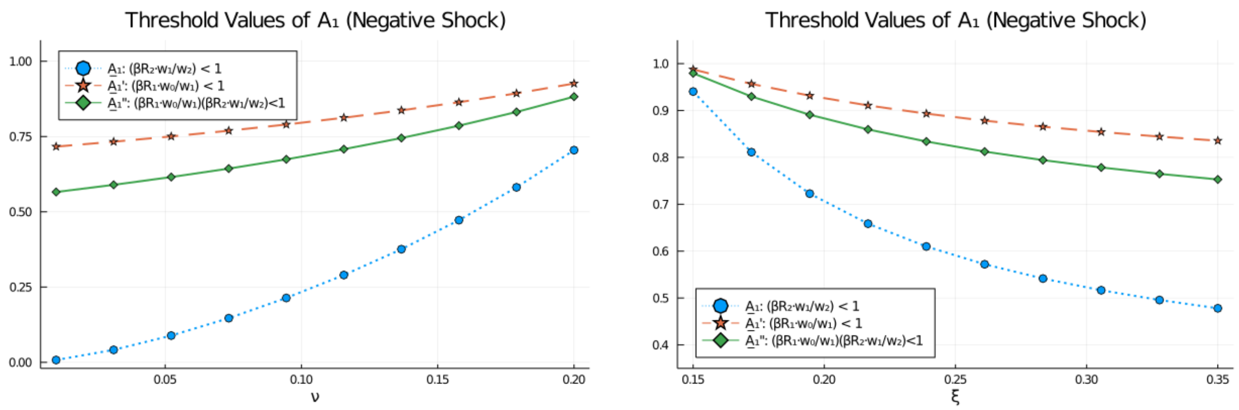


Figure 11: Threshold values of A_1 for a sequence of ν and ξ

$$
\begin{aligned}
\bar{X} = &\Bigg(- \beta(1-\delta)\left(\frac{K_0}{A_0}\right)^{1-\theta}\left[2\beta\theta - (1-\nu)\left[\hat{s} + (1-\hat{\delta})\left(\frac{K_0}{A_0}\right)^{1-\theta}\right]\right] \\
&+ \sqrt{\left\{\beta(1-\delta)\left(\frac{K_0}{A_0}\right)^{1-\theta}\left[2\beta\theta - (1-\nu)\left[\hat{s} + (1-\hat{\delta})\left(\frac{K_0}{A_0}\right)^{1-\theta}\right]\right]\right\}^2 - 4\beta^2(1-\delta)^2\left(\frac{K_0}{A_0}\right)^{2-2\theta}\left(\beta^2\theta^2 - \hat{s}\left[\hat{s} + (1-\hat{\delta})\left(\frac{K_0}{A_0}\right)^{1-\theta}\right]\right)}\ \Bigg) \\
&\Bigg/ \left(2\beta^2(1-\delta)^2\left(\frac{K_0}{A_0}\right)^{2-2\theta}\right), \qquad (325) \\
&\textit{where } \left(\frac{K_0}{A_0}\right)^{1-\theta} = \frac{\xi(1-\theta) + \beta\theta(1-\nu)\left(\xi + \nu + \frac{1}{\beta} - 1\right)}{\left[1 - (1-\delta)\beta(1-\nu)\right]\left(\xi + \nu + \frac{1}{\beta} - 1\right)}.
\end{aligned}
$$

A numerical example in Figure 11 shows that Proposition 24 gives a less tight condition on A_1 ($\frac{\underline{A}_1''}{A_0} < \frac{A_1}{A_0} < 1$) than Proposition 7 ($\frac{\underline{A}_1'}{A_0} < \frac{A_1}{A_0} < 1$).

B.3.2 Numerical Examples

Figure 12 summarizes the discussion about sufficient conditions for $\beta R_{t+1}\frac{w_t}{w_{t+1}} < 1$ at all $t \geq 0$. A blue line with circles represents $\beta R_1 \frac{w_0}{w_1}$ for a range of A_1. Since $\beta R_1 \frac{w_0}{w_1}$ is decreasing in A_1, $A_1 > \underline{A}_1'$ guarantees $\beta R_1 \frac{w_0}{w_1} < 1$.[46] A green line with stars stands for $\beta R_2 \frac{w_1}{w_2}$ for different A_1. We have shown that $A_1 \in (\underline{A}_1, \bar{A}_1)$ guarantees $\beta R_{t+1}\frac{w_t}{w_{t+1}} < 1$ for all $t \geq 1$ (Proposition 6 and 18), where $\underline{A}_1 < \underline{A}_1'$ (Proposition 7). Hence, $A_1 \in (\underline{A}_1', \bar{A}_1)$ is sufficient for $\beta R_{t+1}\frac{w_t}{w_{t+1}} < 1$ for all $t \geq 1$, which is consistent with the greenline being below 1 for any $A_1 \in (\underline{A}_1', \bar{A}_1)$. Finally, a condition $A_1 > \underline{A}_1''$ implies that low-income

[46] Since equation (137) implies

$$\beta R_1 \frac{w_0}{w_1} = \beta \left[\frac{\theta + (1-\delta)\left(\frac{K_1}{A_1}\right)^{1-\theta}}{\hat{s} + (1-\hat{\delta})\left(\frac{K_0}{A_0}\right)^{1-\theta}}\right],$$

with $K_1 = K_0$, $\beta R_1 \frac{w_0}{w_1}$ is decreasing in A_1.

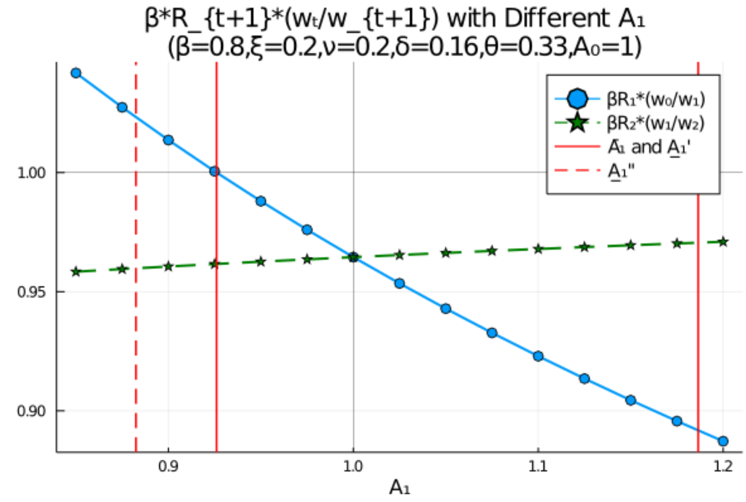


Figure 12: Plot of $\beta R_{t+1}\frac{w_t}{w_{t+1}}$ at $t=0$ and $t=1$

households at $t = 1$ do not save for the next high-income state (Proposition 24). As far as $\underline{A}_1'' > \underline{A}_1$, $A_1 \in (\underline{A}_1'', \bar{A}_1)$ gives a less tight condition than $A_1 \in (\underline{A}_1', \bar{A}_1)$ for the insurance contract to be optimal.

Figure 13 illustrates how the sufficient conditions on A_1 vary as ξ and ν change. In the top left panel, the idiocyncratic shock is *iid* ($\xi + \nu = 1$). As the probability from low-income state to high-income state, ν, increases, the range of A_1 sufficient for $\beta R_{t+1}\frac{w_t}{w_{t+1}} < 1 \forall t \geq 0$ shrinks. An intuition is that aggregate capital supply decreases with ν, which leads to higher interest rate in the steady state. On the other hand, the increase in ξ widens the range of A_1 sufficient for $\beta R_{t+1}\frac{w_t}{w_{t+1}} < 1 \forall t \geq 0$.

We have a conjecture that $\beta R_2 \frac{w_1}{w_2}$ is increasing in A_1 as far as idiosyncratic income shock is either *iid* ($\xi + \nu = 1$) or positively correlated ($\xi + \nu < 1$) over time. The numerical investigation supports this conjecture. First, Figure 14 describes how the plot of $\beta R_2 \frac{w_1}{w_2}$ with respect to A_1 depends on the persistence of income shocks. We see that the slope becomes less steep as the persistence becomes smaller. If income shocks are positively correlated, meaning that agents are likely to stay in the same income state, the slope is positive. Second, we draw many parameters from $(\beta, \xi, \nu, \delta, \theta) \in [0.01, 0.99]^5$ and check in which case we have a negative slope on $\beta R_2 \frac{w_1}{w_2}$ at $A_1 = A_0$. After drawing 30^5 parameters from a uniform grid, we found 4.6 million cases with a negative slope of $\beta R_2 \frac{w_1}{w_2}$ at $A_1 = A_0$. The mean, minimum, and maximum of $\xi + \nu$ are 1.33, 1.03, and 1.98, respectively. Since $\xi + \nu > 1$ for all of these parameters, all cases found with a negative slope have negatively correlated income shocks. Similarly, we draw 30^4 parameters for the iid case ($\xi + \nu = 1$), but none of them has a negative slope. These numerical results infer that $\beta R_2 \frac{w_1}{w_2}$ has a negative slope only when income shocks are negatively correlated.

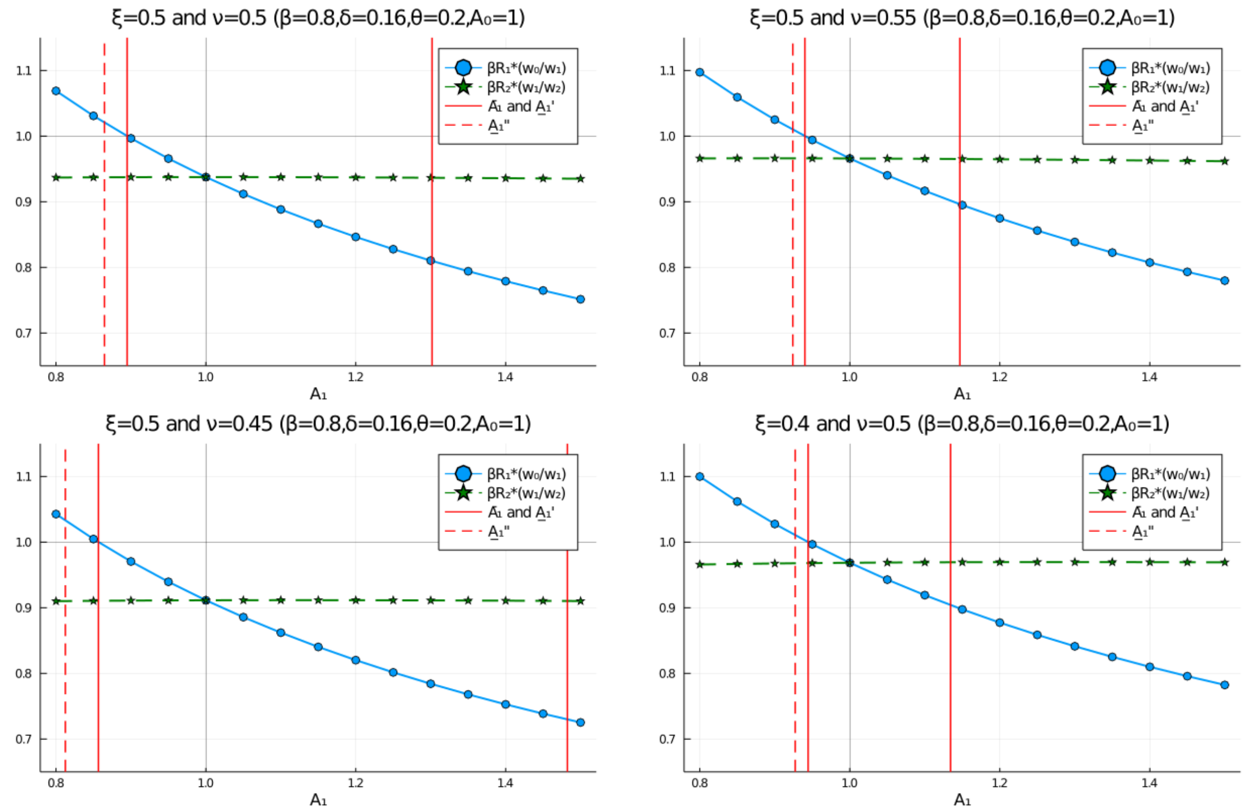


Figure 13: The range of A_1 sufficient for $\beta R_{t+1} \frac{w_t}{w_{t+1}} < 1 \; \forall t \geq 0$

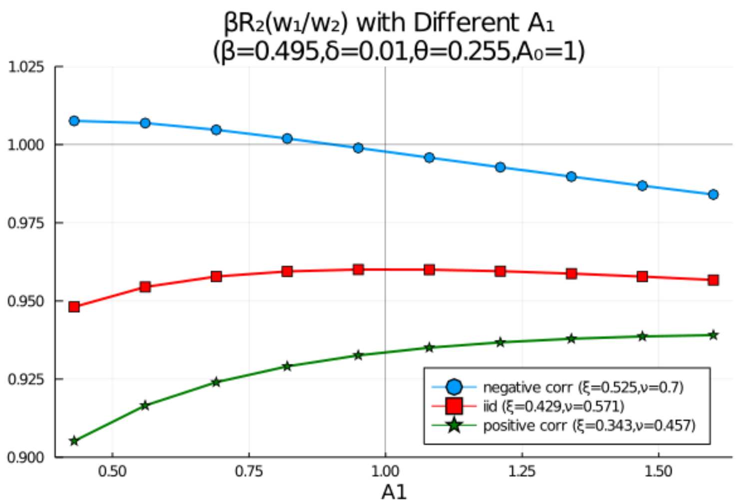


Figure 14: Change in the persistence of income shocks

B.3.3 Comparative Statics

To conclude this section, we discuss comparative statics about the range of A_1, $(\underline{A}_1', \bar{A}_1)$, satisfying the sufficient conditions for $\beta R_{t+1} < \frac{w_t}{w_{t+1}}$ for all $t \geq 0$. In particular, we focus on the comparative statics with respect to ν and ξ, where ν is the probability from low-income state to high-income state and ξ is the probability from high-income state to low-income state.

We have analytical results about the comparative statics of $\bar{A}_1$. Since $\bar{A}_1$ is given by equation (135):

$$\frac{\bar{A}_1}{A_0} = \left(\frac{\frac{1}{\beta} - 1 + \delta}{R_0 - 1 + \delta} \right)^{\frac{1}{1-\theta}},$$

the comparative statics of R_0 gives the comparative statics of $\bar{A}_1$. R_0 is written as:

$$R_0 = \frac{\xi(1-\theta)(1-\delta) + \theta(\xi + \nu + \frac{1}{\beta} - 1)}{\xi(1-\theta) + \beta\theta(1-\nu)(\xi + \nu + \frac{1}{\beta} - 1)},$$

where $\theta(\xi + \nu + \frac{1}{\beta} - 1)$ in the numerator is increasing in ν and $\beta\theta(1-\nu)(\xi + \nu + \frac{1}{\beta} - 1)$ in the denominator is decreasing in ν. Hence, R_0 is increasing in ν, and $\bar{A}_1$ is decreasing in ν. We derive the comparative statics of $\bar{A}_1$ with respect to ξ directly from equation (67):

$$\begin{aligned}
\frac{\bar{A}_1}{A_0} &= \left[\frac{1-\beta(1-\delta)}{\theta} \; \frac{\xi(1-\theta) + (1-\nu)\theta\left[1 - (1-\nu-\xi)\beta\right]}{\left[1-(1-\nu)\beta(1-\delta)\right]\left[1-(1-\nu-\xi)\beta\right]} \right]^{\frac{1}{1-\theta}} \\
&= \left[\frac{1-\beta(1-\delta)}{\theta} \; \frac{\frac{1-\theta}{\frac{1}{\xi}[1-(1-\nu)\beta]+\beta} + (1-\nu)\theta}{\left[1-(1-\nu)\beta(1-\delta)\right]} \right]^{\frac{1}{1-\theta}}
\end{aligned}$$

We see that $\bar{A}_1$ is increasing in ξ.

We don't provide analytical results about comparative statics of $\underline{A}_1'$ with respect to ν and ξ, but Figure 11 implies that $\underline{A}_1'$ is increasing in ν and decreasing in ξ.

B.3.4 Monotone Sequence of $\{A_t\}_{t=1}^{\infty}$

We derive sufficient conditions on a monotone sequence of $\{A_t\}_{t=1}^{\infty}$ such that $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ holds at all $t \geq 1$. It turns our that if $\{\frac{K_t}{A_t}\}_{t=1}^{\infty}$ is also monotone, finding a sufficient condition at $t = 1$ suffices for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 1$. Since the aggregate capital at $t = 1$ (K_1) is pre-determined in the steady state at $t = 0$, we can derive a condition on A_1 in closed form, as we did in Section 5.2.2 for a permanent productivity shock. Once we find a sufficient condition on $\{A_t\}_{t=2}^{\infty}$ for the monotonicity of $\{\frac{K_t}{A_t}\}_{t=1}^{\infty}$, such a condition on $\{A_t\}_{t=2}^{\infty}$ and the condition on A_1 together guarantee $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 1$.

Preparations for the Sufficient Conditions The economy is in a steady state at $t = 0$ with $\beta R^* < 1$ under Assumption 5. At $t = 1$, a new path of $\{A_t\}_{t=1}^{\infty}$ is realized. We want to derive conditions on $\{A_t\}_{t=1}^{\infty}$ that guarantee $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all $t \geq 1$. We first express $\beta R_{t+1}\frac{w_t}{w_{t+1}}$ in terms of capital:

$$\beta R_{t+1}\frac{w_t}{w_{t+1}} = \beta\left[\theta A_{t+1}^{1-\theta}K_{t+1}^{\theta-1} + 1 - \delta\right]\frac{(1-\theta)A_t^{1-\theta}K_t^{\theta}}{(1-\theta)A_{t+1}^{1-\theta}K_{t+1}^{\theta}}. \tag{326}$$

By using the law of motion of capital:

$$\begin{aligned} K_{t+1} &= \left[\frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta) + (1-\nu)\beta\theta\right]A_t^{1-\theta}K_t^{\theta} + (1-\nu)\beta(1-\delta)K_t \\ &=: \hat{s}A_t^{1-\theta}K_t^{\theta} + (1-\hat{\delta})K_t, \end{aligned}$$

we can derive following expressions:

$$\frac{K_{t+1}}{A_{t+1}} = \frac{A_t}{A_{t+1}}\left[\hat{s}\left(\frac{K_t}{A_t}\right)^{\theta} + (1-\hat{\delta})\frac{K_t}{A_t}\right], \tag{327}$$

$$\beta R_{t+1}\frac{w_t}{w_{t+1}} = \beta\left[\frac{\theta + (1-\delta)\left(\frac{K_{t+1}}{A_{t+1}}\right)^{1-\theta}}{\hat{s} + (1-\hat{\delta})\left(\frac{K_t}{A_t}\right)^{1-\theta}}\right]. \tag{328}$$

We first show that the monotonicity of $\{\frac{K_t}{A_t}\}$ give rise to analytically tractable sufficieint conditions for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 1$.

Lemma 22 (Sufficieint condition for $\beta R_{t+1}\frac{w_t}{w_{t+1}} < 1$ at all $t \geq 1$)**.**

1. Suppose $\{A_t\}_{t=1}^{\infty}$ and $\{\frac{K_t}{A_t}\}_{t=1}^{\infty}$ are monotonically increasing over time (i.e., $A_t \leq A_{t+1}$ and $\frac{K_t}{A_t} < \frac{K_{t+1}}{A_{t+1}}$ for all $t \geq 1$). Then, $\beta R_1 < 1$ is sufficient to guarantee $\beta R_{t+1}\frac{w_t}{w_{t+1}} < 1$ for all $t \geq 1$. This gives a condition: $A_1 < \bar{A}_1$.

2. Suppose $\{\frac{K_t}{A_t}\}_{t=1}^{\infty}$ is monotonically decreasing over time (i.e., $\frac{K_{t+1}}{A_{t+1}} < \frac{K_t}{A_t}$ for all $t \geq 1$). Then, the following condition is sufficient to guarantee $\beta R_{t+1}\frac{w_t}{w_{t+1}} < 1$ for all $t \geq 1$:

$$\left(\frac{K_1}{A_1}\right)^{1-\theta} < \frac{1}{\nu(1-\delta)}\left[\frac{\xi(1-\theta)}{1-(1-\nu-\xi)\beta} - \nu\theta\right]. \tag{329}$$

This gives a condition: $A_1 > \underline{A_1}$.

Proof. 1. Since $w_t = (1-\theta)A_t\left(\frac{K_t}{A_t}\right)^{\theta}$, monotonicity of $\{A_t\}_{t=1}^{\infty}$ and $\{\frac{K_t}{A_t}\}_{t=1}^{\infty}$ imply monotonicity of w_t. If A_t and $\frac{K_t}{A_t}$ are weakly increasing over time, $\frac{w_{t+1}}{w_t} \geq 1$ for all $t \geq 1$. Then, $\beta R_{t+1} < 1$ is a sufficient condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$. Since $R_{t+1} = \theta\left(\frac{K_{t+1}}{A_{t+1}}\right)^{\theta-1} + 1 - \delta$, monotone increase of $\frac{K_t}{A_t}$ implies monotone decrease of R_t. Therefore, $\beta R_2 < 1$ is sufficient for $\beta R_{t+1} < 1$ for all $t \geq 1$. Furthermore, $\frac{K_1}{A_1} < \frac{K_2}{A_2}$ implies $R_1 > R_2$. Thus, $\beta R_1 < 1$ is sufficient for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 1$. Proposition 6 shows $\beta R_1 < 1$ if $A_1 < \bar{A}_1$, where

$$\frac{\bar{A}_1}{A_0} = \left[\frac{1-\beta(1-\delta)}{\theta}\,\frac{\xi(1-\theta) + (1-\nu)\theta\left[1-(1-\nu-\xi)\beta\right]}{\left[1-(1-\nu)\beta(1-\delta)\right]\left[1-(1-\nu-\xi)\beta\right]}\right]^{\frac{1}{1-\theta}} > 1.$$

2. Using equation (328), the condition $\beta R_{t+1}\frac{w_t}{w_{t+1}} < 1$ is written as:

$$\beta R_{t+1}\frac{w_t}{w_{t+1}} = \beta \left[\frac{\theta + (1-\delta)\left(\frac{K_{t+1}}{A_{t+1}}\right)^{1-\theta}}{\hat{s} + (1-\hat{\delta})\left(\frac{K_t}{A_t}\right)^{1-\theta}}\right] < 1$$

$$\Leftrightarrow \left(\frac{K_{t+1}}{A_{t+1}}\right)^{1-\theta} - (1-\nu)\left(\frac{K_t}{A_t}\right)^{1-\theta} < \frac{1}{1-\delta}\left[\frac{\xi(1-\theta)}{1-(1-\nu-\xi)\beta} - \nu\theta\right], \tag{330}$$

where we know the RHS is strictly positive under Assumption 5 (Lemma 9). The equation (330) can be written as:

$$\underbrace{\left(\frac{K_{t+1}}{A_{t+1}}\right)^{1-\theta} - \left(\frac{K_t}{A_t}\right)^{1-\theta}}_{<0 \text{ if } \frac{K_{t+1}}{A_{t+1}} < \frac{K_t}{A_t}} + \nu\left(\frac{K_t}{A_t}\right)^{1-\theta} < \frac{1}{1-\delta}\left[\frac{\xi(1-\theta)}{1-(1-\nu-\xi)\beta} - \nu\theta\right].$$

Since $\frac{K_t}{A_t}$ is decreasing over time, $\left(\frac{K_t}{A_t}\right)^{1-\theta}$ is largest at $t = 1$. Since we have:

$$\left(\frac{K_{t+1}}{A_{t+1}}\right)^{1-\theta} - \left(\frac{K_t}{A_t}\right)^{1-\theta} + \nu\left(\frac{K_t}{A_t}\right)^{1-\theta} < \nu\left(\frac{K_t}{A_t}\right)^{1-\theta} \leq \nu\left(\frac{K_1}{A_1}\right)^{1-\theta} \quad \text{for all } t \geq 1,$$

a sufficient condition for (330) at all $t \geq 1$ is given by:

$$\left(\frac{K_1}{A_1}\right)^{1-\theta} < \frac{1}{\nu(1-\delta)}\left[\frac{\xi(1-\theta)}{1-(1-\nu-\xi)\beta} - \nu\theta\right].$$

Proposition 7 shows this is satisfied if $A_1 > \underline{A_1}$, where

$$\underline{A_1}/A_0 := \left[\frac{\beta\nu(1-\delta)}{1-(1-\delta)\beta(1-\nu)}\ \frac{\xi(1-\theta)+\beta\theta(1-\nu)(\xi+\nu+\frac{1}{\beta}-1)}{\xi(1-\theta)-\beta\theta\nu\left[\xi+\nu+\frac{1}{\beta}-1\right]}\right]^{\frac{1}{1-\theta}} < 1.$$

□

Next, we we derive sufficieint conditions for the monotonicity of $\{\frac{K_t}{A_t}\}$. Lemma 23 derives a condition on A_{t+1} as a function of K_t. Since K_t is an endogenous variable, we will use a first-order approximation to derive a sufficieint condition on $\{A_{t+1}\}$ as a function of exogenous parameters.

Lemma 23 (Monotonicity of $\frac{K_t}{A_t}$). *Consider an economy with $\frac{K_t}{A_t}$ at time t, where the steady-state value of capital over productivity is given by $\frac{K^*}{A^*}$. We have $\frac{K_{t+1}}{A_{t+1}} > \frac{K_t}{A_t}$ if and only if*

$$\frac{A_{t+1}}{A_t} < 1 + \hat{\delta}\left[\left(\frac{K^*/A^*}{K_t/A_t}\right)^{1-\theta} - 1\right]. \tag{331}$$

Proof. From a law of motion of capital (327), we know $\frac{K_{t+1}}{A_{t+1}} > \frac{K_t}{A_t}$ if and only if

$$\frac{K_{t+1}}{A_{t+1}} := \frac{A_t}{A_{t+1}} \left[\hat{s} \left(\frac{K_t}{A_t} \right)^{\theta} + (1-\hat{\delta}) \frac{K_t}{A_t} \right] > \frac{K_t}{A_t}$$

By dividing both sides by $\frac{K_t}{A_t}$ $(\neq 0)$, this is equivalent to:

$$\hat{s} \left(\frac{K_t}{A_t} \right)^{\theta-1} + 1 - \hat{\delta} > \frac{A_{t+1}}{A_t}.$$

In the steady state ($K_{t+1} = K_t = K^*$), we have:

$$\hat{s} = \hat{\delta} \left(\frac{K^*}{A^*} \right)^{1-\theta}.$$

By substituing this, we derive:

$$1 + \hat{\delta} \left[\left(\frac{K^*/A^*}{K_t/A_t} \right)^{1-\theta} - 1 \right] > \frac{A_{t+1}}{A_t}.$$

□

Lemma 23 means $\{\frac{K_t}{A_t}\}_{t=1}^{\infty}$ is monotonically increasing if the condition (331) holds for all $t \geq 1$. In order to derive a condition on $\{A_{t+1}\}$ without an endogenous variable K_t, we derive a first order approximation of $\tilde{k}_t := \frac{K_t/A_t}{K^*/A^*}$ and $\hat{k}_t := \frac{K^*/A^*}{K_t/A_t}$. They are used in case of an increasing and decreasing sequence of $\{A_t\}_{t=1}^{\infty}$, respectively.

Lemma 24 (First-Order Approximation)**.**

1. A first-order approximation of $\tilde{k}_t := \frac{K_t/A_t}{K^*/A^*}$ is given by:

$$\tilde{k}_t < 1 - \frac{A_1 - A_0}{A_t} \left[1 - \hat{\delta}(1-\theta) \right]^{t-1} - \sum_{u=1}^{t-1} \frac{A_{t-u+1} - A_{t-u}}{A_t} \left[1 - \hat{\delta}(1-\theta) \right]^{u-1}. \tag{332}$$

2. A first-order approximation of $\hat{k}_t := \frac{K^*/A^*}{K_t/A_t}$ is given by:

$$\hat{k}_t < 1 - \frac{A_t}{A_1} \left(\frac{A_0 - A_1}{A_0} \right) \left[1 - \hat{\delta}(1-\theta) \right]^{t-1} - \sum_{u=1}^{t-1} \frac{A_t}{A_{t-u}} \left(\frac{A_{t-u}}{A_{t-u+1}} - 1 \right) \left[1 - \hat{\delta}(1-\theta) \right]^{u-1}. \tag{333}$$

Proof. 1. We know the law of motion of $\frac{K_t}{A_t}$ by equation (327):

$$\frac{K_{t+1}}{A_{t+1}} = \frac{A_t}{A_{t+1}} \left[\hat{s} \left(\frac{K_t}{A_t} \right)^{\theta} + (1-\hat{\delta}) \frac{K_t}{A_t} \right], \text{ where } \hat{s} = \hat{\delta} \left(\frac{K^*}{A^*} \right)^{1-\theta}.$$

By dividing both hand sides by $\frac{K^*}{A^*}$, we have:

$$\frac{K_{t+1}/A_{t+1}}{K^*/A^*} = \frac{A_t}{A_{t+1}} \left[\hat{\delta} \left(\frac{K_t/A_t}{K^*/A^*} \right)^{\theta} + (1-\hat{\delta}) \left(\frac{K_t/A_t}{K^*/A^*} \right) \right]. \tag{334}$$

Denote $\tilde{k}_t := \frac{K_t/A_t}{K^*/A^*}$ with $\tilde{k}^* = 1$. We will approximate $f(\tilde{k}_t) := \hat{\delta}(\tilde{k}_t)^{\theta} + (1-\hat{\delta})\tilde{k}_t$ by Talor expansion:

$$f(\tilde{k}_t) = f(\tilde{k}^*) + f'(\tilde{k}^*)(\tilde{k}_t - \tilde{k}^*) + f''(\tilde{k}^*)(\tilde{k}_t - \tilde{k}^*)^2 + o(||\tilde{k}_t - \tilde{k}^*||^2), \tag{335}$$

where we have:

$$\begin{aligned}
f(\tilde{k}^*) &= \hat{\delta} + 1 - \hat{\delta} = 1 \\
f'(\tilde{k}_t) &= \hat{\delta}\theta(\tilde{k}_t)^{\theta-1} + 1 - \hat{\delta} \\
\rightarrow\ f'(\tilde{k}^*) &= 1 - \hat{\delta}(1-\theta) \\
f''(\tilde{k}_t) &= \hat{\delta}\theta(\theta-1)(\tilde{k}_t)^{\theta-2} \\
\rightarrow\ f''(\tilde{k}^*) &= -\theta(1-\theta)\hat{\delta}.
\end{aligned}$$

Therefore, we approximate $\tilde{k}_{t+1}$ by:

$$\tilde{k}_{t+1} = \frac{A_t}{A_{t+1}} \left[1 + [1-\hat{\delta}(1-\theta)](\tilde{k}_t - \tilde{k}^*) - \theta(1-\theta)\hat{\delta}(\tilde{k}_t - \tilde{k}^*)^2 + o(||\tilde{k}_t - \tilde{k}^*||^2) \right]. \tag{336}$$

By subtracting $\tilde{k}^*$ $(=1)$ from both sides, we have:

$$\tilde{k}_{t+1} - \tilde{k}^* = \left(\frac{A_t}{A_{t+1}} - 1 \right) + \frac{A_t}{A_{t+1}} \left[1 - \hat{\delta}(1-\theta) \right] (\tilde{k}_t - \tilde{k}^*) - \frac{A_t}{A_{t+1}} \theta(1-\theta)\hat{\delta}(\tilde{k}_t - \tilde{k}^*)^2 + o(||\tilde{k}_t - \tilde{k}^*||^2). \tag{337}$$

Since the third term is strictly negative, we derive the upper bound of $\tilde{k}_t - \tilde{k}^*$ as follows:

$$\begin{aligned}
\tilde{k}_t - \tilde{k}^* &= \frac{A_{t-1}}{A_t} - 1 + \frac{A_{t-1}}{A_t} \left[1 - \hat{\delta}(1-\theta) \right] (\tilde{k}_{t-1} - \tilde{k}^*) - \frac{A_{t-1}}{A_t} \theta(1-\theta)\hat{\delta}(\tilde{k}_{t-1} - \tilde{k}^*)^2 + o(||\tilde{k}_{t-1} - \tilde{k}^*||^2) \\
&< \frac{A_{t-1}}{A_t} \left[1 - \hat{\delta}(1-\theta) \right] (\tilde{k}_{t-1} - \tilde{k}^*) - \left(1 - \frac{A_{t-1}}{A_t} \right) \\
&< \frac{A_{t-2}}{A_t} \left[1 - \hat{\delta}(1-\theta) \right]^2 (\tilde{k}_{t-2} - \tilde{k}^*) - \frac{A_{t-1}}{A_t} \left[1 - \hat{\delta}(1-\theta) \right] \left(1 - \frac{A_{t-2}}{A_{t-1}} \right) - \left(1 - \frac{A_{t-1}}{A_t} \right) \\
&< \frac{A_1}{A_t} \left[1 - \hat{\delta}(1-\theta) \right]^{t-1} (\tilde{k}_1 - \tilde{k}^*) - \sum_{u=1}^{t-1} \frac{A_{t-u+1} - A_{t-u}}{A_t} \left[1 - \hat{\delta}(1-\theta) \right]^{u-1}.
\end{aligned} \tag{338}$$

By substituting $\tilde{k}_1 := \frac{K_1/A_1}{K^*/A^*} = \frac{K_1/A_1}{K_0/A_0} = \frac{A_0}{A_1}$, we derive:

$$\tilde{k}_t < 1 - \frac{A_1 - A_0}{A_t} \left[1 - \hat{\delta}(1-\theta) \right]^{t-1} - \sum_{u=1}^{t-1} \frac{A_{t-u+1} - A_{t-u}}{A_t} \left[1 - \hat{\delta}(1-\theta) \right]^{u-1}.$$

2. To prepare for a sufficient condition in case of a declining sequence of $\{A_t\}$, we derive a Taylor expansion of the law of motion in terms of $\hat{k}_t := \frac{K^*/A^*}{K_t/A_t}$ with $\hat{k}^* = 1$. Equation (334) is written as:

$$\hat{k}_{t+1} = \frac{A_{t+1}}{A_t} \frac{1}{\hat{\delta}(\hat{k}_t)^{-\theta} + (1-\hat{\delta})\hat{k}_t^{-1}}. \tag{339}$$

Define $g(\hat{k}_t) := \frac{1}{\hat{\delta}(\hat{k}_t)^{-\theta}+(1-\hat{\delta})\hat{k}_t^{-1}}$ and Taylor approximate this function.

$$\begin{aligned}
g(\hat{k}^*) &= 1 \\
g'(\hat{k}_t) &= \frac{\theta\hat{\delta}(\hat{k}_t)^{-\theta-1} + (1-\hat{\delta})(\hat{k}_t)^{-2}}{\left[\hat{\delta}(\hat{k}_t)^{-\theta} + (1-\hat{\delta})(\hat{k}_t)^{-1}\right]^2} \\
\rightarrow\ g'(\hat{k}^*) &= 1 - \hat{\delta}(1-\theta) \\
g''(\hat{k}_t) &= -\frac{\theta(\theta+1)\hat{\delta}(\hat{k}_t)^{-\theta-2} + 2(1-\hat{\delta})(\hat{k}_t)^{-3}}{\left[\hat{\delta}(\hat{k}_t)^{-\theta} + (1-\hat{\delta})(\hat{k}_t)^{-1}\right]^2} + 2\frac{\left[\theta\hat{\delta}(\hat{k}_t)^{-\theta-1} + (1-\hat{\delta})(\hat{k}_t)^{-2}\right]^2}{\left[\hat{\delta}(\hat{k}_t)^{-\theta} + (1-\hat{\delta})(\hat{k}_t)^{-1}\right]^3} \\
\rightarrow g''(\hat{k}^*) &= -\left[\theta(\theta+1)\hat{\delta} + 2(1-\hat{\delta})\right] + 2\left[\theta\hat{\delta} + 1 - \hat{\delta}\right] \\
&= \hat{\delta}(1-\theta)\left[(1-\theta)(2\hat{\delta}-1) - 1\right]
\end{aligned}$$

We have $g''(\hat{k}^*) < 0$ if $(1-\theta)(2\hat{\delta}-1) - 1 < 0 \Leftrightarrow 2\hat{\delta} < \frac{1}{1-\theta} + 1$. This is true since $\hat{\delta} < 1$ and $\frac{1}{1-\theta} > 1$. Therefore, Taylor expansion of (339) is given by:

$$\hat{k}_{t+1} - \hat{k}^* = \left(\frac{A_{t+1}}{A_t} - 1\right) + \frac{A_{t+1}}{A_t}\left[1 - \hat{\delta}(1-\theta)\right](\hat{k}_t - \hat{k}^*) + \frac{A_{t+1}}{A_t} g''(\hat{k}^*)(\hat{k}_t - \hat{k}^*)^2 + o(||\hat{k}_t - \hat{k}^*||^2) \tag{340}$$

Since $g''(\hat{k}^*) < 0$,

$$\begin{aligned}
\hat{k}_t - \hat{k}^* &= \left(\frac{A_t}{A_{t-1}} - 1\right) + \frac{A_t}{A_{t-1}}\left[1 - \hat{\delta}(1-\theta)\right](\hat{k}_{t-1} - \hat{k}^*) + \frac{A_t}{A_{t-1}} g''(\hat{k}^*)(\hat{k}_{t-1} - \hat{k}^*)^2 + o(||\hat{k}_{t-1} - \hat{k}^*||^2) \\
&< \frac{A_t}{A_{t-1}}\left[1 - \hat{\delta}(1-\theta)\right](\hat{k}_{t-1} - \hat{k}^*) + \left(\frac{A_t}{A_{t-1}} - 1\right) \\
&< \frac{A_t}{A_{t-2}}\left[1 - \hat{\delta}(1-\theta)\right]^2(\hat{k}_{t-2} - \hat{k}^*) + \frac{A_t}{A_{t-2}}\left(1 - \frac{A_{t-2}}{A_{t-1}}\right)\left[1 - \hat{\delta}(1-\theta)\right] + \frac{A_t}{A_{t-1}}\left(1 - \frac{A_{t-1}}{A_t}\right) \\
&< \frac{A_t}{A_1}\left[1 - \hat{\delta}(1-\theta)\right]^{t-1}(\hat{k}_1 - \hat{k}^*) - \sum_{u=1}^{t-1} \frac{A_t}{A_{t-u}}\left(\frac{A_{t-u}}{A_{t-u+1}} - 1\right)\left[1 - \hat{\delta}(1-\theta)\right]^{u-1}.
\end{aligned}$$

By using $\hat{k}_1 - \hat{k}^* := \frac{K_0/A_0}{K_1/A_1} - 1 = \frac{A_1 - A_0}{A_0}$, we have:

$$\hat{k}_t < 1 - \frac{A_t}{A_1}\left(\frac{A_0 - A_1}{A_0}\right)\left[1 - \hat{\delta}(1-\theta)\right]^{t-1} - \sum_{u=1}^{t-1} \frac{A_t}{A_{t-u}}\left(\frac{A_{t-u}}{A_{t-u+1}} - 1\right)\left[1 - \hat{\delta}(1-\theta)\right]^{u-1}.$$

□

A Sufficient Condition on $\{A_t\}_{t=1}^{\infty}$ First-order approximations of $\tilde{k}_t$ and $\hat{k}_t$ allow us to establish a sufficient condition on $\{A_t\}_{t=2}^{\infty}$ for the monotonicity of $\{\frac{K_t}{A_t}\}_{t=1}^{\infty}$. Then, Lemma 22 gives a condition on A_1 that is sufficient for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 1$. We state a proposition below.

Proposition 25.

1. (Positive Shocks) Consider a weakly increasing path of $\{A_t\}_{t=1}^{\infty}$ converging to A^* ($A_0 < A_1 \leq \cdots \leq A^*$ with $\lim_{t\to\infty} A_t = A^*$) that is unexpectedly realized at $t = 1$. If the sequence of $\{A_t\}$ satisfies the condition (342), both $\{A_t\}$ and $\{\frac{K_t}{A_t}\}$ are monotonically increasing in t. This implies monotone increase of w_t and monotone decline of R_t. Therefore, a condition for $\beta R_1 < 1$ is sufficient to guarantee $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 1$ (Lemma 22). The condition, $A_1 < \bar{A}_1$, is a necessary and sufficient condition for $\beta R_1 < 1$. Therefore, the conditions (341) and (342) together guarantee $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 1$.

$$A_0 < A_1 < \bar{A}_1, \tag{341}$$

$$1 \leq \frac{A_{t+1}}{A_t} < 1 + \hat{\delta}\left[\left(\frac{1}{1 - \frac{A_1 - A_0}{A_t}\left[1 - \hat{\delta}(1-\theta)\right]^{t-1} - \sum_{u=1}^{t-1}\frac{A_{t-u+1} - A_{t-u}}{A_t}\left[1 - \hat{\delta}(1-\theta)\right]^{u-1}}\right)^{1-\theta} - 1\right] \ \forall t \geq 1, \tag{342}$$

$$\text{where } \bar{A}_1 := A_0\left[\frac{1 - \beta(1-\delta)}{\theta}\,\frac{\xi(1-\theta) + (1-\nu)\theta\left[1 - (1-\nu-\xi)\beta\right]}{\left[1 - (1-\nu)\beta(1-\delta)\right]\left[1 - (1-\nu-\xi)\beta\right]}\right]^{\frac{1}{1-\theta}}.$$

2. (Negative shocks) Consider a weakly decreasing path of $\{A_t\}_{t=1}^{\infty}$ converging to A^* ($A_0 > A_1 \geq \cdots \geq A^*$ with $\lim_{t\to\infty} A_t = A^*$) that is unexpectedly realized at $t = 1$. If the sequence of $\{A_t\}$ satisfies the condition (344), $\{\frac{K_t}{A_t}\}$ is monotonically declining in t. This implies that condition (329), $\left(\frac{K_1}{A_1}\right)^{1-\theta} < \frac{1}{\nu(1-\delta)}\left[\frac{\xi(1-\theta)}{1-(1-\nu-\xi)\beta} - \nu\theta\right]$, is sufficient to guarantee $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 1$ (Lemma 22). The condition, $\underline{A}_1 < A_1$, is equivalent to (329). Therefore, the conditions (343) and (344) together guarantee $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 1$.

$$\underline{A}_1 < A_1 < A_0, \tag{343}$$

$$1 - \hat{\delta}\left[1 - \left(1 - \frac{A_t}{A_1}\left(\frac{A_0 - A_1}{A_0}\right)\left[1 - \hat{\delta}(1-\theta)\right]^{t-1} - \sum_{u=1}^{t-1}\frac{A_t}{A_{t-u}}\left(\frac{A_{t-u}}{A_{t-u+1}} - 1\right)\left[1 - \hat{\delta}(1-\theta)\right]^{u-1}\right)^{1-\theta}\right] < \frac{A_{t+1}}{A_t} \leq 1 \ \forall t \geq 1, \tag{344}$$

$$\text{where } \underline{A}_1 := A_0\left[\frac{\beta\nu(1-\delta)}{1 - (1-\delta)\beta(1-\nu)}\,\frac{\xi(1-\theta) + \beta\theta(1-\nu)(\xi + \nu + \frac{1}{\beta} - 1)}{\xi(1-\theta) - \beta\theta\nu\left[\xi + \nu + \frac{1}{\beta} - 1\right]}\right]^{\frac{1}{1-\theta}}.$$

Proof. 1. Lemma 23 states that $\frac{K_{t+1}}{A_{t+1}} > \frac{K_t}{A_t}$ for all $t \geq 1$ if

$$\frac{A_{t+1}}{A_t} < 1 + \hat{\delta}\left[\left(\frac{K^*/A^*}{K_t/A_t}\right)^{1-\theta} - 1\right] \text{ at all } t \geq 1. \tag{345}$$

In Lemma 24, we establish an upper bound on $\tilde{k}_t := \frac{K_t/A_t}{K^*/A^*}$. Therefore, we have:

$$\left(\frac{K^*/A^*}{K_t/A_t}\right)^{1-\theta} > \left[1 - \frac{A_1 - A_0}{A_t}\left[1-\hat{\delta}(1-\theta)\right]^{t-1} - \sum_{u=1}^{t-1}\frac{A_{t-u+1} - A_{t-u}}{A_t}\left[1-\hat{\delta}(1-\theta)\right]^{u-1}\right]^{-(1-\theta)} \tag{346}$$

Hence, the follwing condition is sufficient for condition (345):

$$\frac{A_{t+1}}{A_t} < 1 + \hat{\delta}\left[\left(\frac{1}{1 - \frac{A_1 - A_0}{A_t}\left[1-\hat{\delta}(1-\theta)\right]^{t-1} - \sum_{u=1}^{t-1}\frac{A_{t-u+1}-A_{t-u}}{A_t}\left[1-\hat{\delta}(1-\theta)\right]^{u-1}}\right)^{1-\theta} - 1\right] \quad \forall t \geq 1.$$

Under this condition, $\{\frac{K_t}{A_t}\}_{t=1}^{\infty}$ is monotonically increasing. Thus, Lemma 22 states that condition (341) is sufficient for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 1$.

2. Lemma 23 states that $\frac{K_{t+1}}{A_{t+1}} < \frac{K_t}{A_t}$ for all $t \geq 1$ if

$$\frac{A_{t+1}}{A_t} > 1 + \hat{\delta}\left[\left(\frac{K^*/A^*}{K_t/A_t}\right)^{1-\theta} - 1\right] \text{ at all } t \geq 1. \tag{347}$$

In Lemma 24, we establish an upper bound on $\hat{k}_t := \frac{K^*/A^*}{K_t/A_t}$. Therefore, we have:

$$\left(\frac{K^*/A^*}{K_t/A_t}\right)^{1-\theta} > \left[1 - \frac{A_t}{A_1}\left(\frac{A_0 - A_1}{A_0}\right)\left[1-\hat{\delta}(1-\theta)\right]^{t-1} - \sum_{u=1}^{t-1}\frac{A_t}{A_{t-u}}\left(\frac{A_{t-u}}{A_{t-u+1}} - 1\right)\left[1-\hat{\delta}(1-\theta)\right]^{u-1}\right]^{1-\theta} \tag{348}$$

Hence, the follwing condition is sufficient for condition (347):

$$\frac{A_{t+1}}{A_t} > 1 + \hat{\delta}\left[\left[1 - \frac{A_t}{A_1}\left(\frac{A_0 - A_1}{A_0}\right)\left[1-\hat{\delta}(1-\theta)\right]^{t-1} - \sum_{u=1}^{t-1}\frac{A_t}{A_{t-u}}\left(\frac{A_{t-u}}{A_{t-u+1}} - 1\right)\left[1-\hat{\delta}(1-\theta)\right]^{u-1}\right]^{1-\theta} - 1\right] \quad \forall t \geq 1.$$

Under this condition, $\{\frac{K_t}{A_t}\}_{t=1}^{\infty}$ is monotonically decreasing. Thus, Lemma 22 states that condition (343) is sufficient for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 1$. □

B.3.5 Further Discussion on Corollary 2

As we see in Section 5.3, the Euler equation between $t = 0$ and $t = 1$ for low-income agents holds at the time of MIT shock ($t = 1$), and households do not respond to future anticipated productivity shocks. This is not generally the case, as we discuss below.

CRRA utility As an illustration, we derive what happens under CRRA utility function ($\frac{c^{1-\sigma}-1}{1-\sigma}$ with $\sigma \neq 1$). Low income agents finance their consumption using their state-contingent assets, so the budget constraint is given by:

$$R_t \mathbf{a}_{s,t} = \mathbf{c}_{s,t} + \frac{1-\nu}{R_{t+1}} \mathbf{c}_{s+1,t+1} + \frac{1-\nu}{R_{t+1}} \frac{1-\nu}{R_{t+2}} \mathbf{c}_{s+2,t+2} + \cdots \tag{349}$$

If consumption follows the Euler equation at all future periods:

$$\mathbf{c}_{s+1,t+1} = (\beta R_{t+1})^{\frac{1}{\sigma}} \mathbf{c}_{s,t},$$

the budget constraint is given by:

$$R_t \mathbf{a}_{s,t} = \mathbf{c}_{s,t} \left[1 + (1-\nu) \beta^{\frac{1}{\sigma}} (R_{t+1})^{\frac{1-\sigma}{\sigma}} + (1-\nu)^2 \beta^{\frac{2}{\sigma}} (R_{t+1} R_{t+2})^{\frac{1-\sigma}{\sigma}} + \cdots \right]. \tag{350}$$

Then, we see that today's consumption ($\mathbf{c}_{s,t}$) depends on future interest rate.

Given that consumption today depends on future interest rates, expectations about future productivity affect today's consumption decisions. If a future path of interest rates unexpectedly change at time t, then the Euler equation between t and $t+1$ no longer holds. This implies that MIT shocks and anticipated shocks have different implications for the law of motion of capital if CRRA utility with $\sigma \neq 1$.

Comparison with a Neoclassical Growth Model The two results (the Euler equation at $t = 1$ and the indifference between MIT shocks and anticipated shocks) are not true in a standard neoclassical growth model unless the capital fully depreciates ($\delta = 1$).

Suppose the economy is in a steady state at $t = -5$ with aggreagte productivity A_0. Agents realize the news at $t = -4$ that productivity increases from A_0 to A_1 at $t = 1$ permanently:

$$A_t = \begin{cases} A_0 \text{ if } t \leq 0 \\ A_1 \text{ if } t \geq 1 \end{cases}$$

Transitional dymanics of a standard neoclassical growth model is described by an Euler equation and a resource constraint:

$$\frac{1}{C_t} = \beta R_{t+1} \frac{1}{C_{t+1}}, \tag{351}$$

$$C_t + K_{t+1} = A_t^{1-\theta} K_t^{\theta} + (1-\delta) K_t. \tag{352}$$

Here we continue to assume the logarithmic utility. Since the economy is in a steady state at $t = -5$, $\beta R^* = 1$ implies:

$$\beta \left[\theta A_0^{1-\theta} K_{-5}^{\theta-1} + 1 - \delta \right] = 1$$

$$\Leftrightarrow K_{-5} = A_0 \left(\frac{\theta}{\frac{1}{\beta} - 1 + \delta} \right)^{\frac{1}{1-\theta}}$$

By substituting (352) into (351), the law of motion of capital is derived as:

$$K_{t+2} = A_{t+1}^{1-\theta} K_{t+1}^{\theta} + (1-\delta) K_{t+1} - \beta \underbrace{\left(\theta A_{t+1}^{1-\theta} K_{t+1}^{\theta-1} + 1 - \delta\right)}_{=R_{t+1}} \underbrace{\left[A_t^{1-\theta} K_t^{\theta} + (1-\delta) K_t - K_{t+1}\right]}_{C_t}. \tag{353}$$

At $t = -4$, the aggregate capital is given by $K_{-4} = K_{-5}$, and the capital will eventually converge to a new steady state with:

$$K_{\infty} = A_1 \left(\frac{\theta}{\frac{1}{\beta} - 1 + \delta} \right)^{\frac{1}{1-\theta}}.$$

This terminal condition allows us to derive the sequence of capital numerically. Notice that choice variables at time t (C_t, K_{t+1}) depend on future capital (K_{t+2}), so the agents respond to future anticipated shocks (Figure 15). Intuitively, higher productivity from $t = 1$ onwards allows agents to consume more. Since agents prefer a smooth consumption profile, they start to consume a little more when they realize the news at $t = -4$.

The interest rate does not change at $t = -4$ as $A_{-5} = A_{-4} = A_0$ and $K_{-5} = K_{-4}$ imply $R_{-4} = \theta A_{-4}^{1-\theta} K_{-4}^{\theta-1} + 1 - \delta = R^* = \frac{1}{\beta}$. The response of C_{-4} to the future shock implies that the Euler equation between $t = -5$ and $t = -4$ does not hold:

$$\frac{1}{C_{-5}} \neq \beta R_{-4} \frac{1}{C_{-4}} = \frac{1}{C_{-4}}. \tag{354}$$

An optimality condition requires that the Euler equation between $t = 0$ and $t = 1$ holds.[47]

Full depreciation of capital ($\delta = 1$) If the capital fully depreciates, the economy does not respond to future anticipated shocks. With $\delta = 1$, we have a well-known closed form solution to the law of motion of capital:

$$K_{t+1} = \beta \theta A_t^{1-\theta} K_t^{\theta}, \tag{355}$$

$$C_t = (1 - \beta\theta) A_t^{1-\theta} K_t^{\theta}. \tag{356}$$

[47] We don't prove here that the Euler equation between $t = 0$ and $t = 1$ does not hold if an MIT shock is realized at $t = 1$. However, if the Euler equation holds,

$$\begin{aligned} \frac{1}{C_0} &= \beta R_1 \frac{1}{C_1} \\ \Leftrightarrow C_1 &= \beta \left[\theta A_1^{1-\theta} K_0^{\theta-1} + 1 - \delta\right] \left[A_0^{1-\theta} K_0^{\theta} - \delta K_0\right]. \end{aligned}$$

Since this C_1 does not depend on future capital (K_2), this is generally not the solution to equations (351) and (352). If $\delta = 1$, C_1 can be solved as:

$$C_1 = (1 - \beta\theta) A_1^{1-\theta} K_0^{\theta}.$$

This is the same solution as in the case of anticipated productivity shock, as we will see in equation (356). Therefore, with $\delta = 1$, C_1 derived from the Euler equation between $t = 0$ and $t = 1$ is optimal at the time of MIT shock.

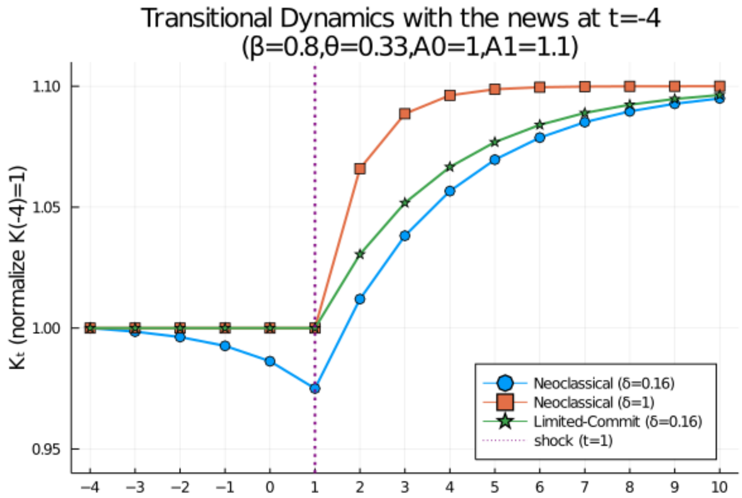


Figure 15: Transitional Dynamics of K_t after an Anticipated Shock

Choice variables at time t (C_t and K_{t+1}) are not dependent on future capital (K_{t+1}). Therefore, the economy does not respond to future anticipated shocks. This means that the Euler equation between $t = -5$ and $t = -4$ continues to hold. An optimality condition requires that the Euler equation between $t = 0$ and $t = 1$ holds as well.

A Limited-Commitment Model As we see in the main section, the aggregate capital follows the law of motion:

$$K_{t+1} = \left[\frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta) + (1-\nu)\beta\theta\right] A_t^{1-\theta} K_t^{\theta} + (1-\nu)\beta(1-\delta)K_t. \tag{357}$$

This implies that the aggregate consumption (C_t) is given by:

$$\begin{aligned} C_t &= A_t^{1-\theta} K_t^{\theta} + (1-\delta)K_t - K_{t+1} \\ &= \left[1 - \frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta) + (1-\nu)\beta\theta\right] A_t^{1-\theta} K_t^{\theta} + \left[1-(1-\nu)\beta\right](1-\delta)K_t. \end{aligned} \tag{358}$$

Since both choice variables do not depend on future capital, the economy does not respond to future anticipated shocks. This means that the Euler equation for low-income agents between $t = -5$ and $t = -4$ still holds after the news is realized as well as the Euler equation between $t = 0$ and $t = 1$.

C Old Stuff That Can Be Erased Once Paper is Done

C.1 The Optimal Consumption Allocation

In this section we will derive the optimal consumption contract for an arbitrary sequence of wages and interest rates, under Assumption 3. In section 5.2 we will present sufficient conditions such that this condition is satisfied along the entire transition path.

It is easier to characterize consumption and asset allocations relative to the aggregate wage w_t. That is, define $\{c_{s,t}, a_{s,t}\}$ through

$$\mathbf{c}_{s,t} = w_t c_{s,t} \tag{359}$$

$$\mathbf{a}_{s,t} = w_t a_{s,t} \tag{360}$$

In Appendix B.2 we show that under Assumption 3, the key optimality conditions are that

1. The shortsale constraint binds if $z_{t+1} = \zeta$ and thus $\mathbf{a}_{0,t+1} = a_{0,t+1} = 0$.
2. The standard complete markets Euler equation holds if $z_{t+1} = 0$ and thus for all $s, t \geq 0$,
$$\frac{w_{t+1} c_{s+1,t+1}}{w_t c_{s,t}} = \beta R_{t+1}$$
3. The transversality condition holds
$$\lim_{j \to \infty} \frac{(1-\nu)^j}{\prod_{\tau=0}^{j-1} R_{t+\tau}} \mathbf{a}_{s+j,t+j} = 0 \tag{361}$$

The sequential markets budget constraints **??** can then be written in wage-deflated form (and exploiting the forms of the transition probabilities) as

$$c_{0,t} + \xi \frac{w_{t+1}}{w_t} a_{1,t+1} = \zeta \tag{362}$$

$$c_{s,t} + (1-\nu) \frac{w_{t+1}}{w_t} a_{s+1,t+1} = R_t a_{s,t} \text{ for } s \geq 1 \tag{363}$$

Using equation 363 we can write $a_{s,t}$ as

$$\begin{aligned} a_{s,t} &= \frac{c_{s,t}}{R_t} + \frac{(1-\nu) w_{t+1}}{R_t w_t} a_{1,t+1} \\ &= \frac{c_{s,t}}{R_t} + \frac{(1-\nu) w_{t+1}}{R_t w_t} \left(\frac{c_{s+1,t+1}}{R_{t+1}} + (1-\nu) \frac{w_{t+2}}{R_{t+1} w_{t+1}} a_{s+2,t+2} \right) \\ &= \frac{c_{s,t}}{R_t} + \frac{(1-\nu) w_{t+1} c_{s+1,t+1}}{w_t R_t R_{t+1}} + \frac{(1-\nu)^2 w_{t+2} c_{s+2,t+2}}{w_t R_t R_{t+1} R_{t+2}} + \ldots + \lim_{j \to \infty} \frac{(1-\nu)^j w_{t+j} a_{s+j,t+j}}{w_t \prod_{\tau=0}^{j-1} R_{t+\tau}} \end{aligned}$$

Using the Euler equations and transversality condition we obtain for all $s \geq 1$,

$$\begin{aligned} R_t a_{s,t} &= c_{s,t} + \frac{(1-\nu) w_t c_{s,t} \beta R_{t+1}}{w_t R_{t+1}} + \frac{(1-\nu)^2 w_t c_{s,t} \beta^2 R_{t+1} R_{t+2}}{w_t R_{t+1} R_{t+2}} + \ldots \\ &= c_{s,t} \sum_{\tau=0}^{\infty} [\beta(1-\nu)]^\tau = \frac{c_{s,t}}{1-\beta(1-\nu)} \\ c_{s,t} &= [1-\beta(1-\nu)]\, R_t a_{s,t} \end{aligned}$$

This equation also implies (exploiting the Euler equation) that

$$\xi \frac{w_{t+1}}{w_t} a_{1,t+1} = \xi \frac{w_{t+1}}{w_t} \frac{c_{1,t+1}/R_{t+1}}{1-\beta(1-\nu)} = \xi \frac{\beta R_{t+1}}{w_t} \frac{w_t c_{0,t}/R_{t+1}}{1-\beta(1-\nu)} = \frac{\xi \beta c_{0,t}}{1-\beta(1-\nu)}$$

and thus exploiting equation 362 we obtain

$$c_{0,t} = \left(\frac{1-\beta(1-\nu)}{1-\beta(1-\nu-\xi)} \right) \zeta \tag{364}$$

The remaining consumption and asset levels directly follow from the consumption Euler equations and the sequential budget constraints. We summarize the equilibrium consumption and asset allocation, for a given sequence of wages and interest rates, in the next proposition:

Proposition 26. *Suppose the sequence of interest rates and wages $\{R_t, w_t\}_{t=0}^{\infty}$ is exogenously given and satisfies Assumption 3. Also assume that the initial asset levels satisfy $\mathbf{a}_{0,0} = 0$ and $0 < \mathbf{a}_{s,0} < w_0 \bar{a}_0$ for $s \geq 1$, where $\bar{a}_0 := \frac{\beta}{1-(1-\nu-\xi)\beta}\zeta$. Then the optimal consumption and asset allocation satisfies, for all t*

$$\begin{aligned} \mathbf{c}_{0,t} &= \left(\frac{1-\beta(1-\nu)}{1-\beta(1-\nu-\xi)} \right) \zeta w_t && (365) \\ \mathbf{c}_{s,t} &= [1-(1-\nu)\beta]\, R_t \mathbf{a}_{s,t} \quad \textit{for } s = 1, 2, \cdots && (366) \\ \mathbf{c}_{s+1,t+1} &= \beta R_{t+1} \mathbf{c}_{s,t} \quad \textit{for } s = 0, 1, \cdots && (367) \\ \mathbf{a}_{0,t} &= 0 && (368) \\ \mathbf{a}_{1,t+1} &= \left(\frac{\beta}{1-(1-\nu-\xi)\beta} \right) \zeta w_t && (369) \\ \mathbf{a}_{s+1,t+1} &= \beta R_t \mathbf{a}_{s,t} \quad \textit{for } s = 1, 2, \cdots && (370) \end{aligned}$$

The wage-deflated allocations satisfy $c_{s,t} = \mathbf{c}_{s,t}/w_t$ and $a_{s,t} = \mathbf{a}_{s,t}/w_t$.

Proof. Appendix B.2 shows that the three properties in the previous page hold under Assumption 3 and the initial conditions. We have derived the optimal contract given the three properties. □

Let us interpret Proposition 26. High-productivity individuals consume a constant share $\kappa := \frac{1-\beta(1-\nu)}{1-\beta(1-\nu-\xi)}$ of their labor income ζw_t, independent of the wage and time period. The remaining fraction $\frac{\beta}{1-\beta(1-\nu-\xi)}$ is spent on insurance $\mathbf{a}_{1,t+1}$ against income falling, at actuarially fair price ξ per unit, and thus at cost $\xi\mathbf{a}_{1,t+1}$. Individuals with zero productivity enter the period with gross capital income $R_t\mathbf{a}_{s,t}$ and split that income in constant proportion between consumption today (share $[1-(1-\nu)\beta]$) and insurance against remaining unproductive (share β times the actuarially fair price $1-\nu$ per unit of insurance).

Remark 2. *Need to discuss which of these optimality conditions hold in period $t=1$ after the MIT shock. I believe all equations hold for all $t \geq 0$ apart from the consumption Euler equation (equation 33), which need not hold between period $t=0$ and $t=1$.*

C.2 Discussion

This needs to be revised; currently has fragments from other parts moved here

Appendix **??** shows that a contract with $c_{h,t} = c_{h,t+1} = c_h$ satisfies the optimality condition if $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$. The intuition is that agents do not have an incentive to save for a high-income state if the rate of return on saving is sufficiently low, or wage at $t+1$ is sufficiently higher than wage at t. We will first assume this condition and derive the transitional dynamics of capital, wage, and interest rate. Then, we check the condition, $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$, to make sure that the contract satisfies optimality conditions.

Since a participation constraint of high-income households binds in every period (in the initial steady state as well as in every period along the transition), we can think of high-income individuals signing a new contract. Zero profits of financial intermediaries then require that the expected cost of the consumption allocation implied by high income today and a sequence of low incomes from tomorrow onwards equals the value of high income today. In other words, high-income agents pay the insurance premium in this period to insure against future low-income states in a way financial intermediaries earn zero profits. The cost of the contract is the sum of consumption today and discounted future consumption conditional on the realization of a low-income state:

Given the consumption contract, we now derive the implied asset holdings. These assets can be interpreted as the capital the intermediary saves for the individual, or as the bank account balance of the individual with the intermediary. The interpretation is that households save in a state-contingent bank account with a financial intermediary that inputs the savings into capital. Crucially, this balance cannot be taken to a competing intermediary if a high-income agent chooses to leave the current contract, and thus a new contract of a high-income individual starts with zero assets.

As in Krueger and Uhlig (2020), in this model risk-averse households seek to insure themselves against idiosyncratic income shocks by signing long-term insurance contracts with risk-neutral financial intermediaries. These intermediaries provide optimal insurance, subject to the limited commitment constraints of individuals. In Appendix **??**, we formulate the contracting problem in recursive form and show that the limited commitment constraint for individuals with currently high income is binding if and only if today's marginal utility from consumption is larger than the marginal utility tomorrow: $\frac{1}{w_t c_t} > \beta R_{t+1} \frac{1}{w_{t+1} c_{t+1}}$. We will demonstrate below that if the constraint of high income-individuals is binding in periods t and $t+1$, the these agents have constant deflated consumption over time, $c_{h,t} = c_{h,t+1} = c_h$. Therefore, a sufficient condition for the limited commitment constraint of high income individuals to bind for all time periods is that $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all t.

The previous proposition provides a complete characterization of the optimal consumption and implied asset allocation, given an arbitrary wage and interest rate path satisfying the condition given in the proposition. It is valid both in a stationary equilibrium (in which case wages and interest rates are constant), as well as along the transition path induced by a surprise change in the path of total factor productivity A_t.

The fact that the optimal consumption contract even after an MIT shock can be stated without first deriving steady state asset holdings should be somewhat surprising. The typical result from a Bewley-style model is that one first solves for the endogenous steady state consumption and asset distribution, and then assets (and its distribution) serve as initial conditions for the surprise-induced transition. Since wages and interest rates change surprisingly between period 0 (the initial steady state) and period 1 (the first period of the transition), the consumption levels implied by the steady state Euler equations are not necessary optimal in period 1 any longer after the MIT shock (since interest rates and wages change surprisingly).

However, in this model individuals with high income realizations sign new contracts in any case in period 1 (and their Euler equations hold with inequality only even in the initial steady state), and individuals with low income are unaffected by the change in the current wage (since their labor income is zero) and have consumption that is independent of future interest rates (due to log-utility) in steady state and along the transition. Thus, their Euler equations hold between period 0 and 1 even in the presence of the MIT shock in between these periods. Thus the consumption allocation characterized in the previous proposition is valid even for period 1, and can be stated independent of the steady state asset distribution.[48]

[48]We make the implicit assumption that after the MIT shock individuals with low income realizations retain their asset position, and thus financial intermediaries continue to make zero expected profits from existing contracts after the MIT shock hits. One could envision alternative ways of renegotiating existing contracts after the MIT shock, but these alternatives would entail expected losses or windfall profits for the intermediaries resulting from the MIT shock.

Denote the continuation utility from a given consumption allocation be given by

$$U_{s,t}(c) = \begin{cases} \log(w_t c_{0,t}) + \beta\left[(1-\xi)U_{0,t+1}(c) + \xi U_{1,t+1}(c)\right] & \text{if } s = 0 \\ \log(w_t c_{s,t}) + \beta\left[\nu U_{0,t+1}(c) + (1-\nu)U_{s+1,t+1}(c)\right] & \text{if } s = 1,2,3,... \end{cases} \tag{371}$$

and the net cost of such an allocation as

$$V_{s,t}(c) = \begin{cases} w_t(c_{0,t} - z) + \frac{1}{1+r_{t+1}}\left[(1-\xi)V_{0,t+1}(c) + \xi V_{1,t+1}(c)\right] & \text{if } s = 0 \\ w_t c_{s,t} + \frac{1}{1+r_{t+1}}\left[\nu V_{0,t+1}(c) + (1-\nu)V_{s+1,t+1}(c)\right] & \text{if } s = 1,2,3,... \end{cases} \tag{372}$$

C.3 Proofs

Proposition ?? (Sequential Market Equilibrium)**.** *Consider a limited-commitment economy described in Section 2. Suppose a sequence of interest rates and wages* $\{R_t(A^t), w_t(A^t)\}_{t\geq 0, A^t}$ *satisfies Assumption 3 and the initial distribution* $\Phi(a_0, z_0)$ *satisfies Assumption 4. Then, individual consumption and asset allocations* $\{\hat{\mathbf{c}}_t(a_0, z^t, A^t), \hat{\mathbf{a}}_{t+1}(a_0, z^{t+1}, A^{t+1})\}$ *given by equations (22) and (23), aggregate consumption and capital* $\{C_t(A^t), K_{t+1}(A^t)\}$ *given by equations (16) and (17), Arrow security price* $\{q(A^{t+1}|A^t)\}$ *given by equation (19), and a sequnce of wages and interest rates* $\{w_t(A^t), R_t(A^t)\}$ *given by equations (14) and (15) constitute a sequential equilibrium.*

In this economy, the law of motion of aggregate capital follows equation (40). In addition, if the initial consumption-asset allocation is given by a simple frame $\{\mathbf{c}_{s,0}, \mathbf{a}_{s,0}\}_{s\geq 0}$ *with probability mass* $\{\phi_s\}_{s\geq 0}$ *in equation (32), then the consumption-asset allocation at any* $t \geq 1$ *is determined by equations (33) and (34).*

Proof. We have checked all the equilibrium conditions except the market clearing condition. As we have seen in Proposition 2, K_{t+1} follows:

$$K_{t+1} = \left[\frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta) + (1-\nu)\beta\theta\right] A_t^{1-\theta}K_t^{\theta} + (1-\nu)\beta(1-\delta)K_t.$$

Aggregate consumption, C_t, is given by:

$$\begin{aligned} C_t &= \int \sum_{z^t} \hat{\mathbf{c}}_t(a_0, z^t, A^t)\pi(z^t)d\Phi(a_0, z_0) \\ &= \int \sum_{z^t} \underbrace{\hat{\mathbf{c}}_t(a_0, z^t, A^t; z_t = \zeta)}_{=w_t(A^t)c_0} \pi(z^t; z_t = \zeta)d\Phi(a_0, z_0) \\ &\quad + \int \sum_{z^t} \underbrace{\hat{\mathbf{c}}_t(a_0, z^t, A^t; z_t = 0)}_{=[1-(1-\nu)\beta]R_t(A^t)\mathbf{a}_t(z^t, A^t; z_t=0)} \pi(z^t; z_t = 0)d\Phi(a_0, z_0) \\ &= w_t(A^t)c_0\frac{\nu}{\xi+\nu} + [1-(1-\nu)\beta]R_t(A^t)K_t \\ &= (1-\theta)A_t^{1-\theta}K_t^{\theta}\frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta} + [1-(1-\nu)\beta]\left[\theta A_t^{1-\theta}K_t^{\theta} + (1-\delta)K_t\right] \end{aligned}$$

By substituing K_{t+1} and C_t, we verify that the goods market clears:

$$C_t + K_{t+1} = K_t^{\theta} A_t^{1-\theta} + (1-\delta)K_t$$

□

Proposition 26 (Optimal Contract given $\{R_t, w_t\}$). *Suppose the sequence of interest rates and wages $\{R_t, w_t\}_{t=0}^{\infty}$ is given and satisfies $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all $t \geq 0$. Assume a participation constraint for high-income agents binds every period. The consumption and implied asset position of each agent at time t are given by:*

$$\begin{aligned}
c_{h,t} &= \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta} z && \textbf{(??)} \\
\mathbf{c}_{s+1,t+1} &= \beta R_{t+1} \mathbf{c}_{s,t} && \textit{for } s = 0, 1, \cdots \\
a_{h,t} &= 0 \\
\mathbf{a}_{1,t+1} &= \frac{\beta}{1-(1-\nu-\xi)\beta} w_t z && (369) \\
\mathbf{a}_{s+1,t+1} &= \beta R_t \mathbf{a}_{s,t} && \textit{for } s = 1, 2, \cdots \\
\mathbf{c}_{s,t} &= \left[1-(1-\nu)\beta\right] R_t \mathbf{a}_{s,t} && \textit{for } s = 1, 2, \cdots .
\end{aligned}$$

where we denote $c_{h,t}$ and $c_{0,t}$ interchangeably.

Proof. Zero profits of financial intermediaries for a contract with high-income agents implies that:

$$w_t z = \mathbf{c}_{h,t} + \xi \frac{1}{R_{t+1}} \mathbf{c}_{1,t+1} + \xi(1-\nu) \frac{1}{R_{t+1}R_{t+2}} \mathbf{c}_{2,t+2} + \xi(1-\nu)^2 \frac{1}{R_{t+1}R_{t+2}R_{t+3}} \mathbf{c}_{3,t+3} \cdots ,$$

where the left hand side is the labor income today, and the right hand side is the discounted sum of future consumption costs. The optimal contract requires that consumption follows a standard Euler equation if the household does not renew a contract (which happens in a low-income state):

$$\frac{1}{\mathbf{c}_{s,t}} = \beta R_{t+1} \frac{1}{\mathbf{c}_{s+1,t+1}}.$$

By substituing this, we obtain $c_{h,t}$ as:

$$c_{h,t} = \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta} z.$$

Since we assume that a participation constraint binds for high-income agents, an implies asset for high-income agents is zero: $a_{h,t} = 0$.

Finally, by using an implies budget constraint:

$$\mathbf{c}_{s,t} + (1-\nu)\mathbf{a}_{s+1,t+1} = R_t \mathbf{a}_{s,t} \text{ for } s = 1, 2, \cdots ,$$

we derive the relation between consumption and implied asset of low-income agents:

$$\mathbf{c}_{s,t} = \left[1 - \beta(1-\nu)\right] R_t \mathbf{a}_{s,t}.$$

An implied asset of low-income agents with $s = 1$ follows by substituing $\mathbf{c}_{s,t}$ with $\mathbf{c}_{1,t} = \beta R_t \mathbf{c}_{h,t-1}$

□

C.4 Stuff from Stationary Model

From the Markov transition matrix, the mass of each type of agents is derived as follows:

$$\phi_s = \begin{cases} \frac{\nu}{\xi+\nu} & \text{if } s = 0 \\ \frac{\nu\xi}{\xi+\nu}(1-\nu)^{s-1} & \text{if } s \geq 1. \end{cases} \tag{373}$$

where s is the number of periods in a low-income state and $s = 0$ denotes the high income state.

Now we derive the aggregate consumption demand. It is the sum of deflated consumption weighted by the mass of each type of agents:

$$\begin{aligned} C(R) &:= \phi_0 c_h + \sum_{s=1}^{\infty} \phi_s c_s \\ &= \frac{\nu}{\xi+\nu} c_h + \sum_{s=1}^{\infty} \frac{\nu\xi}{\xi+\nu}(1-\nu)^{s-1}(\beta R)^s c_h \\ &= \left[\frac{1-(1-\xi-\nu)\beta R}{1-(1-\xi-\nu)\beta}\right]\left[\frac{1-(1-\nu)\beta}{1-(1-\nu)\beta R}\right], \end{aligned} \tag{374}$$

where the last line uses the normalization $\frac{\nu}{\xi+\nu} z = 1$.

Similarly, the aggregate saving, which we call the aggregate capital supply in the general equilibrium, is the sum of the consumption-implied asset positions weighted by the population shares ϕ_s.

$$\begin{aligned} \kappa^s(R) &= \sum_{s=0}^{\infty} \phi_s a_s \\ &= \sum_{s=1}^{\infty} \frac{\nu\xi}{\xi+\nu}(1-\nu)^{s-1}(\beta R)^{s-1} a_1 \\ &= \frac{\xi\beta}{\left[1-(1-\nu)\beta R\right]\left[1-(1-\xi-\nu)\beta\right]}. \end{aligned} \tag{375}$$

We analyze the production side and derive the stationary equilibrium. The representative production firms use labor and capital to produce final goods. Final goods are consumed by households or invested as capital for future production. Given a production function in equation (2), two optimality conditions are satisfied in an equilibrium:

$$w_t = F_L(K, L) \qquad = (1-\theta)A_t^{1-\theta}K_t^{\theta} \ , \tag{376}$$

$$R_t = F_K(K, L) + 1 - \delta \qquad = \theta A_t^{1-\theta}K_t^{\theta-1} + 1 - \delta \ , \tag{377}$$

where the aggregate labor, L, is normalized to one. In a general equilibrium, we have a goods market clearing condition:

$$w_t C_t + K_{t+1} = A_t^{1-\theta}K_t^{\theta} + (1-\delta)K_t \ , \tag{378}$$

$$\text{where } \ C_t = \left[\phi_0 c_{h,t} + \sum_{s=1}^{\infty} \phi_s c_{s,t}\right] .$$

We define a capital market clearning condition as well. In the stationary equilibrium, aggregate variables are constant over time, e.g. $K_{t+1} = K_t = K$. The equation (378) is written as:

$$\begin{aligned} wC(R) &= (1-\theta)A^{1-\theta}K^{\theta} + \left[\theta A^{1-\theta}K^{\theta-1} - \delta\right] K \\ &= w + (R-1)K \\ \therefore \ \frac{C(R)-1}{R-1} &= \frac{K}{w} \end{aligned}$$

The left hand side represents the deflated aggregate capital supply, $\kappa^s(R)$, and the right hand side represents the deflated capital demand, $\kappa^d(R)$. Indeed, we can show $\frac{C(R)-1}{R-1} = \kappa^s(R)$ from equations (374) and (375). The capital market clearing condition is given by:

$$\kappa^s(R) = \kappa^d(R) \tag{379}$$

$$\text{where } \kappa^s(R) = \frac{\xi\beta}{[1-(1-\nu)\beta R]\,[1-(1-\xi-\nu)\beta]}$$

$$\kappa^d(R) = \frac{\theta}{(1-\theta)(R-1+\delta)}. \tag{380}$$

We discuss Walras' Law for a non-stationary environment in Appendix C.8.2. By solving (379), we derive the equilibrium interest rate in a closed form:

$$R^* = \frac{\xi(1-\theta)(1-\delta) + \theta\left(\xi + \nu + \frac{1}{\beta} - 1\right)}{\xi(1-\theta) + \beta\theta(1-\nu)\left(\xi + \nu + \frac{1}{\beta} - 1\right)}. \tag{381}$$

We have derived the equilibrium interest rate and capital in the stationary equilibrium in equations (54 and (55) under the assumption $\beta R < 1$. We discuss a condition for the existence of an equilibrium with $\beta R < 1$. We call an equilibrium with $\beta R < 1$ a partial insurance equilibrium.

In the capital market clearing condition (379), we see that the capital supply function is strictly increasing in $R \in (1-\delta, \frac{1}{\beta})$, and the capital demand function is strictly decreasing in $R \in (1-\delta, \frac{1}{\beta})$. Note that we assume $0 < \xi, \nu, \beta, \theta < 1$ and $0 < \delta \leq 1$ throughout the paper. Therefore, a partial insurance equilibrium is unique if it exists. We also see that the capital demand goes to infinity as an interest rate approaches $1-\delta$ from above:

$$\lim_{R \to 1-\delta} \kappa^d(R) \left(:= \frac{\theta}{(1-\theta)(R-1+\delta)} \right) = +\infty.$$

This means that the excess demand for capital, $\kappa^d(R) - \kappa^s(R)$, is infinite at the limit $R \to 1-\delta$ and is monotonically decreasing in $R \in (1-\delta, \frac{1}{\beta})$. Therefore, an equilibrium with $\beta R < 1$ exists if and only if

$$\kappa^d(R = \frac{1}{\beta}) < \kappa^s(R = \frac{1}{\beta}).$$

This is equivalent to the following condition, which we state as an explicit assumption and impose it henceforth, guaranteeing that the initial stationary equilibrium features partial consumption insurance

C.5 Stuff from the Transition Analysis

C.5.1 A Law of Motion of Aggregate Capital

This subsection derives the law of motion of the aggregate capital stock. This can be done using either the capital market clearing condition or the goods market clearing condition. Both give exactly the same result (as it should, of course), and the discussion of the goods market clearing condition is found in Appendix C.8.3.

From the asset market clearing condition, the aggregate capital stock is equal to the sum of individual asset holdings of households, or equivalently, the assets the financial intermediary holds on behalf of those with currently low, but past high income. Therefore, the capital stock at time $t = 1$, K_{t+1}, is equal to the sum of household saving determined at time t. Because high-income agents save a constant fraction of their labor income (369) and low-income agents save according to equation (**??**), we can compute K_{t+1} as a function of K_t.

Proposition 27 (A Law of Motion of Aggregate Capital)**.** *Consider an economy with one-sided limited commitment. Assume $\beta R_t < \frac{w_{t+1}}{w_t}$ holds at all t. If aggregate capital and productivity at time t are given by K_t and A_t, aggregate capital at time $t+1$ is given by:*

$$K_{t+1} = \left[\frac{\xi\beta(1-\theta)}{1-(1-\nu-\xi)\beta} + (1-\nu)\beta\theta \right] A_t^{1-\theta} K_t^{\theta} + (1-\nu)\beta(1-\delta)K_t. \tag{382}$$

Proof. K_{t+1} is the sum of household savings:

$$
\begin{aligned}
K_{t+1} &= w_{t+1} \sum_{s=1}^{\infty} \phi_s a_{s,t+1} \\
&= \phi_1 w_{t+1} a_{1,t+1} + \sum_{s=2}^{\infty} \phi_s \underbrace{w_{t+1} a_{s,t+1}}_{=\beta R_t w_t a_{s-1,t} \text{ by (??)}} \\
&= \phi_1 \underbrace{\frac{\beta}{1-(1-\nu-\xi)\beta} w_t z}_{=w_{t+1}a_{1,t+1} \text{ by (369)}} + (1-\nu)\beta R_t \underbrace{w_t \sum_{s=1}^{\infty} \phi_s a_{s,t}}_{K_t} \\
&= \underbrace{\frac{\xi\beta}{1-(1-\nu-\xi)\beta} w_t}_{\text{savings by high income}} + \underbrace{(1-\nu)\beta R_t K_t}_{\text{savings by low income}}
\end{aligned}
\tag{383}
$$

By substituting $w_t = (1-\theta)A_t^{1-\theta}K_t^{\theta}$ and $R_t = \theta A_t^{1-\theta}K_t^{\theta-1} + 1 - \delta$, we have the law of motion of aggregate capital in a closed form:

$$
\begin{aligned}
K_{t+1} &= \frac{\xi\beta(1-\theta)}{1-(1-\nu-\xi)\beta} A_t^{1-\theta}K_t^{\theta} + (1-\nu)\beta\left[\theta A_t^{1-\theta}K_t^{\theta} + (1-\delta)K_t\right] \\
&= \left[\frac{\xi\beta(1-\theta)}{1-(1-\nu-\xi)\beta} + (1-\nu)\beta\theta\right] A_t^{1-\theta}K_t^{\theta} + (1-\nu)\beta(1-\delta)K_t.
\end{aligned}
$$

□

As $Y_t = K_t^{\theta}(A_t L_t)^{1-\theta}$ with $L_t = 1$, we can express the law of motion as:

$$K_{t+1} = \hat{s}Y_t + (1-\hat{\delta})K_t \tag{384}$$

where

$$\hat{s} = \frac{\xi\beta(1-\theta)}{1-(1-\nu-\xi)\beta} + (1-\nu)\beta\theta \tag{385}$$

$$\hat{\delta} = 1 - (1-\nu)\beta(1-\delta). \tag{386}$$

Defining the time discount rate through $\beta = \frac{1}{1+\rho}$ we can write

$$\hat{s} = (1-\theta)\frac{\xi}{\xi+\nu+\rho} + \theta\frac{1-\nu}{1+\rho} \approx (1-\theta)\frac{\xi}{\xi+\nu+\rho} + \theta(1-(\nu+\rho)) \tag{387}$$

$$\hat{\delta} \approx \nu + \rho + \delta. \tag{388}$$

This expression resembles the law of motion of capital in a Solow model, but with a depreciation rate that is larger (by $\nu + \rho$) than the physical depreciation rate δ and a saving rate that is an explicit function of the structural parameters of the model and depends negatively on $\nu + \rho$ and positively on the risk of income falling to zero ξ. We discuss the relation to the literature in Section 7.

In a stationary equilibrium, aggregate variables are constant over time, i.e., $K_{t+1} = K_t = K$ and $A_t = A_0$. We verify that stationary aggregate capital derived from the law of motion coincides with the aggregate capital stock in the stationary equilibrium we derived in Section 4. The discussion is found in Appendix C.8.3.

C.5.2 Linear Approximation of Transitional Dynamics

This subsection derives a linear approximation of equation (**??**) around the stationary equilibrium. It gives us an approximation of capital at time t along the transition path. Figures illustrate that this approximation is farily accurate.

Proposition 28. *Consider an economy in a stationary equilibrium at $t = 0$ and a permanent productivity shock at $t = 1$ ($A_t = A_1 \neq A_0$ for all $t \geq 1$). Given that $\beta R_t < \frac{w_{t+1}}{w_t}$ for all $t \geq 1$ (the conditions are given in the next proposition), aggregate capital at time t is approximated by:*

$$K_t = \varphi^{t-1}(K_0 - K^*) + K^*, \tag{389}$$

$$\textit{where } 0 < \varphi := \theta + (1-\theta)(1-\nu)\beta(1-\delta) < 1. \tag{390}$$

Therefore, the interest rate, wage, and consumption of high-type agents at time $t \geq 1$ is approximated by:

$$R_t = \theta A_1^{1-\theta} \left[\varphi^{t-1}(K_0 - K^*) + K^*\right]^{\theta-1} + 1 - \delta \;\; \forall t \geq 1, \tag{391}$$

$$w_t = (1-\theta) A_1^{1-\theta} \left[\varphi^{t-1}(K_0 - K^*) + K^*\right]^{\theta} \;\; \forall t \geq 1, \tag{392}$$

$$\mathbf{c}_{h,t} := w_t c_{h,t} = (1-\theta) A_1^{1-\theta} \left[\varphi^{t-1}(K_0 - K^*) + K^*\right]^{\theta} \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta} z \;\; \forall t \geq 1. \tag{393}$$

Proof. The law of motion of capital is given by (**??**):

$$K_{t+1} - K_t = \left[\left(\frac{K^*}{K_t}\right)^{1-\theta} - 1\right]\left[1-(1-\nu)\beta(1-\delta)\right] K_t.$$

By denoting $\hat{\delta} = 1 - (1-\nu)\beta(1-\delta)$, it is written as:

$$K_{t+1} = \left[\hat{\delta}\left\{\left(\frac{K^*}{K_t}\right)^{1-\theta} - 1\right\} + 1\right] K_t,$$

$$\Leftrightarrow \frac{K_{t+1} - K^*}{K^*} = \left[\hat{\delta}\left\{\left(\frac{K_t}{K^*}\right)^{\theta-1} - 1\right\} + 1\right] \frac{K_t}{K^*} - 1. \tag{394}$$

A first-order Taylor expansion is:

$$f(K_t) \approx f(K^*) + f'(K^*)(K_t - K^*)$$
$$\text{where } f(K_t) = \left[\hat{\delta}\left\{\left(\frac{K_t}{K^*}\right)^{\theta-1} - 1\right\} + 1\right]\frac{K_t}{K^*} - 1,$$
$$f'(K_t) = \hat{\delta}(\theta-1)\left(\frac{K_t}{K^*}\right)^{\theta-2}\frac{1}{K^*}\frac{K_t}{K^*} + \left[\hat{\delta}\left\{\left(\frac{K_t}{K^*}\right)^{\theta-1} - 1\right\} + 1\right]\frac{1}{K^*}.$$

Therefore, the LHS of equation (394) is approximated by:

$$\frac{K_{t+1} - K^*}{K^*} \approx \left[1 - (1-\theta)\hat{\delta}\right]\frac{K_t - K^*}{K^*}. \tag{395}$$

The aggregate capital at time t after the productivity shock at time 1 is given by:

$$K_t \approx \left[1 - (1-\theta)\hat{\delta}\right]^{t-1}(K_1 - K^*) + K^*, \tag{396}$$
$$\text{where } \hat{\delta} = 1 - (1-\nu)\beta(1-\delta).$$

□

φ represents the speed of convergence (i.e., smaller φ implies quicker convergence). Comparative statics with respect to $(\theta, \nu, \beta, \delta)$ are following:

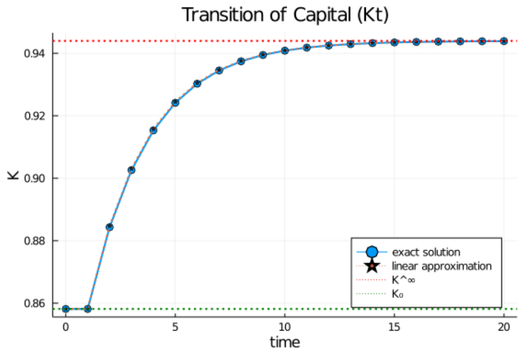
$$\frac{\partial\varphi}{\partial\theta} = 1 - (1-\nu)\beta(1-\delta) > 0,$$
$$\frac{\partial\varphi}{\partial\nu} = -(1-\theta)\beta(1-\delta) < 0,$$
$$\frac{\partial\varphi}{\partial\beta} = (1-\theta)(1-\nu)(1-\delta) > 0,$$
$$\frac{\partial\varphi}{\partial\delta} = -(1-\theta)(1-\nu)\beta < 0.$$

Figure 16 compares the paths of aggregate capital computed by equation (40) and the linear approximation. The approximation is accurate at least for the parameters chosen.

C.6 Path of Aggregate Variables and the Precision of First-Order Approximation

Figure 18 displays the path of aggregate variables, $(A_t, R_t, w_t, \beta R_{t+1}\frac{w_t}{w_{t+1}})$, after a permanent positive shock on A_t and monotonically increasing $\{A_t\}_{t=1}^{\infty}$ realized at $t = 1$, respectively. As we have derived an upper bound on $\tilde{k}_t := \frac{K_t/A_t}{K^*/A^*}$ using a first-order Taylor approximation in Lemma 24, we plot the path of $\tilde{k}_t$ and approximated $\tilde{k}_t$. It turns out that the approximation is farily accurate, as the approximation error ($:= 100 * \left(\frac{\text{approximated } \tilde{k}_t}{\tilde{k}_t} - 1\right)$) is less than 0.015% under the parameters chosen. Figure 19 illustrates the transition path after a negative shock on $\{A_t\}_{t=1}^{\infty}$.

Figure 16: Comparison between Exact Solution and Linear Approximation with parameters: ($\beta = 0.8, \xi = \nu = 0.2, \delta = 0.16, \theta = 0.33, A_0 = 1, A_1 = 1.1$)



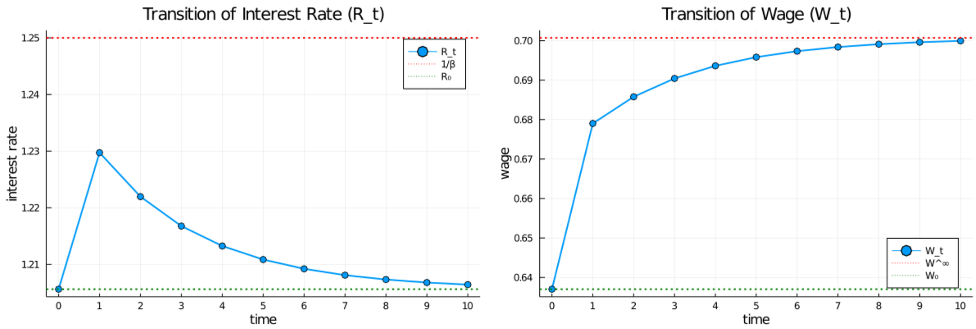


Figure 17: Transitional Dynamics of R_t and w_t

C.7 Stuff from Aggregate Shocks to Productivity

We now consider an economy with stochastic productivity growth, introducing aggregate risk into the economy. Assume that the productivity process $\{A_t\}$ is stochastic, following a probability distribution $\pi(A_{t+1}|A^t)$, which is independent of idiosyncratic shocks. All other elements of the model remain completely unchanged.

In Section C.7.1, we consider a general productivity process, where the probability of A_{t+1} could depend on the entire history of aggregate states $A^t \equiv \{A_0, A_1, \cdots, A_t\}$. We will prove that the insurance contract specified in the deterministic case is still optimal in an economy with aggregate shocks. We proceed with three steps: (i) Define a sequential market equilibrium with aggregate shocks and state our conjecture about household's consumption and saving; (ii) Under the conjecture that households do not save for high-income states, verify that the conjectured allocation satisfies the household's budget constraint, the Euler equation for low-income states, and market cleaning conditions; (iii) Under the assumption, $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all t and for all possible states (A^t, A_{t+1}), verify that households do not save for high-income states. Given that the conjectured price and allocation constitute a sequential market equilibrium, we derive a law of motion of aggregate capital, which remains to be in closed form.

In Section 6.1, we specify a productivity process with *iid* growth rates. Productivity

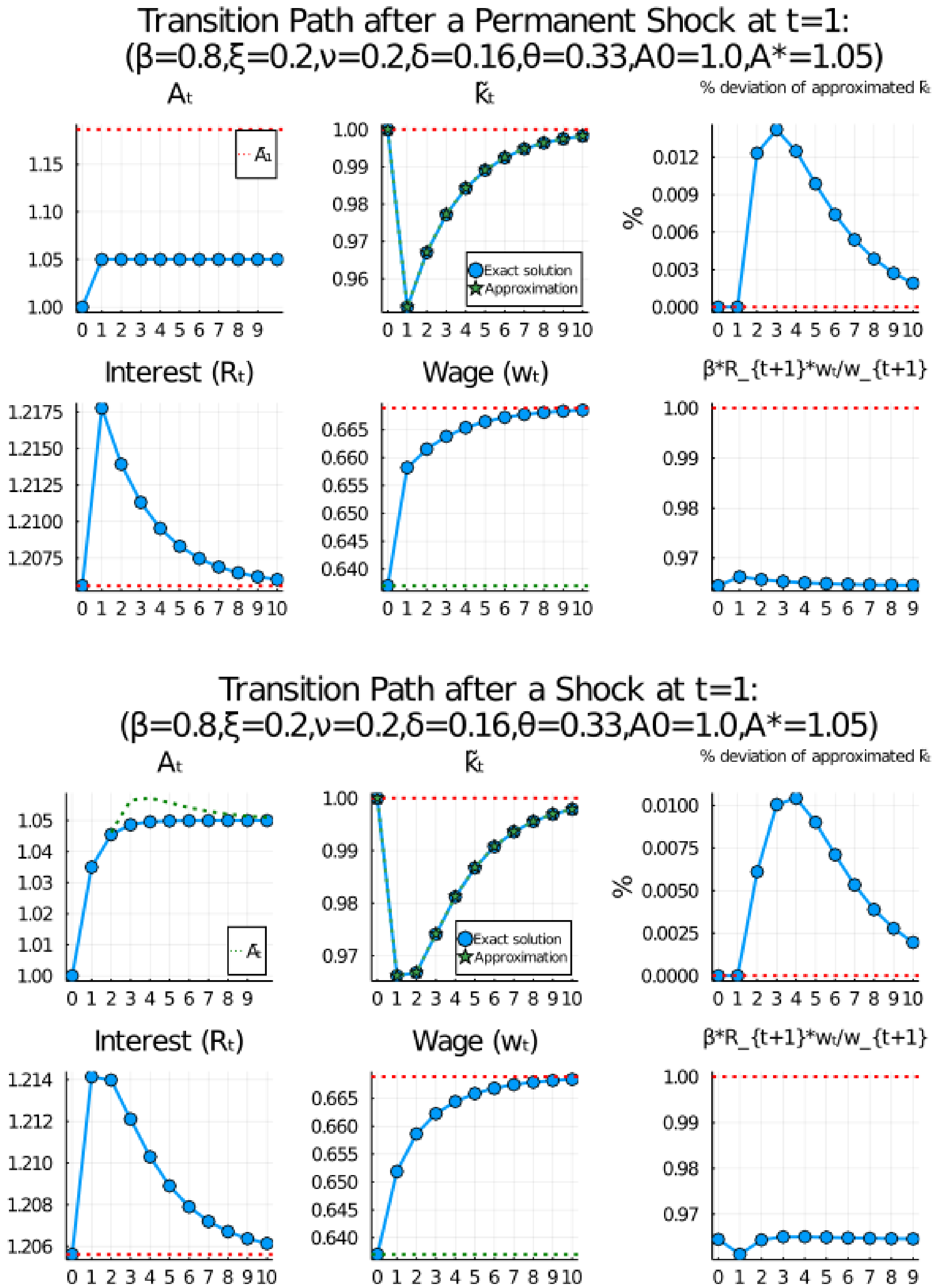


Figure 18: Transition Path after a Positive Shock on $\{A_t\}_{t=1}^{\infty}$, where $\tilde{k}_t := \frac{K_t/A_t}{K^*/A^*}$

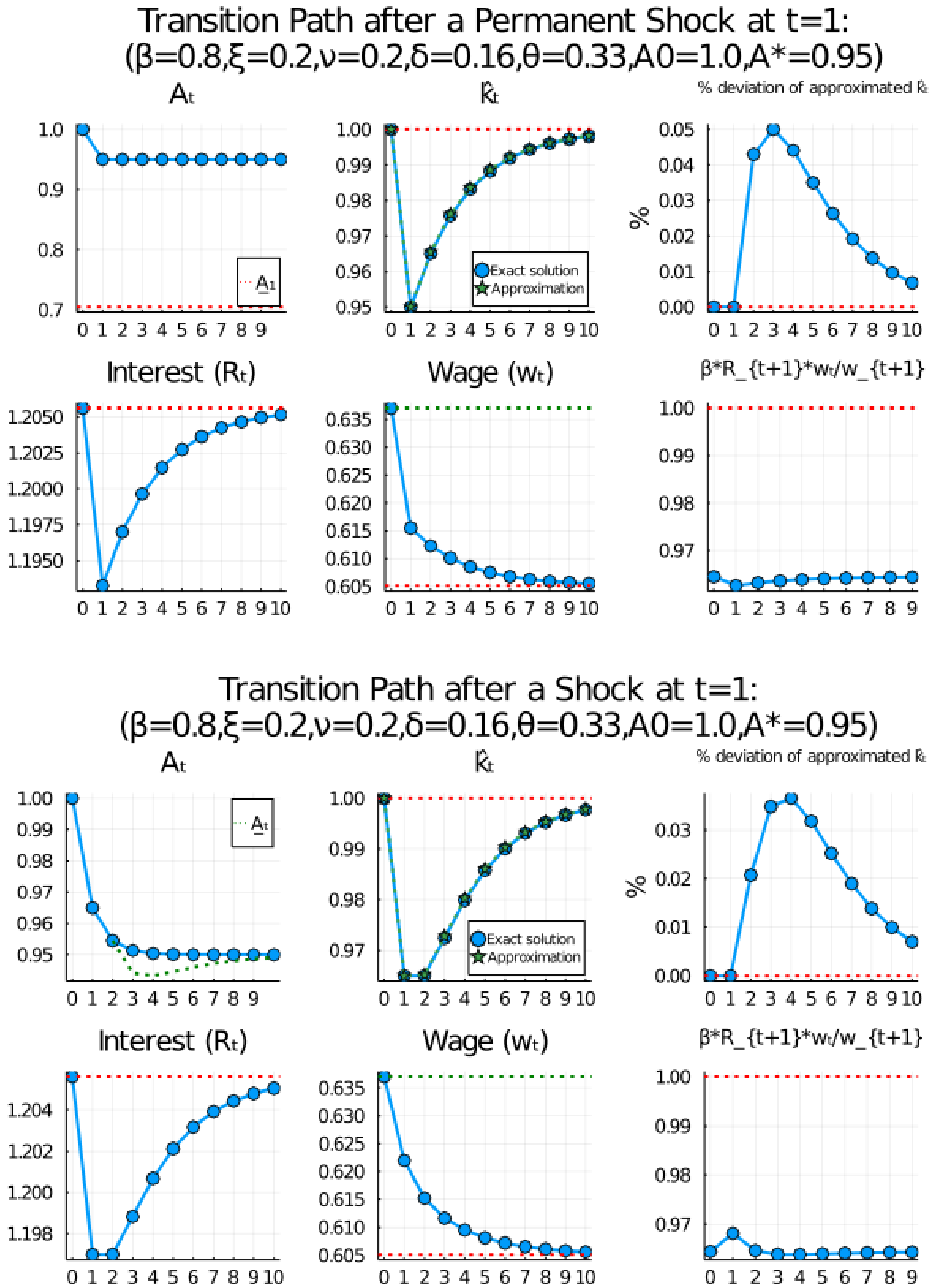


Figure 19: Transition Path after a Negative Shock on $\{A_t\}_{t=1}^{\infty}$, where $\hat{k}_t := \frac{K^*/A^*}{K_t/A_t}$

growth can take two values, $\frac{A_{t+1}}{A_t} \in \{1-\epsilon, 1+\epsilon\}$, with equal probability:

$$\frac{A_{t+1}}{A_t} = \begin{cases} 1+\epsilon \text{ with probability } \frac{1}{2} \\ 1-\epsilon \text{ with probability } \frac{1}{2} \end{cases} \quad \text{for all } t \geq 0 \text{ , } iid \tag{397}$$

Under this specification, we find a sufficient condition on ϵ such that $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ holds for all t and all (A^t, A_{t+1}). Proposition 8 states that under Assumption 5 and Assumption G, $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ holds for all t with probability 1. Corollary 9 states that under Assumption 5, a condition for $\beta R^* < 1$ in a steady state, there exists $\bar{\epsilon} > 0$ such that for all $0 < \epsilon < \bar{\epsilon}$, Assumption G holds, and hence $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ holds for all t and all possible states.

C.7.1 Sequential Market Equilibrium with Stochastic Growth

— This part is moved to Section 2 and 3.

C.7.2 Definition of Sequential Market Equilibrium

We first define a sequential market equilibrium with aggregate shocks. Households' consumption and savings depend on both the history of individual states $z^t = (z_0, z_1, \cdots, z_t)$ and the history of aggregate states $A^t = (A_0, A_1, \cdots, A_t)$.

Definition 2. *For an initial condition $(A_0, K_0, \Phi(a_0, z_0))$, an equilibrium is sequences of wages and interest rates $\{w_t(A^t), R_t(A^t)\}_{t=0,A^t}^{\infty}$, price of Arrow securities $\{q(A_{t+1}|A^t)\}_{t=0,A^t,A_{t+1}}^{\infty}$, aggregate consumption and capital $\{C_t(A^t), K_{t+1}(A^t)\}_{t=0,A^t}^{\infty}$ and individual consumption and asset allocations $\{\hat{\mathbf{c}}_t(a_0, z^t, A^t), \hat{\mathbf{a}}_{t+1}(a_0, z^{t+1}, A^{t+1})\}_{t=0}^{\infty}$ such that*

1. *Given $\{w_t(A^t), R_t(A^t), q(A_{t+1}|A^t)\}_{t=0,A^t,A_{t+1}}^{\infty}$, the household consumption and asset*

allocation $\{\hat{\mathbf{c}}_t(a_0, z^t, A^t), \hat{\mathbf{a}}_{t+1}(a_0, z^{t+1}, A^{t+1})\}$ *solves, for all* (a_0, z_0),[49]

$$\max_{\{\mathbf{c}_t(a_0,z^t,A^t),\mathbf{a}_{t+1}(a_0,z^{t+1},A^{t+1})\}} \sum_{t=0}^{\infty} \sum_{A^t} \sum_{z^t} \beta^t \pi(A^t)\pi(z^t) \log(\mathbf{c}_t(a_0, z^t, A^t)) \tag{398}$$

$s.t.$

$$\mathbf{c}_t(a_0, z^t, A^t) + \sum_{A_{t+1}} \sum_{z_{t+1}} q(A_{t+1}|A^t)\pi(z_{t+1}|z_t)\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) = w_t(A^t)z_t + R_t(A^t)\mathbf{a}_t(a_0, z^t, A^t) \tag{399}$$

$$\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) \geq 0 \tag{400}$$

2. *Factor prices equal marginal products*

$$w_t(A^t) = (1-\theta)A_t \left(\frac{K_t}{A_t}\right)^{\theta} \tag{401}$$

$$R_t(A^t) = \theta \left(\frac{K_t}{A_t}\right)^{\theta-1} + 1 - \delta \tag{402}$$

3. *The goods market and capital market clears*

$$C_t + K_{t+1} = K_t^{\theta}(A_t)^{1-\theta} + (1-\delta)K_t \tag{403}$$

$$K_{t+1} = \int \sum_{z^{t+1}} \hat{\mathbf{a}}_{t+1}(a_0, z^{t+1}, A^{t+1})\pi(z^{t+1})d\Phi(a_0, z_0) \ \ \forall A^{t+1} \tag{404}$$

where

$$C_t = \int \sum_{z^t} \hat{\mathbf{c}}_t(a_0, z^t, A^t)\pi(z^t)d\Phi(a_0, z_0) \tag{405}$$

We make two remarks on the budget constraint (**??**). First, a familiar way of defining Arrow securities may be that households pay a price $q^b(A_{t+1}, z_{t+1}|A^t, z^t)$ and receive one unit of non-deflated consumption goods if the state (A_{t+1}, z_{t+1}) is realized. If we denote such an asset by $\mathbf{b}_{t+1}$, the budget constraint is given by:

$$\mathbf{c}_t(a_0, z^t, A^t) + \sum_{A_{t+1}} \sum_{z_{t+1}} q^b(A_{t+1}, z_{t+1}|A^t, z^t)\mathbf{b}_{t+1}(a_0, z^{t+1}, A^{t+1}) = w_t(A^t)z_t + \mathbf{b}_t(a_0, z^t, A^t) \tag{406}$$

[49] We will follow a convention from now on that whenever we take a summation over aggregate states A_{t+1}, we sum over states with strictly positive probabilies. Mathematically, we denote:

$$\sum_{A^t} \pi(A^t) := \sum_{\{A^t; \pi(A^t)>0\}} \pi(A^t) = 1$$

$$\sum_{A_{t+1}} \pi(A_{t+1}|A^t) := \sum_{\{A_{t+1}; \pi(A_{t+1}|A^t)>0\}} \pi(A_{t+1}|A^t) = 1$$

Because we define an Arrow security that yields $R_{t+1}(A^{t+1})$ at $t+1$, the relation between the two types of Arrow securities is:

$$q^b(A_{t+1}, z_{t+1}|A^t, z^t) = \frac{q^a(A_{t+1}, z_{t+1}|A^t, z^t)}{R_{t+1}(A^{t+1})}, \tag{407}$$

$$\text{and } \mathbf{b}_{t+1}(a_0, z^{t+1}, A^{t+1}) = R_{t+1}(A^{t+1})\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) \tag{408}$$

Since the return on both Arrow securities is given by:

$$\frac{1}{q^b(A_{t+1}, z_{t+1}|A^t, z^t)} = \frac{R_{t+1}(A^{t+1})}{q^a(A_{t+1}, z_{t+1}|A^t, z^t)},$$

two formulations are equivalent. Our formulation of Arrow securities gives a simple expression for the capital market clearing condition (**??**).

Second, we express the price of Arrow security as:

$$q^a(A_{t+1}, z_{t+1}|A^t, z^t) = q(A_{t+1}|A^t)\pi(z_{t+1}|z_t). \tag{409}$$

This holds because aggregate states are independent of idiosyncratic shocks, and households pay an actuarially fair price for the insurance against idiosyncratic shocks.

C.7.3 Conjectured Allocation

We conjecture that households' consumption and saving rules will be the same as in the deterministic case (equations 365–370). Specifically, high-income households consume a constant fraction, $\frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}$, of their labor income, and low-income household's consumption follows the Euler equation. Because of the logarithmic utility function, low-income households consume a constant fraction, $[1-(1-\nu)\beta]$, of their capital income. Since the income and substitution effects cancel out, their saving choice does not depend on Arrow security price or future interest rates.

$$\mathbf{c}_t(z^t, A^t) = \begin{cases} w_t(A^t)c_0, \text{ where } c_0 := \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}\zeta, & \text{if } z_t = \zeta \\ \beta R_t(A^t)\mathbf{c}_{t-1}(z^{t-1}, A^{t-1}), & \text{if } z_t = 0 \end{cases} \tag{410}$$

$$\mathbf{a}_{t+1}(z^{t+1}, A^{t+1}) = \begin{cases} 0 & \text{if } z_{t+1} = \zeta \\ \frac{\beta}{1-(1-\nu-\xi)\beta}\zeta w_t(A^t) & \text{if } z_t = \zeta \text{ and } z_{t+1} = 0 \\ \beta R_t(A^t)\mathbf{a}_t(z^t, A^t) & \text{if } z_t = 0 \text{ and } z_{t+1} = 0 \end{cases} \tag{411}$$

$$\Rightarrow \mathbf{c}_t(z^t, A^t) = [1-(1-\nu)\beta]R_t(A^t)\mathbf{a}_t(z^t, A^t) \text{ if } z_t = 0 \tag{412}$$

C.7.4 Optimality Conditions for Household's Problem

Households' Lagrangian problem:

$$
\begin{aligned}
U(a_0, z_0) = & \max_{\{\mathbf{c}_t(a_0,z^t,A^t),\mathbf{a}_{t+1}(a_0,z^{t+1},A^{t+1})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\sum_{A^t}\sum_{z^t} \beta^t \pi(A^t)\pi(z^t)\log(\mathbf{c}_t(a_0, z^t, A^t)) \\
& + \sum_{t=0}^{\infty}\sum_{A^t}\sum_{z^t} \mu(z^t, A^t)\Bigg[w_t(A^t)z_t + R_t(A^t)\mathbf{a}_t(a_0, z^t, A^t) \\
& \qquad - \mathbf{c}_t(a_0, z^t, A^t) - \sum_{A_{t+1}}\sum_{z_{t+1}} q(A_{t+1}|A^t)\pi(z_{t+1}|z_t)\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) \Bigg] \\
& + \sum_{t=0}^{\infty}\sum_{A^{t+1}}\sum_{z^{t+1}} \beta^t \pi(A^{t+1})\pi(z^{t+1})\lambda(z^{t+1}, A^{t+1})\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}), \qquad (413)
\end{aligned}
$$

FOCs are:

$$
[\mathbf{c}_t(a_0, z^t, A^t)] : \ \beta^t \pi(A^t)\pi(z^t)\frac{1}{\mathbf{c}_t(a_0, z^t, A^t)} = \mu(z^t, A^t) \qquad (414)
$$

$$
\begin{aligned}
[\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1})] : \ & \mu(z^{t+1}, A^{t+1})R_{t+1}(A^{t+1}) + \beta^t \pi(A^{t+1})\pi(z^{t+1})\lambda(z^{t+1}, A^{t+1}) \\
& = \mu(z^t, A^t)q(A_{t+1}|A^t)\pi(z_{t+1}|z_t) \qquad (415)
\end{aligned}
$$

By substituing $\mu(z^t, A^t)$, we obtain the following Kuhn-Tucker condition:

$$
\frac{1}{\mathbf{c}_t(a_0, z^t, A^t)} = \frac{\pi(A_{t+1}|A^t)}{q(A_{t+1}|A^t)}\left[\beta R_{t+1}(A^{t+1})\frac{1}{\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1})} + \lambda(z^{t+1}, A^{t+1})\right] \qquad (416)
$$

$$
\text{where } \lambda(z^{t+1}, A^{t+1})\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) = 0, \ \lambda(z^{t+1}, A^{t+1}) \geq 0, \ \mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) \geq 0. \qquad (417)
$$

Conjecture 1. *Under logarithmic utility,*

$$
q(A_{t+1}|A^t) = \pi(A_{t+1}|A^t) \qquad (418)
$$

Lemma 25. *Suppose $q(A_{t+1}|A^t) = \pi(A_{t+1}|A^t)$. Then, the conjectured allocation (22) and (23) satisfies the household's budget constraint (**??**), the Euler equation for low-income households:*

$$
\frac{1}{\mathbf{c}_t(a_0, z^t, A^t)} = \frac{\pi(A_{t+1}|A^t)}{q(A_{t+1}|A^t)}\beta R_{t+1}(A^{t+1})\frac{1}{\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1}; z_{t+1} = 0)}, \qquad (419)
$$

*and the market clearing conditions (16) and (**??**).*

Proof. See Appendix A.4. □

C.7.5 No Savings for High-Income States

Lemma 7 shows that conjectured Arrow security price and the conjectured consumption and saving (22) and (23) satisfy equilibrium conditions, except that households do not save for a high-income state. The claim that households make no savings for high-income states follow the same logic as in Lemmas 19 and 20.[50] We show in Proposition **??** below that under the assumption, $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all $t \geq 0$ and all possible states A^t, the Kuhn-Tucker conditions are satisfied if households do not save for a high-income state. Therefore, the conjectured price and allocation constitute a sequential market equilibrium.

Proposition 29. *Suppose that an economy is in a steady state at $t = 0$ with $\beta R_0 < 1$ and that sequences of wages and interest rates $\{w_t(A^t), R_{t+1}(A^{t+1})\}_{t=0}^{\infty}$ satisfy $\beta R_{t+1}(A^{t+1}) < \frac{w_{t+1}(A^{t+1})}{w_t(A^t)}$ for all $t \geq 0$ and (A^t, A_{t+1}). Under the conjectured Arrow sequrity price, $q(A_{t+1}|A^t) = \pi(A_{t+1}|A^t)$, the Kuhn-Tucker conditions for high-income states are satisfied at any time $t \geq 0$ if households do not save for a high-income state. Hence, the conjectured allocation (22) and (23), aggregate consumption and capital given by (18) and (**??**), and sequences of prices $\{w_t(A^t), R_{t+1}(A^{t+1}), q(A_{t+1}|A^t)\}_{t=0}^{\infty}$ (14), (15), and (**??**) constitute a sequential market equilibrium.*

Proof. See Appendix A.4. □

C.7.6 The Aggregate Law of Motion

Since households's consumption and saving is given by equations (22) and (23), the law of motion of aggregate capital follows equation (40):

$$K_{t+1} = \left[\frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta) + (1-\nu)\beta\theta\right] A_t^{1-\theta} K_t^{\theta} + (1-\nu)\beta(1-\delta)K_t$$

We have seen that the conjectured allocation (22) and (23) satisfy all the equilibrium conditions if $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ holds for all $t \geq 0$ and all possible states. We will next derive a sufficient condition on the magnitude of ϵ such that $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ is guaranteed for all $t \geq 0$ and all possible states A^t.

C.7.7 Application

Implications for Asset Pricing Since we know the price of Arrow securities, $q(A_{t+1}|A^t) = \pi(A_{t+1}|A^t)$, and the interest rate at each aggregate state, $R_{t+1}(A^{t+1}) = \theta\left(\frac{K_{t+1}}{A_{t+1}}\right)^{\theta-1} + 1 - \delta$,

[50] At $t = 0$, given an initial state ($a_0 = 0, z_0 = \zeta$) or ($a_0 \leq \bar{a}_0 := \frac{\beta}{1-(1-\nu-\xi)\beta}\zeta, z_0 = 0$), households cannot achieve higher utility at $t = 1$ by saving for a high-income state at $t = 1$ if $\beta R_1 \frac{w_0}{w_1} < 1$. Since consumption choice $c_t(a_0, z^t; z_t = \zeta)$ cannot be larger than $c_0 := \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}\zeta$ at any $t \geq 0$, making positive savings for a high-income state at $t = 1$ wouldn't give higher utility at any time $t \geq 1$, under the assumption of $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 0$ and all possible states A^t. By induction, households at any time $t \geq 1$ do not save for a high-income state at $t + 1$, since they enter a state at time $t \geq 1$ with ($a_t = 0, z_t = \zeta$) or ($a_t \leq \bar{a}_0, z_t = 0$).

we can compute the price of any securities. Examples include a risk-free bond that gives a unit of non-deflated consumption goods at $t+1$ regardless of aggregate states and risky capital whose return depends on aggregate state A^{t+1}. We can check:

1. Comparison with the representative agent model:
   conjecture: idiosyncratic risks under a limited commitment lower the return on assets compared to the representative agent model, but it has no impact on risk premium

2. Comparison with Krueger and Lustig (2010):
   the presence of uninsurable idiosyncratic risk lowers the equilibrium risk-free rate, but it has no effect on the price of aggregate risk in equilibrium under key conditions: (i) a continuum of agents, (ii) CRRA utility, (iii) idiosyncratic labor income risk that is independent of aggregate risk, (iv) a constant capital share of income, and (v) solvency constraints or borrowing constraints on total financial wealth that are proportional to aggregate income.

Inequality over a Business Cycle

- The result from a case of permanent shocks:
  With partial depreciation ($\delta < 1$), the consumption inequality expands at the time of a positive productivity shock.

C.7.8 Full Depreciation of Capital ($\delta = 1$)

In Section C.7.1, we made three assumptions: (i) aggregate productivity process $\{A_t\}$ is independent of idiosyncratic shocks, (ii) the economy is in a steady state at $t = 0$ with $\beta R_0 < 1$, (iii) $\beta R_{t+1}(A^{t+1}) < \frac{w_{t+1}(A^{t+1})}{w_t(A^t)}$ is satisfied for all $t \geq 0$ and (A^t, A_{t+1}). Section 6.1 derived a sufficient condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 0$ and A^t.

In this section, we look into the case with $\delta = 1$. Under full depreciation of capital, $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ holds with any productivity process of A_t that is independent of idiosyncratic shocks. In addition, a representative-agent neoclassical growth model has a law of motion of aggregate capital in closed form under $\delta = 1$, which facilitates a sharp comparison between the limited-commitment model and the neoclassical growth model.

The ratio between βR_{t+1} and $\frac{w_{t+1}}{w_t}$ (equation 71) under full depreciation of capital is given by:

$$\frac{\beta R_{t+1}}{w_{t+1}/w_t}\Bigg|_{\delta=1} = \frac{\beta\left[\theta\left\{\frac{A_t}{A_{t+1}}\left[\hat{s}(\tilde{K}_t)^\theta + (1-\hat{\delta})\tilde{K}_t\right]\right\}^{\theta-1} + 1 - \delta\right]}{\left(\frac{A_{t+1}}{A_t}\right)^{1-\theta}\left[\hat{s}(\tilde{K}_t)^{\theta-1} + 1 - \hat{\delta}\right]^\theta}\Bigg|_{\delta=1}$$

$$= \frac{\beta\theta}{\hat{s}} \tag{420}$$

This condition does not depend on $\{A_t\}$ or $\{K_t\}$. Hence, as long as the parameters satisfy $\frac{\beta\theta}{\hat{s}} < 1$, the condition $\beta R_{t+1}(A^{t+1}) < \frac{w_{t+1}(A^{t+1})}{w_t(A^t)}$ is satisfied for any productivity process of $\{A_t\}$. The condition $\frac{\beta\theta}{\hat{s}} < 1$ is equivalent to Assumption 5 with $\delta = 1$:

$$\frac{\theta}{1-\theta}\,\frac{1}{\frac{1}{\beta}} < \frac{\xi}{\nu\left(\frac{1}{\beta} - 1 + \xi + \nu\right)}. \tag{421}$$

Therefore, under Assumption 5 and $\delta = 1$, the conditions $\beta R_0 < 1$ and $\beta R_{t+1}(A^{t+1}) < \frac{w_{t+1}(A^{t+1})}{w_t(A^t)}$ for all $t \geq 0$ and (A^t, A_{t+1}) are satisfied.

The law of motion of capital (40) with $\delta = 1$ is given by:

$$K_{t+1} = \hat{s}A_t^{1-\theta}K_t^\theta \tag{422}$$

$$\text{where } \hat{s} := \frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta) + (1-\nu)\beta\theta$$

$$\Leftrightarrow\ \tilde{K}_{t+1} = \frac{A_t}{A_{t+1}}\hat{s}\tilde{K}_t^\theta. \tag{423}$$

$$\therefore\ \tilde{K}^{*LC} = \hat{s}^{\frac{1}{1-\theta}} \tag{424}$$

In a representative-agent neoclassical growth model with $\delta = 1$, there is a well-known closed form solution to the law of motion of capital:

$$K_{t+1} = \beta\theta A_t^{1-\theta}K_t^\theta \tag{425}$$

$$\Leftrightarrow\ \tilde{K}_{t+1} = \frac{A_t}{A_{t+1}}\beta\theta\tilde{K}_t^\theta. \tag{426}$$

$$\therefore\ \tilde{K}^{*Rep} = (\beta\theta)^{\frac{1}{1-\theta}} \tag{427}$$

As we see above, Assumption 5 under $\delta = 1$ is equivalent to $\beta\theta < \hat{s}$. This means that in the steady state, the limited commitment economy holds more capital, and the steady-state interest rate ($R^* = \theta\tilde{K}^{\theta-1}$) is lower:

$$\tilde{K}^{*Rep} < \tilde{K}^{*LC}, \tag{428}$$

$$R^{*Rep} > R^{*LC}. \tag{429}$$

C.8 Stuff from "Additional Discussions"

C.8.1 Definition of Contract Equilibrium

Instead of a financial market as in the main text, we can imagine competitive financial intermediaries offering long-term consumption insurance contracts, subject to a zero-profit condition. Consider a financial intermediary that signs an agent with current labor productivity z and current assets a in period t. The intermediary maximizes that household's lifetime utility, subject to the limited commitment constraints and subject to a zero-profit condition. To set up the zero-profit condition, define the cost net of labor income of an allocation starting from labor productivity history z^t as

$$V_t(\mathbf{c}, z^t) = \mathbf{c}_t(z^t) - w_t z_t + \frac{\sum_{z_{t+1}} \pi(z_{t+1}|z_t) V_{t+1}(\mathbf{c}, z^{t+1})}{1 + r_{t+1}} \tag{430}$$

Furthermore, for any allocation $\mathbf{c}$ define the continuation utility from history z^t on as

$$U_t(\mathbf{c}, z^t) = \sum_{\tau=t}^{\infty} \sum_{z^\tau|z^t} \beta^{\tau-t} \pi(z^\tau|z^t) \log(\mathbf{c}_t(z^\tau)) \tag{431}$$

Given a process of outside options $\{U_t^{Out}(z^t)\}$ an optimal contract for an agent with initial assets and labor productivity a, z then solves the problem

$$U_0(a_0, z_0) = \max_{\{\mathbf{c}_t(a_0, z^t)\}} \sum_{t=0}^{\infty} \sum_{z^t} \beta^t \pi(z^t) \log(\mathbf{c}_t(a_0, z^t)) \tag{432}$$

$$s.t.$$

$$V_0(\mathbf{c}, z_0) \leq (1 + r_0) a_0 \tag{433}$$

$$U_t(\mathbf{c}, z^t) \geq U_t^{Out}(z_t) \qquad \forall t \geq 0, z^t \tag{434}$$

The equilibrium outside options then satisfy $U_\tau^{Out}(z_\tau) = U_\tau(0, z_\tau)$ for all $\tau \geq 0$ and all $z_\tau \in Z$. Here $U_\tau(0, z_\tau)$ is defined analogously to equation (432): it is equal to maximal lifetime utility that can be obtained from a contract starting with productivity z_τ and no assets, that is, a consumption contract with expected lifetime cost equal to the expected present value of labor income, starting from current productivity z_t, see equation (433). As Krueger and Uhlig (2006) the consumption allocation emerging from the optimal contracting problem and the financial markets formulation with Arrow securities yield the same consumption allocation (at the equilibrium outside options). Furthermore, the mapping between the state-contingent assets and the cost of the consumption contracts are given by

$$V_t(z^t) = (1 + r_t) a_t(z^t) \tag{435}$$

Equilibrium can then be defined analogously to definition **??**.

C.8.2 Walras's Law

We derive the capital market clearing condition from the goods market clearing condition, household's budget constraint, and the pricing functions. The goods market clearing condition is expressed as:

$$\underbrace{w_t C_t}_{\text{consumption}} + \underbrace{K^s_{t+1} - (1-\delta)K^s_t}_{\text{investment}} = \underbrace{A_t^{1-\theta}(K^d_t)^\theta}_{\text{production}}, \tag{378}$$

where I denote the household's saving by K^s and the firm's capital demand by K^d. We want to derive the capital-market clearing condition:

$$K^s_t = K^d_s, \tag{436}$$
$$\text{where } K^s_t = w_t \sum_{s=0}^{\infty} \phi_s a_{s,t}.$$

As we consider the case with $\beta R_t < 1$, the deflated asset position of high-type agents is always zero: $a_{0,t} = 0$. Following equations are the budget constraints of high-income and low-income agents:

$$w_t c_{h,t} = \underbrace{w_t z}_{\text{labor income}} - \underbrace{\xi w_{t+1} a_{1,t+1}}_{\text{saving}},$$
$$w_t c_{s,t} = \underbrace{R_t w_t a_{s,t}}_{\text{capital income}} - \underbrace{(1-\nu) w_{t+1} a_{s+1,t+1}}_{\text{saving}} \text{ for } s = 1, 2, \cdots.$$

The aggregate consumption, $w_t C_t$, is the sum of each agent's consumption:

$$\begin{aligned} w_t C_t &:= \sum_{s=0}^{\infty} \phi_s w_t c_{s,t} \\ &= w_t \underbrace{\phi_0 z}_{\equiv 1} + \underbrace{R_t \sum_{s=1}^{\infty} \phi_s w_t a_{s,t}}_{K^s_t} - \underbrace{w_{t+1}\left[\phi_0 \xi a_{1,t+1} + (1-\nu)\sum_{s=1}^{\infty} \phi_s a_{s+1,t+1}\right]}_{= w_{t+1}\left[\sum_{s=1}^{\infty} \phi_s a_{s,t+1}\right] = K^s_{t+1}}. \end{aligned} \tag{437}$$

Substituting $w_t C_t$ in the goods market clearing condition (378) by (437) yields:

$$\begin{aligned} \underbrace{w_t + R_t K^s_t - K^s_{t+1}}_{= w_t C_t} + K^s_{t+1} - (1-\delta)K^s_t &= A_t^{1-\theta}(K^d_t)^\theta \\ \Leftrightarrow \; w_t + (R_t - 1 + \delta)K^s_t &= A_t^{1-\theta}(K^d_t)^\theta, \\ \text{where } w_t &= (1-\theta)A_t^{1-\theta}(K^d_t)^\theta, \\ R_t &= \theta A_t^{1-\theta}(K^d_t)^{\theta-1} + 1 - \delta. \end{aligned} \tag{438}$$

Plugging w_t and R_t into (438) gives:

$$\begin{aligned}(1-\theta)A_t^{1-\theta}(K_t^d)^\theta + \left[\theta A_t^{1-\theta}(K_t^d)^{\theta-1}\right]K_t^s &= A_t^{1-\theta}(K_t^d)^\theta \\ \Leftrightarrow \left[\theta A_t^{1-\theta}(K_t^d)^{\theta-1}\right]K_t^s &= \theta A_t^{1-\theta}(K_t^d)^\theta \\ \Leftrightarrow K_t^s &= K_t^d.\end{aligned}$$

Therefore, we reached the capital market clearing condition.

C.8.3 Aggregate Law of Motion of Capital

Goods Market We can equivalently derive the law of motion for capital from the goods market clearing condition.

$$K_{t+1} = A_t^{1-\theta}K_t^\theta + (1-\delta)K_t - w_t C_t$$

$$\begin{aligned}\text{where } C_t &= \phi_0 c_{h,t} + \sum_{s=1}^{\infty}\phi_s \underbrace{c_{s,t}}_{=[1-(1-\nu)\beta]R_t a_{s,t}} \\ &= \underbrace{\frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}}_{\phi_0 c_{h,t}} + [1-(1-\nu)\beta]R_t\frac{K_t}{w_t}\end{aligned}$$

$$\begin{aligned}\therefore\ K_{t+1} &= A_t^{1-\theta}K_t^\theta + (1-\delta)K_t - \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}\underbrace{w_t}_{=(1-\theta)A_t^{1-\theta}K_t^\theta} - [1-(1-\nu)\beta]\underbrace{R_t}_{=\theta A_t^{1-\theta}K_t^{\theta-1}+1-\delta}K_t \\ &= (1-\theta)A_t^{1-\theta}K_t^\theta - \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}(1-\theta)A_t^{1-\theta}K_t^\theta + (1-\nu)\beta\left[\theta A_t^{1-\theta}K_t^\theta + (1-\delta)K_t\right] \\ &= \frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta)A_t^{1-\theta}K_t^\theta + (1-\nu)\beta\left[\theta A_t^{1-\theta}K_t^\theta + (1-\delta)K_t\right].\end{aligned}$$

Aggregate Capital in the Stationary Equilibrium In a stationary equilibrium, $K_t = K_{t+1} = K$ and $A_t = A$. By substituting them into (40), we derive the aggregate capital in the stationary equilibrium. We check that this coincides with the one computed before. (40) with $K_t = K_{t+1} = K$ and $A_t = A$ is given by:

$$\begin{aligned}K &= \frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta)A^{1-\theta}K^\theta + (1-\nu)\beta\left[\theta A^{1-\theta}K^\theta + (1-\delta)K\right] \\ \Leftrightarrow\ K^{1-\theta} &= \frac{1}{1-(1-\nu)\beta(1-\delta)}\left[\frac{\xi\beta}{1-(1-\nu-\xi)\beta}(1-\theta) + (1-\nu)\beta\theta\right]A^{1-\theta} \\ \therefore K^* &= A\left[\frac{\xi\beta(1-\theta) + (1-\nu)\beta\theta\left[1-(1-\nu-\xi)\beta\right]}{\left[1-(1-\nu)\beta(1-\delta)\right]\left[1-(1-\nu-\xi)\beta\right]}\right]^{\frac{1}{1-\theta}} \qquad (439)\end{aligned}$$

Now I check that this is consistent with the results in a stationary equilibrium:

$$K^* = A\left(\frac{\theta}{R^*-1+\delta}\right)^{\frac{1}{1-\theta}},\ \text{where } R^* = \frac{\xi(1-\theta)(1-\delta) + \theta\left(\xi+\nu+\frac{1}{\beta}-1\right)}{\xi(1-\theta) + \beta\theta(1-\nu)\left(\xi+\nu+\frac{1}{\beta}-1\right)}.$$

As we have:

$$R^* - 1 + \delta = \frac{\theta\left[1-(1-\delta)\beta(1-\nu)\right](\xi+\nu+\frac{1}{\beta}-1)}{\xi(1-\theta)+\beta\theta(1-\nu)\left(\xi+\nu+\frac{1}{\beta}-1\right)},$$

we obtain

$$K^* = A\left[\frac{\xi(1-\theta)+\beta\theta(1-\nu)\left(\xi+\nu+\frac{1}{\beta}-1\right)}{\left[1-(1-\delta)\beta(1-\nu)\right](\xi+\nu+\frac{1}{\beta}-1)}\right]^{\frac{1}{1-\theta}}. \tag{440}$$

This is the same as (439). Therefore, the level of capital K^* derived from the law of motion (40) is the same as K^* computed in the stationary equilibrium.