

One-Sided Limited Commitment and Aggregate Risk*

Yoshiki Ando

University of Pennsylvania

Dirk Krüger

University of Pennsylvania, CEPR and NBER

Harald Uhlig

University of Chicago, CEPR and NBER

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Abstract

In this paper we study the neoclassical growth model with idiosyncratic income risk and aggregate risk in which risk sharing is endogenously constrained by one-sided limited commitment. Households can trade a full set of contingent claims that pay off depending on both idiosyncratic and aggregate risk, but limited commitment rules out that households sell these assets short. The model results, under suitable restrictions of the parameters of the model, in partial consumption insurance in equilibrium. With log-utility and idiosyncratic income shocks taking two values one of which is zero (e.g., employment and unemployment) we show that the equilibrium can be characterized in closed form, despite the fact that it features a non-degenerate consumption- and wealth distribution. We use the tractability of the model to study, analytically, inequality over the business cycle and asset pricing, and derive conditions under which our model has identical, as well as conditions under which it has lower/higher risk premia than the corresponding representative agent version of the model.

Keywords: Limited commitment, idiosyncratic income risk, neoclassical growth model, transitional dynamics, aggregate risk

J.E.L. classification codes: E21, E23, D15, D31

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1 Introduction

The canonical macroeconomic model with household heterogeneity developed by Bewley (1986), Imrohoroglu (1989), Huggett (1993), and Aiyagari (1994) envisions a large population of households facing idiosyncratic income risk and incomplete asset markets. This model has been used by a vast applied literature studying business cycles (as in Krusell and Smith, 1998), fiscal policy (as in Aiyagari and McGrattan, 1998) and monetary policy (as in Kaplan, Moll, and Violante, 2018). From a theoretical perspective, however, imperfect consumption insurance against income risk (and the resulting non-degenerate wealth distribution) stems from the exogenous restriction of the set of insurance contracts that individuals can enter, restricting explicit insurance that individuals would otherwise choose to trade.¹ As an alternative, imperfect consumption insurance can emerge as the consequence of limited contract enforcement or private information that impedes the full insurance of idiosyncratic risk.

In this paper we seek to integrate a household consumption-saving problem with idiosyncratic income risk and limited contract enforcement into a discrete-time neoclassical production economy, in the same way that Aiyagari (1994) and Krusell and Smith (1998) did for the standard incomplete markets structure. Specifically, households face idiosyncratic and aggregate productivity (and potentially, stochastic depreciation) risk in which risk sharing is endogenously constrained by one-sided limited commitment. Households can trade a full set of contingent claims that pay off depending on both idiosyncratic and aggregate risk, but limited commitment rules out that households sell these assets short. The model results, under suitable restrictions of the parameters of the model, in partial consumption insurance in equilibrium. With log-utility and idiosyncratic income shocks taking two values one of which is zero (e.g., employment and unemployment) we show that the equilibrium can be characterized in closed form, despite the fact that it features a non-degenerate consumption- and wealth distribution which fluctuates over the cycle.

The key to this result is that if interest rates are low and/or future wage growth is high, then households optimally do not purchase contingent claims that pay off in their high idiosyncratic productivity state. Households with low productivity in contrast hold assets, but in the absence of labor income their consumption-saving problem has a simple solution (resembling that of the classic cake-eating problem) in which all low-

¹On the empirical side, Blundell, Pistaferri, and Preston (2008) document a degree of consumption insurance especially with respect to highly persistent or permanent shocks that is difficult to fully rationalize in the standard incomplete markets model, see Kaplan and Violante (2010) or Krueger and Wu (2021). More recent empirical papers confirm the need for models in which household consumption smoothing opportunities extend beyond simple self-insurance, see, e.g., Arellano, Blundell, and Bonhomme (2017), Eika, Mogstad, and Vestad (2020), Chatterjee, Morley, and Singh (2021), Braxton et al. (2021), Commault (2022), and Balke and Lamadon (2022).

productivity households exhibit the same constant saving rate out of their capital income. The constant saving rate is independent of the current (and expectations about future) interest rates, a consequence of logarithmic period utility. Consequently, the model easily aggregates and the aggregate law of motion for capital can be given in closed form, both in the case of an unexpected shock to aggregate productivity as well as in the case of recurrent aggregate productivity fluctuations. We then verify that as long as the productivity fluctuations (expected or unexpected) are not too large (with precise conditions given in terms of exogenous parameters only), then equilibrium interest rates are indeed sufficiently low (and wage growth is sufficiently high) that individuals indeed find it optimal to not purchase contingent claim for the high-productivity state.

Finally, we demonstrate the potential usefulness of our theoretical framework for applied work by studying two applications. First, we show that consumption inequality is procyclical in our model: in response to a positive productivity shock, consumption of high productivity individuals (those at the top of the consumption distribution) increases proportionally with the wage, whereas the increase in consumption of those with only capital income (who are located at the lower end of the consumption distribution) is more sluggish. Thus, consumption inequality increases in response to a positive productivity shock that expands output in the economy.

Second, we investigate whether the presence of idiosyncratic and only partially insured risk changes the asset pricing implication of this production economy, relative to that of the standard complete markets model. We first show that if (and only if) both wages and returns move proportionally with the aggregate shock (which is the case in a version of our model in which capital fully depreciates between periods, or in an endowment version of the model), then the multiplicative equity premium in our economy coincides with that of the representative agent (RA) economy. With less than full (but deterministic) depreciation, capital income becomes less volatile than labor income and the limited commitment model features a *lower* risk premium than the RA economy. However, once we extend the model to permit the depreciation rate to be stochastic as well, not only does capital income turn more volatile than labor income, but now the risk premium in the model with idiosyncratic risk exceeds that of the RA economy, and, depending on the size of idiosyncratic risk, can display a quantitatively sizable return premium on risky capital. It is important to note that we can demonstrate all but the last quantitative results theoretically, owing to the analytical tractability of our model with (only partially insurable) idiosyncratic and aggregate risk.

1.1 Related Literature

Our paper is motivated by the large literature on general equilibrium models with idiosyncratic risk and incomplete markets cited in the previous section. Our limited commitment alternative builds on the theoretical limited commitment literature pioneered by Thomas and Worrall (1988), Kehoe and Levine (1993, 2001), Kocherlakota (1996), Alvarez and Jermann (2000), Ábrahám and Laczó (2018) and, for a continuum economy, Krueger and Perri (2006), Krueger and Perri (2011) and Cole, Krueger, Mailath, and Park (2023). The papers by Krueger, Lustig, and Perri (2008) or Broer (2013) explore the quantitative properties of this class of models.

In Krueger and Uhlig (2006) we studied theoretically and in *partial equilibrium*, a version of this model with one-sided commitment in which households entered into long-term insurance contracts with financial intermediaries and there was no exogenous punishment from default. Rather, households could sign the best available contract with a competing intermediary, with the value of this contract being determined endogenously in equilibrium. Crucially, for the current paper, we showed that the contract equilibrium allocation could alternatively be decentralized in an asset market equilibrium with state-contingent claims and state-contingent shortsale constraints (in the spirit of Alvarez and Jermann, 2000) that are exactly at zero, i.e. ruling it borrowing altogether.

This paper embeds this structure into a general equilibrium production economy and exhibits assumptions that allows for a characterization of equilibrium in closed form, even in the presence of aggregate shocks.² In Krueger and Uhlig (2022) and Krueger, Li, and Uhlig (2023) we construct a limited commitment production economy in continuous time and Poisson income risk to analytically characterize the steady state and transitional dynamics of the model as the solution to a Bernoulli differential equation. The current paper extends this analysis to aggregate shocks and casts it in a perhaps more familiar discrete time setting, allowing for a direct connection and comparison with Alvarez and Jermann (2000)'s analysis of state-contingent shortsale constraints in general equilibrium. Broer (2020) studies the degree of consumption insurance over the business cycle, providing a quantitative comparison of the standard incomplete markets model and the limited commitment model with aggregate shocks.

Finally, we share the focus on analytical tractability in general equilibrium models with idiosyncratic risk and partial consumption insurance with Krueger, Ludwig, and Villalvazo (2021), Achdou, Han, Lasry, Lions, and Moll (2022), and Kocherlakota (2023) who study the standard incomplete markets model. To provide intuition for the under-

²Related general equilibrium analyses also appear in the sovereign debt literature, see, e.g., Hellwig and Lorenzoni (2009) and Martins-da Rocha and Santos (2019), and in general equilibrium models with collateral constraints, such as Gottardi and Kubler (2015).

lying reason that makes our model tractable we also relate our results to that obtained in the two-agent economy with workers and capitalists studied by Moll (2014).

The rest of the paper unfolds as follows. The next section sets up the model and defines equilibrium, and Section 3 contains a general characterization of this equilibrium. In Section 4 we then analyze the stationary equilibrium and the transition path after a one-time unexpected shock. We study the asset pricing implications of the model with aggregate shocks in Section 5. Equipped with the results from our model we then relate these results to related versions of the neoclassical growth model from the existing literature in Section 6. Section 7 concludes, and detailed derivations and proofs are contained in the Appendix.

2 Model

Time is discrete, infinite and indexed by $t = 0, 1, \dots$. The economy is populated by a continuum of households of measure 1 and a representative, competitive production firm.

2.1 Aggregate Risk

We consider an economy with idiosyncratic and potentially also aggregate productivity shocks. Denote by A_t current aggregate total factor productivity, and by $A^t = \{A_0, \dots, A_t\}$ the history of productivity, with associated probability distribution $\pi(A_{t+1}|A^t)$. We treat the initial productivity level A_0 as fixed.³ For most of the paper no further substantive restrictions on the productivity process $\{A_t\}$ need to be imposed,⁴ but for a subset of the results we need to make further assumptions on the productivity process.

Assumption 1. *Let the productivity process $\{A_t\}_{t=0}^\infty$ satisfy one of three assumptions:*

1. *Steady state: $A_t = A_0$ with probability 1, for all $t > 0$.*
2. *MIT shock: at time $t = 0$ productivity is at its steady value A_0 which households believe to last forever. At time $t = 1$ they learn that instead productivity is given by the deterministic sequence $\{A_t\}$.*
3. *Stochastic growth rates: the growth rate of aggregate productivity $\frac{A_{t+1}}{A_t}$ follows a finite state Markov chain.*

³We introduce stochastic depreciation rates into the model in Section 5. Until then, to keep the notation concise, we set up the model (and its notation) only with aggregate shocks to total factor productivity.

⁴For convenience of notation, we assume that A_t takes only positive real values, $A_t \in \mathbb{R}_+$, and that the number of possible states in each period is finite.

2.2 Technology

The production side of the economy is described by a completely standard neoclassical production function of the form

$$Y_t = K_t^\theta (A_t L_t)^{1-\theta} \quad (1)$$

where θ is the capital share and A denotes the (potentially time-varying) level of aggregate (labor-augmenting) productivity. Capital depreciates at rate δ . This production technology is operated by a representative and competitive firm hiring labor and capital at rental rates w_t, r_t , and the standard optimality conditions read as

$$w_t = (1 - \theta) A_t \left(\frac{K_t}{A_t L_t} \right)^\theta \quad (2)$$

$$r_t = \theta \left(\frac{K_t}{A_t L_t} \right)^{\theta-1} - \delta \quad (3)$$

2.3 Idiosyncratic Risk, Household Endowments and Preferences

Individuals are indexed by $i \in [0, 1]$ and in each period t have idiosyncratic stochastic labor productivity $z_{it} \in Z = \{0, \zeta\}$, where $\zeta > 1$ is a parameter. Since the identity of individuals is irrelevant we will suppress the index i whenever there is no scope for confusion and simply write z_t for the current idiosyncratic labor productivity as well as $z^t = (z_0, z_1, \dots, z_t)$ for the history of productivity realizations. The probability of a given productivity history is denoted by $\pi(z^t)$. The distribution of idiosyncratic shocks is assumed to be independent of aggregate shocks.

The idiosyncratic labor productivity process is Markov with time-invariant transition matrix:

$$\pi(z_{t+1}|z_t) = \begin{bmatrix} 1 - \nu & \nu \\ \xi & 1 - \xi \end{bmatrix} \quad (4)$$

where ν is the probability of switching from productivity 0 to productivity ζ and ξ is the probability of switching from ζ to 0. The stationary distribution over labor productivity is then given by $(\psi_l, \psi_h) = \left(\frac{\xi}{\xi + \nu}, \frac{\nu}{\xi + \nu} \right)$, and households are assumed to draw their initial productivity from this stationary distribution (which is then also the cross-sectional distribution of labor productivity at all future dates $t > 0$). We normalize average labor productivity to one, which implies the parameter restriction

$$\frac{\nu}{\xi + \nu} \zeta = 1. \quad (5)$$

This assumption implies that the aggregate supply of labor is $L_t = 1$ for all t . In addition to labor productivity a given household is endowed with initial wealth a_0 , and we denote the cross-section probability measure over wealth and labor productivity by $\Phi(a_0, z_0)$.

Each household has preferences representable by a standard intertemporal utility function $u(c)$ defined over stochastic consumption streams c and given by

$$U(c) = \mathbf{E}_0 \sum_{t=0}^{\infty} \beta^t \log(c_t) \quad (6)$$

with logarithmic period utility function and time discount factor β .

2.4 Financial Markets and Household Budget Constraint

Households face idiosyncratic and aggregate risk and seek to insure against that risk by trading a full set of contingent claims. For analytic convenience we will use contingent claims that pay $R_{t+1}(A^{t+1}) = 1 + r_{t+1}(A^{t+1})$ units of consumption in aggregate history A^{t+1} for an individual with history (A^t, z^t) if and only if tomorrow's idiosyncratic state is z_{t+1} , and we denote by $q_t(A_{t+1}, z_{t+1}|A^t, z^t)$ the price of such a claim, and by $\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1})$ the position of these assets for a household with initial characteristics (a_0, z_0) . The budget constraint of the household then reads as⁵

$$\mathbf{c}_t(a_0, z^t, A^t) + \sum_{A_{t+1}} \sum_{z_{t+1}} q_t(A_{t+1}, z_{t+1}|A^t, z^t) \mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) = w_t(A^t) z_t + R_t(A^t) \mathbf{a}_t(a_0, z^t, A^t) \quad (10)$$

Note the crucial difference to the budget constraint in the standard *incomplete* markets in that we permit agents to trade assets that pay contingent on individual productivity realizations (as in *complete* market models) whereas these assets are by assumption ruled out there.

⁵A perhaps more familiar way of defining Arrow securities is that households pay a price $q^b(A_{t+1}, z_{t+1}|A^t, z^t)$ and receive *one* unit of consumption if the state (A_{t+1}, z_{t+1}) is realized. If we denote such an asset by \mathbf{b}_{t+1} , the budget constraint (10) instead reads as:

$$\mathbf{c}_t(a_0, z^t, A^t) + \sum_{A_{t+1}} \sum_{z_{t+1}} q^b(A_{t+1}, z_{t+1}|A^t, z^t) \mathbf{b}_{t+1}(a_0, z^{t+1}, A^{t+1}) = w_t(A^t) z_t + \mathbf{b}_t(a_0, z^t, A^t) \quad (7)$$

Because we define a contingent claim that yields $R_{t+1}(A^{t+1})$ at $t+1$, the relation between the prices and asset positions of the two types of assets is given by:

$$q^b(A_{t+1}, z_{t+1}|A^t, z^t) = \frac{q^a(A_{t+1}, z_{t+1}|A^t, z^t)}{R_{t+1}(A^{t+1})}, \quad (8)$$

$$\text{and } \mathbf{b}_{t+1}(a_0, z^{t+1}, A^{t+1}) = R_{t+1}(A^{t+1}) \mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) \quad (9)$$

Since the return on both contingent claims is given by:

$$\frac{1}{q^b(A_{t+1}, z_{t+1}|A^t, z^t)} = \frac{R_{t+1}(A^{t+1})}{q^a(A_{t+1}, z_{t+1}|A^t, z^t)},$$

the two formulations are completely equivalent. Our formulation of contingent claims gives a simpler and perhaps more intuitive expression for the capital market clearing condition (16) below, though.

The key distinction between our model and the standard complete markets model is the limited commitment friction, the implementation of which we now turn to. Without any cost of defaulting on incurred state-contingent debt, all household contingent claim positions are required to be non-negative, that is, we impose the constraints $\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) \geq 0$. An alternative and (as shown in Krueger and Uhlig (2006)) equivalent formulation of the limited commitment friction without punishment from default would be to introduce financial intermediaries that offer long-term consumption insurance contracts. These contracts would stipulate potentially fully income-history contingent consumption payments in exchange for delivering all of labor income to the intermediaries whenever the individual is productive. The one-sided limited commitment friction implies that whereas intermediaries can fully commit to long-term contracts, individuals cannot. Specifically, in every period, after having observed current labor productivity, the individual can leave her current contract and sign up with an alternative intermediary at no punishment, obtaining in equilibrium the highest lifetime utility contract that allows an intermediary to break even. The lifetime utility from a newly signed contract is the key (potentially time-varying) endogenous entity in this formulation of the model. In Krueger and Uhlig (2006) we have shown that these two formulations of the one-sided limited commitment friction are equivalent, and we therefore here focus on the financial market formulation, as in Alvarez and Jermann (2000), and with the borrowing limits that are not “too tight” (in their nomenclature) being exactly at zero, given that there is no punishment from default.⁶

The household optimization problem can then be stated as

$$\max_{\{\mathbf{c}_t(a_0, z^t, A^t), \mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1})\}} \sum_{t=0}^{\infty} \sum_{A^t} \sum_{z^t} \beta^t \pi(A^t) \pi(z^t) \log(c_t(a_0, z^t, A^t)) \quad (11)$$

subject to the budget constraints (10) and subject to the shortsale constraints:

$$\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) \geq 0 \quad (12)$$

2.5 Definition of Sequential Market Equilibrium

We now define a sequential market equilibrium with aggregate shocks. Households’ consumption and savings allocations depend on both the history of individual states $z^t = (z_0, z_1, \dots, z_t)$ and the history of aggregate states $A^t = (A_0, A_1, \dots, A_t)$. Aggregate allocations and prices depend on the history of aggregate shocks A^t .

⁶We could also motivate our model as a hybrid alternative (or intermediate) model located right in between the Aiyagari (1994) model with tight borrowing constraints and the complete markets model with a full set of state-contingent claims and natural state contingent borrowing constraints.

Definition 1. For an initial condition $(A_0, K_0, \Phi(a_0, z_0))$, an equilibrium is sequences of wages and interest rates $\{w_t(A^t), R_t(A^t)\}$, prices of contingent claims $\{q_t(A_{t+1}, z_{t+1}|A^t, z^t)\}$, aggregate consumption and capital $\{C_t(A^t), K_{t+1}(A^t)\}$ and individual consumption and asset allocations $\{\mathbf{c}_t(a_0, z^t, A^t), \mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1})\}$ such that

1. Given $\{w_t(A^t), R_t(A^t), q_t(A_{t+1}, z_{t+1}|A^t, z^t)\}_{t=0, A^t, z^t, A_{t+1}, z_{t+1}}^\infty$, the household consumption and asset allocation $\{\mathbf{c}_t(a_0, z^t, A^t), \mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1})\}$, for all initial conditions (a_0, z_0) , maximizes (11) subject to the budget constraints (10) and the shortsale constraints (12).

2. Factor prices equal marginal products

$$w_t(A^t) = (1 - \theta)A_t \left(\frac{K_t(A^{t-1})}{A_t} \right)^\theta \quad (13)$$

$$R_t(A^t) = 1 + \theta \left(\frac{K_t(A^{t-1})}{A_t} \right)^{\theta-1} - \delta \quad (14)$$

3. The goods market and capital market clear

$$C_t(A^t) + K_{t+1}(A^t) = (K_t(A^{t-1}))^\theta (A_t)^{1-\theta} + (1 - \delta)K_t(A^t) \quad (15)$$

$$K_{t+1}(A^t) = \int \sum_{z^{t+1}} \mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) \pi(z^{t+1}) d\Phi(a_0, z_0) \quad \forall A^{t+1} \quad (16)$$

where

$$C_t(A^t) = \int \sum_{z^t} \mathbf{c}_t(a_0, z^t, A^t) \pi(z^t) d\Phi(a_0, z_0) \quad (17)$$

3 Characterization of Equilibrium

This section characterizes a sequential equilibrium with aggregate shocks. We will first derive the optimal household choices of consumption and savings for a given conjectured stochastic process for interest rates and wages, $\{R_t(A^t), w_t(A^t)\}_{t \geq 0, A^t}$ in subsection 3.1, then characterize the equilibrium asset distribution in subsection 3.2, then use both results to confirm that the equilibrium prices of the state-contingent claims have the conjectured form in Section 3.3 and finally determine the aggregate law of motion of the economy in closed form in subsection 3.4.

3.1 Optimal Household Choices

We will construct an equilibrium in which household choices take an especially simple form. As long as the real interest rate is not too high, wages do not fall too fast and households do not start life with too many assets, then they choose not to accumulate

state-contingent assets for the high income state tomorrow and insure against sequences of low idiosyncratic income realizations through contingent asset purchases such that for these households a standard complete markets Euler equation holds. The deviation from the complete markets full insurance allocation (which would result in the equilibrium collapsing to that of a representative agent economy) stems from the fact that currently low-income individuals would want to *borrow* against the high idiosyncratic income state, but are prevented from doing so due to limited commitment to repay their debts. Thus, the best they can do is to set the contingent claim for the high idiosyncratic income state to zero.

The following proposition makes this argument formal. It will require Assumptions 2 - 4 on equilibrium prices and initial conditions. In Section 3.3 below we will verify that Assumption 2 is indeed true in equilibrium.⁷ Assumption 3 on wages and interest rate will be replaced below by assumptions purely on the exogenous fundamentals of the economy, see Assumption 5 for the stationary equilibrium, Assumptions 6 or 7 for the transition, and Assumption G for the economy with aggregate shocks.

Assumption 2 (Contingent Claims Prices). *The prices of contingent claims are given by*⁸

$$q_t(A_{t+1}, z_{t+1}|A^t, z^t) = \pi(z_{t+1}|z_t)\pi(A_{t+1}|A^t). \quad (18)$$

Assumption 3 (No Savings Incentives). *The equilibrium interest rate and wage rate processes, $\{R_t(A^t), w_t(A^t)\}$ satisfy:*

$$\beta R_0(A_0) < 1 \quad (19)$$

$$\beta R_{t+1}(A^{t+1}) < \frac{w_{t+1}(A^{t+1})}{w_t(A^t)} \text{ for all } t \geq 0 \text{ and } A^{t+1} \quad (20)$$

Assumption 4 (Initial Distribution). *The initial distribution over wealth and labor productivity, $\Phi(a_0, z_0)$, satisfies:*

(i) $a_0 = 0$ if $z_0 = \zeta$ (*high-productivity households ($z_0 = \zeta$) initially have zero wealth*).

(ii) $0 < a_0 < \bar{a}_0 := \frac{\beta}{1-(1-\nu-\xi)\beta}\zeta$ if $z_0 = 0$ (*the initial wealth of low-productivity households is strictly positive but not too high*).

⁷That is, we follow a guess-and-verify approach, guessing that asset prices have a specific form in Assumption 2 and then verifying that the optimal household choices imply asset prices in equilibrium that take indeed the form stipulated in Assumption 2.

⁸Recall that these contingent claims pay $R_{t+1}(A^{t+1})$ units of consumption in event history A^{t+1} . Under Assumption 2, the price of a contingent claim that pays one state-contingent unit of consumption then takes the perhaps more familiar form $q_t^b(A_{t+1}, z_{t+1}|A^t, z^t) = \frac{\pi(z_{t+1}|z_t)\pi(A_{t+1}|A^t)}{R_{t+1}(A^{t+1})}$, as, for example, in Krueger and Perri (2006).

We will demonstrate below that under Assumption 3 the steady state of the model will have an associated wealth distribution that satisfies Assumption 4, although the following proposition characterizing optimal household choices does not require the initial distribution to be the steady state distribution (as long as it satisfies Assumption 4).

Proposition 1 (Optimal Household Consumption and Asset Allocation). *Suppose Assumption 2 on contingent claims prices is satisfied and suppose that the sequence of wages and interest rates $\{w_t(A^t), R_t(A^t)\}_{t=0}^\infty$ satisfies the no-savings Assumption 3 and that the initial wealth distribution satisfies Assumption 4. Then the optimal consumption and asset allocation of individual households is given by*

$$\mathbf{c}_t(a_0, z^t, A^t) = \begin{cases} c_0 w_t(A^t), \text{ where } c_0 := \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta} \zeta, & \text{if } z_t = \zeta \\ [1 - (1 - \nu)\beta] R_t(A^t) \mathbf{a}_t(a_0, z^t, A^t) & \text{if } z_t = 0 \end{cases} \quad (21)$$

$$\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) = \begin{cases} 0 & \text{if } z_{t+1} = \zeta \\ \frac{\beta}{1-(1-\nu-\xi)\beta} \zeta w_t(A^t) & \text{if } z_t = \zeta \text{ and } z_{t+1} = 0 \\ \beta R_t(A^t) \mathbf{a}_t(a_0, z^t, A^t) & \text{if } z_t = 0 \text{ and } z_{t+1} = 0 \end{cases} \quad (22)$$

where $\mathbf{a}_0(a_0, z^0, A^0) = w_0(A^0) a_0$.

Proof. See Appendix A.1.1 □

We now discuss the intuition and implications of Proposition 1. First, it is easy to verify that (21) and (22) imply:⁹

$$\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1}) = \beta R_{t+1}(A^{t+1}) \mathbf{c}_t(a_0, z^t, A^t) \text{ if } z_{t+1} = 0. \quad (23)$$

that is, consumption growth between t and $t + 1$ follows a standard complete markets Euler equation for those households that are unproductive in period $t + 1$.¹⁰ In contrast, those that are productive in $t + 1$ (i.e., have $z_{t+1} = \zeta$), satisfy the Euler equation with strict inequality, as one can show by combining Assumption 3 with equation (21), see Lemma 5 in Appendix A.1.1:

$$\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1}) > \beta R_{t+1}(A^{t+1}) \mathbf{c}_t(a_0, z^t, A^t) \text{ if } z_{t+1} = \zeta. \quad (25)$$

⁹The derivation of this result can be found in Appendix A.1.1.

¹⁰Another way to write this is:

$$\frac{1}{\mathbf{c}_t(a_0, z^t, A^t)} = \underbrace{\pi(A_{t+1}, z_{t+1} | A^t, z^t)}_{\text{prob. of } (A_{t+1}, z_{t+1})} \underbrace{\frac{R_{t+1}(A^{t+1})}{q_t(A_{t+1}, z_{t+1} | A^t, z^t)}}_{\text{return on a contingent claim}} \underbrace{\beta \frac{1}{\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1})}}_{\text{discounted marginal utility}} \text{ if } z_{t+1} = 0. \quad (24)$$

Independence of A_{t+1} and z_{t+1} gives $\pi(A_{t+1}, z_{t+1} | A^t, z^t) = \pi(z_{t+1} | z_t) \pi(A_{t+1} | A^t)$, while Assumption 2 gives $q_t(A_{t+1}, z_{t+1} | A^t, z^t) = \pi(z_{t+1} | z_t) \pi(A_{t+1} | A^t)$. Then, equation (24) simplifies to equation (23).

Also note that households with currently positive labor income (i.e., with $z = \zeta$), consume and save a constant fraction of their current labor income, independent of the aggregate shock (history) and independent of current or (expected) future interest rates:

$$\frac{c_t(a_0, z^t, A^t)}{\zeta w_t(A^t)} = \frac{1 - (1 - \nu)\beta}{1 - (1 - \nu - \xi)\beta} \quad (26)$$

$$\frac{a_{t+1}(a_0, (z^t, z_{t+1} = 0), A^{t+1})}{\zeta w_t(A^t)} = \frac{\beta}{1 - (1 - \nu - \xi)\beta}. \quad (27)$$

One aspect of the optimal household allocation that is perhaps surprising is the lack of insurance against aggregate risk. All households are exposed to interest rate risk but only households that will experience a high idiosyncratic productivity shock, $z_{t+1} = \zeta$, will also be exposed to wage risk. Thus, there appears scope for a better allocation of this aggregate risk. The key to Proposition 1 is that under Assumption 3 it is optimal for households to *borrow* against the future ($A_{t+1}, z_{t+1} = \zeta$) state, for all A_{t+1} realizations. But this is precisely what is ruled out by limited commitment, and thus the best they can do is to set the $a_{t+1}(a_0, (z^t, z_{t+1} = \zeta), A^{t+1}) = 0$.

Finally, it is easy to verify that the household choices satisfy the budget constraint. One final noteworthy observation about the optimal allocation is that low-productivity households consume a share $(1 - (1 - \nu)\beta)$ of their cum-interest wealth $R_t a_t$. A household solving the classic “cake-eating problem” (i.e., the optimal intertemporal consumption-savings problem in the absence of risk and labor income starting with a given amount of wealth) would consume a share $(1 - \beta)$ of that wealth. Here that share is higher (by the factor $\beta\nu$) because of the implicit insurance nature of the allocation: upon making a state transition to high productivity the household effectively surrenders her state-contingent wealth, with the wealth redistributed to the low-productivity “stayers”. This is reminiscent of the actuarially fair annuity payments in the presence of mortality risk in the classic papers by Yaari (1965) and Blanchard (1985).

As Proposition 1 indicates, optimal consumption and asset holdings are proportional to wages (either current wages when the household has high productivity today, i.e., $s = 0$) or to the wage when she last was productive. We therefore define, for future reference and use, wage-deflated consumption and asset choices as

$$c_t(a_0, z^t, A^t) = \frac{c_t(a_0, z^t, A^t)}{w_t(A^t)} \quad (28)$$

$$a_{t+1}(a_0, z^{t+1}, A^{t+1}) = \frac{a_{t+1}(a_0, z^{t+1}, A^{t+1})}{w_{t+1}(A^t)} \quad (29)$$

3.2 The Cross-Sectional Distribution

The sequential market equilibrium household consumption-asset allocation has a simple structure. Either the shortsale constraint for a given continuation history z^{t+1} is not

binding, $\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) > 0$, and the standard complete-markets Euler equation

$$\frac{\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1})}{\mathbf{c}_t(a_0, z^t, A^t)} = \beta R_{t+1}(A^{t+1}) \quad (30)$$

applies or the constraint is binding, $\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) = 0$ and the Euler equation turns into an inequality.

The equilibrium consumption and asset allocation then has a simple structure in which individual consumption and assets only depend on the length $s \geq 0$ of the most recent spell of low productivity (where $s = 0$ denotes an agent with currently high productivity), and potentially calendar time t . When productivity is high, the short-sale (limited commitment) constraint is binding and assets are zero. Denote this simple consumption-asset allocation by $\{\mathbf{c}_{s,t}, \mathbf{a}_{s,t}\}_{s,t=0}^{\infty}$ and note that the Markov process for individual productivity implies that the cross-sectional distribution of the waiting times is time invariant and given by

$$\phi_s = \begin{cases} \frac{\nu}{\xi+\nu} & \text{if } s = 0 \\ \frac{\xi}{\xi+\nu} \nu (1-\nu)^{s-1} & \text{if } s = 1, 2, 3, \dots \end{cases} \quad (31)$$

Corollary 1. *Suppose the initial asset distribution is given by $\{\mathbf{a}_{s,0}\}_{s \geq 0}$ with the probability mass of s -agents given by (31), and suppose that Assumptions 2 - 4 are satisfied. Then, the initial consumption allocation $\{\mathbf{c}_{s,0}\}_{s \geq 0}$ is given by equation (21), and the consumption-asset allocation $\{\mathbf{c}_{s,t}, \mathbf{a}_{s,t}\}_{s \geq 0}$ at any $t \geq 1$ is determined by:*

$$\mathbf{c}_{s,t}(A^t) = \begin{cases} \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta} \zeta w_t(A^t) & \text{if } s = 0 \\ \beta R_t(A^t) \mathbf{c}_{s-1,t-1}(A^{t-1}) & \text{if } s \geq 1 \end{cases} \quad (32)$$

$$\mathbf{a}_{s,t}(A^t) = \mathbf{a}_{s,t}(A^{t-1}) = \begin{cases} 0 & \text{if } s = 0 \\ \frac{\beta}{1-(1-\nu-\xi)\beta} \zeta w_{t-1}(A^{t-1}) & \text{if } s = 1 \\ \beta R_{t-1}(A^{t-1}) \mathbf{a}_{s-1,t-1}(A^{t-1}) & \text{if } s \geq 2 \end{cases} \quad (33)$$

The probability mass of s -households is time-invariant and given by (31).

Proof. The initial consumption distribution follows directly from equation (21) in Proposition 1. Apply equations (22) and (23) to the initial allocation $\{\mathbf{c}_{s,0}, \mathbf{a}_{s,0}\}_{s \geq 0}$. \square

Note that this corollary implies that even though the cross-sectional distribution of waiting times remains constant over time and across aggregate shocks, the levels of consumption and assets at these countably many mass points given by (32) and (33) varies with time and aggregate history A^t , but *only* through its effects on aggregate wages and interest rates. Also note that if the consumption-asset allocation is of the

simple form stipulated above, aggregate consumption and assets in the goods market clearing condition and the asset market clearing condition can be written as

$$C_t(A^t) = \sum_{s=0}^{\infty} \phi_s \mathbf{c}_{s,t}(A^t) \quad (34)$$

$$K_{t+1}(A^t) = \sum_{s=0}^{\infty} \phi_s \mathbf{a}_{s,t+1}(A^t) \quad (35)$$

3.3 Confirming the Conjectured Prices of Contingent Claims on Capital Returns

Before we explicitly carry out the aggregation we can verify, with the optimal household allocations in place, that the prices of the contingent claims of capital are of the form stipulated in Assumption 2.

Fix (A^t, z^t) . For all households for which the shortsale constraint for a contingent claim that pays off in aggregate state A^{t+1} is not binding (the positive mass of individuals with idiosyncratic state $z_{t+1} = 0$, i.e., those with $s > 0$ in period $t + 1$), the first order conditions with respect to consumption and the state-contingent asset claim can be combined to obtain (see Appendix A.1.1, equation (110)) the standard complete markets Euler equation:

$$q_t(A_{t+1}, z_{t+1} | A^t, z^t) = \pi(z_{t+1} | z_t) \pi(A_{t+1} | A^t) \beta R_{t+1}(A^{t+1}) \left[\frac{\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1})}{\mathbf{c}_t(a_0, z^t, A^t)} \right]^{-1} \quad (36)$$

For each of these households (with $s > 0$) the optimal consumption allocation satisfies $\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1}) = \beta R_{t+1}(A^{t+1}) \mathbf{c}_t(a_0, z^t, A^t)$ (see equation (23)). Using this result in equation (36) immediately confirms that

$$q_t(A_{t+1}, z_{t+1} | A^t, z^t) = \pi(z_{t+1} | z_t) \pi(A_{t+1} | A^t) \quad (37)$$

That is, in Assumption 2 we have guessed the form of equilibrium asset prices, and now we have verified that the optimal household choices under this guess lead to equilibrium prices consistent with that guess.

Finally, note that at these prices for households with $z_{t+1} = \zeta$ we have (see equation (25))

$$q_t(A_{t+1}, z_{t+1} | A^t, z^t) > \pi(z_{t+1} | z_t) \pi(A_{t+1} | A^t) \beta R_{t+1}(A^{t+1}) \left[\frac{\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1})}{\mathbf{c}_t(a_0, z^t, A^t)} \right]^{-1} \quad (38)$$

These households, at these equilibrium prices, would like to reduce their consumption growth between period t and $t + 1$ by borrowing against the contingency of high productivity tomorrow, but the limited commitment constraints precisely prevent these types of shortsalses.

3.4 Aggregation

Now that we have characterized the cross-sectional distribution of assets, we can use equation (35) to derive the aggregate law of motion for capital. Since the optimal household consumption and asset decisions in Proposition 1 are closed-form expressions of the general equilibrium factor prices (w_t, R_t) , and these in turn are functions only of the aggregate capital stock and aggregate productivity through the first-order conditions of the firm, characterizing the law of motion for the aggregate capital stock K_t is also sufficient to fully characterize the distribution of consumption and assets over time. What is special about this model with nontrivial household heterogeneity is that the model aggregates, in the sense that the capital stock in period $t + 1$ can be expressed exclusively as a function of the aggregate capital stock in period t , despite the fact that the model features a non-trivial consumption and wealth distribution, and that this law of motion of capital can be characterized in closed form. The following proposition is then a straightforward consequence of the results in the previous subsection and proved in Appendix A.1.2.

Proposition 2 (Aggregate Law of Motion for Capital). *Under the assumptions maintained in Proposition 1 (which implies that the household consumption and saving allocations are given by (21) and (22)), the law of motion for the aggregate capital stock is given by:*

$$K_{t+1}(A^t) = \hat{s}A_t^{1-\theta}K_t(A^{t-1})^\theta + (1 - \hat{\delta})K_t(A^{t-1}) \quad (39)$$

where

$$\hat{s} = \frac{\xi\beta(1 - \theta)}{1 - (1 - \nu - \xi)\beta} + (1 - \nu)\beta\theta \quad (40)$$

$$\hat{\delta} = 1 - (1 - \nu)\beta(1 - \delta). \quad (41)$$

As in the classic Solow model the aggregate saving rate \hat{s} is a constant in this model, but in contrast to the Solow model here it is an explicit function of the fundamental parameters capturing income risk at the micro level as well as time preferences and the capital share in production, and one can derive explicit comparative statics with respect to these deep parameters. Note that if $\nu = \xi = 0$ and there is no idiosyncratic risk, then $\hat{s} = \beta\theta$. If furthermore $\delta = 1$, then $\hat{\delta} = 1$ and the model dynamics collapses to that of the standard representative agent stochastic neoclassical growth model, which with log-utility and full depreciation –and only then– has a closed-form solution for the aggregate capital stock. We give this closed form in Section 4.2.1 and will return to a comparison of our model with the neoclassical growth model and the Solow model in Section 6.

We now use the general results to analytically characterize a stationary equilibrium, the transition path after an unexpected transitory or permanent productivity shock and

the equilibrium with aggregate shocks, under specific assumptions on the aggregate productivity process. This allows us to restate Assumption 3 purely in terms of fundamentals and to verify that the steady state asset distribution satisfies Assumption 4.

4 Stationary Equilibrium and Transitional Dynamics

We derive the stationary equilibrium and then study the transitional dynamics of the model from this steady state following an unexpected shock to productivity. The steady state results displayed here are translations of similar findings derived in Krueger and Uhlig (2022) who employ a continuous-time version of the model, and stated here to make the paper self-contained, given that the steady state is the starting point for the analysis of transitions and aggregate fluctuations.

4.1 Stationary Equilibrium

For the purpose of this section we maintain Assumption 1.1 and thus productivity is constant at A_0 . In a stationary equilibrium the wage w , the gross interest rate R and the aggregate capital stock K are constants (over time and across aggregate states).

From Proposition 1 it immediately follows that in steady state the optimal wage-deflated consumption and asset choices, as function of the wait time s , are given as

$$\frac{\mathbf{c}_0}{w} = c_0 = \frac{1 - (1 - \nu)\beta}{1 - (1 - \nu - \xi)\beta} \zeta \quad (42)$$

$$\frac{\mathbf{c}_s}{w} = c_s = (\beta R)^s c_0 \quad \text{for } s = 1, 2, \dots \quad (43)$$

$$\frac{\mathbf{a}_0}{w} = a_0 = 0 \quad (44)$$

$$\frac{\mathbf{a}_1}{w} = a_1 = \frac{\beta}{1 - (1 - \nu - \xi)\beta} \zeta \quad (45)$$

$$\frac{\mathbf{a}_s}{w} = a_s = (\beta R)a_{s-1} = (\beta R)^{s-1} a_1 \quad \text{for } s = 2, 3, \dots \quad (46)$$

High income agents consume a constant fraction of their labor income, and consumption of low income agents drifts down at a constant rate, βR , until they switch to a high income state and renew the contract. As long as $\beta R < 1$, consumption converges to zero in the long run, $\lim_{s \rightarrow \infty} c_s = 0$.

We can of course use the stationary version of the aggregate law of motion (39) in Proposition 2

$$K_0 = \hat{s}(A_0)^{1-\theta} K_0^\theta + (1 - \hat{\delta})K_0 \quad (47)$$

to determine the aggregate steady state capital stock and the associated stationary wage and interest rate, denoted by (K_0, R_0, w_0) (we use the "0" notation because the steady

state will be the initial condition for the transition analysis in Section 4.2). This delivers, in parallel with Proposition 8 in Krueger and Uhlig (2022), but for discrete time:

$$K_0 = A_0 \left(\hat{s}/\hat{\delta} \right)^{\frac{1}{1-\theta}} \quad (48)$$

$$R_0 = \theta A_0^{1-\theta} K_0^{\theta-1} + 1 - \delta = \theta \left(\hat{\delta}/\hat{s} \right) + 1 - \delta \quad (49)$$

$$w_0 = (1 - \theta) A_0^{1-\theta} K_0^\theta = (1 - \theta) A_0 \left(\hat{s}/\hat{\delta} \right)^{\frac{\theta}{1-\theta}} \quad (50)$$

where $(\hat{s}, \hat{\delta})$ were defined in (40) and (41).

To obtain intuition for the aggregate law of motion, and derive a graphical representation of the capital market clearing condition for our model akin to that in the Aiyagari (1994) model, it is instructive to carry out the explicit aggregation in the stationary equilibrium. The capital market clearing condition in the steady state reads as

$$\begin{aligned} K &= \sum_{s=0}^{\infty} \phi_s a_s w = \sum_{s=1}^{\infty} \phi_s a_s w \\ &= \frac{\beta \zeta}{1 - (1 - \nu - \xi) \beta} \frac{\nu \xi}{\xi + \nu} w + \sum_{s=2}^{\infty} \phi_s a_s w \\ &= \frac{\beta \zeta}{1 - (1 - \nu - \xi) \beta} \frac{\nu \xi}{\xi + \nu} w + \beta R (1 - \nu) \sum_{s=2}^{\infty} \phi_{s-1} a_{s-1} w \\ &= \frac{\beta \zeta}{1 - (1 - \nu - \xi) \beta} \frac{\nu \xi}{\xi + \nu} w + \beta R (1 - \nu) K \end{aligned} \quad (51)$$

The first row is due to the fact that saving for the high idiosyncratic state $z = \zeta$ is zero, and thus, $a_0 = 0$. The second row splits the demand for assets (supply of capital) into the part coming from productive agents saving for the low income state (a_1) and the part stemming from unproductive agents rolling over parts of their assets (a_s) for $s > 1$. The third row exploits the optimal asset allocation in equation (46) and the form of the stationary wait-time distribution ϕ_s in equation (31), and the last row uses the capital market clearing condition. Plugging in for (R, w) from the firm's optimality conditions and rearranging delivers back the stationary version of the aggregate law of motion in equation (47).

As in Figure II.b of Aiyagari (1994) and Figure 3a in Krueger and Uhlig (2022), the capital market clearing condition (51) can be used to display the determination of the stationary equilibrium interest rate and capital stock graphically, derive its comparative statics properties and clarify the conditions needed for the existence of a stationary equilibrium with partial consumption insurance. To do so, it is instructive to divide both sides of (51) by the wage w and use the firm's optimality condition with respect to capital, and the normalization of expected productivity to one, (5), to write both sides as a function

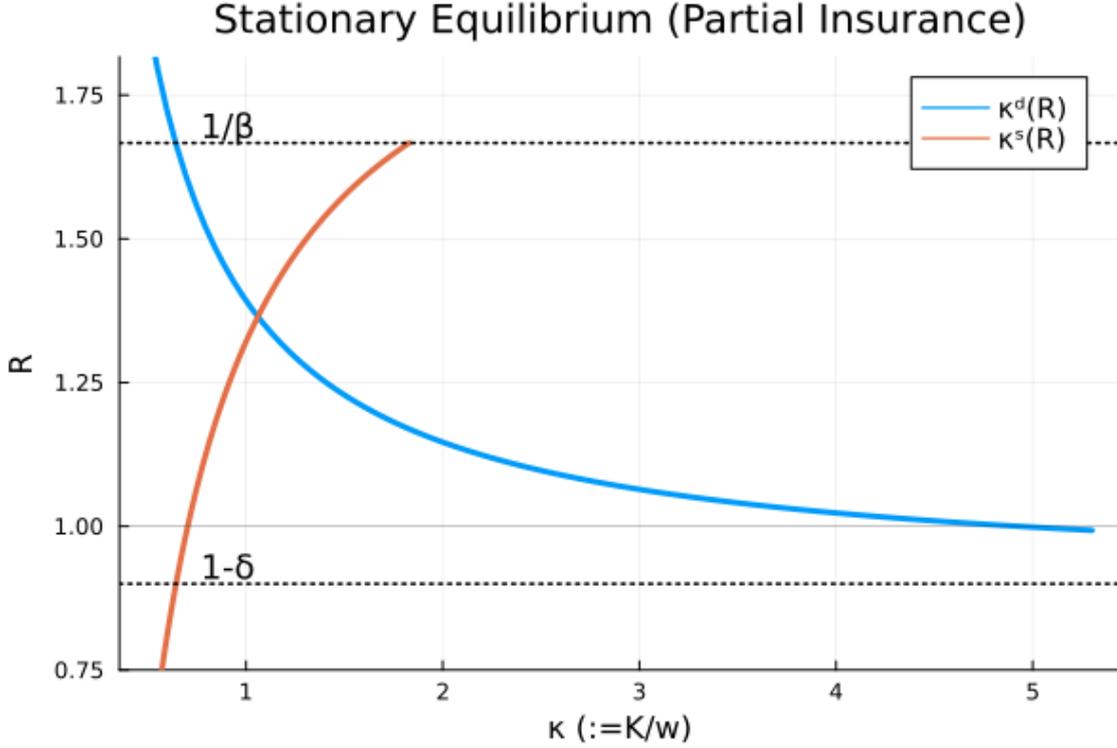


Figure 1: Determination of Market Clearing Interest Rate and Capital Stock. Parameter values are $\beta = 0.6, \delta = 0.1, \theta = 0.33, \xi = 0.5, \nu = 0.2$

of the gross interest rate R . This yields

$$K(R)/w(R) =: \kappa^d(R) = \frac{\theta}{(1-\theta)(R-1+\delta)} = \frac{\xi\beta}{[1-(1-\nu)\beta R][1-(1-\nu-\xi)\beta]} := \kappa^s(R) \quad (52)$$

Figure 1 plots the (wage-normalized) demand for capital $\kappa^d(R)$ by the production firms (the relation between the return and the capital stock determined by the first order condition for capital). It has exactly the same form as in the original Aiyagari (1994) paper, sloping downward, and with the capital stock diverging to ∞ as the net interest rate approaches $-\delta$. The supply of capital κ^s from the household side is finite at $R = 1 - \delta$, strictly increasing in the interest rate (something that is typically hard to prove in the original Aiyagari (1994) model), and also finite at $R = 1/\beta$. Equation (52) also allows to determine unambiguous comparative statics as the parameters of the model shift either the demand curve (in case of the production parameters (θ, δ)) or the supply curve (in case of the idiosyncratic risk parameters (ξ, ν) and the preference parameter β). Finally it also shows that a necessary and sufficient condition for a unique simple stationary partial insurance equilibrium with $R_0 < 1/\beta$ is that $\kappa^d(1/\beta) < \kappa^s(1/\beta)$. This leads to the following Assumption (the counterpart of Assumption 2 in Krueger and Uhlig (2022), but cast in discrete time):

Assumption 5.

$$\frac{\theta}{(1-\theta)\left(\frac{1}{\beta}-1+\delta\right)} < \frac{\xi}{\nu\left(\frac{1}{\beta}-1+\xi+\nu\right)}$$

This assumption is satisfied if the chance of productivity falling ξ and the risk of it not recovering quickly (as given by $1-\nu$) is sufficiently large.¹¹ The assumption insures that $\kappa^d(1/\beta) < \kappa^s(1/\beta)$. Equipped with this assumption, defined purely in terms of exogenous parameters of the model, we can state the following proposition, completely characterizing the stationary equilibrium of the model. It is the discrete-time counterpart to Proposition 8 in Krueger and Uhlig (2022), and here mainly stated because the stationary equilibrium forms the point of departure for the transition and aggregate risk analyses in Sections 4.2 and 5.

Proposition 3 (Stationary Equilibrium). *Suppose Assumption 5 holds. Then, there exists a stationary partial insurance equilibrium in which the capital stock K_0 , interest rate R_0 and wage rate w_0 are given in a closed form:*

$$K_0 = A_0 \left(\hat{s}/\hat{\delta}\right)^{\frac{1}{1-\theta}} = A_0 \left[\frac{\xi(1-\theta) + \beta\theta(1-\nu)\left(\xi + \nu + \frac{1}{\beta} - 1\right)}{[1 - (1-\delta)\beta(1-\nu)]\left(\xi + \nu + \frac{1}{\beta} - 1\right)} \right]^{\frac{1}{1-\theta}} \quad (53)$$

$$R_0 = \frac{\xi(1-\theta)(1-\delta) + \theta\left(\xi + \nu + \frac{1}{\beta} - 1\right)}{\xi(1-\theta) + \beta\theta(1-\nu)\left(\xi + \nu + \frac{1}{\beta} - 1\right)} \quad (54)$$

$$w_0 = (1-\theta)A_0 \left[\frac{\xi(1-\theta) + \beta\theta(1-\nu)\left(\xi + \nu + \frac{1}{\beta} - 1\right)}{[1 - (1-\delta)\beta(1-\nu)]\left(\xi + \nu + \frac{1}{\beta} - 1\right)} \right]^{\frac{\theta}{1-\theta}} \quad (55)$$

The equilibrium interest rate R_0 is strictly increasing in the capital share θ , strictly decreasing in the depreciation rate δ , the time discount factor β as well as the risk of productivity falling ξ and remaining low $1-\nu$ and is independent of productivity A_0 . The capital stock K_0 is strictly increasing in the time discount factor β as well as the risk of productivity falling ξ and remaining low $1-\nu$, strictly decreasing in the depreciation rate δ , and is proportional to the level of productivity A_0 . The comparative statics of w_0 is the same as for K_0 . The simple stationary equilibrium is unique¹² in the sense that there is no other simple

¹¹Defining the time discount rate ρ by $\beta = \frac{1}{1+\rho}$, we can restate the assumption as

$$\frac{\theta}{(1-\theta)(\rho+\delta)} < \frac{\xi}{\nu(\rho+\xi+\nu)}$$

which coincides with the assumption insuring the existence of a stationary equilibrium of the continuous-time model in Krueger and Uhlig (2022).

¹²We cannot rule out stationary equilibria in which allocations are more complex functions of idiosyncratic histories, although we conjecture such equilibria do not exist under Assumption 1.

stationary partial-insurance equilibrium in which the stationary consumption and wealth allocation and its associated cross-sectional distribution is simply a function of wait time s .

Proof. The equations follow directly from the aggregate law of motion (39) at steady state, see Appendix A.2.1 for the derivation. Intuitively, existence and uniqueness follows directly from the monotonicity of $\kappa^d(R)$ and $\kappa^s(R)$ in Figure 1 as well Assumption 5. The comparative statics results with respect to R_0 follow directly from the fact that the curve $\kappa^d(R)$ shifts to the right with an increase in θ and a decrease in δ whereas $\kappa^s(R)$ is independent of these parameters, and the fact that $\kappa^s(R)$ shifts to the right with an increase in $\xi, 1 - \nu, \beta$ and $\kappa^d(R)$ is independent of these parameters. The comparative statics results with respect to K_0 and w_0 then follow from the firm optimality conditions, given the comparative statics with respect to R_0 \square

Assumption 5 is also a necessary condition for the existence of a simple partial insurance equilibrium.¹³ In the next subsection we use the stationary equilibrium (R_0, K_0) and the associated asset distribution determined by the optimal asset allocation (46) and the wait-time distribution ϕ_s in (31) as point of departure to analytically characterize the transition path induced by an unexpected change in the productivity path.

4.2 Transitional Dynamics

Starting from a stationary partial insurance equilibrium with Assumption 5 in place we now assume that at the beginning of period $t = 1$ the economy experiences an unexpected, zero probability shock (a so-called MIT shock) that alters productivity from A_0 to a new deterministic sequence $\{A_t\}_{t=1}^{\infty}$. There are no further surprises about productivity or any other parameters of the economy thereafter; that is, aggregate productivity now satisfies Assumption 1, part 2. Assumption 4 on the initial distribution is now guaranteed to be satisfied since the economy starts at the stationary equilibrium characterized in the previous subsection. Note that we continue to impose the no-savings Assumption 3; in Section 4.2.1 below we provide conditions on the path $\{A_t\}_{t=1}^{\infty}$ sufficient to guarantee that this assumption is indeed satisfied.

The analysis in this subsection follows the work in continuous time of Krueger, Li, and Uhlig (2023) and translates their main results for deterministic MIT transition paths into the discrete-time framework of this paper, but it also permits us to handle both the full-depreciation case (which in turn yields a useful point of comparison to the representative

¹³A violation of Assumption 5 does not exclude the possibility for the existence of a stationary equilibrium with $\beta R \geq 1$. The characterization of optimal consumption and asset allocations in Proposition 1 requires, in a stationary equilibrium, that $\beta R < 1$ and is no longer valid if $\beta R \geq 1$, and thus the ensuing aggregation analysis no longer applies.

agent neoclassical growth model) and the case of permanent productivity declines which are infeasible in continuous time.

The optimal household allocations and the aggregate law of motion for capital are special cases of Propositions 1 and 2, respectively, and thus (recall equation (39)):

$$K_{t+1} = \hat{s}(A_t)^{1-\theta}(K_t)^\theta + (1 - \hat{\delta})K_t \quad (56)$$

where¹⁴

$$\hat{s} = (1 - \nu)\beta\theta + \frac{\xi\beta(1 - \theta)}{1 - (1 - \nu - \xi)\beta} \approx \theta(1 - \nu - \rho) + (1 - \theta) \left[\frac{\xi}{\xi + \nu + \rho} \right] \quad (57)$$

$$\hat{\delta} = 1 - (1 - \nu)\beta(1 - \delta) \approx \nu + \rho + \delta. \quad (58)$$

This law of motion for capital resembles that in the classical Solow growth model, but with a depreciation rate $\hat{\delta}$ that is larger (by $\nu + \rho$) than the physical depreciation rate δ and a saving rate \hat{s} that is an explicit function of the structural parameters of the model and depends negatively on $\nu + \rho$ and positively on the risk of income falling to zero ξ . We discuss the relation to the Solow model and the literature more broadly in Section 6.

Studying the special case of unexpected transitions is useful for three purposes. First, it will allow us, in Subsection 4.2.1, to derive a sufficient condition purely on the productivity process such that the no-savings condition $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ is satisfied for all periods along the transition. Second, we show in Subsection 4.2.2 that the original steady state allocation, chosen under the assumption by households that wages and interest rates will never change, remains optimal in period 1, after the MIT shock has hit and wages and interest rates undergo the unexpected transition. Third, there we will also clarify why the aggregate transition induced by a change in productivity is independent of whether this change is unanticipated (as in the MIT shock thought experiment) or anticipated. This in turn suggests that the model with aggregate shocks in Section 5 remains analytically tractable and retains the same characteristics as the model with unexpected MIT transitions. Finally, as a first application, Subsection 4.2.3 characterizes the evolution of consumption inequality following the MIT shock.

4.2.1 Sufficient Conditions for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ along the Transition Path

Thus far, we have derived the dynamics of the capital stock in equation (39) under the maintained assumption that the limited commitment constraint of households receiving high income is always binding along the transition path, see Assumption 3. Equivalently, phrased in terms of state-contingent asset accumulation, it was assumed that households have an asset position of zero when starting the period with high idiosyncratic productivity. We showed in Proposition 1 that such a contract satisfies the optimality conditions

¹⁴Recall ρ is defined by $\beta = \frac{1}{1+\rho}$ and the approximation assumes that ρ, ξ, ν are sufficiently small.

when $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all $t \geq 0$. Intuitively, when interest rates are low and/or wages are expected to be higher in the future than today, individuals have no incentive to save for the contingency of high idiosyncratic labor productivity tomorrow.

In this subsection, we derive sufficient conditions insuring $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all $t \geq 0$ after a positive and after a negative productivity shock, respectively. Broadly speaking, the shock to total factor productivity A cannot be too large in either direction. Furthermore, if depreciation is 100%, then no further assumptions besides those already made to ensure the existence of partial insurance steady state from which the transition starts are necessary, as we demonstrate next. These results differ in a significant way from the corresponding findings in the continuous-time transition analysis of Krueger, Li, and Uhlig (2023). Whereas the positive permanent shock case delivers a condition that has an identical counterpart in continuous time, the full-depreciation case cannot be analyzed in continuous time. Perhaps more importantly, for negative permanent shocks the no-savings condition cannot be satisfied in continuous time (since the instantaneous growth rate in the wage is infinitely negative), whereas in discrete time we provide exactly such a sufficient condition on the exogenous size of the productivity decline.¹⁵

Full Depreciation With full depreciation, $\delta = \hat{\delta} = 1$, the equilibrium law of motion for capital is given from equation (56), with $\delta = 1$ by

$$K_{t+1} = \hat{s} A_t^{1-\theta} K_t^\theta \quad (59)$$

as long as, along the transition $\beta R_{t+1} < \frac{w_{t+1}}{w_t} < 1$. Lemma 6 in Appendix A.2.2 shows that with full depreciation $R_{t+1} \frac{w_t}{w_{t+1}} = R_0$, for all $t \geq 0$. But since under Assumption 5 there exists a stationary partial insurance equilibrium (with $\beta R_0 < 1$) and by assumption this is the starting point for the unexpected transition, along this transition $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ is guaranteed by Assumption 5, independent of the sequence of productivity levels $\{A_t\}$.

Also note that in the limit, as idiosyncratic risk vanishes (ν and ξ converge to zero), the saving rate \hat{s} in our model approaches that of the standard representative agent model with log-utility and full depreciation $s = \beta\theta$. Finally, with full depreciation the nonlinear first order difference equation in (59) has a closed-form solution since it implies that the log of the capital stock obeys a linear first order difference equation which can easily be solved in closed form. This discussion is summarized in the following proposition.

Proposition 4. *Let Assumption 5 be satisfied and suppose the economy is originally in a partial insurance steady state (K_0, R_0, w_0) and $\{(\mathbf{a}_{s,0}, \mathbf{c}_{s,0})\}_{s=0}^\infty$ characterized in Proposition*

¹⁵On the downside, the first-order difference equation characterizing the dynamics of the equilibrium has no explicit solution whereas the corresponding differential equation in continuous time has. See Section 6 for further discussion and Krueger, Li, and Uhlig (2023) for the continuous time solution.

3. Then the aggregate capital stock in period t of the transition induced by an unexpected change in productivity after period 0 to the sequence $\{A_t\}_{t=1}^{\infty}$ is determined as

$$K_t = \exp \left[(1 - \theta) \left[\sum_{\tau=1}^{t-1} \theta^{t-1-\tau} \log A_{\tau} \right] + \frac{1 - \theta^{t-1}}{1 - \theta} \log \hat{s} + \theta^{t-1} \log K_0 \right], \quad (60)$$

the factor prices (R_t, w_t) are given by the firm's optimality conditions (13) and (14), and individual household allocations $\{(\mathbf{a}_{s,t}, \mathbf{c}_{s,t})\}$ are as stated in Proposition 1, given the dynamics of the capital stock K_t in (60).

Proof. Follows directly from taking logs on both sides of equation (59) and solving the linear first order difference equation for $\log(K_t)$,

$$\log K_{t+1} = \log \hat{s} + (1 - \theta) \log A_t + \theta \log K_t$$

and then exponentiating. This delivers (60). \square

With less than full depreciation, $\delta < 1$, Assumption 5 is not sufficient to insure that the no-savings condition $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ is satisfied for $t > 0$. For arbitrary sequences of productivity it is difficult to establish general conditions purely in terms of the fundamentals of the economy, but in the case of fully *permanent* shocks this is possible since for this case we can establish that the capital stock evolves monotonically over time. To do so it is useful to distinguish positive technology shocks (which induce positive wage growth along the transition and temporarily elevated interest rates) from negative technology shocks (with negative wage growth and depressed interest rates along the transition), since the two cases differ in the restrictiveness of the assumptions needed to ensure that the no-savings condition is satisfied.

Permanent Shocks We first establish the monotone convergence of the capital stock following a permanent shock to productivity.

Proposition 5 (Monotone Convergence of (K_t, R_t, w_t)). *Assume the economy is in a stationary equilibrium associated with aggregate productivity A_0 and associated capital K_0 at time $t = 0$, and suppose at time $t = 1$, productivity unexpectedly and permanently changes to A_1 with $A_1 > A_0$. Furthermore, suppose $\beta R_t < \frac{w_{t+1}}{w_t}$ for all $t \geq 0$ (Assumption 3). Then, aggregate capital K_t and wages w_t monotonically increase and converge to their new stationary equilibrium values, and the interest rate jumps up on impact and then converges monotonically to the old (and new) stationary equilibrium from above:*

$$K_0 = K_1 < K_2 < \dots < K^* = \frac{A_1}{A_0} K_0, \quad (61)$$

$$w_0 < w_1 < w_2 < \dots < w^* = \frac{A_1}{A_0} w_0, \quad (62)$$

$$R_0 < R_1 > R_2 > \dots > R^* = R_0. \quad (63)$$

Symmetrically, following a permanent negative productivity shock $A_t = A_1 < A_0$ for all $t \geq 1$, the aggregate capital stock and wages monotonically decrease along the transition, and the interest rate falls on impact before converging back to the old (and new) stationary equilibrium from below:

$$K_0 = K_1 > K_2 > \dots > K^* = \frac{A_1}{A_0} K_0, \quad (64)$$

$$w_0 > w_1 > w_2 > \dots > w^* = \frac{A_1}{A_0} w_0, \quad (65)$$

$$R_0 > R_1 < R_2 < \dots < R^* = R_0. \quad (66)$$

Proof. See Appendix A.2.3 □

Equipped with this result we can now give sufficient conditions, purely in terms of fundamentals, for the condition $\beta R_t < \frac{w_{t+1}}{w_t}$ to indeed be satisfied for all t . We first consider the case of a positive productivity shock. In this case, the capital stock and thus the wage $w_t = (1 - \theta)A_t^{1-\theta}K_t^\theta$ is monotonically increasing over time, and so $\beta R_{t+1} < 1$ is a sufficient condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$. Furthermore, from the previous proposition the interest rate jumps up at $t = 1$ and then monotonically converges to the (old and new) stationary equilibrium interest rate. Therefore, $\beta R_1 < 1$ guarantees that $\beta R_{t+1} < 1$ and thus $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all $t \geq 0$. The following proposition provides a sufficient condition for this, and is the counterpart to Assumption 3 in Krueger, Li, and Uhlig (2023).

Proposition 6 (Sufficient Condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t} \forall t \geq 0$ after a Positive Shock). *Let Assumption 5 be satisfied and let the economy be in a stationary equilibrium with $\beta R_0 < 1$. After a permanent positive productivity shock at $t = 1$ ($A_t = A_1 > A_0$ for all $t \geq 1$), $\beta R_{t+1} < \frac{w_{t+1}}{w_t} \forall t \geq 0$ is satisfied if the shock is not too large, that is, if $A_1 \in [A_0, \bar{A}_1)$ where the threshold \bar{A}_1 satisfies*

Assumption 6.

$$\frac{\bar{A}_1}{A_0} = \left[\frac{1 - \beta(1 - \delta)}{\theta} \frac{\xi(1 - \theta) + (1 - \nu)\theta [1 - (1 - \nu - \xi)\beta]}{[1 - (1 - \nu)\beta(1 - \delta)][1 - (1 - \nu - \xi)\beta]} \right]^{\frac{1}{1-\theta}} = \left(\frac{\rho + \delta}{\theta} \right)^{\frac{1}{1-\theta}} K_0 > 1. \quad (67)$$

Proof. See Appendix A.2.4. □

Intuitively, if $A_1/A_0 > 1$ is sufficiently small, the initial jump in the interest rate is not too large, and we can guarantee $\beta R_{t+1} < 1$ along the transition path. This, coupled with positive wage growth induced by the positive productivity shock insures that $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all $t \geq 1$ along the transition path, and high-productivity households have no incentive to save for any t , confirming the existence of a simple, no-savings equilibrium.

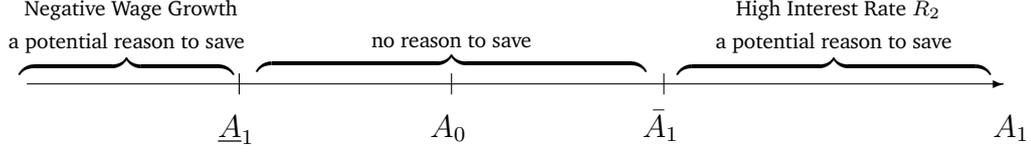


Figure 2: Thresholds for A_1

The case of a negative technology shock is more challenging because wages are declining along the transition (see the previous proposition), and thus high-productivity individuals face stronger incentives to save in anticipation of lower labor income in the future. We can nevertheless give a sufficient condition on the size of the productivity decline that guarantees the no-savings condition be satisfied. This result has no counterpart to the analysis in continuous time.

Proposition 7 (Sufficient Condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t} \forall t \geq 0$ after a Negative Shock). *Let Assumption 5 be satisfied and let the economy be in a stationary equilibrium with $\beta R_0 < 1$. After a permanent negative productivity shock at $t = 1$ ($A_t = A_1 < A_0$ for all $t \geq 1$), $\beta R_{t+1} < \frac{w_{t+1}}{w_t} \forall t \geq 0$ is satisfied if $A_1 \in (\underline{A}_1, A_0]$ holds, where the threshold satisfies*

Assumption 7.

$$\underline{A}_1/A_0 = \left[1 - \nu + \nu \frac{1 - (1 - \delta)\beta(1 - \nu)}{\beta\nu(1 - \delta)} \frac{\xi(1 - \theta) - \beta\theta\nu(\xi + \nu + \frac{1}{\beta} - 1)}{\xi(1 - \theta) + \beta\theta(1 - \nu)(\xi + \nu + \frac{1}{\beta} - 1)} \right]^{\frac{1}{\theta-1}} < 1. \quad (68)$$

Proof. See Appendix A.2.5. □

Figure 2 illustrates Propositions 6 and 7 graphically. Note that the conditions stated in these two propositions are sufficient but not necessary for the household limited commitment constraint to be binding in the high income state. To summarize, Proposition 6 and 7 state that if the permanent productivity shock is not too large, $A_1 \in (\underline{A}_1, \bar{A}_1)$, the condition on interest rate and wage growth, $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$, is satisfied for all $t \geq 0$.

In Appendix B.1.1 we generalize the results in this subsection to arbitrary monotone deterministic and convergent sequences $\{A_t\}_{t=1}^{\infty}$ with $A_0 < A_1 \leq A_2 \leq \dots$ or with $A_0 > A_1 \geq A_2 \geq \dots$ with $\lim_{t \rightarrow \infty} A_t = A^*$. The results are similar to the ones for permanent shocks in that they require that the initial productivity shock A_1 and the subsequent shocks cannot be too large or too small, but the analysis requires a first-order approximation of the capital stock dynamics.

4.2.2 MIT Shocks, Anticipated Shocks and Consumption on Impact

In Proposition 1 we provided a general characterization of the optimal consumption allocation under the no-savings in the high state condition. We now draw out one perhaps unexpected implication of this general characterization in the context of MIT shocks: despite the unexpected change in wages and interest rates starting in period $t = 1$, the optimal consumption allocation of low-productivity individuals satisfies the standard Euler equation even through the surprise period, i.e., between period $t = 0$ and $t = 1$.

Corollary 2 (Consumption at the time of a shock). *Consider an unexpected shock to productivity at $t = 1$ and assume that Assumptions 2, 3, and 4 hold. Then consumption of high-income agents ($c_{h,t}$) is a constant fraction of their income, and consumption of low-income agents ($c_{s,t}$ for $s \geq 1$) satisfies the Euler equation between periods $t = 0$ and $t = 1$:*

$$\begin{aligned} c_{h,1} &= \frac{1 - (1 - \nu)\beta}{1 - (1 - \nu - \xi)\beta} z w_1 \\ c_{s,1} &= \beta R_1 c_{s-1,0} \quad \text{for } s \geq 1. \end{aligned} \tag{69}$$

Proof. Follows directly from the general characterization in Proposition 1. See Appendix A.2.6 for detail. \square

The key to this result is that with log-utility, unconstrained households consume a constant fraction of their assets cum interest, see equation (21):

$$\mathbf{c}_{s,1} := w_1 c_{s,1} = [1 - \beta(1 - \nu)] R_1 \mathbf{a}_{s,1} \quad \forall s \geq 1, \forall t \tag{70}$$

Two observations are crucial. First, current consumption and assets chosen for tomorrow do not depend on *future* interest rates with log-utility.¹⁶ Second, the surprise change in current (period 1) TFP does impact the marginal product of capital and thus R_1 (even though the capital stock in $t = 1$ is predetermined), and therefore consumption $c_{s,1}$ changes in period 1 relative to what the household had planned in the initial steady state ($c_{s,0}$), as equation (70) indicates, even though $\mathbf{a}_{s,1}$ is predetermined from the previous period. But since the impact of R_1 on consumption is proportional, it exactly cancels out with the direct change in R_1 in the Euler equation, and thus the scaled (by R_1) consumption level $c_{s,1}$ continues to satisfy the Euler equation between periods $t = 0, 1$.

In this section we have analyzed the transitional dynamics after an unanticipated productivity shock (MIT shock) at $t = 1$, starting from the initial steady state. Note that the assumption that the TFP changes are completely unanticipated is irrelevant for the

¹⁶We discuss the relation to the literature also exploiting log-utility to obtain analytical tractability in heterogeneous-agent macro models in section 6

transition dynamics. As we saw from equation (70), low-income agents consume a constant fraction of their implied asset position *regardless* of future interest rates, and high-income agents also consume a constant fraction of their labor income. Individual state-contingent savings for the low-idiosyncratic productivity state $\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}; z_{t+1} = 0) = \mathbf{a}_{t+1}(a_0, z^t, A^t; z_{t+1} = 0)$ do not depend on future productivity shocks, either.¹⁷ Therefore, aggregate consumption in the economy is independent of future interest rates and wages, and the law of motion of aggregate capital does not depend on these future prices either. Thus the dynamics of the economy unfolds the same, regardless of whether future productivity shocks are anticipated or unanticipated. Note that these results do *not* hold in our model if households have CRRA utility with $\sigma \neq 1$; they also do not hold in the standard neoclassical growth model, as Appendix B.1.2 shows.

4.2.3 Application 1: Inequality along the Transition

We now use the result characterizing the aggregate transitional dynamics to study, as a first application of the model, how consumption (and wealth) inequality responds to unexpected or (partially) expected aggregate productivity shocks, thereby analyzing how inequality evolves over the business cycle in our model. Our analysis largely follows Section 6 of Krueger, Li, and Uhlig (2023), but in discrete time we can tackle both positive and negative permanent shocks to productivity. We demonstrate four findings: (i) the consumption distribution in the long run is invariant to aggregate productivity A_t ; (ii) With full depreciation of capital ($\delta = 1$), the consumption distribution is time-invariant and independent of aggregate shocks; (iii) If depreciation is partial ($\delta < 1$), consumption inequality expands on impact of a positive permanent productivity shock and then shrinks (and often undershoots) towards the new steady state; (iv) Following a negative productivity shock, the evolution of consumption inequality is symmetric to that of a positive productivity shock.

The basis of our inequality results is Corollary 1 which shows that the consumption distribution follows the simple structure in equation (32) characterizing wage-deflated

¹⁷Recall from equation (22) that:

$$\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}; z_{t+1} = 0) = \begin{cases} \frac{\beta}{1-(1-\nu-\xi)\beta} \zeta w_t(A^t) & \text{if } z_t = \zeta \\ \beta R_t(A^t) \mathbf{a}_t(a_0, z^t, A^t) & \text{if } z_t = 0 \end{cases} = \mathbf{a}_{t+1}(a_0, z^{t+1}, A^t; z_{t+1} = 0)$$

and thus is independent of (expectations of) $R_{t+1}(A^{t+1})$. With log utility, the effect of a higher return on assets on savings in a state with higher TFP (the substitution effect) is completely offset by the lower marginal utility from consumption (the income effect). Thus, households save the same amount regardless of future aggregate productivity. The payoff tomorrow from these assets, $R_{t+1} a_{t+1}$, is dependent on R_{t+1} (and thus changes, surprisingly in the case of the MIT shock, expectedly in the case of foreseen changes in A_{t+1}), but households do not act on these foreseen changes.

consumption $c_{s,t} = \frac{c_{s,t}}{w_t}$ as

$$c_{s,t}(A^t) = \begin{cases} c_0 := \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta}\zeta & \text{if } s = 0 \\ \beta R_t(A^t) \frac{w_{t-1}(A^{t-1})}{w_t(A^t)} c_{s-1,t-1}(A^{t-1}) & \text{if } s \geq 1 \end{cases} \quad (71)$$

where we recall that s is the number of periods an individual has spent in a low-income state since last receiving high income.

Consumption Distribution in the Long Run In the stationary equilibrium, the consumption distribution is determined exclusively by the equilibrium βR^* , which does not depend on productivity A_t . This implies that after aggregate shocks have subsided, the consumption distribution (scaled by the now different wage) will return to the initial stationary distribution in the long run.

Proposition 8 (Consumption Distribution in the Long Run). *Suppose that Assumption 3 holds and that an economy is in a steady state at $t = 0$ with productivity A_0 . Suppose also that after a productivity shock at $t = 1$, aggregate productivity settles down at $\lim_{t \rightarrow \infty} A_t = A_\infty$. Then, the deflated consumption distribution in the long run is the same as the initial distribution:*

$$c_s^* = (\beta R^*)^s c_0 \text{ for } s = 0, 1, 2, \dots,$$

since the steady-state interest rate, R^* , does not depend on productivity A .

Proof. See Appendix A.2.7. □

The Dynamics of the Consumption Distribution with $\delta = 1$ In the case of full depreciation of capital, $\delta = 1$, wages and gross returns are proportional to aggregate productivity A_t , and so are the incomes of the high-productivity individuals with labor income and the low-productivity individuals with capital income. As a consequence, the deflated consumption distribution is constant along the transition path.

Proposition 9. *Suppose Assumption 3 holds and suppose that an economy is in a stationary equilibrium at $t = 0$ with a deflated consumption distribution $\{c_s^*\}_{s \geq 0}$. With full depreciation of capital ($\delta = 1$), the deflated consumption distribution is time-invariant for any sequence of $\{A_t\}_{t \geq 0}$:*

$$c_{s,t}(A^t) = c_s^*, \text{ for any } s \geq 0 \text{ at all } t \geq 0 \text{ and } A^t. \quad (72)$$

Proof. See Appendix A.2.8. □

Note that there is no counterpart to this result in continuous time since the full-depreciation case cannot be analyzed there.

Consumption Inequality after a Positive Shock when $\delta < 1$ With full depreciation, the differential evolution of capital- and labor income, which is the source of changing inequality, is muted. We now study the more realistic case with partial depreciation in which the gross return to capital moves less than one for one with aggregate productivity. The thought experiment is again an MIT shock that hits the economy at the beginning of period $t = 1$ and changes aggregate productivity to a sequence $\{A_t\}_{t \geq 0}$, and the object of interest is the evolution of consumption inequality associated with this shock.¹⁸

The next proposition characterizes the impact of a positive productivity shock on inequality on impact (when assets coming into the period are pre-determined). Inequality widens because wages and thus consumption of high-productivity individuals moves one for one with productivity whereas the gross return and thus consumption of low-productivity individuals increase less than one for one. Consequently, since the latter group has lower consumption to start with, inequality widens on impact with less than full depreciation.

Proposition 10. *Suppose that Assumption 3 holds, and suppose the economy is in a stationary equilibrium at $t = 0$ and hit by a positive productivity shock at $t = 1$ such that $A_1 > A_0$. If $0 < \delta < 1$ and $0 < \theta < 1$, consumption inequality rises between $t = 0$ and $t = 1$:*

$$\begin{aligned} c_{0,t} &= c_0 := \frac{1 - (1 - \nu)\beta}{1 - (1 - \nu - \xi)\beta} \zeta & \forall t \geq 0 \\ c_{s,1} &< c_{s,0} & \forall s \geq 0 \end{aligned} \quad (73)$$

and thus $c_{s,1}/c_{0,1} < c_{s,0}/c_{0,0}$ for all $s > 0$. Furthermore $c_{s,1}/c_{\tilde{s},1} = c_{s,0}/c_{\tilde{s},0}$ for all $s, \tilde{s} > 0$.

Proof. See Appendix A.2.9. □

To illustrate the results in this section, Figure 3 uses a numerical example to illustrate the evolution of consumption inequality in response to a positive (upper panels) and negative (lower panels) aggregate productivity shock. The figures in the left column display the Lorenz curves of the consumption distribution for various time periods, and the right panels show change in the Lorenz curve (in percentage point deviations) relative to the initial steady state distribution, which is also the final steady state consumption distribution (see Proposition 8). As predicted by Proposition 10, in response to a positive technology shock consumption inequality increases and the Lorenz curve shifts out and further away from the 45-degree line. Eventually, as predicted by Proposition 8 the Lorenz curve will return to its original shape as the economy converges to the new steady state associated with permanently higher productivity.¹⁹

¹⁸As discussed in Section 4.2.2, it does not matter whether these productivity shocks are anticipated or unanticipated. Therefore, the same results as in this subsection also go through in a stochastic economy

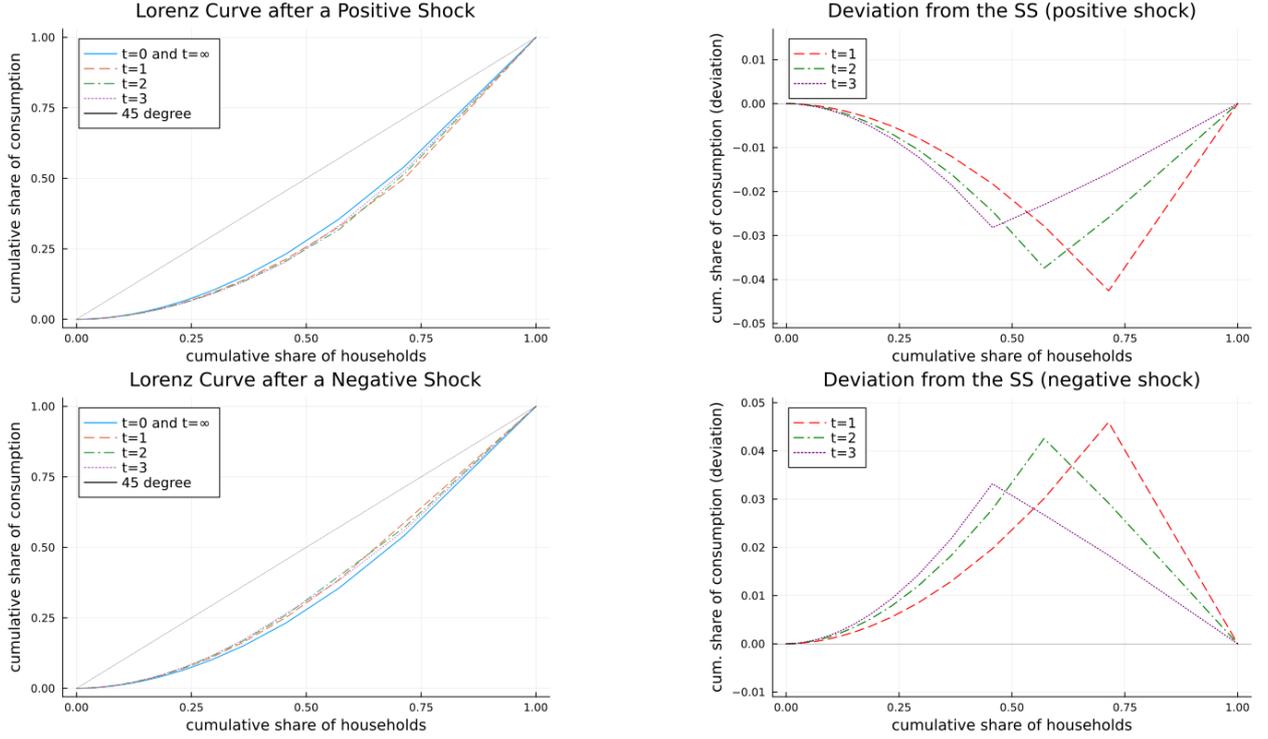


Figure 3: Evolution of the Lorenz Curve after a positive and after a negative productivity shock at $t = 1$. Parameter values are $\beta = 0.6, \delta = 0.1, \theta = 0.33, \xi = 0.5, \nu = 0.2, \frac{A_1}{A_0} \in \{1.5, 1/1.5\}$

Consumption Inequality after a Negative Shock The evolution of the deflated consumption distribution for a permanent negative productivity shock is largely symmetric to that of a positive shock. The corresponding theoretical results are stated in Corollaries 4–6 in Appendix B.1.4. There are two key differences between a positive shock and a negative shock. First, the sufficient condition (derived in Subsection 4.2.1) required for insuring that $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all $t \geq 1$ differs across the two cases. After a positive shock, $\beta R_{t+1} < 1$ is sufficient since wages are increasing along the transition path. In contrast, after a negative shock agents may have an incentive to save even when $\beta R_{t+1} < 1$ since wages are declining over time. Second, at the time of the shock, in $t = 1$, low-income agents may have higher consumption than high-income agents in case of a negative productivity shock because the decline in wages (which is relevant for high-productivity individuals) is larger than the decline in interest rates (which is relevant for low-productivity individuals) if $0 < \delta < 1$.²⁰

when productivity shocks take the specific sample path $\{A_t\}_{t \geq 0}$.

¹⁹We discuss the forces determining the evolution of consumption inequality along the transition in Appendix B.1.3.

²⁰Low-income households may have higher consumption than high-income households at $t = 1$ only if $\beta R_1 > \frac{w_1}{w_0}$ and Assumption 3 is violated in $t = 1$. However, the proposed consumption allocation (stipu-

5 Aggregate Shocks to Productivity and Depreciation

We now consider the economy with stochastic productivity growth, introducing aggregate risk into the economy. In addition, in order to introduce a mechanism by which rates of return to capital are imperfectly correlated with, and more volatile than wages, we introduce shocks to the depreciation rate of capital as well.²¹ Both productivity growth and the depreciation rate now depend on an aggregate state Ω_t , with the history of the state being denoted as $\Omega^t := (\Omega_0, \dots, \Omega_t)$. The aggregate state Ω_t follows a finite state first-order Markov process.

In Section 3, we showed that under the *assumption* of $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all t and for all possible aggregate states we derived the optimal household consumption and asset allocation as well as the aggregate law of motion of aggregate capital. We now verify that under an assumption purely on exogenous parameters of the model this condition is satisfied in the constructed equilibrium, for all t and all Ω^t in the economy with aggregate risk. To establish this result we assume that

$$\frac{A_{t+1}}{A_t}(\Omega_{t+1}) \in \{1 - \epsilon, \dots, 1 + \epsilon\} \quad (74)$$

$$\delta_{t+1}(\Omega_{t+1}) \in \{\bar{\delta} - \varsigma, \dots, \bar{\delta} + \varsigma\} \quad (75)$$

That is, the smallest and the largest realizations of productivity growth are denoted by $1 - \epsilon$ and $1 + \epsilon$, respectively, and the depreciation rate is bounded by $\bar{\delta} - \varsigma$ and $\bar{\delta} + \varsigma$.

Otherwise, there is no change in the rest of the economic environment. The resource constraint continues to be given by:

$$C_t(\Omega^t) + K_{t+1}(\Omega^t) = K_t(\Omega^{t-1})^\theta A_t(\Omega^t)^{1-\theta} + (1 - \delta_t(\Omega_t))K_t(\Omega^t). \quad (76)$$

Since aggregate capital is determined one period ahead, K_{t+1} depends on the history of aggregate state Ω^t up to time t . Factor prices continue to be equal to their marginal product:

$$w_t(\Omega^t) = (1 - \theta) (A_t(\Omega^t))^{1-\theta} (K_t(\Omega^{t-1}))^\theta \quad (77)$$

$$R_t(\Omega^t) = \theta (A_t(\Omega^t))^{1-\theta} (K_t(\Omega^{t-1}))^{\theta-1} + 1 - \delta_t(\Omega_t). \quad (78)$$

The key innovation is that now the return on capital R_t depends on both on the realizations of stochastic productivity and depreciation (A_t, δ_t) . Appendix A.3.2 contains the (omitting zero contingent asset holdings for the high-productivity state) is still optimal as long as beginning of the period assets of low-income households at $t = 1$ are not too large; recall that Assumption 3 is only a sufficient condition. If they are indeed large, then currently low-productivity individuals would optimally save even for the high-productivity state tomorrow.

²¹For examples of papers that have used stochastic depreciation for the same reason, see, e.g. Smetters (2003) and Krueger and Kubler (2006).

complete definition of equilibrium and shows that as long as $\beta R_{t+1}(\Omega^{t+1}) < \frac{w_{t+1}(\Omega^{t+1})}{w_t(\Omega^t)}$ for all t, Ω^{t+1} the previous characterization of equilibrium goes through completely unchanged. We now show theoretically in Section 5.1 that if ϵ and ς are not too large (as stated precisely in Assumption 8 below), then $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ holds for all t and all (Ω^t, Ω_{t+1}) and the model with aggregate shocks has indeed a partial insurance equilibrium of the form characterized in Section 3. In Section 5.2 we then deduce the asset pricing implications of the model. Specifically, there we provide theoretical conditions under which the aggregate risk premium for capital in our model coincides with that obtained in the standard representative agent (RA) economy, and we then show quantitatively under what conditions the risk premium is significantly larger than that in the RA economy.

5.1 A Sufficient Condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$

We introduce the following sufficient condition on the magnitude of aggregate productivity shocks and depreciation shocks.

Assumption 8.

$$\beta \left[\theta \left(\frac{1-\epsilon}{1+\epsilon} \right)^{1-\theta} \frac{\hat{\delta}_{max} + \epsilon}{\hat{s}} + 1 - \delta_{min} \right] < 1 - \epsilon \quad (79)$$

where we recall $\hat{s} = \left[\frac{\xi\beta}{1-(1-\nu-\xi)\beta} (1-\theta) + (1-\nu)\beta\theta \right]$ and $1 - \hat{\delta} = (1-\nu)\beta(1-\delta)$ from equations (57) and (58), and define $\hat{\delta}_{max} := 1 - (1-\nu)\beta(1-\bar{\delta}-\varsigma)$ and $\delta_{min} := \bar{\delta} - \varsigma$.

Proposition 11. *Suppose Assumption 5 holds and that the economy is in the partial-insurance steady state at $t = 0$. Furthermore, suppose that the bounds on productivity growth and depreciation (ϵ, ς) satisfy Assumption 8. Then the condition $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ is satisfied for all $t \geq 1$ with probability 1. Furthermore, there exists an open neighborhood \mathcal{N} around $(\epsilon, \varsigma) = (0, 0)$ such that Assumption 8 is satisfied for all $(\epsilon, \varsigma) \in \mathcal{N}$.*

Proof. See Appendix A.3.2. □

To derive the sufficient condition (79), we proceed in four steps, stated in the forms of Lemmas in the appendix. First, defining $\tilde{K}_t := \frac{K_t}{A_t}$ as the productivity-adjusted capital stock, and using the aggregate law of motion for the capital stock (39), we can write both $\beta R_{t+1}(\Omega^{t+1})$ and $\frac{w_{t+1}(\Omega^{t+1})}{w_t(\Omega^t)}$ as functions exclusively of productivity-adjusted capital \tilde{K}_t and the growth rate of productivity $\frac{A_{t+1}}{A_t}$, since the aggregate law of motion (39) can be rewritten as

$$\tilde{K}_{t+1} = \frac{A_t}{A_{t+1}} \left(\hat{s} \tilde{K}_t^\theta + (1 - \hat{\delta}) \tilde{K}_t \right) \quad (80)$$

and noting that $R_{t+1} = \theta \tilde{K}_{t+1}^{\theta-1} + 1 - \delta$ and $\frac{w_{t+1}}{w_t} = \frac{A_{t+1}}{A_t} \left(\frac{\tilde{K}_{t+1}}{\tilde{K}_t} \right)^\theta$.

Second, holding \tilde{K}_t fixed, the condition is most easily violated for the smallest $\frac{A_{t+1}}{A_t}$ and the smallest δ_{t+1} . Third, the interest rate $\beta R_{t+1}(\Omega^{t+1})$ is strictly decreasing in \tilde{K}_t and wage growth is strictly decreasing in \tilde{K}_t . Fourth, we can establish a lower and an upper bound on the stochastic process $\{\tilde{K}_t\}$. The sufficient condition then insures that

$$\beta R_{t+1}(A^{t+1}) \Big|_{\tilde{K}_t = \tilde{K}^{min}, \frac{A_{t+1}}{A_t} = 1 - \epsilon, \delta = \bar{\delta} - \varsigma} < \frac{w_{t+1}(A^{t+1})}{w_t(A^t)} \Big|_{\tilde{K}_t = \tilde{K}^{max}, \frac{A_{t+1}}{A_t} = 1 - \epsilon} \quad (81)$$

which implies that the condition $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ is satisfied for all t with probability 1.

Note that the existence of an open neighborhood \mathcal{N} in Proposition 11 insures that the set of parameters for which the proposition applies is non-empty. Furthermore, the sufficient condition (79) can be readily be checked numerically, and in our asset pricing application below we ensure that it is satisfied for all parameterizations of the model we consider.

5.2 Application 2: Asset Pricing

We now examine the asset pricing implications of the limited-commitment model with idiosyncratic and aggregate risk. Specifically, we contrast the risk-free rate and the risk premium in our model with those in the representative agent model.

In Section 5.2.1, we derive the risk-free rate and the risk premium of holding capital in the two models. In Section 5.2.2, we show that if capital always fully depreciates ($\bar{\delta} = 1, \varsigma = 0$), the limited-commitment model has a lower risk-free rate than the corresponding representative-agent model, but an identical risk premium. In Section 5.2.3 we discuss why the same conclusion does not hold if one deviates from $\delta \equiv 1$. There we argue that in an endowment economy and the production economy with $\delta \equiv 1$, the available resources in the economy as well as gross capital income and labor income are all perfectly correlated with aggregate productivity $A_t^{1-\theta}$, whereas with less than perfect depreciation and deterministic depreciation, gross capital income (and thus the income of low-productivity households) moves *less* than one for one with productivity $A_t^{1-\theta}$, resulting in a lower equity premium in our model. Finally, in Section 5.2.4 we show that with stochastic capital depreciation capital income becomes more volatile than labor income, and (for sufficiently large depreciation shocks), the risk premium in our model exceeds that in the representative agent economy, and can account for a significant share of the empirically observed risk premium if idiosyncratic risk is sufficiently large.

5.2.1 The Risk-Free Rate and the Risk Premium in the Two Models

We now derive the price of risk-free bonds $q^B(\Omega^t)$ and the multiplicative risk premium $1 + \lambda_t(\Omega^t)$ on capital, defined as the ratio between the expected gross return on risky capital

and the gross risk-free rate, in the limited-commitment model (Lemma 1) and in the representative-agent model (Lemma 2). To do so, recall that the stochastic gross return on buying one unit of capital in node Ω^t and holding it to node Ω^{t+1} (and then selling it) is denoted by $R_{t+1}(\Omega^{t+1})$. Furthermore, for future reference we denote aggregate consumption by $C_t(\Omega^t)$.

Lemma 1. *In the limited commitment model, the price of risk-free bonds and the risk premium of holding capital at aggregate state Ω^t is given by:*

$$q_t^{B,LC}(\Omega^t) = \mathbb{E}_t \left[\frac{1}{R_{t+1}(\Omega^{t+1})} \right] \quad (82)$$

$$1 + \lambda_t^{LC}(\Omega^t) := \frac{\mathbb{E}_t[R_{t+1}(\Omega^{t+1})]}{\mathbb{E}_t[1/q^B(\Omega^t)]} = \mathbb{E}_t[R_{t+1}(\Omega^{t+1})] \mathbb{E}_t \left[\frac{1}{R_{t+1}(\Omega^{t+1})} \right] > 1. \quad (83)$$

Here $\mathbb{E}_t[\cdot]$ denotes the expectation conditional on Ω^t , that is, $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\Omega^t]$

Proof. See Appendix A.3.3. □

Lemma 2. *In the representative agent model, the price of risk-free bonds and the risk premium at aggregate state Ω^t is given by:*

$$q_t^{B,RA}(\Omega^t) = \mathbb{E}_t \left[\beta \frac{C_t(\Omega^t)}{C_{t+1}(\Omega^{t+1})} \right] \quad (84)$$

$$1 + \lambda_t^{RA}(\Omega^t) := \frac{\mathbb{E}_t[R_{t+1}(\Omega^{t+1})]}{\mathbb{E}_t[1/q^B(\Omega^t)]} = \mathbb{E}_t[R_{t+1}(\Omega^{t+1})] \mathbb{E}_t \left[\beta \frac{C_t(\Omega^t)}{C_{t+1}(\Omega^{t+1})} \right]. \quad (85)$$

Proof. See Appendix A.3.3. □

5.2.2 Economy with Full Depreciation of Capital

In the case of full depreciation, for both the limited-commitment economy and the representative agent economy the aggregate law of motion of capital is given in closed form, which allows us to derive the risk-free rate and the risk premium explicitly. We show that the limited-commitment model has a lower risk-free rate but an identical risk premium compared to the representative agent model. That is, idiosyncratic and only partially insurable risk drives down the return on all assets but leaves the multiplicative risk premium unaffected.

Proposition 12. *Consider the economy with full depreciation of capital ($\delta = 1$ with probability 1, that is $\bar{\delta} = 1, \varsigma = 0$). Let Assumption 5 be satisfied and the economy be initially in a partial insurance steady state. Given the same history of aggregate shocks Ω^t , the risk-free rate is lower in the limited-commitment model than in the representative-agent model:*

$$\frac{1}{q_t^{B,LC}(\Omega^t)} < \frac{1}{q_t^{B,RA}(\Omega^t)} \text{ for all } \Omega^t. \quad (86)$$

Table 1: Asset Pricing in the Two Economies with $\delta = 1$

	LC	RA
Low of Motion	$K_{t+1}^{LC} = \hat{s}^{LC} A_t^{1-\theta} K_t^\theta$	$K_{t+1}^{RA} = \beta\theta A_t^{1-\theta} K_t^\theta$
Return on Capital	$R_{t+1}^{LC} = \theta \left(\frac{K_{t+1}^{LC}}{A_{t+1}} \right)^{\theta-1}$	$R_{t+1}^{RA} = \theta \left(\frac{K_{t+1}^{RA}}{A_{t+1}} \right)^{\theta-1}$
Risk-Free Rate $\left(\frac{1}{q^B(\Omega^t)} \right)$	$\frac{1}{\mathbb{E}_t[1/R_{t+1}^{LC}(\Omega^{t+1})]}$	$\frac{1}{\mathbb{E}_t[1/R_{t+1}^{RA}(\Omega^{t+1})]}$
Risk Premium $\left(\frac{\mathbb{E}_t[R_{t+1}]}{1/q^B(\Omega^t)} \right)$	$\mathbb{E}_t[A_{t+1}^{1-\theta}] \mathbb{E}_t\left[\frac{1}{A_{t+1}^{1-\theta}}\right]$	$\mathbb{E}_t[A_{t+1}^{1-\theta}] \mathbb{E}_t\left[\frac{1}{A_{t+1}^{1-\theta}}\right]$

The risk premium is positive, and is the same in the two models and is given by:

$$1 + \lambda_t(\Omega^t) = \mathbb{E}_t[A_{t+1}^{1-\theta}] \mathbb{E}_t\left[\frac{1}{A_{t+1}^{1-\theta}}\right] > 1. \quad (87)$$

If productivity growth $\frac{A_{t+1}}{A_t}$ follows an iid process, then the common risk premium is constant over time and across states, $1 + \lambda_t(\Omega^t) = 1 + \lambda$.

Proof. See Appendix A.3.4. □

Table 1 summarizes the asset pricing results with full depreciation. In this case, *both* models have a constant aggregate saving rate, but the key is that the one in the limited-commitment (LC) model is higher than the representative agent (RA) model: $\hat{s}^{LC} > \beta\theta = s^{RA}$ under Assumption 5. Under that assumption, in steady state $R_0^{LC} < R_0^{RA} = 1/\beta$ and thus $K_0^{LC} > K_0^{RA}$. Since in both models, $K_0 = A_0 s^{\frac{1}{1-\theta}}$, we have that $\hat{s}^{LC} > s^{RA}$, and thus the saving rate and thus the capital stock is higher in the LC economy, driving down the risk-free rate below that in the RA economy.²² However, since all returns move proportionally in both models, the risk premium is the same in the models.

5.2.3 Intuition: Different Risk Premia if $\delta \neq 1$

We have seen that the limited commitment model has a lower interest rate but the same risk premium as the representative agent economy if $\delta = 1$. Here we provide intuition

²²This statement is conditional on the initial capital stock and the sequence of productivity shocks being the same when comparing both models. Assumption 5 and $\delta \equiv 1$ then imply $\beta R^{LC} < 1$ and $\beta R_{t+1}^{LC}(\Omega^{t+1}) < \frac{w_{t+1}^{LC}(\Omega^{t+1})}{w_t^{LC}(\Omega^t)}$ for any $(t, \Omega^t, \Omega_{t+1})$, and thus the no-savings condition is satisfied with probability one. With a constant saving rate $s \in \{\hat{s}^{LC}, s^{RA} := \beta\theta\}$, the capital stock $K_t(\Omega^t)$ after any productivity history can be characterized in closed-form as

$$\log K_t = (1 + \theta + \dots + \theta^{t-2}) \log s + (1 - \theta) \left[\sum_{\tau=1}^{t-1} \theta^{\tau-1} \log A_{t-\tau} \right] + \theta^{t-1} \log K_0. \quad (88)$$

It follows directly that the limited commitment commitment economy has more capital after *every* sequence of productivity shocks than the RA economy, further depressing returns below those of the RA economy. See Appendix B.2.1 for the details.

(using the analogy with an endowment economy, See Proposition 13) for why this is the case and why the same result does not go through in a production economy unless $\delta \equiv 1$. Key to this result is that if two economies have stochastic discount factors that only differ by a non-stochastic (but possibly time-varying) constant, then they will have different risk-free rates but a common multiplicative equity premium. This is the content of Proposition 14 below.

Endowment Economy To build intuition, we first derive this result in an endowment economy in which the exogenous aggregate endowment is equal to aggregate consumption $\{C_t(\Omega^t)\}_{t,\Omega^t}$ and the risky asset is now a Lucas tree that pays a share α of the aggregate endowment at all times (and whose price we denote as $q_t(\Omega^t)$). The remaining fraction $1 - \alpha$ of the aggregate endowment is “labor income” which is subject to exactly the same idiosyncratic shocks (that wash out in the aggregate) as in the production economy studied thus far.

In this economy we obtain the same asset pricing results as in the production economy with $\delta = 1$: idiosyncratic risk and limited commitment drives down the risk-free rate but leaves the equity premium unchanged relative to the representative agent version of the model. This is the content of the next proposition which is proved in Appendix A.3.5.

Proposition 13. *In the endowment economy, assume that the parameters satisfy the assumption corresponding to Assumption 5 for the production economy:*

$$\frac{\alpha}{(1 - \alpha) \left(\frac{1}{\beta} - 1 \right)} < \frac{\xi}{\nu \left(\frac{1}{\beta} - 1 + \xi + \nu \right)}, \quad (89)$$

so that the no-savings Assumption 3 is satisfied. Then the risk-free rate (the inverse of the price of a risk-free bond), is lower in the limited commitment economy than in the representative agent economy:

$$\frac{1}{q_t^{B,LC}(\Omega^t)} < \frac{1}{q_t^{B,RA}(\Omega^t)} \text{ for all } \Omega^t. \quad (90)$$

The risk premium on a claim to risky aggregate consumption is the same in both economies and is given by:

$$1 + \lambda_t(\Omega^t) = \mathbb{E}_t[C_{t+1}(\Omega^{t+1})] \mathbb{E}_t \left[\frac{1}{C_{t+1}(\Omega^{t+1})} \right] > 1 \quad (91)$$

If the growth rate of the exogenous endowment process, $\frac{C_{t+1}}{C_t}$, is iid, then the risk premium $1 + \lambda_t$ is constant over time and across states of the world.

Krueger and Lustig (2010) show that when the stochastic discount factors in two

models differ only by a non-random multiplicative term²³ then they have the same (multiplicative) risk premium (but typically different risk-free rates). Recall that a stochastic discount factor is a stochastic process that satisfies the condition $\mathbb{E}_t [m_{t,t+1}(\Omega^{t+1})R_t^j(\Omega^{t+1})] = 1$ for any asset j that is being traded in the economy and has an equilibrium one-period return $R_t^j(\Omega^{t+1})$. In our applications the two assets are a risk-free one-period bond and risky capital (risky equity in the endowment economy). For the representative agent economy the stochastic discount factor is given by $m_{t,t+1}^{RA}(\Omega^{t+1}) = \beta \frac{C_t(\Omega^t)}{C_{t+1}(\Omega^{t+1})}$ and in our limited commitment economy it is given by $m_{t,t+1}^{LC}(\Omega^{t+1}) = \beta \frac{c_t(z^t, \Omega^t)}{c_{t+1}(z^{t+1}, \Omega^{t+1})}$ for those agents whose limited commitment constraint is not binding between nodes (z^t, Ω^t) and (z^{t+1}, Ω^{t+1}) and who all share the same consumption growth rate. The following proposition characterizes situations in which a) the SDF's in both models can be characterized in terms of the growth rate of aggregate resources in the economy (the sum between capital and labor income) and b) the growth rate of aggregate resources in both models is proportional to each other. If both a) and b) are true (as in the endowment economy and the production economy with full depreciation), this is sufficient for the SDF's to be proportional and the risk premia to be identical. When a) or b) fails (as in the production economy with less than full depreciation) this conclusion is no longer true.

Proposition 14. *Denote total resources available in the economy at aggregate node Ω^t by $\Upsilon_t(\Omega^t)$, given by*

$$\Upsilon_t(\Omega^t) = \begin{cases} C_t(\Omega^t) & \text{in the endowment economy} \\ K_t(\Omega^{t-1})^\theta A_t(\Omega^t)^{1-\theta} + (1 - \delta(\Omega_t))K_t(\Omega^{t-1}) & \text{in the production economy} \end{cases} \quad (92)$$

1. *In both the endowment economy and the production economy with $\delta(\Omega_t) \equiv 1$, all unconstrained agents consume a non-random fraction of total resources. The stochastic discount factor in both the representative agent- and the limited commitment model is proportional to $\frac{\Upsilon_t}{\Upsilon_{t+1}}$ and satisfies:*

$$m_{t,t+1}(\Omega^{t+1}) = \gamma_t \frac{\Upsilon_t(\Omega^t)}{\Upsilon_{t+1}(\Omega^{t+1})}, \quad (93)$$

The factor γ_t is potentially time-varying and differs across the two models, but does not depend on Ω_{t+1} .

2. *In the endowment economy and the production economy with $\delta = 1$, aggregate resource growth $\frac{\Upsilon_t}{\Upsilon_{t+1}}$ is proportional between the two models, i.e., there is a non-random*

²³They demonstrate that the aggregate risk premium in an endowment economy with idiosyncratic and aggregate risk is the same with complete markets (i.e., the representative agent model) and when insurance against idiosyncratic risk is absent by assumption, i.e., the standard incomplete markets model.

sequence of numbers γ'_t (in the endowment economy, $\gamma'_t \equiv 1$) satisfying:

$$\frac{\Upsilon_t^{LC}(\Omega^t)}{\Upsilon_{t+1}^{LC}(\Omega^{t+1})} = \gamma'_t \frac{\Upsilon_t^{RA}(\Omega^t)}{\Upsilon_{t+1}^{RA}(\Omega^{t+1})}, \quad (94)$$

Thus, there exists another non-random sequence of factors γ''_t satisfying:

$$\frac{m_{t,t+1}^{LC}(\Omega^{t+1})}{m_{t,t+1}^{RA}(\Omega^{t+1})} = \gamma''_t := \gamma'_t \frac{\gamma_t^{LC}}{\gamma_t^{RA}} \quad (95)$$

Consequently, in both the endowment and the production economy with $\delta = 1$, the limited commitment model with idiosyncratic risk has the same risk premium as the representative agent model.

Proof. See Appendix A.3.6. □

If the two models do not have multiplicative stochastic discount factors, i.e., if γ''_t in equation (95) depends on Ω^{t+1} , then the risk premia are typically different. In the production economy with $\delta \neq 1$, total resources are not proportional to the aggregate productivity shock $A_t^{1-\theta}$ due to the non-depreciated capital term $(1 - \delta(\Omega_t))K$. Hence, equation (94) does not hold, the stochastic discount factors in the two models are not proportional and the risk premia will typically differ.

In the left panel of Figure 4 we display the risk premium in the representative agent economy and for various calibrations of the limited commitment economy, plotted against the depreciation rate δ , which we take as deterministic for the time being, since the argument we make in this section does not depend on depreciation being stochastic. We observe that risk premium is *higher* in the RA economy than in our model, but converges in the two models as $\delta \rightarrow 1$ (as theory predicts). As argued above, in our model the individuals pricing the assets are the currently low-productivity households with positive capital holdings, and it is their consumption growth that is relevant for the equity premium. In the RA economy it is the consumption growth of the representative household. The right panel shows that the relevant consumption growth is more volatile in the RA economy, pushing up the equity premium above the limited commitment economy. In both models, the premium is quantitatively small, however, as is expected from the huge literature on the equity premium puzzle.

Why is consumption growth of the representative agent more volatile than that of unconstrained agents in the limited commitment model? Consumption of the representative agent is a combination of wage income and asset income. Both are stochastic due to aggregate productivity shocks, but the part $1 - \delta$ in the gross capital return R is deterministic, making gross capital income less volatile than labor income, unless $\delta = 1$. In our model consumption of those pricing the asset is financed exclusively from (the less volatile) capital income and thus the risk premium is smaller. That is, the equity

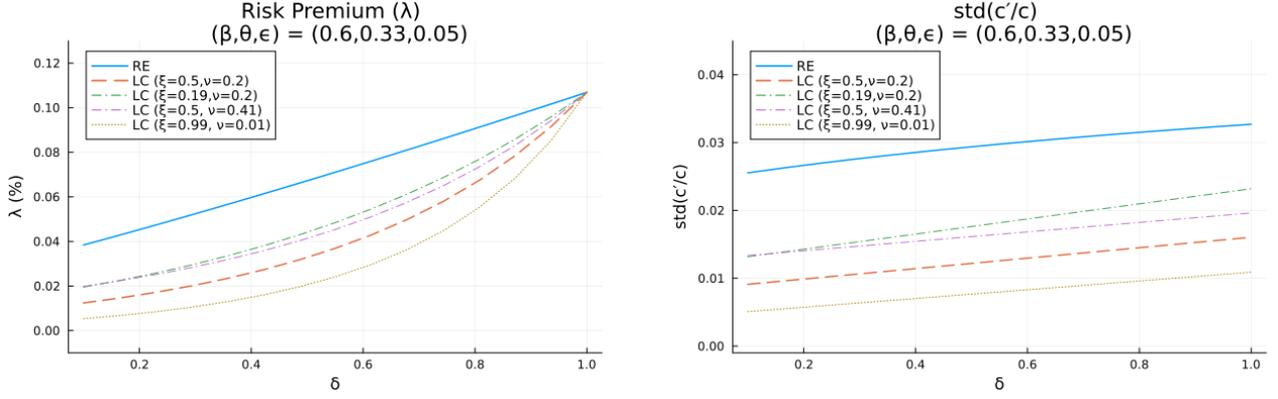


Figure 4: Risk Premium and Consumption Growth as a Function of δ

premium puzzle deepens in our economy, relative to the standard representative agent production economies studied in the literature.

5.2.4 Stochastic Depreciation

The analytical tractability of our model also gives a clear indication of why risk premium is smaller in our model. Capital income of those participating in the capital market is not volatile enough. This is precisely the “problem” stochastic depreciation is meant to fix. We now demonstrate quantitatively that with this model element the risk premium not only can become larger than in the RA economy, but that for sufficiently large idiosyncratic risk the model can account for a significant share of the empirically observed premium.

To do so, we calibrate the idiosyncratic shocks such that the fraction of the working population (i.e., high-income agents) is $2/3$ and the auto-correlation of idiosyncratic productivity shocks is 0.9 . This gives $(\xi, \nu) = (0.066, 0.033)$. Table 2 presents basic asset pricing statistics for the economy with stochastic depreciation, and the top left panel of Figure 5 displays how the premium varies with the size of the depreciation shocks, in two parameterizations of the limited commitment economy that differ in the extent of idiosyncratic risk²⁴ as well as in the RA economy. The domain of the shocks is restricted such that Assumption 8 is always satisfied.

The remaining panels then decompose the multiplicative risk premium

$$\lambda_t = -Cov(R_{t+1}, \beta c_t/c_{t+1}) = -\beta \text{std}(R_{t+1}) \text{std}(c_t/c_{t+1}) \text{corr}(R_{t+1}, c_t/c_{t+1}) \quad (96)$$

into its components. Here c_t, c_{t+1} refers to consumption of the representative agent (i.e., aggregate consumption) in the RA economy, and consumption of the capital owners (the

²⁴The baseline calibration gives $(\xi, \nu) = (0.066, 0.033)$. The variance of idiosyncratic shocks is ξ/ν , so higher ξ (e.g., $\xi = 0.99$) implies larger idiosyncratic risk. The largest depreciation shocks ς satisfying Assumption 8 is 0.042 with $(\xi, \nu) = (0.066, 0.033)$ and 0.155 with $(\xi, \nu) = (0.99, 0.033)$.

	$\xi = 0.066$		$\xi = 0.99$			Data
	$\varsigma = 0$	$\varsigma = 0.04$	$\varsigma = 0$	$\varsigma = 0.045$	$\varsigma = 0.15$	
$\mathbb{E}[R_{t+1}] - 1$ (%)	16.2	16.2	2.9	2.9	2.9	7%
$1/q_t(\Omega_t) - 1$ (%)	16.2	16.1	2.9	2.7	0.7	1%
λ_t (%)	0.00023	0.119	0.00007	0.191	2.17	6%
$\text{std}(R_{t+1})$	0.002	0.04	0.001	0.045	0.15	0.17
$\text{std}\left(\frac{C_{t+1}}{C_t}\right)$	0.005	0.014	0.002	0.035	0.117	0.036
$\text{corr}\left(R_{t+1}, \frac{C_{t+1}}{C_t}\right)$	1.0	0.954	1.0	0.999	1.0	0.371
$\text{corr}\left(R_{t+1}, \frac{w_{t+1}}{w_t}\right)$	1.0	0.044	1.0	0.019	0.006	–

Table 2: Moments of the return on risky capital (R_{t+1}), risk-free rate ($1/q_t$), aggregate consumption growth (C_{t+1}/C_t) in the limited-commitment model and in the data. The data on $\text{std}(R_{t+1})$, $\text{std}\left(\frac{C_{t+1}}{C_t}\right)$, $\text{corr}\left(R_{t+1}, \frac{C_{t+1}}{C_t}\right)$ are computed based on Table 14.3.1 in Ljungqvist and Sargent (2018). Parameters are $(\beta, \nu, \bar{\delta}, \theta, \epsilon) = (0.8, 0.033, 0.1, 0.33, 0.01)$. The productivity growth rate and depreciation rate are independent and can take two values with probability $1/2$. The economy is in steady state at t .

unconstrained, low labor productivity individuals) in the limited commitment economy. The figure shows that the key for raising the risk premium in the limited commitment model (relative to the representative agent economy, and towards an empirically plausible value) is to raise the volatility of consumption growth of the capital owners (see the lower left panel of Figure 5), which in turn requires the depreciation shocks (and thus the volatility of gross capital returns) to be sizeable (see the upper right panel of Figure 5). With large depreciation shocks and large idiosyncratic risk (second to last column of Table 2) the model generates a risk premium of more than two percent (see the third row of Table 2), whereas the corresponding RA economy displays a premium of only 0.7 percent. Crucially, it does so with an empirically plausible volatility of gross capital returns (see the fourth row of the table). In the absence of stochastic depreciation the premium in both economies is close to zero (but larger in the RA economy, as discussed in the previous section). These calculations are meant to be illustrative of the potential of our model to generate sizeable risk premia for owning capital, rather than provide a full quantification, which in turn would require an extension to more than two idiosyncratic productivity states (at the expense of analytical tractability of the model).

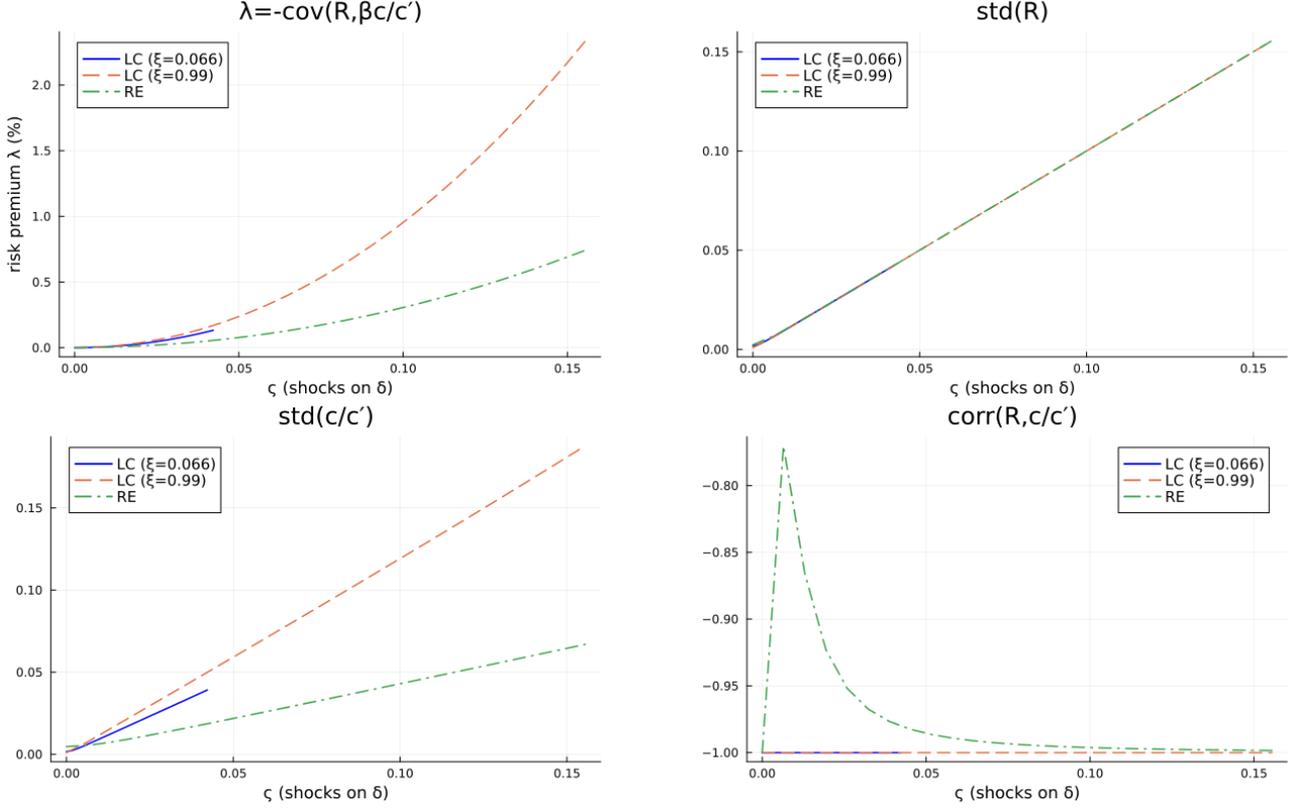


Figure 5: Decomposition of Risk Premium $\lambda = -\beta \text{std}(R) \text{std}(c/c') \text{corr}(R, c/c')$. Parameter values are $(\beta, \nu, \bar{\delta}, \theta, \epsilon) = (0.8, 0.033, 0.1, 0.33, 0.01)$.

6 Where Does the Tractability of the Model Come From?

Equipped with the theoretical results from our model we are now in a position to more succinctly relate the model results to the most related contributions in the literature. Recall that we obtain a closed form for the steady state, the transitional dynamics of aggregate capital after an MIT shock as well as the law of motion for this capital stock even in the presence of aggregate productivity shocks, despite the fact that the economy features idiosyncratic risk that is only partially insurable. This is a surprising result given that a neoclassical growth model without a commitment constraint does not have a closed-form solution unless full depreciation is assumed. We now relate our model to other well-studied versions of general equilibrium models with neoclassical production. To do so, recall that the equilibrium law of motion in our model is given by

$$K_{t+1} = \hat{s}Y_t + (1 - \hat{\delta})K_t \quad \text{where} \quad (97)$$

$$\hat{s} = \theta(1 - \nu)\beta + \frac{(1 - \theta)\xi\beta}{1 - (1 - \nu - \xi)\beta} \approx \theta(1 - \nu - \rho) + (1 - \theta) \left[\frac{\xi}{\xi + \nu + \rho} \right] \quad (98)$$

$$\hat{\delta} = 1 - (1 - \nu)\beta(1 - \delta) \approx \nu + \rho + \delta. \quad (99)$$

where the aggregate saving rate \hat{s} is a weighted average of the saving rate $(1 - \nu)\beta$ of capital owners out of their capital income (which is a share θ of total income $Y_t = (K_t)^\theta (A_t)^{1-\theta}$) and the saving rate $\frac{\xi\beta}{1-(1-\nu-\xi)\beta}$ of productive workers out of their labor income (a share $1 - \theta$ of aggregate income).

In this section we demonstrate three results. First, we show that an economy with neoclassical production and two types of households, hand-to-mouth workers that earn all labor income and consume it all in every period, and capitalists that own the capital stock and finance all consumption from capital income, has a law of motion similar to (97) but a different (still constant) saving rate and effective depreciation rate. Second, as the transition probabilities of income shocks approach zero: $\xi \rightarrow 0$ and $\nu \rightarrow 0$, the equilibrium law of motion in our economy converges to that of the two-agent economy. This is the content of Section 6.1 and allows us to relate our model to that of Moll (2014). Third, in Section 6.2 we explain why the limit of our model (as $\xi, \nu \rightarrow 0$) is not the standard neoclassical growth model (which does not in general have a constant aggregate saving rate), unless there is full depreciation (the case we already analyzed in Section 4.2.1).

6.1 An Economy with Capitalists and Workers

Consider a two-agent (henceforth abbreviated 2A) neoclassical growth economy with a representative worker and a representative capital owner. A variant of such a model, albeit in continuous time, was studied in Moll (2014), with similar structures appearing in Hornstein and Uhlig (2000) and Danthine and Donaldson (2002) as well.²⁵ A hand-to-mouth worker earns labor income and consumes all of this income in each period, by assumption. A capital owner receives only capital income, solves a dynamic consumption saving problem, whose solution is to consume a $1 - \beta$ fraction of her gross capital income, the optimal consumption-saving rule with log-utility. Therefore, consumption of both groups is given by:

$$C^{worker} = w_t, \quad (100)$$

$$C^{owner} = (1 - \beta)R_t K_t. \quad (101)$$

and the goods market clearing condition in this economy can be written as

$$\begin{aligned} K_{t+1} &= A_t^{1-\theta} K_t^\theta + (1 - \delta)K_t - C^{worker} - C^{owner} \\ &= A_t^{1-\theta} K_t^\theta + (1 - \delta)K_t - (1 - \theta)A_t^{1-\theta} K_t^\theta - (1 - \beta)[\theta A_t^{1-\theta} K_t^\theta + (1 - \delta)K_t] \\ &= \beta\theta A_t^{1-\theta} K_t^\theta + \beta(1 - \delta)K_t = \hat{s}^{2A} Y_t + (1 - \hat{\delta}^{2A})K_t \end{aligned} \quad (102)$$

²⁵The most relevant version of Moll (2014), for the purpose of this paper, is contained in an online appendix, available here: <https://benjaminmoll.com/wp-content/uploads/2019/07/capitalists-workers.pdf>.

This aggregate law of motion coincides qualitatively with equation (97) for our limited commitment model, and furthermore, as can readily be verified from equations (98) and (99), $\lim_{\xi, \nu \rightarrow 0} \hat{s} = \hat{s}^{2A}$ as well as $\lim_{\xi, \nu \rightarrow 0} \hat{\delta} = \hat{\delta}^{2A}$. When we take the limit, we assume that the ratio $\kappa := \xi/\nu$ remains fixed so that the share of low (and high) productivity individuals $\psi_l = \frac{\xi}{\xi+\nu} = \frac{\kappa}{\kappa+1}$ remains well-defined and constant and aggregate labor input $L = \frac{\nu\zeta}{\xi+\nu} = \frac{\zeta}{\kappa+\nu}$ remains at 1. The aggregate law of motion also coincides with the original Solow model, but in both models studied here the saving rate is not some exogenous behavioral constant s , but rather a function of the deep technology and preference parameters of the model.

This result also clarifies the key model elements that leads to an aggregate law of motion that can be stated in closed form, as pointed out by Moll (2014): (i) the separation of individuals into “workers” and “capitalists”, (ii) that the period utility of the savers (the capitalists) is logarithmic and (iii) that workers cannot or do not save.²⁶

In our limited commitment model, the equilibrium allocation has a similar structure to the two-agent model discussed above. Workers (high-productivity individuals) do save, but at a constant rate, and all capitalists (low-productivity individuals) also have the same saving rate

$$C^{worker} = \phi_0 \mathbf{c}_{0,t} = \phi_0 \frac{(1 - (1 - \nu)\beta)\zeta}{1 - (1 - \nu - \xi)\beta} w_t, \quad (103)$$

$$C^{owner} = \sum_{s \geq 1} \phi_s \mathbf{c}_{s,t} = [1 - (1 - \nu)\beta] R_t \sum_{s \geq 1} \phi_s \mathbf{a}_{s,t} = [1 - (1 - \nu)\beta] R_t K_t. \quad (104)$$

and thus the same aggregation result as in the two agent model obtains, but with different effective aggregate saving and depreciation rate ($\hat{s}, \hat{\delta}$) (which converge to the saving and depreciation rates in the two agent as idiosyncratic risk vanishes). The key difference to Moll (2014)’s analysis is that in our model the identity of workers and capitalists is not fixed but evolves stochastically with the realization of idiosyncratic productivity z_t .

6.2 The Standard Neoclassical Growth Model

In general, the aggregate law of motion of capital in the standard neoclassical growth model without idiosyncratic risk does not have a closed-form solution (unless $\delta = 1$, see Section 4.2.1) because the representative agent has two sources of income, both wage

²⁶Note that if the model is cast in continuous time, then the corresponding law of motion forms a Bernoulli differential equation which has an explicit solution (that is, the equilibrium capital path can be given in closed form), as does the original Solow model, as pointed out by Jones (2000). His unpublished note is available here: <https://web.stanford.edu/~chadj/closedform.pdf>. We exploit this result in the continuous time limited commitment model of Krueger, Li, and Uhlig (2023). Unless $\delta = 1$, no such explicit solution of the first-order nonlinear difference equation (102) is available in discrete time.

income and capital income, and typically will not consume (and save) a constant fraction of that aggregate income.

The results in the previous section might thus appear puzzling, since our model without idiosyncratic risk is precisely the representative agent neoclassical growth model.²⁷ However, it is important to note that the equilibrium allocation in our model discussed in the previous section was derived under the no-savings assumption (Assumption 3 for the general case and Assumption 5 for the steady state), and this assumption is in general violated for the standard neoclassical growth model (whose steady state will have $\beta R = 1$ rather than $\beta R < 1$). But if βR were equal to 1, then also in our limited commitment economy in the stationary equilibrium all households consume the same, the consumption distribution is degenerate and the model indeed collapses to the standard neoclassical growth model (see Krueger and Uhlig (2022) for the knife-edge conditions under which this case emerges).²⁸

7 Conclusion

In this paper we have developed an analytically tractable macroeconomic model with idiosyncratic risk and endogenously incomplete market cast in discrete time, and have derived its macroeconomic, distributional and asset pricing implications. In our environment individuals can trade a full set of state-contingent claims, as in the standard complete markets model, but face tight shortsale constraints in that they cannot borrow, as typically the case in the standard incomplete markets model. Thus, we have hoped to supply a natural hybrid alternative to both these benchmark models in which, despite featuring a nondegenerate income-, consumption- and wealth distribution, all relevant

²⁷Appendix B.3.2 discusses this point in greater detail. It shows that both the standard representative agent model without any heterogeneity and a representative-agent like model with $\nu = \xi = 0$ (but well-defined $\kappa = \xi/\nu \in (0, \infty)$) in which there are two permanent productivity types that face no idiosyncratic risk and can freely borrow (recall in our model they cannot) satisfy the standard representative agent Euler equation for aggregate consumption (and the aggregate resource constraint). These two equations do not result in a closed-form aggregate law of motion for capital of the form (97), as is the case in our model, unless there is full depreciation, $\delta = 1$.

²⁸One might then question whether no-saving condition can be satisfied at all if ν, ξ are close to zero. Focusing on steady state, rewrite Assumption 5 as:

$$\frac{\theta}{(1 - \theta) \left(\frac{1}{\beta} - 1 + \delta \right)} < \frac{\xi/\nu}{\frac{1}{\beta} - 1 + \nu(\xi/\nu + 1)} = \frac{\kappa}{\frac{1}{\beta} - 1 + \nu(\kappa + 1)}$$

As long as the relative hazard rate $\kappa = \xi/\nu$ remains constant and is sufficiently large (that, there is "enough" idiosyncratic income risk) as $\nu, \xi \rightarrow 0$, then Assumption 5 remains satisfied, making the limit studied in the previous section meaningful. The same is true along a transition path as long as it starts from a capital stock at or above the steady state capital and the productivity shock is not too large.

model properties can be characterized theoretically, including the evolution of inequality over the business cycle and the risk premium on holding capital.

Future work using this general partial insurance general equilibrium framework needs to establish the applied and empirical relevance of the model. On the applied policy side, in the current framework the only asset available to be traded by the private sector is productive physical capital. The model lends itself naturally to the analysis of government debt policy (and fiscal policy more generally). Given the focus on partial private insurance of idiosyncratic risk in the current paper, a natural question is whether the provision of better public insurance, either in the direct form of progressive income taxation, or indirectly, by expanding government debt, improves risk sharing or crowds out private insurance so strongly as to potentially be counter-productive, as in the work by Golosov and Tsyvinski (2007), Krueger and Perri (2011) or Park (2014).

On the empirical side, the model contains sharp predictions for the change of the consumption distribution in the presence of aggregate productivity shocks, and more generally, for the change in the joint distribution of income, consumption and wealth in response to aggregate shocks affecting wages and rates of return to capital. Of particular interest are shocks that impact wages (and thus workers) and interest rates (and thus capital owners) asymmetrically. We leave evaluating these predictions to future work.

References

- Ábrahám, Árpád and Sarolta Laczó. 2018. “Efficient risk sharing with limited commitment and storage.” *The Review of Economic Studies* 85 (3):1389–1424.
- Achdou, Yves, Jiequn Han, Jean-Michel Lasry, Pierre-Louis Lions, and Benjamin Moll. 2022. “Income and wealth distribution in macroeconomics: A continuous-time approach.” *The Review of Economic Studies* 89 (1):45–86.
- Aiyagari, S Rao. 1994. “Uninsured idiosyncratic risk and aggregate saving.” *The Quarterly Journal of Economics* 109 (3):659–684.
- Aiyagari, S Rao and Ellen R McGrattan. 1998. “The optimum quantity of debt.” *Journal of Monetary Economics* 42 (3):447–469.
- Alvarez, Fernando and Urban J Jermann. 2000. “Efficiency, equilibrium, and asset pricing with risk of default.” *Econometrica* 68 (4):775–797.
- Arellano, Manuel, Richard Blundell, and Stéphane Bonhomme. 2017. “Earnings and consumption dynamics: a nonlinear panel data framework.” *Econometrica* 85 (3):693–734.
- Balke, Neele and Thibaut Lamadon. 2022. “Productivity Shocks, Long-Term Contracts, and Earnings Dynamics.” *American Economic Review* 112 (7):2139–77.
- Barro, R and Xavier Sala-i Martin. 2004. *Economic growth second edition*. MIT Press.
- Bewley, Truman. 1986. “Stationary monetary equilibrium with a continuum of independently fluctuating consumers.” *Contributions to mathematical economics in honor of Gérard Debreu* :79–102.
- Blanchard, Olivier. 1985. “Debt, deficits, and finite horizons.” *Journal of Political Economy* 93:223–247.
- Blundell, Richard, Luigi Pistaferri, and Ian Preston. 2008. “Consumption inequality and partial insurance.” *American Economic Review* 98 (5):1887–1921.
- Braxton, J Carter, Kyle F Herkenhoff, Jonathan L Rothbaum, and Lawrence Schmidt. 2021. “Changing income risk across the US skill distribution: Evidence from a generalized Kalman filter.” *National Bureau of Economic Research Working Paper* 29567.
- Broer, Tobias. 2013. “The wrong shape of insurance? What cross-sectional distributions tell us about models of consumption smoothing.” *American Economic Journal: Macroeconomics* 5 (4):107–140.

- . 2020. “Consumption insurance over the business cycle.” *Working Paper* .
- Chatterjee, Arpita, James Morley, and Aarti Singh. 2021. “Estimating household consumption insurance.” *Journal of Applied Econometrics* 36 (5):628–635.
- Cole, Harold, Dirk Krueger, George Mailath, and Yena Park. 2023. “Trust in Risk Sharing: A Double-Edged Sword.” *Review of Economic Studies* forthcoming.
- Commault, Jeanne. 2022. “Does Consumption Respond to Transitory Shocks?” *American Economic Journal: Macroeconomics* 14 (2):96–122.
- Danthine, Jean-Pierre and John Donaldson. 2002. “Labour relations and asset returns.” *Review of Economic Studies* 69 (1):41–64.
- Eika, Lasse, Magne Mogstad, and Ola L Vestad. 2020. “What can we learn about household consumption expenditure from data on income and assets?” *Journal of Public Economics* 189:104163.
- Golosov, Mikhail and Aleh Tsyvinski. 2007. “Optimal taxation with endogenous insurance markets.” *The Quarterly Journal of Economics* 122 (2):487–534.
- Gottardi, Piero and Felix Kubler. 2015. “Dynamic competitive economies with complete markets and collateral constraints.” *The Review of Economic Studies* 82 (3):1119–1153.
- Hellwig, Christian and Guido Lorenzoni. 2009. “Bubbles and self-enforcing debt.” *Econometrica* 77 (4):1137–1164.
- Hornstein, Andreas and Harald Uhlig. 2000. “What is the real story for interest rate volatility?” *German Economic Review* 1 (1):43–67.
- Huggett, Mark. 1993. “The risk-free rate in heterogeneous-agent incomplete-insurance economies.” *Journal of Economic Dynamics and Control* 17 (5-6):953–969.
- Imrohoroğlu, Ayşe. 1989. “Cost of business cycles with indivisibilities and liquidity constraints.” *Journal of Political Economy* 97 (6):1364–1383.
- Kaplan, Greg, Benjamin Moll, and Giovanni L Violante. 2018. “Monetary policy according to HANK.” *American Economic Review* 108 (3):697–743.
- Kaplan, Greg and Giovanni L Violante. 2010. “How much consumption insurance beyond self-insurance?” *American Economic Journal: Macroeconomics* 2 (4):53–87.
- Kehoe, Timothy J and David K Levine. 1993. “Debt-constrained asset markets.” *The Review of Economic Studies* 60 (4):865–888.

- . 2001. “Liquidity constrained markets versus debt constrained markets.” *Econometrica* 69 (3):575–598.
- Kocherlakota, Narayana R. 1996. “Implications of efficient risk sharing without commitment.” *The Review of Economic Studies* 63 (4):595–609.
- . 2023. “Public debt bubbles in heterogeneous agent models with tail risk.” *International Economic Review* 64 (2):491–509.
- Krueger, Dirk and Felix Kubler. 2006. “Pareto-improving social security reform when financial markets are incomplete!?” *American Economic Review* 96 (3):737–755.
- Krueger, Dirk, Fulin Li, and Harald Uhlig. 2023. “Neoclassical Growth Transition Dynamics with One-Sided Commitment.” *Manuscript* .
- Krueger, Dirk, Alexander Ludwig, and Sergio Villalvazo. 2021. “Optimal taxes on capital in the OLG model with uninsurable idiosyncratic income risk.” *Journal of Public Economics* 201:104491.
- Krueger, Dirk and Hanno Lustig. 2010. “When is market incompleteness irrelevant for the price of aggregate risk (and when is it not)?” *Journal of Economic Theory* 145 (1):1–41.
- Krueger, Dirk, Hanno Lustig, and Fabrizio Perri. 2008. “Evaluating asset pricing models with limited commitment using micro consumption data.” *Journal of the European Economic Association Papers and Proceedings* 6 (2-3):715–726.
- Krueger, Dirk and Fabrizio Perri. 2006. “Does income inequality lead to consumption inequality? Evidence and theory.” *The Review of Economic Studies* 73 (1):163–193.
- . 2011. “Public versus private risk sharing.” *Journal of Economic Theory* 146 (3):920–956.
- Krueger, Dirk and Harald Uhlig. 2006. “Competitive risk sharing contracts with one-sided commitment.” *Journal of Monetary Economics* 53 (7):1661–1691.
- . 2022. “Neoclassical growth with long-term one-sided commitment contracts.” *National Bureau of Economic Research Working Paper* 30518.
- Krueger, Dirk and Chunzan Wu. 2021. “Consumption insurance against wage risk: Family labor supply and optimal progressive income taxation.” *American Economic Journal: Macroeconomics* 13 (1):79–113.
- Krusell, Per. 2014. “Real macroeconomic theory.” *Manuscript* .

- Krusell, Per and Anthony A Smith, Jr. 1998. "Income and wealth heterogeneity in the macroeconomy." *Journal of Political Economy* 106 (5):867–896.
- Ljungqvist, Lars and Thomas J Sargent. 2018. *Recursive macroeconomic theory*. MIT Press.
- Martins-da Rocha, V Filipe and Mateus Santos. 2019. "Self-enforcing debt and rational bubbles." Available at SSRN: <https://ssrn.com/abstract=3169229> .
- Moll, Benjamin. 2014. "Productivity losses from financial frictions: Can self-financing undo capital misallocation?" *American Economic Review* 104 (10):3186–3221.
- Park, Yena. 2014. "Optimal taxation in a limited commitment economy." *The Review of Economic Studies* 81 (2):884–918.
- Smetters, Kent. 2003. "Trading with the unborn: A new perspective on capital income taxation."
- Thomas, Jonathan and Tim Worrall. 1988. "Self-enforcing wage contracts." *The Review of Economic Studies* 55 (4):541–554.
- Yaari, Menahem. 1965. "Uncertain lifetime, life insurance, and the theory of the consumer." *The Review of Economic Studies* 32 (1):137–150.

Appendix

A Proofs of Propositions

A.1 Proofs: Section 3 (Characterization of Equilibrium)

A.1.1 Proof of Proposition 1

We prove Proposition 1 in several steps. First, we propose a candidate optimal consumption and asset allocation. We then show in a sequence of steps that this proposed allocation is indeed an optimal choice of the household. To do so, Lemma 3 derives the Kuhn-Tucker conditions for the household optimization problem (11)–(12) for given a sequence of prices $\{R_t(A^t), w_t(A^t), q_t(A_{t+1}, z_{t+1}|A^t, z^t)\}_{t \geq 0, A^t, A_{t+1}, z^t, z_{t+1}}$. Then, Lemmas 4 and 5 show that the proposed allocation satisfies the household's budget constraint and the Kuhn-Tucker conditions. Finally, Proposition 1 shows that since the proposed allocation satisfies the Kuhn-Tucker conditions and a transversality condition, it is an optimal choice of the maximization problem.

We conjecture that, under the maintained assumptions on prices stipulating sufficiently low interest rates/sufficiently high wage growth, individuals have no incentives to save for the high-income state tomorrow, and for the low-income state tomorrow consumption and asset choices are governed by a standard complete-markets Euler equation. That is, we conjecture that the optimal household consumption-asset choice is given by:

$$\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) = \begin{cases} 0 & \text{if } z_{t+1} = \zeta \\ \frac{\beta}{1-(1-\nu-\xi)\beta} \zeta w_t(A^t) & \text{if } z_t = \zeta \text{ and } z_{t+1} = 0 \\ \beta R_t(A^t) \mathbf{a}_t(a_0, z^t, A^t) & \text{if } z_t = 0 \text{ and } z_{t+1} = 0 \end{cases} \quad (22)$$

$$\mathbf{c}_t(a_0, z^t, A^t) = \begin{cases} w_t(A^t) c_0, \text{ where } c_0 := \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta} \zeta, & \text{if } z_t = \zeta \\ [1 - (1 - \nu)\beta] R_t(A^t) \mathbf{a}_t(a_0, z^t, A^t) & \text{if } z_t = 0 \end{cases} \quad (21)$$

where $\mathbf{a}_0(a_0, z^0, A^0) = w_0(A^0) a_0$ are the initial asset holdings of the household (an exogenous initial condition).

It is straightforward to verify that (22) and (21) imply that under the proposed allocation for currently low-income individuals ($z_t = 0$) the standard complete markets Euler equation for consumption holds (a fact that will be useful below for some of the derivations):

$$\mathbf{c}_t(a_0, z^t, A^t) = \beta R_t(A^t) \mathbf{c}_{t-1}(a_0, z^{t-1}, A^{t-1}). \quad (23)$$

To see this consider first an individual with $z_{t-1} = 0$. Then

$$\begin{aligned}\mathbf{a}_t(a_0, z^t, A^t) &= \beta R_{t-1}(A^{t-1}) \mathbf{a}_{t-1}(a_0, z^{t-1}, A^{t-1}) \\ &= \beta R_{t-1}(A^{t-1}) \frac{\mathbf{c}_{t-1}(a_0, z^{t-1}, A^{t-1})}{[1 - (1 - \nu)\beta] R_{t-1}(A^{t-1})},\end{aligned}$$

where the first line follows from equation (22), while the second line stems from equation (21). Therefore

$$\begin{aligned}\mathbf{c}_t(a_0, z^t, A^t) &= [1 - (1 - \nu)\beta] R_t(A^t) \mathbf{a}_t(a_0, z^t, A^t) \\ &= \beta R_t(A^t) \mathbf{c}_{t-1}(a_0, z^t, A^t).\end{aligned}$$

Now consider an individual with $z_{t-1} = \zeta$. Then

$$\begin{aligned}\mathbf{a}_t(a_0, z^t, A^t) &= \frac{\beta}{1 - (1 - \nu - \xi)\beta} \zeta w_{t-1}(A^{t-1}) \\ &= \frac{\beta}{1 - (1 - \nu)\beta} \mathbf{c}_{t-1}(a_0, z^{t-1}, A^{t-1}) \\ \therefore \mathbf{c}_t(a_0, z^t, A^t) &= [1 - (1 - \nu)\beta] R_t(A^t) \mathbf{a}_t(a_0, z^t, A^t) \\ &= \beta R_t(A^t) \mathbf{c}_{t-1}(a_0, z^t, A^t).\end{aligned}$$

We now derive the Kuhn-Tucker condition for the household maximization problem in the following Lemma.

Lemma 3. *Given a sequence of prices $\{R_t(A^t), w_t(A^t), q_t(A_{t+1}, z_{t+1}|A^t, z^t)\}_{t \geq 0, A^t, A_{t+1}}$, FOCs to the household's optimization problem (11)–(12) give the following Kuhn-Tucker condition:*

$$\frac{\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1})}{\mathbf{c}_t(a_0, z^t, A^t)} = \frac{\pi(A_{t+1}|A^t)\pi(z_{t+1}|z_t)}{q_t(A_{t+1}, z_{t+1}|A^t, z^t)} [\beta R_{t+1}(A^{t+1}) + \lambda(a_0, z^{t+1}, A^{t+1}) \mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1})] \quad (105)$$

$$\text{with } \lambda(a_0, z^{t+1}, A^{t+1}) \mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) = 0, \lambda(a_0, z^{t+1}, A^{t+1}) \geq 0, \mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) \geq 0, \quad (106)$$

where $\lambda(a_0, z^{t+1}, A^{t+1})$ denotes a Lagrangian multiplier for a shortsale constraint at state (z^{t+1}, A^{t+1}) .

Proof. Households' Lagrangian problem is given by:

$$\begin{aligned}
U(a_0, z_0) = & \max_{\{\mathbf{c}_t(a_0, z^t, A^t), \mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{A^t} \sum_{z^t} \beta^t \pi(A^t) \pi(z^t) \log(\mathbf{c}_t(a_0, z^t, A^t)) \\
& + \sum_{t=0}^{\infty} \sum_{A^t} \sum_{z^t} \mu(z^t, A^t) \left[w_t(A^t) z_t + R_t(A^t) \mathbf{a}_t(a_0, z^t, A^t) \right. \\
& \quad \left. - \mathbf{c}_t(a_0, z^t, A^t) - \sum_{A_{t+1}} \sum_{z_{t+1}} q_t(A_{t+1}, z_{t+1} | A^t, z^t) \mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) \right] \\
& + \sum_{t=0}^{\infty} \sum_{A^{t+1}} \sum_{z^{t+1}} \beta^t \pi(A^{t+1}) \pi(z^{t+1}) \lambda(a_0, z^{t+1}, A^{t+1}) \mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}), \tag{107}
\end{aligned}$$

where $\mu(z^t, A^t)$ and $\lambda(a_0, z^{t+1}, A^{t+1})$ are Lagrangian multipliers for budget constraints and shortsale constraints. FOCs with respect to $\mathbf{c}_t(a_0, z^t, A^t)$ and $\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1})$ are:

$$[\mathbf{c}_t(a_0, z^t, A^t)] : \beta^t \pi(A^t) \pi(z^t) \frac{1}{\mathbf{c}_t(a_0, z^t, A^t)} = \mu(z^t, A^t) \tag{108}$$

$$\begin{aligned}
[\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1})] : & \mu(z^{t+1}, A^{t+1}) R_{t+1}(A^{t+1}) + \beta^t \pi(A^{t+1}) \pi(z^{t+1}) \lambda(a_0, z^{t+1}, A^{t+1}) \\
& = \mu(z^t, A^t) q_t(A_{t+1}, z_{t+1} | A^t, z^t) \tag{109}
\end{aligned}$$

By substituting $\mu(z^t, A^t)$, we obtain the following Kuhn-Tucker condition:

$$\frac{1}{\mathbf{c}_t(a_0, z^t, A^t)} = \frac{\pi(A_{t+1} | A^t) \pi(z_{t+1} | z_t)}{q_t(A_{t+1}, z_{t+1} | A^t, z^t)} \left[\beta R_{t+1}(A^{t+1}) \frac{1}{\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1})} + \lambda(a_0, z^{t+1}, A^{t+1}) \right] \tag{110}$$

where $\lambda(a_0, z^{t+1}, A^{t+1}) \mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) = 0$, $\lambda(a_0, z^{t+1}, A^{t+1}) \geq 0$, $\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) \geq 0$.

Here we use the conditional probability: $\pi(A_{t+1} | A^t) = \frac{\pi(A^{t+1})}{\pi(A^t)}$, where $A^{t+1} = (A^t, A_{t+1})$, and $\pi(z_{t+1} | z^t) = \frac{\pi(z^{t+1})}{\pi(z^t)}$, where $z^{t+1} = (z^t, z_{t+1})$. Because the idiosyncratic shocks follow Markov, only the current state z_t matters for the probability of z_{t+1} . Hence, $\pi(z_{t+1} | z^t) = \pi(z_{t+1} | z_t)$. Since $\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1})$ takes non-zero value (otherwise the utility would be negative infinite), (110) can be expressed as:

$$\frac{\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1})}{\mathbf{c}_t(a_0, z^t, A^t)} = \frac{\pi(A_{t+1} | A^t) \pi(z_{t+1} | z_t)}{q_t(A_{t+1}, z_{t+1} | A^t, z^t)} [\beta R_{t+1}(A^{t+1}) + \lambda(a_0, z^{t+1}, A^{t+1}) \mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1})]$$

□

The next two lemmas show that the conjectured allocation satisfies the budget constraint and the Kuhn-Tucker condition for a future low-income state.

Lemma 4. *Suppose Assumption 2 on contingent claim prices is satisfied. Then, the allocation defined in equations (21) and (22) satisfies the household's budget constraint (10)*

and the Euler equation between the current state and a future low-income state (and hence the Kuhn-Tucker condition 105):

$$\frac{1}{\mathbf{c}_t(a_0, z^t, A^t)} = \beta R_{t+1}(A^{t+1}) \frac{1}{\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1}; z_{t+1} = 0)} \text{ for all } A_{t+1}. \quad (111)$$

Proof. We first check the budget constraint. In a high-income state ($z_t = \zeta$), substituting the conjectured consumption and asset choice (21) and (22) into equation (10) gives:

$$\begin{aligned} w_t(A^t)c_0 + \sum_{A_{t+1}} \sum_{z_{t+1}} q_t(A_{t+1}, z_{t+1}|A^t, z^t) \frac{\beta}{1 - (1 - \nu - \xi)\beta} \zeta w_t(A^t) &= w_t(A^t)\zeta \\ \therefore w_t(A^t) \left[\frac{1 - (1 - \nu)\beta}{1 - (1 - \nu - \xi)\beta} \zeta + \sum_{A_{t+1}} \pi(A_{t+1}|A^t) \underbrace{\pi(z_{t+1} = 0|z_t = \zeta)}_{=\xi} \frac{\beta}{1 - (1 - \nu - \xi)\beta} \zeta \right] &= w_t(A^t)\zeta \end{aligned}$$

where the second line uses Assumption 2: $q_t(A_{t+1}, z_{t+1}|A^t, z^t) = \pi(A_{t+1}|A^t)\pi(z_{t+1}|z_t)$.

The equality holds since $\sum_{A_{t+1}} \pi(A_{t+1}|A^t) = 1$.

In a low-income state ($z_t = 0$), equation (10) becomes:

$$\begin{aligned} [1 - (1 - \nu)\beta]R_t(A^t)\mathbf{a}_t(z^t, A^t) &+ \sum_{A_{t+1}} \sum_{z_{t+1}} \pi(A_{t+1}|A^t) \underbrace{\pi(z_{t+1} = 0|z_t = 0)}_{=1-\nu} \beta R_t(A^t)\mathbf{a}_t(z^t, A^t) = R_t(A^t)\mathbf{a}_t(z^t, A^t) \\ \Leftrightarrow R_t(A^t)\mathbf{a}_t(z^t, A^t) - (1 - \nu)\beta \left[1 - \sum_{A_{t+1}} \pi(A_{t+1}|A^t) \right] R_t(A^t)\mathbf{a}_t(z^t, A^t) &= R_t(A^t)\mathbf{a}_t(z^t, A^t) \end{aligned}$$

This holds with equality since $\sum_{A_{t+1}} \pi(A_{t+1}|A^t) = 1$.

Second, we examine the Euler equation for low-income households.²⁹ The condition on Lagrange multipliers (106) implies $\lambda(a_0, z^{t+1}, A^{t+1}) = 0$, since $\lambda(a_0, z^{t+1}, A^{t+1})\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) = 0$ and $\mathbf{a}_{t+1}(a_0, z^{t+1}, A^{t+1}) > 0$ in the proposed allocation. Under $q_t(A_{t+1}, z_{t+1}|A^t, z^t) = \pi(A_{t+1}|A^t)\pi(z_{t+1}|z_t)$, the Kuhn-Tucker condition (105) is given by:

$$\frac{1}{\mathbf{c}_t(a_0, z^t, A^t)} = \beta R_{t+1}(A^{t+1}) \frac{1}{\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1}; z_{t+1} = 0)}.$$

\mathbf{c}_{t+1} at a low-income state given by (21) satisfies this Euler equation. \square

The claim that households make no savings for high-income states could be shown by induction.³⁰ Here, we instead show that under Assumption 3: $\beta R_{t+1}(A^{t+1}) < \frac{w_{t+1}(A^{t+1})}{w_t(A^t)}$ at all $t \geq 0$ and A^{t+1} , the Kuhn-Tucker conditions for high-income states are satisfied if households do not save for high-income states.

²⁹Since households always save for a low-income state, the Euler equation holds with equality. If households enter a low-income state with zero assets, their period utility would be negative infinite.

³⁰At $t = 0$, given an initial state ($a_0 = 0, z_0 = \zeta$) or ($a_0 \leq \bar{a}_0 := \frac{\beta}{1 - (1 - \nu - \xi)\beta} \zeta, z_0 = 0$), households cannot achieve higher utility at $t = 1$ by saving for a high-income state at $t = 1$ if $\beta R_1 \frac{w_0}{w_1} < 1$. This is because the Euler equation implies that consumption at $t = 1$ would be lower than consumption c_0 in the

Lemma 5. *Suppose Assumption 2 on contingent claim prices is satisfied and suppose that the sequence of wages and interest rates $\{w_t(A^t), R_t(A^t)\}_{t=0}^\infty$ satisfies the no-savings Assumption 3 and that the initial wealth distribution satisfies Assumption 4. Then, the allocation defined in equations (21) and (22) satisfies the Kuhn-Tucker conditions for high-income states at any time $t \geq 0$. It also implies equation (25):*

$$\mathbf{c}_{t+1}(a_0, z^{t+1}, A^{t+1}) > \beta R_{t+1}(A^{t+1}) \mathbf{c}_t(a_0, z^t, A^t) \text{ if } z_{t+1} = \zeta.$$

Proof. We check the Kuhn-Tucker conditions, (105) and (106), for a high-income state ($z_{t+1} = \zeta$) under Assumption 2, $q_t(A_{t+1}, z_{t+1}|A^t, z^t) = \pi(A_{t+1}|A^t)\pi(z_{t+1}|z_t)$, and Assumption 4, $\beta R_{t+1}(A^{t+1}) < \frac{w_{t+1}(A^{t+1})}{w_t(A^t)}$. Substituting $\mathbf{c}_{t+1}(z^{t+1}, A^{t+1}; z_{t+1} = \zeta) = w_{t+1}(A^{t+1})c_0$ and $\mathbf{a}_{t+1}(z^{t+1}, A^{t+1}) = 0$ into equation (105) gives:

$$\begin{aligned} \frac{1}{w_t(A^t)c_t(a_0, z^t, A^t)} &= \beta R_{t+1}(A^{t+1}) \frac{1}{w_{t+1}(A^{t+1})c_{t+1}(a_0, z^{t+1}, A^{t+1}; z_{t+1} = \zeta)} + \lambda(a_0, z^{t+1}, A^{t+1}) \\ &\Leftrightarrow \underbrace{\beta R_{t+1}(A^{t+1}) \frac{w_t(A^t)}{w_{t+1}(A^{t+1})}}_{<1} \frac{c_t(a_0, z^t, A^t)}{c_0} + \lambda(a_0, z^{t+1}, A^{t+1}) w_t(A^t) c_t(a_0, z^t, A^t) = 1 \end{aligned}$$

As long as $c_t(a_0, z^t, A^t) \leq c_0$ for all (a_0, z^t, A^t) , which we will show below, the Lagrangian multiplier $\lambda(a_0, z^{t+1}, A^{t+1})$ that solves equation (105) satisfies $\lambda(a_0, z^{t+1}, A^{t+1}) > 0$. Then, the Kuhn-Tucker condition for a high-income state is satisfied. Specifically, the Kuhn-Tucker condition for a high-income state with $\lambda(a_0, z^{t+1}, A^{t+1}) > 0$ implies:

$$\frac{1}{\mathbf{c}_t(z^t, A^t)} > \beta R_{t+1}(A^{t+1}) \frac{1}{\mathbf{c}_{t+1}(z^{t+1}, A^{t+1}; z_{t+1} = \zeta)}. \quad (112)$$

This gives equation (25). Because the marginal utility of consumption at time t is higher than the discounted marginal utility of consumption at state (z^{t+1}, A^{t+1}) with $z_{t+1} = \zeta$, households do not have incentives to save for a high-income state.

Under $\beta R_{t+1}(A^{t+1}) < \frac{w_{t+1}(A^{t+1})}{w_t(A^t)}$ for all (t, A^t, A_{t+1}) , we will show that $c_t(a_0, z^t, A^t) \leq c_0$ for all (a_0, z^t, A^t) . We prove by induction. At $t = 0$, the initial wealth distribution satisfies Assumption 4. Given the consumption rule (21) and $\beta R_0 < 1$, the initial consumption conjectured allocation if the household were to save for $t = 1$. From $t = 1$ onward, since consumption choice $c_t(a_0, z^t; z_t = \zeta)$ cannot be larger than c_0 at any $t \geq 0$ (otherwise the consumption profile will violate the Euler equation or budget constraints in future low-income states), positive savings for a high-income state at $t = 1$ wouldn't give higher utility at any time $t \geq 1$, under Assumption 3. By induction, households at any time $t \geq 1$ do not save for a high-income state at $t + 1$, since they start a period $t \geq 1$ with $(a_t = 0, z_t = \zeta)$ or $(a_t \leq \bar{a}_0, z_t = 0)$.

satisfies $c_0(a_0, z_0, A_0) \leq c_0$ for all (a_0, z_0) , as we see the following:

$$\begin{aligned} w_0 c_0(a_0, z_0 = 0, A_0) &=: \mathbf{c}_0(a_0, z_0 = 0, A_0) = [1 - (1 - \nu)\beta]R_0(A_0)\mathbf{a}_0(a_0, z_0, A_0) \\ &< [1 - (1 - \nu)\beta]R_0(A_0)w_0 \frac{\beta}{1 - (1 - \nu - \xi)\beta} \zeta \\ &< \beta R_0 c_0 w_0 \\ &< c_0 w_0 \end{aligned}$$

Suppose $c_t(a_0, z^t, A^t) \leq c_0$ for all (a_0, z^t, A^t) at time $t \geq 0$. $c_{t+1}(a_0, z^{t+1}, A^{t+1})$ is given by equation (21):

$$c_{t+1}(a_0, z^{t+1}, A^{t+1}) = \begin{cases} c_0 & \text{if } z_{t+1} = \zeta \\ \beta R_{t+1}(A^{t+1}) \frac{w_t(A^t)}{w_{t+1}(A^{t+1})} c_t(a_0, z^t, A^t) & \text{if } z_{t+1} = 0 \end{cases}$$

Since $\beta R_{t+1}(A^{t+1}) \frac{w_t(A^t)}{w_{t+1}(A^{t+1})} < 1$ and $c_t(a_0, z^t, A^t) \leq c_0$, we have $c_{t+1}(a_0, z^{t+1}, A^{t+1}) \leq c_0$ for all (c_0, z^{t+1}, A^{t+1}) . \square

Finally, given that the conjectured allocation satisfies the Kuhn-Tucker conditions, we prove that the conjectured allocation is indeed optimal.

Proposition 1 (Optimal Household Consumption and Asset Allocation).

Proof. In Lemmas 4 and 5, we have shown that under Assumptions 3 and 4, the conjectured allocation (21)–(22) satisfies the budget constraints and the Kuhn-Tucker conditions. The shortsale constraints are also satisfied. Now we want to show that the conjectured allocation maximizes the objective (11) under the constraints (10) and (12). We apply the standard proof (e.g., Sims (2006)³¹ and Krusell (2014)) to our setup. The upshot is that since the utility function is concave and the constraint set is convex (the constraint $\mathbf{a}_{t+1}(z^{t+1}, A^{t+1}) \geq 0$ is linear in $\mathbf{a}_{t+1}(z^{t+1}, A^{t+1})$), the Kuhn-Tucker conditions and a transversality condition (126) are jointly sufficient for optimality.

We will show that the conjectured allocation (21)–(22), denoted by $(\{\mathbf{c}_t^*(z^t, A^t), \mathbf{a}_{t+1}^*(z^{t+1}, A^{t+1})\})$, gives (weakly) higher expected utility than any other feasible allocations:

$$\lim_{T \rightarrow \infty} \sum_{t=0}^T \beta^t \mathbb{E} [\log(\mathbf{c}_t^*(z^t, A^t))] \geq \lim_{T \rightarrow \infty} \sum_{t=0}^T \beta^t \mathbb{E} [\log(\mathbf{c}_t(z^t, A^t))], \quad (113)$$

where feasible allocations $(\{\mathbf{c}_t(z^t, A^t), \mathbf{a}_{t+1}(z^{t+1}, A^{t+1})\})$ satisfy the budget constraints and the shortsale constraints. Since $\mathbf{c}_t(z^t, A^t)$ is uniquely determined by the budget con-

³¹The lecture note is available on his website: <http://sims.princeton.edu/yftp/Macro2010/r1lg.pdf>

straint given $(\mathbf{a}_t(z^t, A^t), \{\mathbf{a}_{t+1}(z^{t+1}, A^{t+1})\}_{z^{t+1}, A^{t+1}})$, denote:

$$\begin{aligned} u_t(\mathbf{a}_t, \{\mathbf{a}_{t+1}\}) &:= \log(\mathbf{c}_t(z^t, A^t)) \\ &= \log\left(w_t(A^t)z_t + R_t(A^t)\mathbf{a}_t(z^t, A^t)\right. \\ &\quad \left. - \sum_{A^{t+1}} \sum_{z^{t+1}} \pi(A^{t+1}|A^t)\pi(z^{t+1}|z_t)\mathbf{a}_{t+1}(z^{t+1}, A^{t+1})\right). \end{aligned}$$

With this notation, the Kuhn-Tucker condition implies the following:³²

$$D_2 u_t(\mathbf{a}_t, \{\mathbf{a}_{t+1}\}) + \beta \mathbb{E}_t [D_1 u_{t+1}(\mathbf{a}_{t+1}, \{\mathbf{a}_{t+2}\})] + \mathbb{E}_t [\lambda(z^{t+1}, A^{t+1})] = 0 \quad (119)$$

³²Lagrangian is given by:

$$\begin{aligned} \mathcal{L} &= \sum_{t=0}^{\infty} \beta^t \sum_{A^t, z^t} \pi(z^t)\pi(A^t)u_t(\mathbf{a}_t(z^t, A^t), \{\mathbf{a}_{t+1}(z^{t+1}, A^{t+1})\}_{z^{t+1}, A^{t+1}|z^t, A^t}) \\ &\quad + \sum_{t=0}^{\infty} \beta^t \sum_{A^{t+1}, z^{t+1}} \pi(z^{t+1})\pi(A^{t+1})\lambda_{t+1}(z^{t+1}, A^{t+1}) \\ &=: \sum_{t=0}^{\infty} \beta^t \mathbb{E} [u_t(\mathbf{a}_t, \{\mathbf{a}_{t+1}\})] + \sum_{t=0}^{\infty} \beta^t \mathbb{E} [\lambda_{t+1}] \end{aligned}$$

From Lemma 5, we know that the Kuhn-Tucker condition is satisfied for any (z^{t+1}, A^{t+1}) :

$$\frac{1}{\mathbf{c}_t(z^t, A^t)} = \beta R_{t+1}(A^{t+1}) \frac{1}{\mathbf{c}_{t+1}(z^{t+1}, A^{t+1})} + \lambda(z^{t+1}, A^{t+1}) \quad (114)$$

This implies, since $\sum_{z^{t+1}, A^{t+1}|z^t, A^t} \pi(z^{t+1}|z^t)\pi(A^{t+1}|A^t) = 1$,

$$\frac{1}{\mathbf{c}_t(z^t, A^t)} = \sum_{z^{t+1}, A^{t+1}|z^t, A^t} \pi(z^{t+1}|z^t)\pi(A^{t+1}|A^t) \left[\beta R_{t+1}(A^{t+1}) \frac{1}{\mathbf{c}_{t+1}(z^{t+1}, A^{t+1})} + \lambda(z^{t+1}, A^{t+1}) \right] \quad (115)$$

$$=: \mathbb{E}_t \left[\beta R_{t+1}(A^{t+1}) \frac{1}{\mathbf{c}_{t+1}(z^{t+1}, A^{t+1})} \right] + \mathbb{E}_t [\lambda(z^{t+1}, A^{t+1})] \quad (116)$$

We define the derivative of the flow utility as:

$$\begin{aligned} D_1 u_t(\mathbf{a}_t, \{\mathbf{a}_{t+1}\}) &:= \frac{\partial}{\partial \mathbf{a}_t} u_t(\mathbf{a}_t, \{\mathbf{a}_{t+1}\}) \\ &= R_t(A^t) \frac{1}{\mathbf{c}_t(z^t, A^t)} \end{aligned} \quad (117)$$

$$\begin{aligned} D_2 u_t(\mathbf{a}_t, \{\mathbf{a}_{t+1}\}) &:= \sum_{z^{t+1}, A^{t+1}} \frac{\partial}{\partial \mathbf{a}_{t+1}(z^{t+1}, A^{t+1})} u_t(\mathbf{a}_t(z^t, A^t), \{\mathbf{a}_{t+1}(z^{t+1}, A^{t+1})\}) \\ &= - \sum_{z^{t+1}, A^{t+1}} \pi(z^{t+1}|z^t)\pi(A^{t+1}|A^t) \frac{1}{\mathbf{c}_t(z^t, A^t)} \\ &= - \frac{1}{\mathbf{c}_t(z^t, A^t)} \end{aligned} \quad (118)$$

Therefore, by substituting them into (116), we obtain:

$$D_2 u_t(\mathbf{a}_t, \{\mathbf{a}_{t+1}\}) + \beta \mathbb{E}_t [D_1 u_{t+1}(\mathbf{a}_{t+1}, \{\mathbf{a}_{t+2}\})] + \mathbb{E}_t [\lambda(z^{t+1}, A^{t+1})] = 0.$$

Define the difference in the sum of expected utility up to time T:

$$\tilde{V}_T(\mathbf{a}) := \sum_{t=0}^T \beta^t \mathbb{E} [u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) - u_t(\mathbf{a}_t, \{\mathbf{a}_{t+1}\})] \quad (120)$$

We will show that:

$$\lim_{T \rightarrow \infty} \tilde{V}_T(\mathbf{a}) \geq 0 \quad (121)$$

Since $\log(\cdot)$ is a concave function, we have:³³

$$\begin{aligned} \mathbb{E} \left[\sum_{t=0}^T \beta^t u_t(\mathbf{a}_t, \{\mathbf{a}_{t+1}\}) \right] &\geq \mathbb{E} \left[\sum_{t=0}^T \beta^t \left\{ u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) + D_1 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) \cdot (\mathbf{a}_t - \mathbf{a}_t^*) \right. \right. \\ &\quad \left. \left. + D_2 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) \cdot (\mathbf{a}_{t+1} - \mathbf{a}_{t+1}^*) \right\} \right] \end{aligned} \quad (122)$$

Using this, we obtain:

$$\begin{aligned} \tilde{V}_T(\mathbf{a}) &\geq \mathbb{E} \left[\sum_{t=0}^T \beta^t \left\{ D_1 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) \cdot (\mathbf{a}_t^* - \mathbf{a}_t) + D_2 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) \cdot (\mathbf{a}_{t+1}^* - \mathbf{a}_{t+1}) \right\} \right] \\ &= D_1 u_0(\mathbf{a}_0^*, \{\mathbf{a}_1^*\}) \cdot (\mathbf{a}_0^* - \mathbf{a}_0) \\ &\quad + \mathbb{E} \left[\sum_{t=0}^{T-1} \beta^t \left[D_2 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) + \beta \mathbb{E}_t [D_1 u_{t+1}(\mathbf{a}_{t+1}^*, \{\mathbf{a}_{t+2}^*\})] \right] \cdot (\mathbf{a}_{t+1}^* - \mathbf{a}_{t+1}) \right] \\ &\quad + \beta^T \mathbb{E} [D_2 u_T(\mathbf{a}_T^*, \{\mathbf{a}_{T+1}^*\}) \cdot (\mathbf{a}_{T+1}^* - \mathbf{a}_{T+1})] \end{aligned} \quad (123)$$

where the first term is zero given the same initial condition $\mathbf{a}_0^* = \mathbf{a}_0$. The second term is non-negative.³⁴ This is because the Kuhn-Tucker condition implies (119):

$$D_2 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) + \mathbb{E}_t[\lambda_{t+1}^*] + \beta \mathbb{E}_t[D_1 u_{t+1}(\mathbf{a}_{t+1}^*, \{\mathbf{a}_{t+2}^*\})] = 0$$

³³Here we denote:

$$D_2 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) \cdot (\mathbf{a}_{t+1} - \mathbf{a}_{t+1}^*) := \sum_{z^{t+1}, A^{t+1}} \frac{\partial}{\partial \mathbf{a}_{t+1}(z^{t+1}, A^{t+1})} u_t(\mathbf{a}_t(z^t, A^t), \{\mathbf{a}_{t+1}(z^{t+1}, A^{t+1})\}) \cdot (\mathbf{a}_{t+1}(z^{t+1}, A^{t+1}) - \mathbf{a}_{t+1}^*(z^{t+1}, A^{t+1}))$$

³⁴Here we have applied the law of iterated expectations:

$$\begin{aligned} &\mathbb{E} \left[\left[D_2 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) + \beta D_1 u_{t+1}(\mathbf{a}_{t+1}^*, \{\mathbf{a}_{t+2}^*\}) \right] \cdot (\mathbf{a}_{t+1}^* - \mathbf{a}_{t+1}) \right] \\ &= \mathbb{E} \left[\mathbb{E}_t \left[\left[D_2 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) + \beta D_1 u_{t+1}(\mathbf{a}_{t+1}^*, \{\mathbf{a}_{t+2}^*\}) \right] \cdot (\mathbf{a}_{t+1}^* - \mathbf{a}_{t+1}) \right] \right] \\ &= \mathbb{E} \left[\left[D_2 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) + \beta \mathbb{E}_t [D_1 u_{t+1}(\mathbf{a}_{t+1}^*, \{\mathbf{a}_{t+2}^*\})] \right] \cdot (\mathbf{a}_{t+1}^* - \mathbf{a}_{t+1}) \right] \end{aligned}$$

and thus, the second term is larger or equal to zero:³⁵

$$\mathbb{E} \left[\sum_{t=0}^{T-1} \beta^t \left(-\mathbb{E}_t[\lambda_{t+1}^*] \right) \cdot (\mathbf{a}_{t+1}^* - \mathbf{a}_{t+1}) \right] = \mathbb{E} \left[\sum_{t=0}^{T-1} \beta^t \underbrace{\mathbb{E}_t[\lambda_{t+1}^*] \mathbf{a}_{t+1}}_{\geq 0} \right] - \mathbb{E} \left[\sum_{t=0}^{T-1} \beta^t \underbrace{\mathbb{E}_t[\lambda_{t+1}^*] \mathbf{a}_{t+1}^*}_{=0} \right] \geq 0$$

Therefore, (121) is satisfied if the third term is non-negative:

$$\lim_{T \rightarrow \infty} \beta^T \mathbb{E} \left[D_2 u_T(\mathbf{a}_T^*, \{\mathbf{a}_{T+1}^*\}) \cdot (\mathbf{a}_{T+1}^* - \mathbf{a}_{T+1}) \right] \geq 0. \quad (124)$$

Using the Kuhn-Tucker condition again, this is equivalent to:

$$\lim_{T \rightarrow \infty} \beta^T \mathbb{E} \left[\left\{ \mathbb{E}_T[\lambda_{T+1}^*] + \beta \mathbb{E}_T[D_1 u_{T+1}(\mathbf{a}_{T+1}^*, \{\mathbf{a}_{T+2}^*\})] \right\} \cdot (\mathbf{a}_{T+1}^* - \mathbf{a}_{T+1}) \right] \geq 0. \quad (125)$$

Since $\lambda_{T+1}^*(z^{T+1}, A^{T+1}) \mathbf{a}_{T+1}(z^{T+1}, A^{T+1}) \geq 0$, $\lambda_{T+1}^*(z^{T+1}, A^{T+1}) \mathbf{a}_{T+1}^*(z^{T+1}, A^{T+1}) = 0$ for all (z^{T+1}, A^{T+1}) , and $\beta D_1 u_{T+1}(\mathbf{a}_{T+1}^*, \{\mathbf{a}_{T+2}^*\}) \mathbf{a}_{T+1} \geq 0$, the following is sufficient for (121):

$$\lim_{t \rightarrow \infty} \beta^t \mathbb{E} \left[D_1 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) \mathbf{a}_t^* \right] = 0, \quad (126)$$

$$\text{where } D_1 u_t(\mathbf{a}_t^*, \{\mathbf{a}_{t+1}^*\}) = R_t(A^t) \frac{1}{\mathbf{c}_t^*(z^t, A^t)}.$$

Equation (126) is a transversality condition. Our conjectured allocation satisfies:

$$R_t(A^t) \frac{\mathbf{a}_t^*(z^t, A^t)}{\mathbf{c}_t^*(z^t, A^t)} = \begin{cases} 0 & \text{if } z_t = \zeta \\ \frac{1}{1-(1-\nu)\beta} & \text{if } z_t = 0 \end{cases} \quad (127)$$

Hence, (126) is satisfied in the conjectured allocation. Therefore, the conjectured allocation gives (weakly) higher expected utility than any other feasible allocations and maximizes the objective. \square

A.1.2 Proof of Proposition 2

Proposition 2 (A Law of Motion of Aggregate Capital).

³⁵Here we denote:

$$\mathbb{E}_t[\lambda_{t+1}^*] \mathbf{a}_{t+1} := \sum_{z^{t+1}, A^{t+1}} \pi(z^{t+1}|z^t) \pi(A^{t+1}|A^t) \lambda^*(z^{t+1}, A^{t+1}) \mathbf{a}_{t+1}(z^{t+1}, A^{t+1})$$

Proof. Aggregate saving is the sum of individual savings.

$$\begin{aligned}
K_{t+1} &= \int \sum_{z^{t+1}} \hat{\mathbf{a}}_{t+1}(a_0, z^{t+1}, A^{t+1}) \pi(z^{t+1}) d\Phi(a_0, z_0) \\
&= \underbrace{\pi(z^{t+1}; z_t = \zeta, z_{t+1} = 0)}_{=\xi} \underbrace{\pi(z^t; z_t = \zeta)}_{=\frac{\nu}{\xi+\nu}} \hat{\mathbf{a}}_{t+1}(a_0, z^{t+1}, A^{t+1}; z_t = \zeta, z_{t+1} = 0) \\
&\quad + \int \sum_{z^{t+1}} \underbrace{\pi(z^{t+1}; z_t = 0, z_{t+1} = 0)}_{=1-\nu} \hat{\mathbf{a}}_{t+1}(a_0, z^{t+1}, A^{t+1}; z_t = 0, z_{t+1} = 0) d\Phi(a_0, z_0) \\
&= \xi \frac{\beta}{1 - (1 - \nu - \xi)\beta} w_t(A^t) + (1 - \nu)\beta R_t(A^t) \underbrace{\int \sum_{z^t} \hat{\mathbf{a}}_t(a_0, z^t, A^t; z_t = 0) d\Phi(a_0, z_0)}_{=K_t},
\end{aligned}$$

The second line decomposes the summation into four groups by current and next states: $(z_t = \zeta, z_{t+1} = 0)$, $(z_t = 0, z_{t+1} = 0)$, $(z_t = \zeta, z_{t+1} = \zeta)$, and $(z_t = 0, z_{t+1} = \zeta)$. It sums up only the first two groups, since households do not save for a high-income state. The third line follows the households' saving rule (22). To obtain $K_t = \int \sum_{z^t} \hat{\mathbf{a}}_t(a_0, z^t, A^t; z_t = 0) d\Phi(a_0, z_0)$, note that high-income households have zero savings at the initial period ($t = 0$) and that households do not save for a high-income state at $t \geq 1$. Hence, aggregate savings at any time $t \geq 0$ is the sum of savings by low-income households ($z_t = 0$).

By substituting, $w_t(A^t) = (1 - \theta)(A_t)^{1-\theta} K_t^\theta$ and $R_t(A^t) = \theta(A_t)^{1-\theta} K_t^{\theta-1} + 1 - \delta$, we obtain equation (39). □

A.2 Proofs: Section 4 (Stationary Equilibrium and Transitional Dynamics)

A.2.1 Subsection 4.1: Stationary Equilibrium

Proposition 3 (Stationary Equilibrium).

Proof. We first show that under Assumption 5, we can always find a stationary equilibrium with $\beta R_0 < 1$, i.e., the existence of a partial insurance equilibrium. Since we consider a stationary equilibrium, aggregate productivity A_0 is constant over time and across aggregate states.

Suppose an interest rate satisfies $\beta R_0 < 1$. We will later verify that the equilibrium interest rate indeed satisfies $\beta R_0 < 1$ under Assumption 5. Under $\beta R_0 < 1$, households' consumption and asset choices are given by equations (42)–(46). Hence, aggregate capital supply is given by equation (51). In order for the capital market to clear, the interest

rate satisfies the equation (52):

$$K(R)/w(R) =: \kappa^d(R) = \frac{\theta}{(1-\theta)(R-1+\delta)} = \frac{\xi\beta}{[1-(1-\nu)\beta R][1-(1-\nu-\xi)\beta]} := \kappa^s(R).$$

If the equilibrium interest rate R_0 that solves (52) satisfies $\beta R_0 < 1$, then we show the existence of a partial insurance equilibrium with $\beta R_0 < 1$.

Note that the wage-normalized capital demand is positive infinite in the limit $R \rightarrow 1 - \delta$:

$$\lim_{R \rightarrow 1-\delta} \kappa^d(R) \left(:= \frac{\theta}{(1-\theta)(R-1+\delta)} \right) = +\infty,$$

and $\kappa^d(R)$ is strictly decreasing in $R \in (1 - \delta, \frac{1}{\beta})$. Also, the wage-normalized capital supply is finite at $R = 1 - \delta$:

$$\lim_{R \rightarrow 1-\delta} \kappa^s(R) \left(:= \frac{\xi\beta}{[1-(1-\nu)\beta R][1-(1-\nu-\xi)\beta]} \right) < \infty,$$

since $0 < 1 - (1-\nu)\beta(1-\delta) < 1$ and $0 < 1 - (1-\nu-\xi)\beta < 1$. $\kappa^s(R)$ is strictly increasing in $R \in (1 - \delta, \frac{1}{\beta})$. This means that the excess demand for capital, $\kappa^d(R) - \kappa^s(R)$, is positive infinite at the limit $R \rightarrow 1 - \delta$ and (strictly) monotonically decreasing in $R \in (1 - \delta, \frac{1}{\beta})$. Therefore, an equilibrium interest rate with $\beta R_0 < 1$ exists if:

$$\kappa^d(R) - \kappa^s(R) < 0 \text{ at } R = \frac{1}{\beta}.$$

This condition is equivalent to Assumption 5. Hence, under Assumption 5, there exists a partial insurance equilibrium with $\beta R_0 < 1$.

From the FOCs for a representative firm (equations 13 and 14), we have:

$$K_0 = A_0 \left(\frac{\theta}{R_0 - 1 + \delta} \right)^{\frac{1}{1-\theta}},$$

$$w_0 = (1-\theta)(A_0)^{1-\theta}(K_0)^\theta.$$

By substituting the equilibrium interest rate, we obtain the equilibrium capital and the wage.

Comparative statics are straightforward. As discussed in the main text, $\kappa^d(R)$ is strictly increasing in θ and strictly decreasing in δ , while $\kappa^s(R)$ does not depend on θ or δ . Given that $\kappa^d(R) - \kappa^s(R)$ is strictly decreasing in R , higher θ implies higher R_0 , and higher δ implies lower R_0 . On the other hand, since $\kappa^s(R)$ is strictly increasing in β , ξ , and $1 - \nu$,³⁶ while $\kappa^d(R)$ does not depend on β , ξ , or $1 - \nu$. Thus, higher ξ and $1 - \nu$

³⁶ $\kappa^s(R)$ is strictly increasing in ξ because the derivative with respect to ξ is strictly positive:

$$\frac{\partial}{\partial \xi} \kappa^s(R) = \frac{\beta}{[1-(1-\nu)\beta R][1-(1-\nu-\xi)\beta]} \left[1 - \frac{\xi}{\frac{1}{\beta} - 1 + \nu + \xi} \right] > 0$$

implies lower R_0 . Since K_0 and w_0 are negatively related with R_0 , we have the opposite comparative statics with respect to $(\beta, \xi, 1 - \nu)$. To show that K_0 is decreasing in δ , we show that $R_0 - 1 + \delta$ is increasing in δ :

$$\begin{aligned} R_0 - (1 - \delta) &= \frac{\xi(1 - \theta)(1 - \delta) + \theta \left(\xi + \nu + \frac{1}{\beta} - 1 \right)}{\xi(1 - \theta) + \beta\theta(1 - \nu) \left(\xi + \nu + \frac{1}{\beta} - 1 \right)} - (1 - \delta) \\ &= \frac{\theta \left(\xi + \nu + \frac{1}{\beta} - 1 \right)}{\xi(1 - \theta) + \beta\theta(1 - \nu) \left(\xi + \nu + \frac{1}{\beta} - 1 \right)} - \frac{\beta\theta(1 - \nu) \left(\xi + \nu + \frac{1}{\beta} - 1 \right)}{\xi(1 - \theta) + \beta\theta(1 - \nu) \left(\xi + \nu + \frac{1}{\beta} - 1 \right)} (1 - \delta) \end{aligned}$$

From the FOCs for representative firms, we see that K_0 (and hence w_0) is decreasing in $R_0 - 1 + \delta$ and hence increasing in δ .

Finally, given the aggregate capital supply function derived from the optimal households' consumption and asset allocation that yields a simple equilibrium, since $\kappa^d(R) - \kappa^s(R)$ is strictly monotonically decreasing in $R \in (1 - \delta, \frac{1}{\beta})$, the solution to an equation (52) is unique if it exists. \square

A.2.2 Subsection 4.2.1: Full Depreciation of Capital

Lemma 6. *Suppose Assumption 5 holds. Consider transitional dynamics after an aggregate productivity shock. With full depreciation of capital ($\delta = 1$), $R_{t+1} \frac{w_t}{w_{t+1}} = R_0 = \frac{\theta}{\hat{s}}$ for all $t \geq 1$. Furthermore, $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ is satisfied for all $t \geq 1$.*

Proof. With $\delta = 1$,

$$R_{t+1} = \theta \left(\frac{A_{t+1}}{K_{t+1}} \right)^{1-\theta} \quad \text{and} \quad w_{t+1} = (1 - \theta) A_{t+1}^{1-\theta} K_{t+1}^\theta.$$

Thus,

$$\begin{aligned} \frac{R_{t+1}}{w_{t+1}} &= \frac{\theta}{1 - \theta} \frac{1}{K_{t+1}} \\ &= \frac{\theta}{1 - \theta} \frac{1}{\hat{s} A_t^{1-\theta} K_t^\theta}. \end{aligned} \tag{128}$$

With $w_t = (1 - \theta) A_t^{1-\theta} K_t^\theta$, one obtains:

$$R_{t+1} \frac{w_t}{w_{t+1}} = \frac{\theta}{\hat{s}}. \tag{129}$$

This must be true in steady state. Thus,

$$R_{t+1} \frac{w_t}{w_{t+1}} = R_0 < \frac{1}{\beta}. \tag{130}$$

\square

A.2.3 Monotone Convergence

Proposition 5 (Monotone Convergence of (K_t, R_t, w_t)).

Proof. Consider a positive permanent shock, $A_t = A_1 \forall t \geq 1$ with $A_1 > A_0$. Denote K^* as the new stationary equilibrium capital associated with A_1 . We know from equation (48) that $K_0 < K^*$ since $A_0 < A_1$. We want to show that given $K_t < K^*$, capital in the next period satisfies $K_t < K_{t+1} < K^*$ at any $t \geq 1$, implying a monotone convergence of capital to the new stationary equilibrium capital.

The law of motion of capital is given by equation (39). First, we show $K_t < K_{t+1}$ at any $t \geq 1$, by using $\hat{s} = \hat{\delta} \left(\frac{K^*}{A_1} \right)^{1-\theta}$,

$$\begin{aligned} K_{t+1} - K_t &= \hat{s}A_1^{1-\theta}K_t^\theta + (1 - \hat{\delta})K_t - K_t \\ &= \hat{\delta} \underbrace{\left[\left(\frac{K^*}{K_t} \right)^{1-\theta} - 1 \right]}_{>0 \text{ given } K_t < K^*} K_t > 0. \end{aligned}$$

Because $\hat{\delta} > 0$, the increment in capital ($K_{t+1} - K_t$) is strictly positive until K_t converges to K^* . Second, we show $K_{t+1} < K^*$ at any $t \geq 1$:

$$\begin{aligned} K_{t+1} - K^* &= \hat{s}A_1^{1-\theta}K_t^\theta + (1 - \hat{\delta})K_t - K^* \\ &= \hat{\delta} \left(\frac{K_t}{K^*} \right)^\theta K^* + (1 - \hat{\delta})K_t - K^* \\ &= \hat{\delta} \underbrace{\left[\left(\frac{K_t}{K^*} \right)^\theta - 1 \right]}_{<0 \text{ if } K_t < K^*} K^* + (1 - \hat{\delta}) \underbrace{(K_t - K^*)}_{<0 \text{ if } K_t < K^*} < 0 \end{aligned}$$

We have shown that if $K_t < K^*$, $K_t < K_{t+1} < K^*$. This holds for all $t = 1, 2, \dots$ as we start from $K_1 = K_0 < K^*$. Therefore, we have $K_1 < K_2 < \dots < K_t < \dots < K^*$.

The wage and interest rate follows the FOCs:

$$\begin{aligned} w_t &= (1 - \theta)A_t^{1-\theta}K_t^\theta, \\ R_t &= \theta A_t^{1-\theta}K_t^{\theta-1} + 1 - \delta \text{ for all } t \geq 1. \end{aligned}$$

Given $K_1 = K_0$, both w_t and R_t jump up at $t = 1$. From $t = 2$ onwards, since $0 < \theta < 1$, $K_t < K_{t+1}$ implies $w_t < w_{t+1}$ and $R_t > R_{t+1}$. Therefore, the monotone convergence of capital implies a monotone convergence of wages and interest rates.

In case of a negative shock, the inequality holds in the opposite direction. □

A.2.4 Sufficient Condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t} \forall t \geq 0$ after a Positive Shock

Proposition 6 (Sufficient Condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ after a Positive Shock).

Proof. Since wages, $w_t = (1 - \theta)A_t^{1-\theta}K_t^\theta$, are monotonically increasing after a positive productivity shock, $\beta R_{t+1} < 1$ is a sufficient condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$. After a positive productivity shock at $t = 1$, the interest rate jumps and monotonically converges to the one in a stationary equilibrium, see Proposition 5. Therefore, $\beta R_1 < 1$ guarantees that $\beta R_{t+1} < 1$ for all $t \geq 0$.

A condition for $\beta R_1 < 1$ follows directly from the expression for R_1 and the fact that $K_1 = K_0 = A_0 \left(\frac{\theta}{R_0 - 1 + \delta} \right)^{\frac{1}{1-\theta}}$ is predetermined from the initial stationary equilibrium

$$R_1 = \theta A_1^{1-\theta} K_0^{\theta-1} + 1 - \delta, \quad (131)$$

and therefore $\beta R_1 < 1$ if

$$R_1 = \left(\frac{A_1}{A_0} \right)^{1-\theta} (R_0 - 1 + \delta) + 1 - \delta < \frac{1}{\beta} \quad (132)$$

$$\frac{A_1}{A_0} < \left(\frac{\frac{1}{\beta} - 1 + \delta}{R_0 - 1 + \delta} \right)^{\frac{1}{1-\theta}} \quad (133)$$

$$\frac{A_1}{A_0} < \left[\frac{1 - \beta(1 - \delta)}{\theta} \frac{\xi(1 - \theta) + (1 - \nu)\theta[1 - (1 - \nu - \xi)\beta]}{[1 - (1 - \nu)\beta(1 - \delta)][1 - (1 - \nu - \xi)\beta]} \right]^{\frac{1}{1-\theta}}. \quad (134)$$

This gives the threshold stated in the proposition. Since $R_0 < 1/\beta$, equation (133) implies that $\bar{A}_1 > A_0$. Since K_0 is given by (48), \bar{A}_1/A_0 can be written as:

$$\frac{\bar{A}_1}{A_0} = \left[\frac{1 - \beta(1 - \delta)}{\beta\theta} \right]^{\frac{1}{1-\theta}} K_0 \quad (135)$$

Substituting $\rho = \frac{1}{\beta} - 1$ leads to equation (67). \square

A.2.5 Sufficient Condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t} \forall t \geq 0$ after a Negative Shock

In this subsection, we derive a sufficient condition on the magnitude of a negative productivity shock ($A_1 < A_0$ and $A_t = A_1$ for all $t \geq 1$) such that $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ is satisfied for all $t \geq 0$.

We first derive a sufficient condition for $\beta R_{t+1} \frac{w_t}{w_{t+1}} < 1 \forall t \geq 1$. After a negative shock, the aggregate capital monotonically declines and converges to a new stationary equilibrium. Using this property, we derive a lower bound on A_1 that guarantees $\beta R_{t+1} \frac{w_t}{w_{t+1}} < 1 \forall t \geq 1$ (Lemma 9).

The condition in Lemma 9 ($A_1 \in (A'_1, A_0]$) does not guarantee $\beta R_1 < \frac{w_1}{w_0}$. If $\beta R_1 > \frac{w_1}{w_0}$, low-income households may consume more than high-income households at $t = 1$. This gives rise to a possibility that low-income households at $t = 1$ have an incentive to save for a high-income state at $t = 2$. Hence, we derive a condition for $\beta R_1 < \frac{w_1}{w_0}$, which is sufficient to prevent this possibility (Proposition 7).³⁷

Condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at $t \geq 0$ After a negative shock, the aggregate capital monotonically decreases and converges to a new stationary equilibrium. Therefore, $\left(\frac{K_{t+1}}{A_1}\right)^{1-\theta} < \left(\frac{K_t}{A_1}\right)^{1-\theta}$ for all $t \geq 1$. A sufficient condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t} \forall t \geq 1$ is then written as:

$$\left[\left(\frac{K_t}{A_1}\right)^{1-\theta} - (1-\nu) \left(\frac{K_t}{A_1}\right)^{1-\theta} \right] = \nu \left(\frac{K_t}{A_1}\right)^{1-\theta} < \frac{1}{1-\delta} \left[\frac{\xi(1-\theta)}{1-(1-\nu-\xi)\beta} - \theta\nu \right] \text{ for all } t \geq 1.$$

As $\frac{K_1}{A_1} > \frac{K_2}{A_1} > \dots > \frac{K^*}{A_1}$, it is sufficient to satisfy the condition at time $t = 1$. By solving this inequality, we have:

$$\left(\frac{A_0}{A_1}\right)^{1-\theta} < \left[\frac{1-(1-\delta)\beta(1-\nu)}{\beta\nu(1-\delta)} \right] \left[\frac{\xi(1-\theta) - \beta\theta\nu \left[\xi + \nu + \frac{1}{\beta} - 1 \right]}{\xi(1-\theta) + \beta\theta(1-\nu)\left(\xi + \nu + \frac{1}{\beta} - 1\right)} \right]. \quad (136)$$

Lemma 7. *Assumption 5 implies $\xi(1-\theta) > \beta\theta\nu(\xi + \nu + \frac{1}{\beta} - 1)$. Hence, the right hand side of inequality (136) is strictly positive.*

Proof. We restate Assumption 5:

$$\frac{\theta}{(1-\theta) \left[\frac{1}{\beta} - 1 + \delta \right]} < \frac{\xi}{\nu \left[\frac{1}{\beta} - 1 + \xi + \nu \right]}.$$

As all terms are positive under $0 < \beta < 1$, it is equivalent to:

$$\xi(1-\theta)[1 - \beta(1-\delta)] > \beta\theta\nu \left[\frac{1}{\beta} - 1 + \xi + \nu \right].$$

Since $0 < \beta(1-\delta) < 1$, we have:

$$\xi(1-\theta) > \xi(1-\theta)[1 - \beta(1-\delta)] > \beta\theta\nu\left(\xi + \nu + \frac{1}{\beta} - 1\right). \quad (137)$$

Therefore, all terms in the right hand side of inequality (136) are strictly positive. In the limit $\delta \rightarrow 1$, it is positive infinity: $\lim_{\delta \rightarrow 1} \left[\frac{1-(1-\delta)\beta(1-\nu)}{\beta\nu(1-\delta)} \right] \left[\frac{\xi(1-\theta) - \beta\theta\nu \left[\xi + \nu + \frac{1}{\beta} - 1 \right]}{\xi(1-\theta) + \beta\theta(1-\nu)\left(\xi + \nu + \frac{1}{\beta} - 1\right)} \right] = +\infty$. \square

³⁷We claim that the fact that low-income households consume more than high-income households is per se not a problem. Instead, we can derive a sufficient condition for low-income households at $t = 1$ not to save for the next high-income state. This condition will be less tight than the condition in Proposition 7.

Lemma 8. Under Assumption 5, the right hand side of inequality (136) is strictly larger than 1. Hence, there exists a negative productivity shock A_1 with $A_1 < A_0$ such that (136) holds. Define \underline{A}_1 such that (136) holds with equality. Then $\underline{A}_1 < A_0$.

Proof. Consider $0 < \delta < 1$. The first and second terms in the right hand side of (136) are expressed as:

$$\left[\frac{1 - (1 - \delta)\beta(1 - \nu)}{\beta\nu(1 - \delta)} \right] = 1 + \frac{\frac{1}{\beta(1 - \delta)} - 1}{\nu} \quad (138)$$

$$\left[\frac{\xi(1 - \theta) - \beta\theta\nu \left[\xi + \nu + \frac{1}{\beta} - 1 \right]}{\xi(1 - \theta) + \beta\theta(1 - \nu) \left(\xi + \nu + \frac{1}{\beta} - 1 \right)} \right] = \frac{1}{1 + \frac{\beta\theta \left(\xi + \nu + \frac{1}{\beta} - 1 \right)}{\xi(1 - \theta) - \beta\theta\nu \left[\xi + \nu + \frac{1}{\beta} - 1 \right]}}. \quad (139)$$

Therefore, the product is larger than 1 if:

$$\frac{\frac{1}{\beta(1 - \delta)} - 1}{\nu} > \frac{\beta\theta \left(\xi + \nu + \frac{1}{\beta} - 1 \right)}{\xi(1 - \theta) - \beta\theta\nu \left[\xi + \nu + \frac{1}{\beta} - 1 \right]} \quad (140)$$

Since all terms are positive (remember Lemma 7), this is equivalent to:

$$\begin{aligned} & \left(\frac{1}{\beta(1 - \delta)} - 1 \right) \xi(1 - \theta) - \frac{\theta\nu}{(1 - \delta)} \left(\xi + \nu + \frac{1}{\beta} - 1 \right) + \beta\theta\nu \left(\xi + \nu + \frac{1}{\beta} - 1 \right) > \beta\theta\nu \left(\xi + \nu + \frac{1}{\beta} - 1 \right) \\ \Leftrightarrow & \left(\frac{1 - \beta(1 - \delta)}{\beta(1 - \delta)} \right) \xi(1 - \theta) > \frac{\theta\nu}{(1 - \delta)} \left(\xi + \nu + \frac{1}{\beta} - 1 \right) \\ \Leftrightarrow & \frac{\xi}{\nu \left(\xi + \nu + \frac{1}{\beta} - 1 \right)} > \frac{\theta}{(1 - \theta) \left(\frac{1}{\beta} - 1 + \delta \right)} \end{aligned}$$

The last condition is equivalent to Assumption 5. Therefore, the right hand side of inequality (136) is strictly greater than 1 for any $0 < \delta < 1$. If $\delta = 1$, the right hand side of (136) goes to positive infinity. Put together, this means that for any $0 < \delta \leq 1$, there exists $A_1 < A_0$ such that the condition (136) holds.

We solve for A_1 such that (136) holds with equality:

$$\underline{A}_1 = A_0 \left[\frac{\beta\nu(1 - \delta)}{1 - (1 - \delta)\beta(1 - \nu)} \frac{\xi(1 - \theta) + \beta\theta(1 - \nu) \left(\xi + \nu + \frac{1}{\beta} - 1 \right)}{\xi(1 - \theta) - \beta\theta\nu \left[\xi + \nu + \frac{1}{\beta} - 1 \right]} \right]^{\frac{1}{1 - \theta}}.$$

Because $\left[\frac{\beta\nu(1 - \delta)}{1 - (1 - \delta)\beta(1 - \nu)} \frac{\xi(1 - \theta) + \beta\theta(1 - \nu) \left(\xi + \nu + \frac{1}{\beta} - 1 \right)}{\xi(1 - \theta) - \beta\theta\nu \left[\xi + \nu + \frac{1}{\beta} - 1 \right]} \right] < 1$, $\underline{A}_1 < A_0$. \square

We use the two lemmas to prove a proposition.

Lemma 9 (Sufficient Condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for $t \geq 1$ after a Negative Shock). Let Assumption 5 be satisfied and let the economy be in a stationary equilibrium with $\beta R_0 < 1$.

After a negative and permanent productivity shock at $t = 1$ ($A_t = A_1 < A_0$ for all $t \geq 1$), $\beta R_{t+1} < \frac{w_{t+1}}{w_t} \forall t \geq 1$ is satisfied if $A_1 \in (\underline{A}_1, A_0]$ holds, where the threshold satisfies

$$\underline{A}_1/A_0 = \left[\frac{\beta\nu(1-\delta)}{1-(1-\delta)\beta(1-\nu)} \frac{\xi(1-\theta) + \beta\theta(1-\nu)(\xi + \nu + \frac{1}{\beta} - 1)}{\xi(1-\theta) - \beta\theta\nu \left[\xi + \nu + \frac{1}{\beta} - 1 \right]} \right]^{\frac{1}{1-\theta}} < 1. \quad (141)$$

Proof. We want to derive a sufficient condition for $\beta R_{t+1} \frac{w_t}{w_{t+1}} < 1 \forall t \geq 1$. We focus on the case of $0 < \delta < 1$, because with full depreciation of capital ($\delta = 1$), $\beta R_{t+1} \frac{w_t}{w_{t+1}} = \beta R_0 < 1 \forall t \geq 1$ (Lemma 6). After substituting interest rate and wages and using the law of motion of capital (39), $\beta R_{t+1} \frac{w_t}{w_{t+1}}$ can be written as shown below. The monotonicity of capital, $K_{t+1} < K_t$, implies the following.

$$\begin{aligned} \beta R_{t+1} \frac{w_t}{w_{t+1}} &= \beta \left[\frac{\theta + (1-\delta) \left(\frac{K_{t+1}}{A_1} \right)^{1-\theta}}{\frac{\xi\beta}{1-(1-\nu-\xi)\beta} (1-\theta) + (1-\nu)\beta\theta + (1-\nu)\beta(1-\delta) \left(\frac{K_t}{A_1} \right)^{1-\theta}} \right] \\ &< \beta \left[\frac{\theta + (1-\delta) \left(\frac{K_t}{A_1} \right)^{1-\theta}}{\frac{\xi\beta}{1-(1-\nu-\xi)\beta} (1-\theta) + (1-\nu)\beta\theta + (1-\nu)\beta(1-\delta) \left(\frac{K_t}{A_1} \right)^{1-\theta}} \right] \quad \text{for } t \geq 1 \end{aligned}$$

The last line is less than 1 if

$$\begin{aligned} \beta\theta + \beta(1-\delta) \left(\frac{K_t}{A_1} \right)^{1-\theta} &< \frac{\xi\beta}{1-(1-\nu-\xi)\beta} (1-\theta) + (1-\nu)\beta\theta + (1-\nu)\beta(1-\delta) \left(\frac{K_t}{A_1} \right)^{1-\theta} \\ \Leftrightarrow \nu\beta(1-\delta) \left(\frac{K_t}{A_1} \right)^{1-\theta} &< \frac{\xi\beta}{1-(1-\nu-\xi)\beta} (1-\theta) - \nu\beta\theta \end{aligned}$$

Because $K_1 > K_2 > \dots$, this condition is satisfied for all $t \geq 1$ if it is satisfied at time $t = 1$:

$$\left(\frac{K_1}{A_1} \right)^{1-\theta} < \frac{1}{\nu(1-\delta)} \left[\frac{\xi(1-\theta)}{1-(1-\nu-\xi)\beta} - \nu\theta \right]. \quad (142)$$

Aggregate capital at time $t = 1$ is predetermined at $t = 0$:

$$K_1 = K_0 = A_0 \left[\frac{\xi(1-\theta) + \beta\theta(1-\nu) \left(\xi + \nu + \frac{1}{\beta} - 1 \right)}{[1-(1-\delta)\beta(1-\nu)] \left(\xi + \nu + \frac{1}{\beta} - 1 \right)} \right]^{\frac{1}{1-\theta}}.$$

Substituting K_1 into equation (142) yields:

$$\left(\frac{A_0}{A_1} \right)^{1-\theta} < \left[\frac{1-(1-\delta)\beta(1-\nu)}{\beta\nu(1-\delta)} \right] \left[\frac{\xi(1-\theta) - \beta\theta\nu \left[\xi + \nu + \frac{1}{\beta} - 1 \right]}{\xi(1-\theta) + \beta\theta(1-\nu) \left(\xi + \nu + \frac{1}{\beta} - 1 \right)} \right]. \quad (143)$$

We obtain equation (141) by solving for A_1 . When $A_1 = \underline{A}_1$, the equation holds with equality. Lemma 8 shows that $\underline{A}_1 < A_0$ under Assumption 5. \square

Condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at $t = 0$ Lemma 9 shows that after a negative and permanent productivity shock at $t = 1$, $\beta R_{t+1} < \frac{w_{t+1}}{w_t} \forall t \geq 1$ is satisfied if $A_1 \in (\underline{A}_1, A_0]$ holds. However, this condition does not guarantee $\beta R_1 < \frac{w_1}{w_0}$. If $\beta R_1 > \frac{w_1}{w_0}$, the argument in Lemma 5 breaks down, i.e., $c_1(a_0, z^1, A^1) \leq c_0$ may not hold for some a_0 . Then, low-income households at $t = 1$ may have the incentive to save for the next high-income state. In such a case, the contract stipulated in Proposition 1 may not be optimal. Hence, we derive a sufficient condition for $\beta R_1 < \frac{w_1}{w_0}$. It turns out that the sufficient condition for $\beta R_1 < \frac{w_1}{w_0}$, given by $A_1 \in (\underline{A}_1, A_0]$, implies the sufficient condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t} \forall t \geq 1$, i.e., $A_1 \in (\underline{A}_1, A_0]$, derived in Lemma 9. Therefore, $A_1 \in (\underline{A}_1, A_0]$ is a sufficient condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t} \forall t \geq 0$.

Proposition 7 (Sufficient Condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t} \forall t \geq 0$ after a Negative Shock).

Proof. $\beta R_{t+1} \frac{w_t}{w_{t+1}}$ can be written as:

$$\beta R_{t+1} \frac{w_t}{w_{t+1}} = \beta \left[\frac{\theta + (1 - \delta) \left(\frac{K_{t+1}}{A_{t+1}} \right)^{1-\theta}}{\hat{s} + (1 - \hat{\delta}) \left(\frac{K_t}{A_t} \right)^{1-\theta}} \right]$$

At $t = 0$, $\beta R_{t+1} \frac{w_t}{w_{t+1}} < 1$ is equivalent to:

$$\beta\theta + \beta(1 - \delta) \left(\frac{K_1}{A_1} \right)^{1-\theta} < \hat{s} + (1 - \hat{\delta}) \left(\frac{K_0}{A_0} \right)^{1-\theta},$$

where $\hat{s} = \frac{\xi\beta(1 - \theta)}{1 - (1 - \nu - \xi)\beta} + (1 - \nu)\beta\theta$ and $1 - \hat{\delta} = (1 - \nu)\beta(1 - \delta)$.

The condition can be written as:

$$(1 - \delta) \left(\frac{K_0}{A_0} \right)^{1-\theta} \left[\left(\frac{A_0}{A_1} \right)^{1-\theta} - (1 - \nu) \right] < \frac{\xi(1 - \theta)}{1 - (1 - \nu - \xi)\beta} - \nu\theta$$

$$\Leftrightarrow \left(\frac{A_0}{A_1} \right)^{1-\theta} < 1 - \nu + \frac{1}{1 - \delta} \left(\frac{K_0}{A_0} \right)^{\theta-1} \left[\frac{\xi(1 - \theta)}{1 - (1 - \nu - \xi)\beta} - \nu\theta \right], \quad (144)$$

$$\text{where } \frac{K_0}{A_0} = \left[\frac{\xi(1 - \theta) + \beta\theta(1 - \nu) \left(\xi + \nu + \frac{1}{\beta} - 1 \right)}{[1 - (1 - \delta)\beta(1 - \nu)] \left(\xi + \nu + \frac{1}{\beta} - 1 \right)} \right]^{\frac{1}{1-\theta}}.$$

Using the following derivations:

$$\begin{aligned}
& \frac{1}{1-\delta} \left(\frac{K_0}{A_0} \right)^{\theta-1} \left[\frac{\xi(1-\theta)}{1-(1-\nu-\xi)\beta} - \nu\theta \right] \\
&= \frac{1}{1-\delta} \left[\frac{[1-(1-\delta)\beta(1-\nu)](\xi + \nu + \frac{1}{\beta} - 1)}{\xi(1-\theta) + \beta\theta(1-\nu)(\xi + \nu + \frac{1}{\beta} - 1)} \right] \left[\frac{\xi(1-\theta) - \nu\theta\beta(\xi + \nu + \frac{1}{\beta} - 1)}{\beta(\xi + \nu + \frac{1}{\beta} - 1)} \right] \\
&= \nu \frac{1-(1-\delta)\beta(1-\nu)}{\beta\nu(1-\delta)} \frac{\xi(1-\theta) - \nu\theta\beta(\xi + \nu + \frac{1}{\beta} - 1)}{\xi(1-\theta) + \beta\theta(1-\nu)(\xi + \nu + \frac{1}{\beta} - 1)} \\
&= \nu \left(\frac{A'_1}{A_0} \right)^{\theta-1},
\end{aligned}$$

the condition for $\beta R_1 < \frac{w_1}{w_0}$ is given by:

$$\begin{aligned}
\left(\frac{A_0}{A_1} \right)^{1-\theta} &< 1 - \nu + \nu \left(\frac{A'_1}{A_0} \right)^{\theta-1} \\
\therefore \frac{A_1}{A_0} &> \left[1 - \nu + \nu \left(\frac{A'_1}{A_0} \right)^{\theta-1} \right]^{\frac{1}{\theta-1}} =: \frac{A_1}{A_0}, \tag{145}
\end{aligned}$$

$$\text{where } \frac{A'_1}{A_0} < \frac{A_1}{A_0} < 1. \tag{146}$$

We obtain the last inequality (146), since $0 < \nu < 1$ and $0 < \frac{A'_1}{A_0} < 1$. Hence, $A_1 \in (\underline{A}_1, A_0]$ implies $A_1 \in (\underline{A}'_1, A_0]$. By substituting $\frac{A'_1}{A_0}$ using equation (141), we obtain equation (68) in the statement. \square

A.2.6 Subsection 4.2.2: Consumption on Impact

Corollary 2 (Consumption at the time of a shock).

Proof. At $t = 1$, the asset distribution is predetermined and given by $\{\mathbf{a}_{s,t}\}_{s=0,t=1}^{\infty}$ that satisfies Assumption 4. In particular, we consider the case in which $\{\mathbf{a}_{s,t}\}_{s=0,t=1}^{\infty}$ follows a stationary distribution given by equations (44)–(46), where w_0 follows (50). Note that households purchased contingent assets $\{\mathbf{a}_{s,t}(A^t)\}_{s=0,t=1}^{\infty}$ at $t = 0$, expecting the steady-state productivity for all future periods with probability 1, $A_t = A^*$ for all $t \geq 1$, but an unanticipated aggregate state is realized at $t = 1$. We assume that households hold assets at $t = 1$ as if the anticipated aggregate state (A^*) is realized.

At time $t = 1$, a deterministic sequence of productivity $\{A_t\}_{t=1}^{\infty}$ is unexpectedly realized. The corresponding sequence of prices $\{q_t(A_{t+1}, z_{t+1}|A^t, z^t), w_t(A^t), R_t(A^t)\}_{A^t, z^t, t \geq 1}$ satisfies Assumptions 2 and 3. Then, by Proposition 1, the optimal consumption at $t = 1$ is given by (21). Equation (23) implies that the consumption satisfies the Euler equation between $t = 0$ and $t = 1$. \square

A.2.7 Subsection 4.2.3: Consumption Distribution in the Long Run

Proposition 8 (Consumption Distribution in the Long Run).

Proof. As we see in Proposition 5, in the transitional dynamics, aggregate capital will monotonically converge to a new steady state. The interest rate also converges to a steady-state interest rate.

In a stationary equilibrium, the deflated consumption of low-income agents is determined by:

$$c_s = \beta R^* c_{s-1}.$$

Hence, the consumption distribution is characterized by:

$$c_s = (\beta R^*)^s c_0 \text{ for } s = 0, 1, 2, \dots,$$

where the mass of each agent is given by equation (31):

$$\phi_s = \begin{cases} \frac{\nu}{\xi + \nu} & \text{if } s = 0 \\ \frac{\nu \xi}{\xi + \nu} (1 - \nu)^{s-1} & \text{if } s \geq 1. \end{cases}$$

Because the equilibrium interest rate does not depend on productivity A , shown in equation (54):

$$R^* = \frac{\xi(1 - \theta)(1 - \delta) + \theta \left(\xi + \nu + \frac{1}{\beta} - 1 \right)}{\xi(1 - \theta) + \beta\theta(1 - \nu) \left(\xi + \nu + \frac{1}{\beta} - 1 \right)},$$

the deflated consumption distribution is the same across stationary equilibria with different A .

□

A.2.8 Consumption Distribution with $\delta = 1$

Proposition 9.

Proof. The stationary distribution at $t = 0$ is given by:

$$c_s^* = (\beta R^*)^s c_0, \text{ where } c_0 := \frac{1 - (1 - \nu)\beta}{1 - (1 - \nu - \xi)\beta} \zeta.$$

Note that with $\delta = 1$, the interest rate in the stationary equilibrium (54) is given by:

$$\begin{aligned} R^* &= \frac{\theta \left(\xi + \nu + \frac{1}{\beta} - 1 \right)}{\xi(1 - \theta) + \beta\theta(1 - \nu) \left(\xi + \nu + \frac{1}{\beta} - 1 \right)} \\ &= \frac{\theta}{\hat{s}}. \end{aligned} \tag{147}$$

From $t = 1$ onwards, the deflated consumption distribution evolves according to equation (71):

$$c_{s,t}(A^t) = \begin{cases} c_0 & \text{if } s = 0 \\ \beta R_t(A^t) \frac{w_{t-1}(A^{t-1})}{w_t(A^t)} c_{s-1,t-1}(A^{t-1}) & \text{if } s \geq 1 \end{cases}$$

Lemma 6 shows that:

$$\beta R_t(A^t) \frac{w_{t-1}(A^{t-1})}{w_t(A^t)} = \frac{\beta\theta}{\hat{s}}$$

for any $t \geq 1$ and (A^{t-1}, A_t) . Combined with equation (147), this means that:

$$c_{s,t}(A^t) = \begin{cases} c_0 & \text{if } s = 0 \\ \beta R^* c_{s-1,t-1}(A^{t-1}) & \text{if } s \geq 1 \end{cases}$$

Hence, starting from the stationary distribution at $t = 0$, the deflated consumption distribution is time-invariant:

$$c_{s,t}(A^t) = c_s^* \text{ for all } t \geq 0 \text{ and } A^t.$$

□

A.2.9 Inequality after a Positive Shock with $\delta < 1$

Proposition 10.

Proof. Proposition 1 shows that in a high-income state, $c_{0,t} = c_0$. We derive that $c_{s,1} < c_{s,0}$ for all $s \geq 1$ if $A_1 > A_0$ and $\delta < 1$, meaning that the deflated consumption of low-income agents at time $t = 1$ is lower than the deflated consumption in a stationary equilibrium ($t = 0$) for all $s \geq 1$. Using equation (32) in Corollary 1,

$$\begin{aligned} w_1 c_{s,1} &= \beta R_1 w_0 c_{s-1,0} \\ \text{while } c_{s,0} &= \beta R_0 c_{s-1,0}, \end{aligned}$$

we have:

$$\begin{aligned} c_{s,1} &= \frac{w_0 R_1}{w_1 R_0} c_{s,0}, \\ \text{where } R_t &= \theta A_t^{1-\theta} K_t^{\theta-1} + 1 - \delta, \\ w_t &= (1 - \theta) A_t^{1-\theta} K_t^\theta. \\ \therefore c_{s,1} &= \left(\frac{A_0^{1-\theta}}{A_1^{1-\theta}} \right) \frac{\theta A_1^{1-\theta} K_0^{\theta-1} + 1 - \delta}{\theta A_0^{1-\theta} K_0^{\theta-1} + 1 - \delta} c_{s,0} \\ &= \frac{\theta A_0^{1-\theta} K_0^{\theta-1} + (1 - \delta) \left(\frac{A_0}{A_1} \right)^{1-\theta}}{\theta A_0^{1-\theta} K_0^{\theta-1} + 1 - \delta} c_{s,0} \end{aligned}$$

We use the fact that $K_1 = K_0$ as the aggregate capital at $t = 1$, K_1 , is predetermined at time $t = 0$. Given that $0 < \delta < 1$, $0 < \theta < 1$, and $A_1 > A_0$, we have $(1-\delta) \left(\frac{A_0}{A_1}\right)^{1-\theta} < 1-\delta$. Therefore, $c_{s,1} < c_{s,0}$ for all $s \geq 1$.

Next, we show that consumption ratio between two low-income agents at $t = 1$ is the same as at $t = 0$ (stationary equilibrium):

$$\frac{c_{s,1}}{c_{\tilde{s},1}} = \frac{c_{s,0}}{c_{\tilde{s},0}} \text{ for any } s \geq 1 \text{ and } \tilde{s} \geq 1 \quad (148)$$

By equation (71),

$$\begin{aligned} c_{s,1} &= \beta R_1 \frac{w_0}{w_1} c_{s-1,0} \quad \text{if } s \geq 1 \\ c_{\tilde{s},1} &= \beta R_1 \frac{w_0}{w_1} c_{\tilde{s}-1,0} \quad \text{if } \tilde{s} \geq 1 \end{aligned}$$

This implies that:

$$\frac{c_{s,1}}{c_{\tilde{s},1}} = \frac{c_{s-1,0}}{c_{\tilde{s}-1,0}} \text{ for any } s \geq 1 \text{ and } \tilde{s} \geq 1. \quad (149)$$

At time $t = 0$, in a stationary equilibrium, the consumption distribution follows:

$$c_{s,0} = (\beta R_0)^s c_0 \text{ for any } s \geq 0$$

Hence,

$$\frac{c_{s-1,0}}{c_{\tilde{s}-1,0}} = \frac{(\beta R_0)^{s-1} c_0}{(\beta R_0)^{\tilde{s}-1} c_0} = \frac{(\beta R_0)^s c_0}{(\beta R_0)^{\tilde{s}} c_0} = \frac{c_{s,0}}{c_{\tilde{s},0}} \text{ for any } s \geq 1 \text{ and } \tilde{s} \geq 1 \quad (150)$$

By combining (149) and (150), we obtain (148) □

A.3 Proofs: Section 5 (Aggregate Shocks)

A.3.1 A Sufficient Condition in an Economy with Stochastic Productivity

Here we derive a sufficient condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ in an economy with stochastic aggregate productivity but without stochastic depreciation. The condition is on the magnitude of aggregate growth rate shocks ϵ . A sufficient condition in an economy with stochastic depreciation is derived in Section A.3.2. We first state two Lemmas that are useful to find a sufficient condition on ϵ .

Lemma 10. Define $\tilde{K}_t := \frac{K_t}{A_t}$. The law of motion of capital (39) is expressed as:

$$\tilde{K}_{t+1} = \frac{A_t}{A_{t+1}} \left[\hat{s} \tilde{K}_t^\theta + (1 - \hat{\delta}) \tilde{K}_t \right]. \quad (151)$$

Given the law of motion and the aggregate productivity process (74), assuming that \tilde{K}_t takes a positive finite value, the maximum value and the minimum value of $\frac{K_t}{A_t}$ is given by:

$$\tilde{K}^{max} = \left(\frac{\hat{s}}{\hat{\delta} - \epsilon} \right)^{\frac{1}{1-\theta}} \quad \text{and} \quad \tilde{K}^{min} = \left(\frac{\hat{s}}{\hat{\delta} + \epsilon} \right)^{\frac{1}{1-\theta}} \quad (152)$$

Proof. Deflating the law of motion of capital (39) by A_{t+1} gives equation (151). Note that the RHS of equation (151) is increasing in \tilde{K}_t and decreasing in A_{t+1} . \tilde{K}^{min} in equation (152) solves:

$$\tilde{K}^{min} = \frac{1}{1 + \epsilon} \left[\hat{s} \left(\tilde{K}^{min} \right)^\theta + (1 - \hat{\delta}) \tilde{K}^{min} \right].$$

A comparison of (151) to (152) shows that $\tilde{K}_t \geq \tilde{K}^{min}$ and $A_{t+1} \leq (1 + \epsilon)A_t$ implies $\tilde{K}_{t+1} \geq \tilde{K}^{min}$. Likewise, \tilde{K}^{max} solves:

$$\tilde{K}^{max} = \frac{1}{1 - \epsilon} \left[\hat{s} \left(\tilde{K}^{max} \right)^\theta + (1 - \hat{\delta}) \tilde{K}^{max} \right].$$

The symmetric argument shows that $\tilde{K}_t \leq \tilde{K}^{max}$ and $A_{t+1} \geq (1 - \epsilon)A_t$ implies $\tilde{K}_{t+1} \leq \tilde{K}^{max}$. □

Lemma 11.

1. The constraint $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ is most likely to be violated at the time of the lowest negative shock ($\frac{A_{t+1}}{A_t} = 1 - \epsilon$).
2. If $\tilde{K}_t = \tilde{K}^{min}$, $\tilde{K}_{t+1} = \frac{A_t}{A_{t+1}}(1 + \epsilon)\tilde{K}^{min}$. If $\tilde{K}_t = \tilde{K}^{max}$, $\tilde{K}_{t+1} = \frac{A_t}{A_{t+1}}(1 - \epsilon)\tilde{K}^{max}$.
3. Suppose $\frac{A_{t+1}}{A_t} = 1 - \epsilon$ and $\tilde{K}_t \in [\tilde{K}^{min}, \tilde{K}^{max}]$. R_{t+1} is highest at $\tilde{K}_t = \tilde{K}^{min}$. $\frac{w_{t+1}}{w_t}$ is lowest at $\tilde{K}_t = \tilde{K}^{max}$

Proof. 1. Note that

$$R_{t+1} = \theta \tilde{K}_{t+1}^{\theta-1} + 1 - \delta,$$

$$\text{and } \frac{w_{t+1}}{w_t} = \frac{A_{t+1}}{A_t} \left(\frac{\tilde{K}_{t+1}}{\tilde{K}_t} \right)^\theta = \left(\frac{A_{t+1}}{A_t} \right)^{1-\theta} \left(\hat{s} \tilde{K}_t^{\theta-1} + 1 - \hat{\delta} \right)^\theta.$$

Then, one could multiply $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ with $\left(\frac{A_t}{A_{t+1}} \right)^{1-\theta}$ to rewrite the proof as:

$$\beta \left[\theta \left(\frac{K_{t+1}}{A_t} \right)^{\theta-1} + (1 - \delta) \left(\frac{A_t}{A_{t+1}} \right)^{1-\theta} \right] < \left(\hat{s} \tilde{K}_t^{\theta-1} + 1 - \hat{\delta} \right)^\theta. \quad (153)$$

The left hand side is strictly decreasing in $\frac{A_{t+1}}{A_t}$ unless $\delta = 1$. Therefore, the condition is easier to be violated if $\frac{A_{t+1}}{A_t} = 1 - \epsilon$. Although the condition does not depend on $\frac{A_{t+1}}{A_t}$

under $\delta = 1$, it is still sufficient to check the condition at the time of negative shock ($\frac{A_{t+1}}{A_t} = 1 - \epsilon$) for any $0 < \delta \leq 1$.

2. \tilde{K}_{t+1} is given by (151):

$$\tilde{K}_{t+1} = \frac{A_t}{A_{t+1}} \left[\hat{s} \tilde{K}_t^\theta + (1 - \hat{\delta}) \tilde{K}_t \right].$$

If $\tilde{K}_t = \tilde{K}^{min}$ and $\frac{A_{t+1}}{A_t} = 1 + \epsilon$, we know:

$$\begin{aligned} \tilde{K}_{t+1} &= \frac{1}{1 + \epsilon} \left[\hat{s} (\tilde{K}^{min})^\theta + (1 - \hat{\delta}) \tilde{K}^{min} \right] = \tilde{K}^{min} \\ \therefore \left[\hat{s} (\tilde{K}^{min})^\theta + (1 - \hat{\delta}) \tilde{K}^{min} \right] &= (1 + \epsilon) \tilde{K}^{min} \end{aligned}$$

Hence, if $\tilde{K}_t = \tilde{K}^{min}$, we derive:

$$\tilde{K}_{t+1} = \frac{A_t}{A_{t+1}} (1 + \epsilon) \tilde{K}^{min} \quad (154)$$

Similarly, if $\tilde{K}_t = \tilde{K}^{max}$, we have

$$\tilde{K}_{t+1} = \frac{A_t}{A_{t+1}} (1 - \epsilon) \tilde{K}^{max}. \quad (155)$$

3. R_{t+1} is decreasing in \tilde{K}_{t+1} , since R_{t+1} is given by:

$$R_{t+1} = \theta \tilde{K}_{t+1}^{\theta-1} + 1 - \delta.$$

\tilde{K}_{t+1} is determined by (151):

$$\tilde{K}_{t+1} = \frac{A_t}{A_{t+1}} \left[\hat{s} \tilde{K}_t^\theta + (1 - \delta) \tilde{K}_t \right].$$

Given $\frac{A_t}{A_{t+1}} = \frac{1}{1 - \epsilon}$, \tilde{K}_{t+1} is increasing in \tilde{K}_t . Therefore, R_{t+1} is decreasing in \tilde{K}_t and takes the maximum value at $\tilde{K}_t = \tilde{K}^{min}$.

$\frac{w_{t+1}}{w_t}$ is given by:

$$\begin{aligned} \frac{w_{t+1}}{w_t} &= \frac{A_{t+1}}{A_t} \left(\frac{\tilde{K}_{t+1}}{\tilde{K}_t} \right)^\theta \\ &= \left(\frac{A_{t+1}}{A_t} \right)^{1-\theta} \left[\hat{s} \tilde{K}_t^{\theta-1} + 1 - \hat{\delta} \right]^\theta, \end{aligned}$$

which is decreasing in \tilde{K}_t . Therefore, $\frac{w_{t+1}}{w_t}$ takes the minimum value at $\tilde{K}_t = \tilde{K}^{max}$. \square

Now we introduce the following assumption. For it, recall the definitions of $\hat{s} = \left[\frac{\xi\beta}{1-(1-\nu-\xi)\beta} (1 - \theta) + (1 - \nu)\beta\theta \right]$ and $1 - \hat{\delta} = (1 - \nu)\beta(1 - \delta)$.

Assumption G.

$$\beta \left[\theta \left(\frac{1 - \epsilon}{1 + \epsilon} \right)^{1 - \theta} \frac{\hat{\delta} + \epsilon}{\hat{s}} + 1 - \delta \right] < 1 - \epsilon \quad (156)$$

Proposition 15. *Suppose Assumptions 5 hold (which insures the existence of a partial insurance steady state) and that the economy is in this steady state at $t = 0$ (as described in Proposition 3). Furthermore assume that Assumption G is satisfied for the aggregate productivity process in (74). Then the condition $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ holds for all $t \geq 1$ with probability 1. Furthermore, there exist an $\bar{\epsilon} > 0$ such that for all $0 \leq \epsilon < \bar{\epsilon}$, Assumption G is satisfied.*

Proof. Since the economy is in a steady state at $t = 0$ and Assumption 5 is satisfied, the initial capital K_0 and the associated interest rate R_0 satisfy the following:

$$\begin{aligned} \tilde{K}^{\min} &< \frac{K_0}{A_0} = \left(\frac{\hat{s}}{\hat{\delta}} \right)^{\frac{1}{1-\theta}} < \tilde{K}^{\max} \\ \beta R_0 &< 1. \end{aligned}$$

Moreover, the initial wealth distribution $\{a_{0,s}\}_{s=0}^{\infty}$ is given by the stationary distribution described in equations (44)–(46), which satisfies Assumption 4.

Suppose sequences of all possible prices $\{w_t(A^t), R_t(A^t), q_t(A_{t+1}, z_{t+1}|A^t, z^t)\}_{A^t, A_{t+1}, z^t, z_{t+1}}$ satisfy Assumptions 2 and 3. Then, the household's optimal consumption and asset allocation is given by equation (21) and (22) in Proposition 1, and the law of motion of capital follows equation (39) in Proposition 2. We want to verify that under Assumption G, Assumptions 2 and 3 are indeed satisfied in an equilibrium. See Subsection 3.3 for the claim that the prices of the contingent claims are of the form stipulated in Assumption 2. Since $\beta R_0 < 1$ is satisfied in the steady state, our focus in this Proposition is to show that:

$$\beta R_{t+1} < \frac{w_{t+1}}{w_t} \text{ for all } t \geq 0 \text{ and } A^{t+1}.$$

By Lemma 10, $\tilde{K}^{\min} < \tilde{K}_0 < \tilde{K}^{\max}$ implies $\tilde{K}^{\min} < \tilde{K}_t < \tilde{K}^{\max}$ for all $t \geq 1$. We saw in Lemma 11 that the condition $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ is most likely to be violated at the time of the lowest negative shock. Given the negative shock, R_{t+1} achieves the maximum if $\tilde{K}_t = \tilde{K}^{\min}$, and $\frac{w_{t+1}}{w_t}$ takes the minimum if $\tilde{K}_t = \tilde{K}^{\max}$. If the maximum of βR_{t+1} is smaller than the minimum of $\frac{w_{t+1}}{w_t}$ with $\frac{A_{t+1}}{A_t} = 1 - \epsilon$, the condition is satisfied for all \tilde{K}_t and $\frac{A_{t+1}}{A_t}$. By imposing those, we have a sufficient condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$, where

$$\begin{aligned} \beta R_{t+1} &< \frac{w_{t+1}}{w_t} \\ \Leftrightarrow \beta \left[\theta \tilde{K}_{t+1}^{\theta-1} + 1 - \delta \right] &< \left(\frac{A_{t+1}}{A_t} \right)^{1-\theta} \left[\hat{s} \tilde{K}_t^{\theta-1} + 1 - \hat{\delta} \right]^{\theta}. \end{aligned}$$

\tilde{K}_{t+1} in the left hand side takes the minimum at $\tilde{K}_{t+1} = \frac{A_t}{A_{t+1}}(1 + \epsilon)\tilde{K}^{min}$. \tilde{K}_t in the right hand side takes the maximum at $\tilde{K}_t = \tilde{K}^{max}$. $\frac{A_{t+1}}{A_t}$ is given by $1 - \epsilon$. Therefore, a sufficient condition is given by:

$$\beta \left[\theta \left[\frac{A_t}{A_{t+1}}(1 + \epsilon)\tilde{K}^{min} \right]^{\theta-1} + 1 - \delta \right] < \left(\frac{A_{t+1}}{A_t} \right)^{1-\theta} \left[\hat{s}(\tilde{K}^{max})^{\theta-1} + 1 - \hat{\delta} \right]^\theta \quad (157)$$

By substituting \tilde{K}^{max} and \tilde{K}^{min} given by equations (152) in Lemma 10, we obtain Assumption G. Under this condition, $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all $t \geq 0$ and all states $\frac{A_{t+1}}{A_t} \in [1 - \epsilon, 1 + \epsilon]$.

Now, we prove the existence of $\bar{\epsilon}$. If $\epsilon = 0$ in Assumption G, we have:

$$\beta \left[\theta \frac{\hat{\delta}}{\hat{s}} + 1 - \delta \right] < 1. \quad (158)$$

In the steady state, \tilde{K} satisfies:

$$\tilde{K}^* = \left(\frac{\hat{\delta}}{\hat{s}} \right)^{\frac{1}{1-\theta}}. \quad (159)$$

Therefore, equation (158) is equivalent to:

$$\begin{aligned} \beta \left[\theta(\tilde{K}^*)^{\theta-1} + 1 - \delta \right] &< 1 \\ \Leftrightarrow \beta R^* &< 1. \end{aligned}$$

This is equivalent to Assumption 5. This means that Assumption G is satisfied in an open neighborhood of $\epsilon = 0$, since Assumption G is continuous in ϵ .

We show that Assumption G becomes monotonically more restrictive as ϵ increases. Assumption G is equivalent to:

$$\left[\theta \left(\frac{1 + \epsilon}{1 - \epsilon} \right)^\theta \left(\frac{\hat{\delta} + \epsilon}{1 + \epsilon} \right) \frac{1}{\hat{s}} + \frac{1 - \delta}{1 - \epsilon} \right] < 1.$$

$\left(\frac{1 + \epsilon}{1 - \epsilon} \right)$ and $\frac{1 - \delta}{1 - \epsilon}$ are strictly increasing in ϵ . $\frac{\hat{\delta} + \epsilon}{1 + \epsilon}$ is weakly increasing in ϵ for all $0 < \hat{\delta} \leq 1$. Therefore, the left hand side is strictly increasing in ϵ . This means that the condition becomes tighter as ϵ increases. Hence, there exists $\bar{\epsilon} > 0$ such that Assumption G is satisfied for all $0 \leq \epsilon < \bar{\epsilon}$. \square

The following is an alternative statement if we don't assume that the economy is in a steady state at $t = 0$.

Corollary 3. *Define*

$$\tilde{K}^{max} = \left(\frac{\hat{s}}{\hat{\delta} - \epsilon} \right)^{\frac{1}{1-\theta}} \quad \text{and} \quad \tilde{K}^{min} = \left(\frac{\hat{s}}{\hat{\delta} + \epsilon} \right)^{\frac{1}{1-\theta}} \quad (160)$$

Suppose Assumptions 4 and 5 hold (which insure the existence of a partial insurance steady state), that the initial aggregate capital at $t = 0$ satisfies $\min \left\{ \tilde{K}^{\min}, \left(\frac{\theta}{\frac{1}{\beta} - 1 + \delta} \right)^{\frac{1}{1-\theta}} \right\} < \tilde{K}_0 < \tilde{K}^{\max}$. Furthermore assume that Assumption G is satisfied for the aggregate productivity process in (74). Then the condition $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ holds for all t with probability 1. Furthermore, there exist an $\bar{\epsilon} > 0$ such that for all $0 \leq \epsilon < \bar{\epsilon}$, Assumption G is satisfied.

Proof. The same logic goes through as in Proposition 15. Since the economy may not be in a steady state at $t = 0$, the initial capital K_0 must satisfy that:

$$\tilde{K}^{\min} < \frac{K_0}{A_0} < \tilde{K}^{\max}$$

$$\text{and } \frac{K_0}{A_0} > \left[\frac{\theta}{\frac{1}{\beta} - 1 + \delta} \right]^{\frac{1}{1-\theta}}.$$

The second condition guarantees that $\beta R_0 < 1$. Assumption 4 on the initial wealth distribution is also introduced. \square

A.3.2 A Sufficient Condition in an Economy with Stochastic Depreciation

We define an economy with stochastic depreciation and derive a sufficient condition on fundamental parameters ensuring that the economy is always in a partial insurance equilibrium.

Sequential Market equilibrium We introduce aggregate state Ω_t and its entire history $\Omega^t := (\Omega_0, \dots, \Omega_t)$ that determine aggregate productivity $A_t(\Omega^t)$ and depreciation rate $\delta_t(\Omega_t)$.³⁸ Ω_t follows finite state first-order Markov. Capital depreciation rate δ_t is stochastic and is realized at the beginning of each period. The sequential market equilibrium in the limited-commitment model is defined as follows.

Definition 2. For an initial condition $(A_0(\Omega_0), \delta_0(\Omega_0), K_0, \Phi(a_0, z_0))$, an equilibrium is sequences of wages and interest rates $\{w_t(\Omega^t), R_t(\Omega^t)\}$, prices of contingent claims $\{q_t(\Omega_{t+1}, z_{t+1} | \Omega^t, z^t)\}$, aggregate consumption and capital $\{C_t(\Omega^t), K_{t+1}(\Omega^t)\}$ and individual consumption and asset allocations $\{\mathbf{c}_t(a_0, z^t, \Omega^t), \mathbf{a}_{t+1}(a_0, z^{t+1}, \Omega^{t+1})\}$ such that

1. Given $\{w_t(\Omega^t), R_t(\Omega^t), q_t(\Omega_{t+1}, z_{t+1} | \Omega^t, z^t)\}_{t=0, \Omega^t, z^t, \Omega_{t+1}, z_{t+1}}^\infty$, the household consumption and asset allocation $\{\mathbf{c}_t(a_0, z^t, \Omega^t), \mathbf{a}_{t+1}(a_0, z^{t+1}, \Omega^{t+1})\}$, for all initial conditions (a_0, z_0) , maximizes the expected lifetime utility:

$$\max_{\{\mathbf{c}_t(a_0, z^t, \Omega^t), \mathbf{a}_{t+1}(a_0, z^{t+1}, \Omega^{t+1})\}} \sum_{t=0}^{\infty} \sum_{\Omega^t} \sum_{z^t} \beta^t \pi(\Omega^t) \pi(z^t) \log(\mathbf{c}_t(a_0, z^t, \Omega^t)) \quad (161)$$

³⁸Aggregate productivity growth rate $g_{A,t+1} := \frac{A_{t+1}}{A_t}$ and depreciation rates δ_{t+1} depend only on Ω_{t+1} .

subject to the budget constraints:

$$\begin{aligned} \mathbf{c}_t(a_0, z^t, \Omega^t) + \sum_{\Omega^{t+1}} \sum_{z_{t+1}} q_t(\Omega_{t+1}, z_{t+1} | \Omega^t, z^t) \mathbf{a}_{t+1}(a_0, z^{t+1}, \Omega^{t+1}) \\ = w_t(\Omega^t) z_t + R_t(\Omega^t) \mathbf{a}_t(a_0, z^t, \Omega^t) \end{aligned} \quad (162)$$

and the shortsale constraints:

$$\mathbf{a}_{t+1}(a_0, z^{t+1}, \Omega^{t+1}) \geq 0. \quad (163)$$

2. Factor prices equal marginal products

$$w_t(\Omega^t) = (1 - \theta) A_t(\Omega^t) \left(\frac{K_t(\Omega^{t-1})}{A_t(\Omega^t)} \right)^\theta \quad (164)$$

$$R_t(\Omega^t) = \theta \left(\frac{K_t(\Omega^{t-1})}{A_t(\Omega^t)} \right)^{\theta-1} + 1 - \delta_t(\Omega_t) \quad (165)$$

3. The goods market and capital market clear

$$C_t(\Omega^t) + K_{t+1}(\Omega^t) = (K_t(\Omega^{t-1}))^\theta A_t(\Omega^t)^{1-\theta} + (1 - \delta_t(\Omega_t)) K_t(\Omega^{t-1}) \quad (166)$$

$$K_{t+1}(\Omega^t) = \int \sum_{z^{t+1}} \mathbf{a}_{t+1}(a_0, z^{t+1}, \Omega^{t+1}) \pi(z^{t+1}) d\Phi(a_0, z_0) \quad \forall \Omega^{t+1} \quad (167)$$

where

$$C_t(\Omega^t) = \int \sum_{z^t} \mathbf{c}_t(a_0, z^t, \Omega^t) \pi(z^t) d\Phi(a_0, z_0) \quad (168)$$

The price of contingent claims and the no-saving condition are given by the following.

Assumption 2' (Contingent Claims Prices).

$$q_t(\Omega_{t+1}, z_{t+1} | \Omega^t, z^t) = \pi(\Omega_{t+1} | \Omega^t) \pi(z_{t+1} | z^t) \quad (169)$$

Assumption 3' (No Savings Incentives).

$$\beta R_0(\Omega_0) < 1 \quad (170)$$

$$\beta R_{t+1}(\Omega^{t+1}) < \frac{w_{t+1}(\Omega^{t+1})}{w_t(\Omega^t)} \text{ for all } t \geq 0 \text{ and } \Omega^{t+1} \quad (171)$$

Optimal Allocation We conjecture that the optimal consumption and asset allocation of individual households is given by

$$\mathbf{c}_t(a_0, z^t, \Omega^t) = \begin{cases} \bar{c} w_t(\Omega^t), \text{ where } \bar{c} := \frac{1-(1-\nu)\beta}{1-(1-\nu-\xi)\beta} \zeta, & \text{if } z_t = \zeta \\ [1 - (1 - \nu)\beta] R_t(\Omega^t) \mathbf{a}_t(a_0, z^t, \Omega^t) & \text{if } z_t = 0 \end{cases} \quad (172)$$

$$\mathbf{a}_{t+1}(a_0, z^{t+1}, \Omega^{t+1}) = \begin{cases} 0 & \text{if } z_{t+1} = \zeta \\ \frac{\beta}{1-(1-\nu-\xi)\beta} \zeta w_t(\Omega^t) & \text{if } z_t = \zeta \text{ and } z_{t+1} = 0 \\ \beta R_t(\Omega^t) \mathbf{a}_t(a_0, z^t, \Omega^t) & \text{if } z_t = 0 \text{ and } z_{t+1} = 0. \end{cases} \quad (173)$$

Given Assumptions 2' and 3', the household's problem (161)–(163) is unchanged from the household's problem defined in Section 2.5 in the presence of stochastic depreciation rate. Therefore, Proposition 1 implies that the household allocation (172) and (173) is optimal under Assumptions 2', 3', and 4.³⁹

The aggregate law of motion is given by:

$$K_{t+1}(\Omega^t) = \hat{s}A_t(\Omega^t)^{1-\theta}K_t(\Omega^{t-1})^\theta + (1 - \hat{\delta}_t(\Omega_t))K_t(\Omega^{t-1}), \quad (176)$$

where $\hat{\delta}_t(\Omega_t)$ depends on time-varying $\delta(\Omega_t)$:

$$\hat{\delta}_t(\Omega_t) = 1 - (1 - \nu)\beta(1 - \delta_t(\Omega_t)). \quad (177)$$

Now we prove Proposition 11. The procedure closely follows that of Proposition 15.

Proposition 11 .

Proof. We want to find a sufficient condition on (ϵ, ς) such that the following holds:

$$\beta R_{t+1}(\Omega^{t+1}) < \frac{w_{t+1}(\Omega^{t+1})}{w_t(\Omega^t)} \text{ for any } (t, \Omega^t, \Omega_{t+1}). \quad (178)$$

First, as in Lemma 10, the minimum and maximum productivity-adjusted capital in the economy, \tilde{K}^{\min} and \tilde{K}^{\max} , are given as follows:

$$\tilde{K}^{\min} := \left(\frac{\hat{s}}{\hat{\delta}_{\max} + \epsilon} \right)^{\frac{1}{1-\theta}} \quad (179)$$

$$\tilde{K}^{\max} := \left(\frac{\hat{s}}{\hat{\delta}_{\min} - \epsilon} \right)^{\frac{1}{1-\theta}} \quad (180)$$

where

$$\hat{\delta} = 1 - (1 - \nu)\beta(1 - \delta) \quad (181)$$

with $\delta_{\min} = \bar{\delta} - \varsigma$ and $\delta_{\max} = \bar{\delta} + \varsigma$. This is because a higher depreciation rate (i.e., $\hat{\delta} = \hat{\delta}_{\max}$) implies lower aggregate capital, and vice versa. To guarantee $\tilde{K}^{\max} > 0$, the parameters must satisfy $\hat{\delta}_{\min} - \epsilon > 0$. This is satisfied under Assumption 8, since

$$\begin{aligned} 1 - \epsilon &> \beta \left[\theta \left(\frac{1 - \epsilon}{1 + \epsilon} \right)^{1-\theta} \frac{\hat{\delta}_{\max} + \epsilon}{\hat{s}} + 1 - \delta_{\min} \right] \\ &> \beta(1 - \delta_{\min}) \\ &> 1 - \hat{\delta}_{\min}. \end{aligned}$$

³⁹Verifying Assumption 2' is immediate, as the Euler equation of unconstrained agents implies:

$$q_t(\Omega_{t+1}, z_{t+1} | \Omega^t, z^t) = \pi(\Omega_{t+1} | \Omega^t) \pi(z_{t+1} | z^t) \beta R(\Omega^{t+1}) \frac{\mathbf{c}_t(a_0, z^t, \Omega^t)}{\mathbf{c}_{t+1}(a_0, z^{t+1}, \Omega^{t+1})} \quad (174)$$

$$\text{where } \mathbf{c}_{t+1}(a_0, z^{t+1}, \Omega^{t+1}) = \beta R(\Omega^{t+1}) \mathbf{c}_t(a_0, z^t, \Omega^t) \quad (175)$$

Notice that δ_{\min} can take a negative value, in which case capital appreciates over time. The condition $\hat{\delta}_{\min} - \epsilon > 0$ is equivalent to:

$$\delta_{\min} > 1 - \frac{1 - \epsilon}{(1 - \nu)\beta}. \quad (182)$$

Since $(1 - \nu)\beta < 1$, δ_{\min} can be negative in equation (182).

Second, since wage growth $\frac{w_{t+1}}{w_t}$ is not impacted by δ_{t+1} while the interest rate R_{t+1} is decreasing in δ_{t+1} , the condition (178) is more likely to be violated if δ_{t+1} is smaller. Hence, a sufficient condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at any $(t, \Omega^t, \Omega_{t+1})$ is given by:

$$\beta R_{t+1}(\Omega^{t+1}) \Big|_{\tilde{K}_t = \tilde{K}^{\min}, \frac{A_{t+1}}{A_t} = 1 - \epsilon, \delta_{t+1} = \delta_{\min}} < \frac{w_{t+1}(\Omega^{t+1})}{w_t(\Omega^t)} \Big|_{\tilde{K}_t = \tilde{K}^{\max}, \frac{A_{t+1}}{A_t} = 1 - \epsilon}. \quad (183)$$

By substituting the interest rate, $R_{t+1}(\Omega^{t+1})$, and wage growth rate, $\frac{w_{t+1}(\Omega^{t+1})}{w_t(\Omega^t)}$, and imposing \tilde{K}^{\min} and \tilde{K}^{\max} in equations (179) and (180), the sufficient condition that corresponds to equation (157) in Proposition 15 is given by:

$$\beta \left[\theta \left(\frac{1 + \epsilon}{1 - \epsilon} \tilde{K}^{\min} \right)^{\theta-1} + 1 - \delta_{\min} \right] < (1 - \epsilon)^{1-\theta} \left[\hat{s}(\tilde{K}^{\max})^{\theta-1} + 1 - \hat{\delta}_{\min} \right]^{\theta}. \quad (184)$$

Solving this inequality gives the sufficient condition (79).

Now we show the existence of an open neighborhood \mathcal{N} around $(\epsilon, \varsigma) = (0, 0)$ such that Assumption 8 is satisfied for all $(\epsilon, \varsigma) \in \mathcal{N}$. As shown in Proposition 15, Assumption 8 is satisfied if $(\epsilon, \varsigma) = (0, 0)$, since the condition (79) is equivalent to Assumption 5 if $(\epsilon, \varsigma) = (0, 0)$. Proposition 15 implies that condition (79) is more likely to be violated as ϵ increases. Likewise, since the left-hand side of inequality (79) is monotonically increasing in ς , Assumption 8 is more likely to be violated as ς increases. Since the condition (79) is continuous in (ϵ, ς) , there exists an open neighborhood \mathcal{N} such that condition (79) is satisfied for all $(\epsilon, \varsigma) \in \mathcal{N}$.

Put differently, given a fixed $\epsilon \geq 0$ that satisfies Assumption G, there exist $\bar{\varsigma} > 0$ such that Assumption 8 is satisfied for all $0 \leq \varsigma < \bar{\varsigma}$. In this case, $(\epsilon, \bar{\varsigma})$ solves the condition (79) with equality. Likewise, given a fixed $\varsigma \geq 0$ that satisfies Assumption 8 with $\epsilon = 0$, there exists $\bar{\epsilon} > 0$ such that Assumption 8 is satisfied for all $0 \leq \epsilon < \bar{\epsilon}$. In this case, $(\bar{\epsilon}, \varsigma)$ solves the condition (79) with equality. □

A.3.3 Subsection 5.2.1: The Risk-Free Rate and the Risk Premium

Lemma 1 .

Proof. We first derive a pricing kernel that allows us to compute the price of any securities, including the price of risk-free bonds $q^B(\Omega^t)$ and the price of risky capital $q^K(\Omega^t)$. A

pricing kernel is defined as the price of one unit of non-deflated consumption goods at time $t + 1$ in a state Ω^{t+1} conditional on the state at t being Ω^t :⁴⁰

$$Q(\Omega^{t+1}|\Omega^t) = \beta \frac{u'(\mathbf{c}_{t+1}(\Omega^{t+1}))}{u'(\mathbf{c}_t(\Omega^t))} \pi(\Omega^{t+1}|\Omega^t). \quad (185)$$

Since we assume a logarithmic utility function, the marginal utility of consumption is given by:

$$u'(\mathbf{c}_t) = \frac{1}{\mathbf{c}_t}.$$

In addition, in the limited commitment model, consumption of asset holders (i.e., households in a low-income state at $t + 1$) follows the Euler equation:

$$\mathbf{c}_{t+1}(\Omega^{t+1}; z_{t+1} = 0) = \beta R_{t+1}(\Omega^{t+1}) \mathbf{c}_t(\Omega^t).$$

Therefore, the pricing kernel in the limited-commitment model is given by:

$$Q(\Omega^{t+1}|\Omega^t) = \frac{1}{R_{t+1}(\Omega^{t+1})} \pi(\Omega^{t+1}|\Omega^t). \quad (186)$$

Given this pricing kernel, we can derive the price of any securities. The price of risk-free bonds that yield one unit of consumption at $t + 1$ regardless of the aggregate state Ω^{t+1} is given by:

$$\begin{aligned} q^B(\Omega^t) &= \sum_{\Omega^{t+1}|\Omega^t} Q(\Omega^{t+1}|\Omega^t) \cdot 1 \\ &= \sum_{\Omega^{t+1}|\Omega^t} \frac{1}{R_{t+1}(\Omega^{t+1})} \pi(\Omega^{t+1}|\Omega^t) \\ &= \mathbb{E}_t \left[\frac{1}{R_{t+1}(\Omega^{t+1})} \right]. \end{aligned} \quad (187)$$

The price of risky assets that yields $R_{t+1}(\Omega^{t+1})$ depending on the aggregate state Ω^{t+1} is given by:

$$\begin{aligned} q^K(\Omega^t) &= \sum_{\Omega^{t+1}|\Omega^t} Q(\Omega^{t+1}|\Omega^t) R_{t+1}(\Omega^{t+1}) \\ &= \sum_{\Omega^{t+1}|\Omega^t} \frac{1}{R_{t+1}(\Omega^{t+1})} \pi(\Omega^{t+1}|\Omega^t) R_{t+1}(\Omega^{t+1}) = 1. \end{aligned} \quad (188)$$

Now we compute the expected return on these two assets. The expected rate of return on risk-free bonds is given by:

$$\mathbb{E}_t \left[\frac{1}{q^B(\Omega^t)} \right] = \frac{1}{\sum_{\Omega^{t+1}|\Omega^t} \frac{1}{R_{t+1}(\Omega^{t+1})} \pi(\Omega^{t+1}|\Omega^t)} =: \frac{1}{\mathbb{E}_t[1/R_{t+1}(\Omega^{t+1})]}. \quad (189)$$

⁴⁰See Ljungqvist and Sargent (2018) p.270

The expected rate of return on risky assets is given by:

$$\mathbb{E}_t \left[\frac{R_{t+1}(\Omega^{t+1})}{1} \right] = \mathbb{E}_t [R_{t+1}(\Omega^{t+1})]. \quad (190)$$

Hence, the risk premium is given by:

$$1 + \lambda_t^{LC} := \frac{\mathbb{E}_t[R_{t+1}(\Omega^{t+1})]}{\mathbb{E}_t[1/q^B(\Omega^t)]} = \mathbb{E}_t[R_{t+1}(\Omega^{t+1})] \mathbb{E}_t \left[\frac{1}{R_{t+1}(\Omega^{t+1})} \right] > 1. \quad (191)$$

The risk premium is strictly larger than 1 because $R_{t+1}(\Omega^{t+1})$ is a non-trivial random variable and Jensen's inequality holds with strict inequality.⁴¹ \square

Lemma 2 .

Proof. As before, the pricing kernel is given by:

$$\begin{aligned} Q(\Omega^{t+1}|\Omega^t) &= \beta \frac{u'(\mathbf{c}_{t+1}(\Omega^{t+1}))}{u'(\mathbf{c}_t(\Omega^t))} \pi(\Omega^{t+1}|\Omega^t) \\ &= \beta \frac{C_t(\Omega^t)}{C_{t+1}(\Omega^{t+1})} \pi(\Omega^{t+1}|\Omega^t). \end{aligned} \quad (193)$$

The second line holds since consumption by a unit measure of representative households is the same as the aggregate consumption.

Then, the price of bonds is given by:

$$\begin{aligned} q^B(\Omega^t) &= \sum_{\Omega^{t+1}|\Omega^t} Q(\Omega^{t+1}|\Omega^t) \cdot 1 \\ &= \mathbb{E}_t \left[\beta \frac{C_t(\Omega^t)}{C_{t+1}(\Omega^{t+1})} \right] \end{aligned} \quad (194)$$

The risk premium is given by:

$$1 + \lambda_t^{Rep} := \frac{\mathbb{E}_t[R_{t+1}(\Omega^{t+1})]}{\mathbb{E}_t[1/q^B(\Omega^t)]} = \mathbb{E}_t[R_{t+1}(\Omega^{t+1})] \mathbb{E}_t \left[\beta \frac{C_t(\Omega^t)}{C_{t+1}(\Omega^{t+1})} \right] \quad (195)$$

Note that λ_t^{Rep} can be positive or negative, depending on the covariance between $R_{t+1}(\Omega^{t+1})$ and $\beta \frac{C_t(\Omega^t)}{C_{t+1}(\Omega^{t+1})}$. A sequence of aggregate consumption C_t follows the Euler equation:

$$\frac{1}{C_t(\Omega^t)} = \beta \mathbb{E}_t \left[R_{t+1}(\Omega^{t+1}) \frac{1}{C_{t+1}(\Omega^{t+1})} \right] \quad (196)$$

⁴¹If $g(\cdot)$ is a convex function, $\mathbb{E}[g(X)] \geq g(\mathbb{E}[X])$, where X is a random variable. Equality holds only if $P(g(X) = a + bX) = 1$, where $a + bX$ is tangent to $g(\cdot)$ at $\mathbb{E}[X]$. The Jensen's inequality implies:

$$\mathbb{E}_t \left[\frac{1}{R_{t+1}(\Omega^{t+1})} \right] > \frac{1}{\mathbb{E}_t [R_{t+1}(\Omega^{t+1})]}. \quad (192)$$

This gives:

$$\begin{aligned} 1 &= \mathbb{E}_t \left[\beta R_{t+1}(\Omega^{t+1}) \frac{C_t(\Omega^t)}{C_{t+1}(\Omega^{t+1})} \right] \\ &= \mathbb{E}_t [R_{t+1}(\Omega^{t+1})] \mathbb{E}_t \left[\beta \frac{C_t(\Omega^t)}{C_{t+1}(\Omega^{t+1})} \right] + \text{cov}_t \left(R_{t+1}(\Omega^{t+1}), \beta \frac{C_t(\Omega^t)}{C_{t+1}(\Omega^{t+1})} \right). \end{aligned} \quad (197)$$

If the covariance term is negative, the risk premium in the representative agent model is positive. \square

A.3.4 Subsection 5.2.2: Economy with $\delta = 1$

Proposition 12.

Proof. The economy is in a steady state at time $t = 0$. Since the aggregate capital in the steady state is given by:

$$K_0 = A_0 s^{\frac{1}{1-\theta}}, \text{ where } s \in \{s^{\text{LC}}, s^{\text{RE}}\}, \quad (198)$$

and the saving rate, s , is higher in the limited-commitment model:

$$\beta\theta =: s^{\text{RE}} < s^{\text{LC}} := \beta\theta + \underbrace{(1-\theta)\nu \left[\frac{\xi}{\nu \left(\frac{1}{\beta} - 1 + \xi + \nu \right)} - \frac{\theta}{(1-\theta)\frac{1}{\beta}} \right]}_{>0 \text{ under Assumption 5 and } \delta=1}, \quad (199)$$

the limited-commitment model starts with larger initial capital. As discussed in Appendix B.2.1, the capital stock $K_t(\Omega^t)$ after any productivity history is characterized in closed-form as

$$\log K_t = (1 + \theta + \dots + \theta^{t-2}) \log s + (1 - \theta) \left[\sum_{\tau=1}^{t-1} \theta^{\tau-1} \log A_{t-\tau} \right] + \theta^{t-1} \log K_0. \quad (200)$$

Hence, given the history of aggregate shocks, the limited-commitment model always has larger capital than the representative agent model. This implies:

$$K_t^{\text{RE}}(\Omega^t) < K_t^{\text{LC}}(\Omega^t) \text{ for all } \Omega^t \quad (201)$$

$$\text{and } R_t^{\text{RE}}(\Omega^t) > R_t^{\text{LC}}(\Omega^t) \text{ for all } \Omega^t \quad (202)$$

Under full depreciation of capital, the price of risk-free bonds is given by the following expression in both models:

$$q^B(\Omega^t) = \mathbb{E}_t \left[\frac{1}{R_{t+1}(\Omega^{t+1})} \right]. \quad (203)$$

This can be shown by substituting the closed form of aggregate consumption in the representative-agent model, $C_t^{Rep} = (1 - \beta\theta)A_t^{1-\theta}K_t^\theta$, into equation (84):

$$\begin{aligned}
\mathbb{E}_t \left[\beta \frac{C_t^{Rep}(\Omega^t)}{C_{t+1}^{Rep}(\Omega^{t+1})} \right] &= \mathbb{E}_t \left[\beta \frac{(1 - \beta\theta)A_t^{1-\theta}K_t^\theta}{(1 - \beta\theta)A_{t+1}^{1-\theta}K_{t+1}^\theta} \right] \\
&= \mathbb{E}_t \left[\frac{\beta A_t^{1-\theta}K_t^\theta}{K_{t+1}A_{t+1}^{1-\theta}K_{t+1}^{\theta-1}} \right] \text{ where } K_{t+1} = \beta\theta A_t^{1-\theta}K_t^\theta \\
&= \mathbb{E}_t \left[\frac{1}{\theta A_{t+1}^{1-\theta}K_{t+1}^{\theta-1}} \right] = \mathbb{E}_t \left[\frac{1}{R_{t+1}^{Rep}(\Omega^{t+1})} \right]
\end{aligned} \tag{204}$$

Hence, higher return on capital in the representative agent model at any state Ω^t implies lower price of risk-free bonds and higher risk-free rate.

As we derived in Section 5.2.1, the risk premium in the two economies is given by:

$$\begin{aligned}
1 + \lambda_t^{LC} &= \mathbb{E}_t [R_{t+1}^{LC}(\Omega^{t+1})] \mathbb{E}_t \left[\frac{1}{R_{t+1}^{LC}(\Omega^{t+1})} \right] \\
1 + \lambda_t^{Rep} &= \mathbb{E}_t [R_{t+1}^{Rep}(\Omega^{t+1})] \mathbb{E}_t \left[\beta \frac{C_t^{Rep}(\Omega^t)}{C_{t+1}^{Rep}(\Omega^{t+1})} \right]
\end{aligned} \tag{205}$$

We explicitly derive the risk premium for each economy. In the limited-commitment economy, the risk premium is:

$$\begin{aligned}
1 + \lambda_t^{LC} &= \mathbb{E}_t [R_{t+1}^{LC}(\Omega^{t+1})] \mathbb{E}_t \left[\frac{1}{R_{t+1}^{LC}(\Omega^{t+1})} \right] \\
&= \mathbb{E}_t \left[\theta (K_{t+1}^{LC})^{\theta-1} (A_{t+1})^{1-\theta} \right] \mathbb{E}_t \left[\frac{1}{\theta (K_{t+1}^{LC})^{\theta-1} (A_{t+1})^{1-\theta}} \right] \\
&= \mathbb{E}_t \left[\theta (\hat{s} A_t^{1-\theta} K_t^\theta)^{\theta-1} (A_{t+1})^{1-\theta} \right] \mathbb{E}_t \left[\frac{1}{\theta (\hat{s} A_t^{1-\theta} K_t^\theta)^{\theta-1} (A_{t+1})^{1-\theta}} \right] \\
&= \mathbb{E}_t [A_{t+1}^{1-\theta}] \mathbb{E}_t \left[\frac{1}{A_{t+1}^{1-\theta}} \right]
\end{aligned} \tag{206}$$

In the representative agent economy,

$$\begin{aligned}
1 + \lambda_t^{Rep} &= \mathbb{E}_t [R_{t+1}^{Rep}(\Omega^{t+1})] \mathbb{E}_t \left[\beta \frac{C_t^{Rep}(\Omega^t)}{C_{t+1}^{Rep}(\Omega^{t+1})} \right] \\
&= \mathbb{E}_t \left[\theta (K_{t+1}^{Rep})^{\theta-1} (A_{t+1})^{1-\theta} \right] \mathbb{E}_t \left[\beta \frac{(1 - \beta\theta)A_t^{1-\theta}K_t^\theta}{(1 - \beta\theta)A_{t+1}^{1-\theta}(K_{t+1}^{Rep})^\theta} \right] \\
&= \mathbb{E}_t [A_{t+1}^{1-\theta}] \mathbb{E}_t \left[\frac{1}{A_{t+1}^{1-\theta}} \right]
\end{aligned} \tag{207}$$

This shows that the risk premium is the same between the two models under full depreciation of capital.

Note that the risk premium can be expressed as:

$$1 + \lambda_t = \mathbb{E}_t \left[\left(\frac{A_{t+1}}{A_t} \right)^{1-\theta} \right] \mathbb{E}_t \left[\left(\frac{A_t}{A_{t+1}} \right)^{1-\theta} \right]$$

If the aggregate growth rate follows an *iid* process,

$$\mathbb{E}_t \left[\left(\frac{A_{t+1}}{A_t} \right)^{1-\theta} \right] := \mathbb{E} \left[\left(\frac{A_{t+1}}{A_t} \right)^{1-\theta} \mid \Omega^t \right] = \mathbb{E} \left[\left(\frac{A_{t+1}}{A_t} \right)^{1-\theta} \right] \text{ for any } (t, \Omega^t).$$

Hence, the risk premium is constant over time and across aggregate states Ω^t . □

A.3.5 Subsection 5.2.3: Endowment Economy

In an endowment economy, aggregate consumption $\{C_t(\Omega^t)\}_{t,\Omega^t}$ is exogenous and depends only on the aggregate state Ω_t . Households face idiosyncratic income shocks and draw $z_t \in \{0, \zeta\}$ each period that follows a Markov transition probability as before. Households trade a state contingent Lucas tree $\sigma_{t+1}(\sigma_0, z^{t+1}, \Omega^{t+1})$ that delivers dividends depending on aggregate states Ω^{t+1} at time $t + 1$. In the limited commitment model, households are subject to a tight borrowing constraint. In a standard complete market model, the borrowing constraint never binds. Since we obtain the same stochastic discount factor in the standard complete market model and the representative agent model, we call such an economy a representative agent model.

A sequential market equilibrium is defined as in the production economy.

Definition 3. Given the price of Lucas tree $\{q_t^{LT}(\Omega^t)\}_{t=0,\Omega^t}^\infty$ and aggregate endowment $\{C_t(\Omega^t)\}_{t=0,\Omega^t}^\infty$, the household allocation $\{\mathbf{c}_t(\sigma_0, z^t, \Omega^t), \sigma_{t+1}(\sigma_0, z^{t+1}, \Omega^{t+1})\}$ solves, for all (σ_0, z_0) ,

$$\max_{\{\mathbf{c}_t(\sigma_0, z^t, \Omega^t), \sigma_{t+1}(\sigma_0, z^{t+1}, \Omega^{t+1})\}} \sum_{t=0}^{\infty} \sum_{\Omega^t} \sum_{z^t} \beta^t \pi(\Omega^t) \pi(z^t) \log(\mathbf{c}_t(\sigma_0, z^t, \Omega^t)) \quad (208)$$

subject to budget constraints and limited-commitment constraints:

$$\begin{aligned} \mathbf{c}_t(\sigma_0, z^t, \Omega^t) + \sum_{\Omega_{t+1}} \sum_{z_{t+1}} \pi(z_{t+1} | z_t) \pi(\Omega_{t+1} | \Omega^t) q_t^{LT}(\Omega^t) \sigma_{t+1}(\sigma_0, z^{t+1}, \Omega^{t+1}) \\ = \underbrace{[1 - \alpha] C_t(\Omega^t) z_t}_{\text{labor income}} + \underbrace{[\alpha C_t(\Omega^t) + q_t^{LT}(\Omega^t)] \sigma_t(\sigma_0, z^t, \Omega^t)}_{\text{dividends}} \end{aligned} \quad (209)$$

$$\sigma_{t+1}(\sigma_0, z^{t+1}, \Omega^{t+1}) \geq 0 \quad (210)$$

The goods market and Lucas tree market clear

$$\sum_{z^t} \int \pi(z^t) \mathbf{c}_t(\sigma_0, z^t, \Omega^t) d\Phi(\sigma_0, z_0) = C_t(\Omega^t) \quad (211)$$

$$\sum_{z^t} \int \pi(z^t) \sigma_t(\sigma_0, z^t, \Omega^t) d\Phi(\sigma_0, z_0) = 1. \quad (212)$$

Factor prices are given by:

$$w_t(\Omega^t) = (1 - \alpha)C_t(\Omega^t) \quad (213)$$

$$R_t(\Omega^t) = \frac{\alpha C_t(\Omega^t) + q_t^{LT}(\Omega^t)}{q_{t-1}^{LT}(\Omega^{t-1})} \quad (214)$$

In the sequential equilibrium, a conjectured optimal allocation is:

$$\mathbf{c}_{0,t} = \left(\frac{1 - (1 - \nu)\beta}{1 - (1 - \nu - \xi)\beta} \right) [1 - \alpha]C_t(\Omega^t)\zeta \quad (215)$$

$$\mathbf{c}_{s,t} = [1 - (1 - \nu)\beta] [\alpha C_t(\Omega^t) + q_t^{LT}(\Omega^t)]\sigma_{s,t} \quad \text{for } s = 1, 2, \dots \quad (216)$$

$$= \beta R_t \mathbf{c}_{s-1,t-1} \quad (217)$$

$$\sigma_{0,t} = 0$$

$$q_t^{LT}(\Omega^t)\sigma_{1,t+1} = \left(\frac{\beta}{1 - (1 - \nu - \xi)\beta} \right) [1 - \alpha]C_t(\Omega^t)\zeta \quad (218)$$

$$q_t^{LT}(\Omega^t)\sigma_{s+1,t+1} = \beta [\alpha C_t(\Omega^t) + q_t^{LT}(\Omega^t)]\sigma_{s,t} \quad \text{for } s = 1, 2, \dots \quad (219)$$

Equation (217) is derived using equations (216) and (219).⁴²

Lemma 12. Consider the limited-commitment model with exogenous aggregate endowment $\{C_t(\Omega^t)\}_{t,\Omega^t}$. Lucas tree yields α fraction of aggregate endowment at all t and Ω^t and is priced at $q_t^{LT}(\Omega^t)$ at state Ω^t . Under Assumption 3:

$$\beta R_{t+1}(\Omega^{t+1}) < \frac{w_{t+1}(\Omega^{t+1})}{w_t(\Omega^t)} \quad \text{for all } t, \Omega^t, \Omega_{t+1}, \quad (220)$$

and the initial condition on the household's share of Lucas tree:

$$\sigma_0 = 0 \quad \text{if } z_0 = \zeta \quad (221)$$

$$0 < \sigma_0 < \bar{\sigma}_0 := \frac{(1 - \alpha)}{1 - (1 - \nu - \xi)\beta} \frac{1}{\alpha + \bar{q}} \zeta \quad \text{if } z_0 = 0 \quad (222)$$

where \bar{q} is given by:

$$\bar{q} := \frac{\xi(1 - \alpha) + \beta(1 - \nu)\alpha(\xi + \nu + \frac{1}{\beta} - 1)}{[1 - \beta(1 - \nu)](\xi + \nu + \frac{1}{\beta} - 1)}, \quad (223)$$

⁴²Substituting the following two equations

$$\sigma_{s+1,t+1} = \beta \frac{[\alpha C_t(\Omega^t) + q_t^{LT}(\Omega^t)]}{q_t^{LT}(\Omega^t)} \sigma_{s,t}$$

$$\sigma_{s,t} = \frac{\mathbf{c}_{s,t}}{[1 - (1 - \nu)\beta] [\alpha C_t(\Omega^t) + q_t^{LT}(\Omega^t)]}$$

into equation (216) at $t + 1$:

$$\mathbf{c}_{s+1,t+1} = [1 - (1 - \nu)\beta] [\alpha C_{t+1}(\Omega^{t+1}) + q_{t+1}^{LT}(\Omega^{t+1})]\sigma_{s+1,t+1}$$

the conjectured household allocation (215)–(219) and the price of Lucas tree:

$$q_t^{LT} = \bar{q} C_t \quad (224)$$

satisfy the budget constraint, Kuhn-Tucker conditions, and the market clearing condition. The interest rate, defined as the return on Lucas tree, is given by:

$$R_{t+1}(\Omega^{t+1}) = \frac{\alpha + \bar{q} C_{t+1}(\Omega^{t+1})}{\bar{q} C_t(\Omega^t)}. \quad (225)$$

Under the assumption on the parameters:

$$\frac{\alpha}{(1-\alpha)\left(\frac{1}{\beta}-1\right)} < \frac{\xi}{\nu\left(\frac{1}{\beta}-1+\xi+\nu\right)}, \quad (89)$$

equation (220) is satisfied for all $(t, \Omega^t, \Omega_{t+1})$.

Proof. First, we verify that the budget constraint is satisfied. In a high-income state, the household earns labor income but does not hold the share of Lucas tree. She purchases a share only for a next high-income state. Hence,

$$\begin{aligned} & \mathbf{c}_t(\sigma_0, z^t, \Omega^t) + \sum_{\Omega_{t+1}} \sum_{z_{t+1}} \pi(z_{t+1}|z_t) \pi(\Omega_{t+1}|\Omega^t) q_t^{LT}(\Omega^t) \sigma_{t+1}(\sigma_0, z^{t+1}, \Omega^{t+1}) - [1-\alpha]C_t(\Omega^t)\zeta \\ &= \left(\frac{1 - (1-\nu)\beta}{1 - (1-\nu-\xi)\beta} \right) [1-\alpha]C_t(\Omega^t)\zeta - \xi \left(\frac{\beta}{1 - (1-\nu-\xi)\beta} \right) [1-\alpha]C_t(\Omega^t)\zeta - [1-\alpha]C_t(\Omega^t)\zeta \\ &= 0. \end{aligned}$$

In a low-income state, the household earns zero labor income and receives dividends from the Lucas tree. The budget constraint holds with equality:

$$\begin{aligned} & \mathbf{c}_t(\sigma_0, z^t, \Omega^t) + \sum_{\Omega_{t+1}} \sum_{z_{t+1}} \pi(z_{t+1}|z_t) \pi(\Omega_{t+1}|\Omega^t) q_t^{LT}(\Omega^t) \sigma_{t+1}(\sigma_0, z^{t+1}, \Omega^{t+1}) - [\alpha C_t(\Omega^t) + q_t^{LT}(\Omega^t)] \sigma_t(\sigma_0, z^t, \Omega^t) \\ &= [1 - (1-\nu)\beta] [\alpha C_t(\Omega^t) + q_t^{LT}(\Omega^t)] \sigma_t(\sigma_0, z^t, \Omega^t) + (1-\nu)\beta [\alpha C_t(\Omega^t) + q_t^{LT}(\Omega^t)] \sigma_t(\sigma_0, z^t, \Omega^t) \\ &\quad - [\alpha C_t(\Omega^t) + q_t^{LT}(\Omega^t)] \sigma_t(\sigma_0, z^t, \Omega^t) \\ &= 0. \end{aligned}$$

Second, we verify that the conjectured allocation satisfies the Kuhn-Tucker conditions:

$$\frac{1}{\mathbf{c}_t(\sigma_0, z^t, \Omega^t)} = \beta R_{t+1}(\Omega^t) \frac{1}{\mathbf{c}_{t+1}(\sigma_0, z^{t+1}, \Omega^{t+1})} \text{ if } \sigma_{t+1}(\sigma_0, z^{t+1}, \Omega^{t+1}) > 0 \quad (226)$$

$$\frac{1}{\mathbf{c}_t(\sigma_0, z^t, \Omega^t)} > \beta R_{t+1}(\Omega^t) \frac{1}{\mathbf{c}_{t+1}(\sigma_0, z^{t+1}, \Omega^{t+1})} \text{ if } \sigma_{t+1}(\sigma_0, z^{t+1}, \Omega^{t+1}) = 0 \quad (227)$$

In a low-income state at $t + 1$, equation (217) implies that equation (226) is satisfied with equality. If the household is in a high-income state at time t and $t + 1$, equation (227) is satisfied if:

$$\begin{aligned} & \mathbf{c}_{t+1}(\sigma_0, z^{t+1}, \Omega^{t+1}) > \beta R_{t+1}(\Omega^{t+1}) \mathbf{c}_t(\sigma_0, z^t, \Omega^t) \\ \Leftrightarrow & \left(\frac{1 - (1 - \nu)\beta}{1 - (1 - \nu - \xi)\beta} \right) [1 - \alpha] C_{t+1}(\Omega^{t+1}) \zeta > \beta R_{t+1}(\Omega^{t+1}) \left(\frac{1 - (1 - \nu)\beta}{1 - (1 - \nu - \xi)\beta} \right) [1 - \alpha] C_t(\Omega^t) \zeta \\ \Leftrightarrow & \beta R_{t+1}(\Omega^{t+1}) < \frac{(1 - \alpha) C_{t+1}(\Omega^{t+1})}{(1 - \alpha) C_t(\Omega^t)} =: \frac{w_{t+1}(\Omega^{t+1})}{w_t(\Omega^t)} \end{aligned}$$

Therefore, under Assumption 3, this condition is satisfied. We now want to show that $\beta R_{t+1} \mathbf{c}_t(\sigma_0, z^t, \Omega^t; z_t = 0) < \mathbf{c}_{t+1}(\sigma_0, z^{t+1}, \Omega^{t+1}; z_{t+1} = \zeta)$ for all $(\sigma_0, z^{t+1}, \Omega^{t+1})$, meaning that a low-income household does not have an incentive to save for a next high-income state. This is shown by claiming that under Assumption 3 and the initial condition (222), low-income households never have higher consumption than high-income households. Therefore, the Kuhn-Tucker conditions are satisfied between a low-income state at t and a high-income state at $t + 1$ as well.

Remember that:

$$\begin{aligned} \mathbf{c}_t(\sigma_0, z^t, \Omega^t; z_t = \zeta) &= \left(\frac{1 - (1 - \nu)\beta}{1 - (1 - \nu - \xi)\beta} \right) [1 - \alpha] C_t(\Omega^t) \zeta \\ \mathbf{c}_t(\sigma_0, z^t, \Omega^t; z_t = 0) &= [1 - (1 - \nu)\beta] [\alpha C_t(\Omega^t) + q_t^{LT}(\Omega^t)] \sigma_t(\sigma_0, z^t, \Omega^t) \end{aligned}$$

with $q_t^{LT}(\Omega^t) = \bar{q} C_t$. We have $\mathbf{c}_t(\sigma_0, z^t, \Omega^t; z_t = 0) < \mathbf{c}_t(\sigma_0, z^t, \Omega^t; z_t = \zeta)$ if:

$$\begin{aligned} [1 - (1 - \nu)\beta] [\alpha C_t(\Omega^t) + q_t^{LT}(\Omega^t)] \sigma_t(\sigma_0, z^t, \Omega^t) &< \left(\frac{1 - (1 - \nu)\beta}{1 - (1 - \nu - \xi)\beta} \right) [1 - \alpha] C_t(\Omega^t) \zeta \\ \Leftrightarrow \sigma_t(\sigma_0, z^t, \Omega^t) &< \frac{(1 - \alpha)}{1 - (1 - \nu - \xi)\beta} \frac{1}{\alpha + \bar{q}} \zeta =: \bar{\sigma}_0 \end{aligned}$$

In the initial period, $\sigma_t(\sigma_0, z^t, \Omega^t) < \bar{\sigma}_0$ is assumed as in equation (222). At time $t = 1$ onwards, $\sigma_{s+1,t+1}$ for $s \geq 0$ follows:

$$\begin{aligned} \sigma_{s+1,t+1} &= \beta \frac{\alpha C_t + q_t^{LT}}{q_t^{LT}} \sigma_{s,t} \\ &= \beta \frac{q_{t-1}^{LT}}{q_t^{LT}} \frac{\alpha C_t + q_t^{LT}}{q_{t-1}^{LT}} \sigma_{s,t} \\ &= \beta \underbrace{\frac{w_{t-1}}{w_t}}_{<1} R_{t-1} \sigma_{s,t}. \end{aligned}$$

This implies that the share of Lucas tree satisfies $\sigma_{s+1,t+1} < \bar{\sigma}_0$ if $\sigma_{s,t} < \bar{\sigma}_0$ for all $s \geq 1$ and $t \geq 0$. Hence, we only need to show that $\sigma_{1,t} < \bar{\sigma}_0$ at all $t \geq 1$. As in equation (218):

$$\sigma_{1,t+1} = \left(\frac{\beta}{1 - (1 - \nu - \xi)\beta} \right) [1 - \alpha] \frac{C_t(\Omega^t)}{q_t^{LT}(\Omega^t)} \zeta.$$

Using $q_t^{LT}(\Omega^t) = \bar{q}C_t$ and $\beta^{\frac{\alpha+\bar{q}}{q}} < 1$, we have $\sigma_{1,t+1} < \bar{\sigma}_0 := \frac{(1-\alpha)}{1-(1-\nu-\xi)\beta} \frac{1}{\alpha+\bar{q}}\zeta$. Therefore, $\sigma_t(\sigma_0, z^t, \Omega^t) < \bar{\sigma}_0$ is satisfied for any $(\sigma_0, z^t, \Omega^t)$. We have shown that low-income households never have higher consumption than high-income households. Assumption 3 implies that (227) holds with strict inequality.

Third, we verify that the market-clearing condition (212) is satisfied. This is done by deriving the equilibrium price of Lucas tree that clears the market. Consider that C_t is realized at time t . A market clearing condition for the share of Lucas tree at $t + 1$ is:

$$\begin{aligned} 1 &= \sum_{s=1}^{\infty} \phi_s \sigma_{s,t+1} \\ &= \frac{\nu\xi}{\xi + \nu} \sigma_{1,t+1} + \sum_{s=2}^{\infty} \frac{\nu\xi}{\xi + \nu} (1 - \nu)^{s-1} \sigma_{s,t+1} \end{aligned} \quad (228)$$

From equations (218)–(219), we know:

$$\sigma_{1,t+1} = \left(\frac{\beta}{1 - (1 - \nu - \xi)\beta} \right) \frac{[1 - \alpha]C_t(\Omega^t)}{q_t^{LT}(\Omega^t)} \zeta \quad (229)$$

$$\sigma_{s,t+1} = \beta \frac{\alpha C_t(\Omega^t) + q_t^{LT}(\Omega^t)}{q_t^{LT}(\Omega^t)} \sigma_{s-1,t} \quad \text{for } s \geq 2 \quad (230)$$

By substituting into (228), we have:

$$\begin{aligned} 1 &= \frac{\nu\xi}{\xi + \nu} \left(\frac{\beta}{1 - (1 - \nu - \xi)\beta} \right) \frac{[1 - \alpha]C_t(\Omega^t)}{q_t^{LT}(\Omega^t)} \zeta \\ &\quad + \beta \left[\frac{\alpha C_t(\Omega^t) + q_t^{LT}(\Omega^t)}{q_t^{LT}(\Omega^t)} \right] \underbrace{\sum_{s=2}^{\infty} \frac{\nu\xi}{\xi + \nu} (1 - \nu)^{s-1} \sigma_{s-1,t}}_{=(1-\nu) \sum_{s=1}^{\infty} \frac{\nu\xi}{\xi + \nu} (1-\nu)^{s-1} \sigma_{s,t} = 1-\nu} \end{aligned} \quad (231)$$

By solving this, we obtain:

$$q_t^{LT}(\Omega^t) = \underbrace{\frac{\xi(1 - \alpha) + \beta(1 - \nu)\alpha(\xi + \nu + \frac{1}{\beta} - 1)}{[1 - \beta(1 - \nu)](\xi + \nu + \frac{1}{\beta} - 1)}}_{\text{denote as } \bar{q}} C_t(\Omega^t) \quad (232)$$

Given $q_t^{LT}(\Omega^t)$, $R_{t+1}(\Omega^{t+1})$ is defined as:

$$\begin{aligned} R_{t+1}(\Omega^{t+1}) &:= \frac{\alpha C_{t+1}(\Omega^{t+1}) + q_{t+1}^{LT}(\Omega^{t+1})}{q_t^{LT}(\Omega^t)} \\ &= \frac{\alpha + \bar{q} C_{t+1}(\Omega^{t+1})}{\bar{q} C_t(\Omega^t)} \end{aligned}$$

Therefore, Assumption (220) is satisfied if:

$$\beta \frac{\alpha + \bar{q} C_{t+1}}{\bar{q} C_t} < \frac{(1 - \alpha)C_{t+1}}{(1 - \alpha)C_t}.$$

Solving this inequality yields a condition in terms of parameters (89). □

Given the sequential equilibrium, we will show in Proposition 13 that when aggregate consumption $C_t(\Omega^t)$ is exogenous and follows a common stochastic process in the two models (the limited-commitment model and the representative-agent model), the limited commitment model has a higher bond price:

$$q_t^{B,LC}(\Omega^t) > q_t^{B,Rep}(\Omega^t) \text{ for all } \Omega^t, \quad (233)$$

and hence a lower risk-free rate.⁴³ To do so, we derive the interest rate in the endowment economy, as is standard in the literature (e.g., Ljungqvist and Sargent, 2018).

The Interest Rate in the Representative Agent Economy In the representative agent model, the interest rate, defined as the return on Lucas tree as before, is given by:

$$R_{t+1}^{Rep}(\Omega^{t+1}|\Omega^t) := \frac{\alpha C_{t+1}(\Omega^{t+1}) + q_{t+1}^{LT}(\Omega^{t+1})}{q_t^{LT}(\Omega^t)}. \quad (234)$$

The price of Lucas tree, which yields α fraction of aggregate endowment, follows:

$$q_t^{Lucas}(\Omega^t) = \sum_{\Omega^{t+1}} Q(\Omega^{t+1}|\Omega^t) [\alpha C_{t+1}(\Omega^{t+1}) + q_{t+1}^{Lucas}(\Omega^{t+1})], \quad (235)$$

$$\text{where } Q(\Omega^{t+1}|\Omega^t) = \beta \frac{C_t(\Omega^t)}{C_{t+1}(\Omega^{t+1})} \pi(\Omega^{t+1}|\Omega^t)$$

Using recursion of the equation, the price of Lucas tree is expressed as:

$$q_t^{Lucas}(\Omega^t) = \frac{1}{u'(C_t(\Omega^t))} \left[\mathbb{E}_t \sum_{j=1}^{\infty} \beta^j u'(C_{t+j}(\Omega^{t+j})) \alpha C_{t+j}(\Omega^{t+j}) + \mathbb{E}_t \lim_{k \rightarrow \infty} \beta^k u'(C_{t+k}(\Omega^{t+k})) q_{t+k}(\Omega^{t+k}) \right]. \quad (236)$$

The last term must be zero to clear the market. Under a logarithmic utility function, which gives $u'(C_{t+j})C_{t+j} = 1$, the price of Lucas tree is proportional to the current aggregate endowment:

$$q_t^{Lucas}(\Omega^t) = \frac{\beta}{1-\beta} \alpha C_t(\Omega^t). \quad (237)$$

Imposing this into equation (234) gives:

$$R_{t+1}^{Rep}(\Omega^{t+1}|\Omega^t) = \frac{1}{\beta} \frac{C_{t+1}(\Omega^{t+1})}{C_t(\Omega^t)} \quad (238)$$

⁴³Note that a labor share in the endowment economy, $1 - \alpha \in (0, 1)$, is constant over time and across states. The share of labor income in the total resources available in the economy is also constant in a production economy with $\delta = 1$. However, in a production economy with $\delta < 1$, the share:

$$\frac{w_t L}{A_t^{1-\theta} K_t^\theta + (1-\delta)K_t} = \frac{(1-\theta)A_t^{1-\theta} K_t^\theta}{A_t^{1-\theta} K_t^\theta + (1-\delta)K_t}$$

is not constant. Since one of the key assumptions in Krueger and Lustig (2010) is violated, the conjecture (the same risk premium) may not be true in such an economy. We will come back to this point later.

Proposition 13.

Proof. In the representative agent model (complete market model), we have seen that under logarithmic utility, the interest rate is given by (234):

$$R_{t+1}^{Rep}(\Omega^{t+1}) = \frac{1}{\beta} \frac{C_{t+1}(\Omega^{t+1})}{C_t(\Omega^t)}$$

In the limited commitment model, the interest rate is given by (225):

$$R_{t+1}^{LC}(\Omega^{t+1}) = \frac{\alpha + \bar{q}}{\bar{q}} \frac{C_{t+1}(\Omega^{t+1})}{C_t(\Omega^t)}. \quad (239)$$

The interest rates are proportional to consumption growth $\frac{C_{t+1}(\Omega^{t+1})}{C_t(\Omega^t)}$ in the two economies. In a partial insurance equilibrium, parameters are assumed to satisfy (89), which is equivalent to:

$$\beta \frac{\alpha + \bar{q}}{\bar{q}} < 1. \quad (240)$$

Therefore, we have:

$$\therefore R_{t+1}^{LC}(\Omega^{t+1}) < R_{t+1}^{Rep}(\Omega^{t+1}).$$

The limited commitment model has a lower interest rate. The price of risk-free bond is given by:

$$q_t^{LC}(\Omega^t) = \mathbb{E}_t \left[\frac{1}{R_{t+1}^{LC}(\Omega^{t+1})} \right] = \mathbb{E}_t \left[\frac{\bar{q}}{\alpha + \bar{q}} \frac{C_t(\Omega^t)}{C_{t+1}(\Omega^{t+1})} \right] \quad (241)$$

$$q_t^{Rep}(\Omega^t) = \mathbb{E}_t \left[\beta \frac{C_t(\Omega^t)}{C_{t+1}(\Omega^{t+1})} \right] \quad (242)$$

Since $\beta < \frac{\bar{q}}{\alpha + \bar{q}}$, the limited-commitment model has a higher price of risk-free bonds and a lower risk-free rate.⁴⁴

⁴⁴In the limited commitment model,

$$\frac{1}{R_{t+1}^{LC}(\Omega^{t+1})} = \beta \frac{\mathbf{c}_t(\Omega^t)}{\mathbf{c}_{t+1}(\Omega^{t+1}; z_{t+1} = 0)} > \beta \frac{C_t^{LC}(\Omega^t)}{C_{t+1}^{LC}(\Omega^{t+1})}, \quad (243)$$

since $\mathbf{c}_{t+1}(\Omega^{t+1}; z_{t+1} = \zeta) > \mathbf{c}_{t+1}(\Omega^{t+1}; z_{t+1} = 0)$. Formally, $C_{t+1}^{LC} > \beta R_{t+1}^{LC} C_t^{LC}$ follows:

$$\begin{aligned} C_{t+1} &:= \sum_{s=0}^{\infty} \phi_s \mathbf{c}_{s,t+1} \\ &= \phi_0 \underbrace{\mathbf{c}_{0,t+1}}_{> \beta R_{t+1} \mathbf{c}_{0,t}} + \sum_{s=1}^{\infty} \phi_s \underbrace{\mathbf{c}_{s,t+1}}_{= \beta R_{t+1} \mathbf{c}_{s-1,t}} \\ &> \beta R_{t+1} \left[\phi_0 \mathbf{c}_{0,t} + \sum_{s=1}^{\infty} \phi_s \underbrace{\mathbf{c}_{s-1,t}}_{> \mathbf{c}_{s,t} \text{ under Assumption 1}} \right] \\ &> \beta R_{t+1} \left[\phi_0 \mathbf{c}_{0,t} + \sum_{s=1}^{\infty} \phi_s \mathbf{c}_{s,t} \right] = \beta R_{t+1} C_t. \end{aligned}$$

Remember that the risk premium in the two economies is given by:

$$1 + \lambda_t^{Rep} = \mathbb{E}_t[R_{t+1}^{Rep}(\Omega^{t+1})] \mathbb{E}_t \left[\beta \frac{C_t(\Omega^t)}{C_{t+1}(\Omega^{t+1})} \right]$$

$$1 + \lambda_t^{LC} = \mathbb{E}_t[R_{t+1}^{LC}(\Omega^{t+1})] \mathbb{E}_t \left[\frac{1}{R_{t+1}^{LC}(\Omega^{t+1})} \right].$$

Given the equilibrium interest rates, the risk premium in the two economies is given by:

$$1 + \lambda_t^{Rep} = \mathbb{E}_t[C_{t+1}(\Omega^{t+1})] \mathbb{E}_t \left[\frac{1}{C_{t+1}(\Omega^{t+1})} \right] > 1 \quad (244)$$

$$1 + \lambda_t^{LC} = \mathbb{E}_t[C_{t+1}(\Omega^{t+1})] \mathbb{E}_t \left[\frac{1}{C_{t+1}(\Omega^{t+1})} \right] > 1 \quad (245)$$

Therefore, we see that the two economies have the same risk premium. If the growth rate of exogenous consumption $\frac{C_{t+1}}{C_t}$ follows an *iid* process, the risk premium $1 + \lambda_t$ is constant over time:

$$1 + \lambda_t = \mathbb{E} \left[\frac{C_{t+1}(\Omega^{t+1})}{C_t(\Omega^t)} \middle| \Omega^t \right] \mathbb{E} \left[\frac{C_t(\Omega^t)}{C_{t+1}(\Omega^{t+1})} \middle| \Omega^t \right]$$

$$= \mathbb{E} \left[\frac{C_{t+1}(\Omega^{t+1})}{C_t(\Omega^t)} \right] \mathbb{E} \left[\frac{C_t(\Omega^t)}{C_{t+1}(\Omega^{t+1})} \right] = 1 + \lambda$$

□

A.3.6 Intuition with Multiplicative Stochastic Discount Factors

Proposition 14.

Proof. First, we show that equation (93) holds in both models. Consider the representative agent model. We want to show that the stochastic discount factor satisfies:

$$m_{t,t+1}^{Rep}(\Omega^{t+1}) := \beta \frac{C_t^{Rep}(\Omega^t)}{C_{t+1}^{Rep}(\Omega^{t+1})} = \gamma_t \frac{\Upsilon_t(\Omega^t)}{\Upsilon_{t+1}(\Omega^{t+1})} \quad (246)$$

for some non-random variable γ_t . In the endowment economy, since the aggregate consumption is exogenous:

$$\Upsilon_t(\Omega^t) = C_t(\Omega^t), \quad (247)$$

Equation (246) is satisfied with $\gamma_t = \beta$. In the production economy with $\delta = 1$, the constant saving rate implies that:

$$C_t^{Rep}(\Omega^t) = (1 - \beta\theta) A_t^{1-\theta} K_t^\theta$$

$$= (1 - \beta\theta) \Upsilon_t(\Omega^t). \quad (248)$$

Again, equation (246) is satisfied with $\gamma_t = \beta$.

In the limited commitment model, we want to show that the SDF satisfies:

$$m_{t,t+1}^{\text{LC}}(\Omega^{t+1}) := \beta \frac{\mathbf{c}_t(z^t, \Omega^t)}{\mathbf{c}_{t+1}(z^{t+1}, \Omega^{t+1}; z_{t+1} = 0)} = \gamma_t \frac{\Upsilon_t(\Omega^t)}{\Upsilon_{t+1}(\Omega^{t+1})} \quad (249)$$

for some γ_t that does not depend on A_{t+1} . Note that the limited-commitment constraint does not bind for a low-income agent at $t + 1$.

From the Euler equation of low-income agents, we have:

$$\beta \frac{\mathbf{c}_t(z^t, \Omega^t)}{\mathbf{c}_{t+1}(z^{t+1}, \Omega^{t+1}; z_{t+1} = 0)} = \frac{1}{R_{t+1}^{\text{LC}}(\Omega^{t+1})}. \quad (250)$$

We have seen in equation (225) that in the endowment economy,

$$R_{t+1}^{\text{LC}}(\Omega^{t+1}) = \frac{\alpha + \bar{q}}{\bar{q}} \frac{C_{t+1}(\Omega^{t+1})}{C_t(\Omega^t)} \quad (251)$$

holds. In the production economy with $\delta = 1$,

$$\begin{aligned} R_{t+1}^{\text{LC}}(\Omega^{t+1}) &= \theta K_{t+1}^{\theta-1} A_{t+1}^{1-\theta} \\ &= \theta \frac{K_{t+1}^\theta A_{t+1}^{1-\theta}}{\hat{s}^{\text{LC}} K_t^\theta A_t^{1-\theta}} \\ &= \frac{\theta}{\hat{s}^{\text{LC}}} \frac{\Upsilon_{t+1}}{\Upsilon_t} \end{aligned} \quad (252)$$

Hence, equation (249) is satisfied with a non-random γ_t in both cases in the limited commitment model.

Second, we prove (94). In the endowment economy, it is trivial since the total resources are exogenous:

$$\Upsilon_t^{\text{Rep}}(\Omega^t) = \Upsilon_t^{\text{LC}}(\Omega^t) = C_t(\Omega^t).$$

In the production economy with $\delta = 1$, given the constant saving rate in the two models, we obtain:

$$\frac{\Upsilon_t(\Omega^t)}{\Upsilon_{t+1}(\Omega^{t+1})} = \frac{K_t^\theta A_t^{1-\theta}}{K_{t+1}^\theta A_{t+1}^{1-\theta}} = \frac{K_t^\theta A_t^{1-\theta}}{(\hat{s} K_t^\theta A_t^{1-\theta})^\theta A_{t+1}^{1-\theta}} \text{ where } \hat{s} \in \{\beta\theta, \hat{s}^{\text{LC}}\}. \quad (253)$$

$\frac{\Upsilon_t(\Omega^t)}{\Upsilon_{t+1}(\Omega^{t+1})}$ are proportional to $\frac{1}{A_{t+1}^{1-\theta}}$ in the two models. Hence, there exists γ'_t in equation (94) that does not depend on Ω_{t+1} . Equation (95) directly follows from equations (93) and (94).

Finally, we proceed to the last statement of the proposition. In the endowment economy, the proof follows Theorem 4.2 in Krueger and Lustig (2010). The risk premium is defined as:

$$1 + \lambda_t := \frac{\mathbb{E}_t [R_{t,1}[\{e_{t+k}\}]]}{R_{t,1}[1]}, \quad (254)$$

where $R_{t,1}[\{e_{t+k}\}]$ is the one-period return of holding a claim $\{e_{t+k}\}_{k \geq 1}$ from time t to $t+1$. The Lucas tree yields α fraction of endowment in the economy, so $e_{t+k} = \alpha \Upsilon_{t+k}$ for all $k \geq 1$ in both models. Their derivation shows that the risk premium can be expressed as a weighted sum of risk premia on strips:

$$1 + \lambda_t = \sum_{k=1}^{\infty} \frac{\omega_k}{1/\mathbb{E}_t[m_{t,t+1}]} \mathbb{E}_t [R_{t,1}[e_{t+k}]], \quad (255)$$

$$\text{where } \omega_k = \frac{\mathbb{E}_t[m_{t,t+k}e_{t+k}]}{\sum_{j=1}^{\infty} \mathbb{E}_t[m_{t,t+j}e_{t+j}]}$$

If $m_{t,t+1}^{LC} = \gamma_t'' m_{t,t+1}^{Rep}$, where γ_t'' is a non-random multiplicative term,

$$\begin{aligned} 1 + \lambda_t^{LC} &:= \frac{\mathbb{E}_t [R_{t,1}^{LC}[e_{t+k}]]}{1/\mathbb{E}_t[m_{t,t+1}^{LC}]} = \frac{\mathbb{E}_t \left[\frac{E_{t+1}[m_{t+1,t+k}^{LC} e_{t+k}]}{E_t[m_{t,t+k}^{LC} e_{t+k}]} \right]}{1/\mathbb{E}_t[m_{t,t+1}^{LC}]} \\ &= \frac{\mathbb{E}_t \left[\frac{E_{t+1}[\gamma_{t+1}'' \dots \gamma_{t+k-1}'' m_{t+1,t+k}^{Rep} e_{t+k}]}{E_t[\gamma_t'' \gamma_{t+1}'' \dots \gamma_{t+k-1}'' m_{t,t+k}^{Rep} e_{t+k}]} \right]}{1/\mathbb{E}_t[\gamma_t m_{t,t+1}^{Rep}]} = \frac{\mathbb{E}_t [R_{t,1}^{Rep}[e_{t+k}]]}{1/\mathbb{E}_t[m_{t,t+1}^{Rep}]} =: 1 + \lambda_t^{Rep} \end{aligned} \quad (256)$$

holds for all $k \geq 1$, where $m_{t,t+k} = m_{t,t+1} m_{t+1,t+2} \dots m_{t+k-1,t+k}$. Since ω_k is also the same between the two models, the two models have the same risk premium.

In the production economy, one unit of capital purchased at t yields $\theta K_{t+1}^{\theta-1} A_{t+1}^{1-\theta} + 1 - \delta$ at time $t+1$. By substituting $e_{t+1} = \theta K_{t+1}^{\theta-1} A_{t+1}^{1-\theta} + 1 - \delta$ and $e_{t+k} = 0$ for all $k \geq 2$, the risk premium is given by:

$$\begin{aligned} 1 + \lambda_t &:= \frac{\mathbb{E}_t [R_{t,1}[\{e_{t+k}\}]]}{R_{t,1}[1]} \\ &= \frac{\mathbb{E}_t \left[\frac{E_{t+1}[\theta K_{t+1}^{\theta-1} A_{t+1}^{1-\theta} + 1 - \delta]}{E_t[m_{t,t+1}(\theta K_{t+1}^{\theta-1} A_{t+1}^{1-\theta} + 1 - \delta)]} \right]}{1/\mathbb{E}_t[m_{t,t+1}]} \\ &= \frac{\mathbb{E}_t[\theta K_{t+1}^{\theta-1} A_{t+1}^{1-\theta} + 1 - \delta] \mathbb{E}_t[m_{t,t+1}]}{\mathbb{E}_t[(\theta K_{t+1}^{\theta-1} A_{t+1}^{1-\theta} + 1 - \delta) m_{t,t+1}]} \end{aligned} \quad (257)$$

If $\delta = 1$, this is simplified to:

$$1 + \lambda_t = \frac{\mathbb{E}_t[A_{t+1}^{1-\theta}] \mathbb{E}_t[m_{t,t+1}]}{\mathbb{E}_t[A_{t+1}^{1-\theta} m_{t,t+1}]}, \quad (258)$$

where we use the fact that K_{t+1} is determined at t and is canceled out in the previous equation. If $m_{t,t+1}^{LC} = \gamma_t'' m_{t,t+1}^{Rep}$ holds, the representative agent model and the limited commitment model have the same risk premium. \square

The proof illustrates that the two models have different risk premia if $\delta \neq 1$ in the production economy. First, if $\delta \neq 1$, the risk premium depends on K_{t+1} , which follows a

different law of motion in the two models. Hence, even multiplicative stochastic discount factors would not imply the same risk premium. Second, the stochastic discount factors are not proportional in the two models if $\delta \neq 1$. As we saw in (252) and (253), under full depreciation of capital, the interest rate is proportional to aggregate resources $\frac{Y_{t+1}}{Y_t}$, which is in turn proportional to $A_{t+1}^{1-\theta}$. This is not generally the case if $\delta \neq 1$. In equations (256) and (258), we have seen that the multiplicative stochastic discount factors ($m_{t,t+1}^{LC} = \gamma_t'' m_{t,t+1}^{Rep}$) imply $1 + \lambda_t^{LC} = 1 + \lambda_t^{Rep}$. If the SDFs are not proportional, the term γ_t'' does not cancel out, so the risk premia are generally different across the two models.

B Additional Discussions

B.1 Additional Discussion about Transitional Dynamics

B.1.1 Monotone Sequence of $\{A_t\}_{t=1}^{\infty}$

We derive sufficient conditions on a monotone sequence of $\{A_t\}_{t=1}^{\infty}$ converging to A^* such that $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ holds at all $t \geq 0$. We first show that if $\{\frac{K_t}{A_t}\}_{t=1}^{\infty}$ is also monotone, $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at $t = 0$ is sufficient for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 0$. Since the aggregate capital at $t = 1$ (K_1) is pre-determined at $t = 0$ (in a steady state), we can derive a condition on A_1 in closed form, as we did in Section 4.2.1 for a permanent productivity shock. We then derive a sufficient condition on $\{A_t\}_{t=2}^{\infty}$ for the monotonicity of $\{\frac{K_t}{A_t}\}_{t=1}^{\infty}$. The condition on $\{A_t\}_{t=2}^{\infty}$ and the condition on A_1 together guarantee $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 0$.

Preparations for the Sufficient Conditions The economy is assumed to be in a steady state at $t = 0$, where $\beta R_0 < 1$ holds under Assumption 5. At $t = 1$, a new path of $\{A_t\}_{t=1}^{\infty}$ is realized. We will derive conditions on $\{A_t\}_{t=1}^{\infty}$ that guarantee $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ for all $t \geq 0$. We first express $\beta R_{t+1} \frac{w_t}{w_{t+1}}$ in terms of capital:

$$\beta R_{t+1} \frac{w_t}{w_{t+1}} = \beta [\theta A_{t+1}^{1-\theta} K_{t+1}^{\theta-1} + 1 - \delta] \frac{(1-\theta) A_t^{1-\theta} K_t^\theta}{(1-\theta) A_{t+1}^{1-\theta} K_{t+1}^\theta}. \quad (259)$$

By using the law of motion of capital, $K_{t+1} = \hat{s} A_t^{1-\theta} K_t^\theta + (1 - \hat{\delta}) K_t$, we derive following expressions:

$$\frac{K_{t+1}}{A_{t+1}} = \frac{A_t}{A_{t+1}} \left[\hat{s} \left(\frac{K_t}{A_t} \right)^\theta + (1 - \hat{\delta}) \frac{K_t}{A_t} \right], \quad (260)$$

$$\beta R_{t+1} \frac{w_t}{w_{t+1}} = \beta \left[\frac{\theta + (1 - \delta) \left(\frac{K_{t+1}}{A_{t+1}} \right)^{1-\theta}}{\hat{s} + (1 - \hat{\delta}) \left(\frac{K_t}{A_t} \right)^{1-\theta}} \right]. \quad (261)$$

We first show that the monotonicity of $\{\frac{K_t}{A_t}\}$ give rise to analytically tractable sufficient conditions for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 0$.

Lemma 13 (Sufficient condition for $\beta R_{t+1} \frac{w_t}{w_{t+1}} < 1$ at all $t \geq 0$).

1. Suppose $\{A_t\}_{t=1}^\infty$ and $\{\frac{K_t}{A_t}\}_{t=1}^\infty$ are monotonically increasing over time (i.e., $A_t \leq A_{t+1}$ and $\frac{K_t}{A_t} < \frac{K_{t+1}}{A_{t+1}}$ for all $t \geq 1$). Then, $A_1 < \bar{A}_1$ is sufficient for $\beta R_{t+1} \frac{w_t}{w_{t+1}} < 1$ at any $t \geq 0$, where

$$\frac{\bar{A}_1}{A_0} = \left[\frac{1 - \beta(1 - \delta)}{\theta} \frac{\xi(1 - \theta) + (1 - \nu)\theta [1 - (1 - \nu - \xi)\beta]}{[1 - (1 - \nu)\beta(1 - \delta)][1 - (1 - \nu - \xi)\beta]} \right]^{\frac{1}{1-\theta}} > 1.$$

2. Suppose $\{A_t\}_{t=1}^\infty$ and $\{\frac{K_t}{A_t}\}_{t=1}^\infty$ are monotonically decreasing over time (i.e., $A_t \geq A_{t+1}$ and $\frac{K_t}{A_t} > \frac{K_{t+1}}{A_{t+1}}$ for all $t \geq 1$). Then, $A_1 > \underline{A}_1$ is sufficient for $\beta R_{t+1} \frac{w_t}{w_{t+1}} < 1$ at any $t \geq 0$, where

$$\frac{\underline{A}_1}{A_0} = \left[1 - \nu + \nu \frac{1 - (1 - \delta)\beta(1 - \nu)}{\beta\nu(1 - \delta)} \frac{\xi(1 - \theta) - \beta\theta\nu(\xi + \nu + \frac{1}{\beta} - 1)}{\xi(1 - \theta) + \beta\theta(1 - \nu)(\xi + \nu + \frac{1}{\beta} - 1)} \right]^{\frac{1}{\theta-1}} < 1.$$

Proof. 1. Since $w_t = (1 - \theta)A_t \left(\frac{K_t}{A_t}\right)^\theta$, monotonicity of $\{A_t\}_{t=1}^\infty$ and $\{\frac{K_t}{A_t}\}_{t=1}^\infty$ imply monotonicity of w_t . If A_t and $\frac{K_t}{A_t}$ are weakly increasing over time, $\frac{w_{t+1}}{w_t} \geq 1$ for all $t \geq 1$. Then, $\beta R_{t+1} < 1$ is a sufficient condition for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$. Since $R_{t+1} = \theta \left(\frac{K_{t+1}}{A_{t+1}}\right)^{\theta-1} + 1 - \delta$, monotone increase of $\frac{K_t}{A_t}$ implies monotone decrease of R_t . Therefore, $\beta R_2 < 1$ is sufficient for $\beta R_{t+1} < 1$ for all $t \geq 1$. Furthermore, $\frac{K_1}{A_1} < \frac{K_2}{A_2}$ implies $R_1 > R_2$. Thus, $\beta R_1 < 1$ is sufficient for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 1$. $\beta R_1 < 1$ also implies $\beta R_1 < \frac{w_1}{w_0}$. Proposition 6 shows $\beta R_1 < 1$ if $A_1 < \bar{A}_1$.

2. Using equation (261), the condition $\beta R_{t+1} \frac{w_t}{w_{t+1}} < 1$ is written as:

$$\begin{aligned} \beta R_{t+1} \frac{w_t}{w_{t+1}} &= \beta \left[\frac{\theta + (1 - \delta) \left(\frac{K_{t+1}}{A_{t+1}}\right)^{1-\theta}}{\hat{s} + (1 - \hat{\delta}) \left(\frac{K_t}{A_t}\right)^{1-\theta}} \right] < 1 \\ \Leftrightarrow \left(\frac{K_{t+1}}{A_{t+1}}\right)^{1-\theta} - (1 - \nu) \left(\frac{K_t}{A_t}\right)^{1-\theta} &< \frac{1}{1 - \delta} \left[\frac{\xi(1 - \theta)}{1 - (1 - \nu - \xi)\beta} - \nu\theta \right], \end{aligned} \quad (262)$$

where we know the RHS is strictly positive under Assumption 5 (Lemma 7). The equation (262) can be written as:

$$\underbrace{\left(\frac{K_{t+1}}{A_{t+1}}\right)^{1-\theta} - \left(\frac{K_t}{A_t}\right)^{1-\theta}}_{< 0 \text{ if } \frac{K_{t+1}}{A_{t+1}} < \frac{K_t}{A_t}} + \nu \left(\frac{K_t}{A_t}\right)^{1-\theta} < \frac{1}{1 - \delta} \left[\frac{\xi(1 - \theta)}{1 - (1 - \nu - \xi)\beta} - \nu\theta \right].$$

Since $\frac{K_t}{A_t}$ is decreasing over time, $\left(\frac{K_t}{A_t}\right)^{1-\theta}$ is largest at $t = 1$. Since we have:

$$\left(\frac{K_{t+1}}{A_{t+1}}\right)^{1-\theta} - \left(\frac{K_t}{A_t}\right)^{1-\theta} + \nu \left(\frac{K_t}{A_t}\right)^{1-\theta} < \nu \left(\frac{K_t}{A_t}\right)^{1-\theta} \leq \nu \left(\frac{K_1}{A_1}\right)^{1-\theta} \text{ for all } t \geq 1,$$

a sufficient condition for (262) at all $t \geq 1$ is given by:

$$\left(\frac{K_1}{A_1}\right)^{1-\theta} < \frac{1}{\nu(1-\delta)} \left[\frac{\xi(1-\theta)}{1-(1-\nu-\xi)\beta} - \nu\theta \right].$$

A condition on A_1 for this inequality is given in Proposition 9 (i.e., $A_1 > \underline{A}'_1$). Proposition 7 shows that $A_1 > \underline{A}_1$ is sufficient for $A_1 > \underline{A}'_1$ and $\beta R_1 < \frac{w_1}{w_0}$. Therefore, it is sufficient for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 0$. \square

Next, we derive sufficient conditions for the monotonicity of $\{\frac{K_t}{A_t}\}$. Lemma 14 derives a condition on A_{t+1} as a function of K_t . Since K_t is an endogenous variable, we will use a first-order approximation to derive a sufficient condition on $\{A_{t+1}\}$ as a function of exogenous parameters.

Lemma 14 (Monotonicity of $\frac{K_t}{A_t}$). *Consider an economy with $\frac{K_t}{A_t}$ at time t , where the steady-state value of capital over productivity is given by $\frac{K^*}{A^*}$. We have $\frac{K_{t+1}}{A_{t+1}} > \frac{K_t}{A_t}$ if and only if*

$$\frac{A_{t+1}}{A_t} < 1 + \hat{\delta} \left[\left(\frac{K^*/A^*}{K_t/A_t} \right)^{1-\theta} - 1 \right]. \quad (263)$$

Proof. From a law of motion of capital (260), we know $\frac{K_{t+1}}{A_{t+1}} > \frac{K_t}{A_t}$ if and only if

$$\frac{K_{t+1}}{A_{t+1}} := \frac{A_t}{A_{t+1}} \left[\hat{s} \left(\frac{K_t}{A_t} \right)^\theta + (1-\hat{\delta}) \frac{K_t}{A_t} \right] > \frac{K_t}{A_t}$$

By dividing both sides by $\frac{K_t}{A_t}$ ($\neq 0$), this is equivalent to:

$$\hat{s} \left(\frac{K_t}{A_t} \right)^{\theta-1} + 1 - \hat{\delta} > \frac{A_{t+1}}{A_t}.$$

In the steady state ($K_{t+1} = K_t = K^*$), we have:

$$\hat{s} = \hat{\delta} \left(\frac{K^*}{A^*} \right)^{1-\theta}.$$

By substituting this, we derive:

$$1 + \hat{\delta} \left[\left(\frac{K^*/A^*}{K_t/A_t} \right)^{1-\theta} - 1 \right] > \frac{A_{t+1}}{A_t}.$$

\square

Lemma 14 means $\{\frac{K_t}{A_t}\}_{t=1}^\infty$ is monotonically increasing if the condition (263) holds for all $t \geq 1$. In order to derive a condition on $\{A_{t+1}\}$ without an endogenous variable K_t , we derive a first order approximation of $\tilde{k}_t := \frac{K_t/A_t}{K^*/A^*}$ and $\hat{k}_t := \frac{K^*/A^*}{K_t/A_t}$. They are used in case of an increasing and decreasing sequence of $\{A_t\}_{t=1}^\infty$, respectively.

Lemma 15 (First-Order Approximation).

1. A first-order approximation of $\tilde{k}_t := \frac{K_t/A_t}{K^*/A^*}$ is given by:

$$\tilde{k}_t < 1 - \frac{A_1 - A_0}{A_t} \left[1 - \hat{\delta}(1 - \theta)\right]^{t-1} - \sum_{u=1}^{t-1} \frac{A_{t-u+1} - A_{t-u}}{A_t} \left[1 - \hat{\delta}(1 - \theta)\right]^{u-1}. \quad (264)$$

2. A first-order approximation of $\hat{k}_t := \frac{K^*/A^*}{K_t/A_t}$ is given by:

$$\hat{k}_t < 1 - \frac{A_t}{A_1} \left(\frac{A_0 - A_1}{A_0}\right) \left[1 - \hat{\delta}(1 - \theta)\right]^{t-1} - \sum_{u=1}^{t-1} \frac{A_t}{A_{t-u}} \left(\frac{A_{t-u}}{A_{t-u+1}} - 1\right) \left[1 - \hat{\delta}(1 - \theta)\right]^{u-1}. \quad (265)$$

Proof. 1. We know the law of motion of $\frac{K_t}{A_t}$ by equation (260):

$$\frac{K_{t+1}}{A_{t+1}} = \frac{A_t}{A_{t+1}} \left[\hat{s} \left(\frac{K_t}{A_t}\right)^\theta + (1 - \hat{\delta}) \frac{K_t}{A_t} \right], \text{ where } \hat{s} = \hat{\delta} \left(\frac{K^*}{A^*}\right)^{1-\theta}.$$

By dividing both hand sides by $\frac{K^*}{A^*}$, we have:

$$\frac{K_{t+1}/A_{t+1}}{K^*/A^*} = \frac{A_t}{A_{t+1}} \left[\hat{\delta} \left(\frac{K_t/A_t}{K^*/A^*}\right)^\theta + (1 - \hat{\delta}) \left(\frac{K_t/A_t}{K^*/A^*}\right) \right]. \quad (266)$$

Denote $\tilde{k}_t := \frac{K_t/A_t}{K^*/A^*}$ with $\tilde{k}^* = 1$. We will approximate $f(\tilde{k}_t) := \hat{\delta}(\tilde{k}_t)^\theta + (1 - \hat{\delta})\tilde{k}_t$ by Talor expansion:

$$f(\tilde{k}_t) = f(\tilde{k}^*) + f'(\tilde{k}^*)(\tilde{k}_t - \tilde{k}^*) + f''(\tilde{k}^*)(\tilde{k}_t - \tilde{k}^*)^2 + o(\|\tilde{k}_t - \tilde{k}^*\|^2), \quad (267)$$

where we have:

$$\begin{aligned} f(\tilde{k}^*) &= \hat{\delta} + 1 - \hat{\delta} = 1 \\ f'(\tilde{k}_t) &= \hat{\delta}\theta(\tilde{k}_t)^{\theta-1} + 1 - \hat{\delta} \\ \rightarrow f'(\tilde{k}^*) &= 1 - \hat{\delta}(1 - \theta) \\ f''(\tilde{k}_t) &= \hat{\delta}\theta(\theta - 1)(\tilde{k}_t)^{\theta-2} \\ \rightarrow f''(\tilde{k}^*) &= -\theta(1 - \theta)\hat{\delta}. \end{aligned}$$

Therefore, we approximate \tilde{k}_{t+1} by:

$$\tilde{k}_{t+1} = \frac{A_t}{A_{t+1}} \left[1 + [1 - \hat{\delta}(1 - \theta)](\tilde{k}_t - \tilde{k}^*) - \theta(1 - \theta)\hat{\delta}(\tilde{k}_t - \tilde{k}^*)^2 + o(\|\tilde{k}_t - \tilde{k}^*\|^2) \right]. \quad (268)$$

By subtracting $\tilde{k}^* (= 1)$ from both sides, we have:

$$\tilde{k}_{t+1} - \tilde{k}^* = \left(\frac{A_t}{A_{t+1}} - 1\right) + \frac{A_t}{A_{t+1}} \left[1 - \hat{\delta}(1 - \theta)\right] (\tilde{k}_t - \tilde{k}^*) - \frac{A_t}{A_{t+1}} \theta(1 - \theta)\hat{\delta}(\tilde{k}_t - \tilde{k}^*)^2 + o(\|\tilde{k}_t - \tilde{k}^*\|^2). \quad (269)$$

Since the third term is strictly negative, we derive the upper bound of $\tilde{k}_t - \tilde{k}^*$ as follows:

$$\begin{aligned}
\tilde{k}_t - \tilde{k}^* &= \frac{A_{t-1}}{A_t} - 1 + \frac{A_{t-1}}{A_t} \left[1 - \hat{\delta}(1 - \theta) \right] (\tilde{k}_{t-1} - \tilde{k}^*) - \frac{A_{t-1}}{A_t} \theta (1 - \theta) \hat{\delta} (\tilde{k}_{t-1} - \tilde{k}^*)^2 + o(\|\tilde{k}_{t-1} - \tilde{k}^*\|^2) \\
&< \frac{A_{t-1}}{A_t} \left[1 - \hat{\delta}(1 - \theta) \right] (\tilde{k}_{t-1} - \tilde{k}^*) - \left(1 - \frac{A_{t-1}}{A_t} \right) \\
&< \frac{A_{t-2}}{A_t} \left[1 - \hat{\delta}(1 - \theta) \right]^2 (\tilde{k}_{t-2} - \tilde{k}^*) - \frac{A_{t-1}}{A_t} \left[1 - \hat{\delta}(1 - \theta) \right] \left(1 - \frac{A_{t-2}}{A_{t-1}} \right) - \left(1 - \frac{A_{t-1}}{A_t} \right) \\
&< \frac{A_1}{A_t} \left[1 - \hat{\delta}(1 - \theta) \right]^{t-1} (\tilde{k}_1 - \tilde{k}^*) - \sum_{u=1}^{t-1} \frac{A_{t-u+1} - A_{t-u}}{A_t} \left[1 - \hat{\delta}(1 - \theta) \right]^{u-1}. \quad (270)
\end{aligned}$$

By substituting $\tilde{k}_1 := \frac{K_1/A_1}{K^*/A^*} = \frac{K_1/A_1}{K_0/A_0} = \frac{A_0}{A_1}$, we derive:

$$\tilde{k}_t < 1 - \frac{A_1 - A_0}{A_t} \left[1 - \hat{\delta}(1 - \theta) \right]^{t-1} - \sum_{u=1}^{t-1} \frac{A_{t-u+1} - A_{t-u}}{A_t} \left[1 - \hat{\delta}(1 - \theta) \right]^{u-1}.$$

2. To prepare for a sufficient condition in case of a declining sequence of $\{A_t\}$, we derive a Taylor expansion of the law of motion in terms of $\hat{k}_t := \frac{K^*/A^*}{K_t/A_t}$ with $\hat{k}^* = 1$. Equation (266) is written as:

$$\hat{k}_{t+1} = \frac{A_{t+1}}{A_t} \frac{1}{\hat{\delta}(\hat{k}_t)^{-\theta} + (1 - \hat{\delta})\hat{k}_t^{-1}}. \quad (271)$$

Define $g(\hat{k}_t) := \frac{1}{\hat{\delta}(\hat{k}_t)^{-\theta} + (1 - \hat{\delta})\hat{k}_t^{-1}}$ and Taylor approximate this function.

$$\begin{aligned}
g(\hat{k}^*) &= 1 \\
g'(\hat{k}_t) &= \frac{\theta \hat{\delta}(\hat{k}_t)^{-\theta-1} + (1 - \hat{\delta})(\hat{k}_t)^{-2}}{\left[\hat{\delta}(\hat{k}_t)^{-\theta} + (1 - \hat{\delta})(\hat{k}_t)^{-1} \right]^2} \\
\rightarrow g'(\hat{k}^*) &= 1 - \hat{\delta}(1 - \theta) \\
g''(\hat{k}_t) &= -\frac{\theta(\theta + 1)\hat{\delta}(\hat{k}_t)^{-\theta-2} + 2(1 - \hat{\delta})(\hat{k}_t)^{-3}}{\left[\hat{\delta}(\hat{k}_t)^{-\theta} + (1 - \hat{\delta})(\hat{k}_t)^{-1} \right]^2} + 2\frac{\left[\theta \hat{\delta}(\hat{k}_t)^{-\theta-1} + (1 - \hat{\delta})(\hat{k}_t)^{-2} \right]^2}{\left[\hat{\delta}(\hat{k}_t)^{-\theta} + (1 - \hat{\delta})(\hat{k}_t)^{-1} \right]^3} \\
\rightarrow g''(\hat{k}^*) &= -\left[\theta(\theta + 1)\hat{\delta} + 2(1 - \hat{\delta}) \right] + 2\left[\theta \hat{\delta} + 1 - \hat{\delta} \right] \\
&= \hat{\delta}(1 - \theta) \left[(1 - \theta)(2\hat{\delta} - 1) - 1 \right]
\end{aligned}$$

We have $g''(\hat{k}^*) < 0$ if $(1 - \theta)(2\hat{\delta} - 1) - 1 < 0 \Leftrightarrow 2\hat{\delta} < \frac{1}{1 - \theta} + 1$. This is true since $\hat{\delta} < 1$ and $\frac{1}{1 - \theta} > 1$. Therefore, Taylor expansion of (271) is given by:

$$\hat{k}_{t+1} - \hat{k}^* = \left(\frac{A_{t+1}}{A_t} - 1 \right) + \frac{A_{t+1}}{A_t} \left[1 - \hat{\delta}(1 - \theta) \right] (\hat{k}_t - \hat{k}^*) + \frac{A_{t+1}}{A_t} g''(\hat{k}^*) (\hat{k}_t - \hat{k}^*)^2 + o(\|\hat{k}_t - \hat{k}^*\|^2) \quad (272)$$

Since $g''(\hat{k}^*) < 0$,

$$\begin{aligned}
\hat{k}_t - \hat{k}^* &= \left(\frac{A_t}{A_{t-1}} - 1 \right) + \frac{A_t}{A_{t-1}} \left[1 - \hat{\delta}(1 - \theta) \right] (\hat{k}_{t-1} - \hat{k}^*) + \frac{A_t}{A_{t-1}} g''(\hat{k}^*) (\hat{k}_{t-1} - \hat{k}^*)^2 + o(\|\hat{k}_{t-1} - \hat{k}^*\|^2) \\
&< \frac{A_t}{A_{t-1}} \left[1 - \hat{\delta}(1 - \theta) \right] (\hat{k}_{t-1} - \hat{k}^*) + \left(\frac{A_t}{A_{t-1}} - 1 \right) \\
&< \frac{A_t}{A_{t-2}} \left[1 - \hat{\delta}(1 - \theta) \right]^2 (\hat{k}_{t-2} - \hat{k}^*) + \frac{A_t}{A_{t-2}} \left(1 - \frac{A_{t-2}}{A_{t-1}} \right) \left[1 - \hat{\delta}(1 - \theta) \right] + \frac{A_t}{A_{t-1}} \left(1 - \frac{A_{t-1}}{A_t} \right) \\
&< \frac{A_t}{A_1} \left[1 - \hat{\delta}(1 - \theta) \right]^{t-1} (\hat{k}_1 - \hat{k}^*) - \sum_{u=1}^{t-1} \frac{A_t}{A_{t-u}} \left(\frac{A_{t-u}}{A_{t-u+1}} - 1 \right) \left[1 - \hat{\delta}(1 - \theta) \right]^{u-1}.
\end{aligned}$$

By using $\hat{k}_1 - \hat{k}^* := \frac{K_0/A_0}{K_1/A_1} - 1 = \frac{A_1 - A_0}{A_0}$, we have:

$$\hat{k}_t < 1 - \frac{A_t}{A_1} \left(\frac{A_0 - A_1}{A_0} \right) \left[1 - \hat{\delta}(1 - \theta) \right]^{t-1} - \sum_{u=1}^{t-1} \frac{A_t}{A_{t-u}} \left(\frac{A_{t-u}}{A_{t-u+1}} - 1 \right) \left[1 - \hat{\delta}(1 - \theta) \right]^{u-1}.$$

□

A Sufficient Condition on $\{A_t\}_{t=1}^{\infty}$ First-order approximations of \tilde{k}_t and \hat{k}_t allow us to establish a sufficient condition on $\{A_t\}_{t=2}^{\infty}$ for the monotonicity of $\{\frac{K_t}{A_t}\}_{t=1}^{\infty}$. Then, Lemma 13 gives a condition on A_1 that is sufficient for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 0$. We state a proposition below.

Proposition 16.

1. (Positive Shocks) Suppose Assumption 5 holds and the economy is in a steady state at $t = 0$. Consider a weakly increasing path of $\{A_t\}_{t=1}^{\infty}$ converging to A^* ($A_0 < A_1 \leq \dots \leq A^*$ with $\lim_{t \rightarrow \infty} A_t = A^*$) that is unexpectedly realized at $t = 1$. If the sequence of $\{A_t\}$ satisfies the condition (274), both $\{A_t\}$ and $\{\frac{K_t}{A_t}\}$ are monotonically increasing in t . Then, $A_1 < \bar{A}_1$ is sufficient for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 0$ (Lemma 13). Therefore, the conditions (273) and (274) together guarantee $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 0$.

$$A_0 < A_1 < \bar{A}_1, \tag{273}$$

$$1 \leq \frac{A_{t+1}}{A_t}$$

$$< 1 + \hat{\delta} \left[\left(\frac{1}{1 - \frac{A_1 - A_0}{A_t} \left[1 - \hat{\delta}(1 - \theta) \right]^{t-1} - \sum_{u=1}^{t-1} \frac{A_{t-u+1} - A_{t-u}}{A_t} \left[1 - \hat{\delta}(1 - \theta) \right]^{u-1}} \right)^{1-\theta} - 1 \right] \quad \forall t \geq 1, \tag{274}$$

$$\text{where } \bar{A}_1 := A_0 \left[\frac{1 - \beta(1 - \delta)}{\theta} \frac{\xi(1 - \theta) + (1 - \nu)\theta[1 - (1 - \nu - \xi)\beta]}{[1 - (1 - \nu)\beta(1 - \delta)][1 - (1 - \nu - \xi)\beta]} \right]^{\frac{1}{1-\theta}}.$$

2. (Negative shocks) Suppose Assumption 5 holds and the economy is in a steady state at $t = 0$. Consider a weakly decreasing path of $\{A_t\}_{t=1}^{\infty}$ converging to A^* ($A_0 > A_1 \geq \dots \geq A^*$

with $\lim_{t \rightarrow \infty} A_t = A^*$) that is unexpectedly realized at $t = 1$. If the sequence of $\{A_t\}$ satisfies the condition (276), $\{\frac{K_t}{A_t}\}$ is monotonically declining in t . This implies that $\underline{A}_1 < A_1$ is sufficient to guarantee $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 0$ (Lemma 13). Therefore, the conditions (275) and (276) together guarantee $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 0$.

$$\underline{A}_1 < A_1 < A_0, \quad (275)$$

$$1 - \hat{\delta} \left[1 - \left(1 - \frac{A_t}{A_1} \left(\frac{A_0 - A_1}{A_0} \right) [1 - \hat{\delta}(1 - \theta)]^{t-1} - \sum_{u=1}^{t-1} \frac{A_t}{A_{t-u}} \left(\frac{A_{t-u}}{A_{t-u+1}} - 1 \right) [1 - \hat{\delta}(1 - \theta)]^{u-1} \right)^{1-\theta} \right] < \frac{A_{t+1}}{A_t} \leq 1 \quad \forall t \geq 1, \quad (276)$$

$$\text{where } \underline{A}_1 := A_0 \left[1 - \nu + \nu \frac{1 - (1 - \delta)\beta(1 - \nu)}{\beta\nu(1 - \delta)} \frac{\xi(1 - \theta) - \beta\theta\nu(\xi + \nu + \frac{1}{\beta} - 1)}{\xi(1 - \theta) + \beta\theta(1 - \nu)(\xi + \nu + \frac{1}{\beta} - 1)} \right]^{\frac{1}{\theta-1}}.$$

Proof. 1. Lemma 14 states that $\frac{K_{t+1}}{A_{t+1}} > \frac{K_t}{A_t}$ for all $t \geq 1$ if

$$\frac{A_{t+1}}{A_t} < 1 + \hat{\delta} \left[\left(\frac{K^*/A^*}{K_t/A_t} \right)^{1-\theta} - 1 \right] \text{ at all } t \geq 1. \quad (277)$$

In Lemma 15, we establish an upper bound on $\tilde{k}_t := \frac{K_t/A_t}{K^*/A^*}$. Therefore, we have:

$$\left(\frac{K^*/A^*}{K_t/A_t} \right)^{1-\theta} > \left[1 - \frac{A_1 - A_0}{A_t} [1 - \hat{\delta}(1 - \theta)]^{t-1} - \sum_{u=1}^{t-1} \frac{A_{t-u+1} - A_{t-u}}{A_t} [1 - \hat{\delta}(1 - \theta)]^{u-1} \right]^{-(1-\theta)} \quad (278)$$

Hence, the following condition is sufficient for condition (277):

$$\frac{A_{t+1}}{A_t} < 1 + \hat{\delta} \left[\left(\frac{1}{1 - \frac{A_1 - A_0}{A_t} [1 - \hat{\delta}(1 - \theta)]^{t-1} - \sum_{u=1}^{t-1} \frac{A_{t-u+1} - A_{t-u}}{A_t} [1 - \hat{\delta}(1 - \theta)]^{u-1}} \right)^{1-\theta} - 1 \right] \quad \forall t \geq 1.$$

Under this condition, $\{\frac{K_t}{A_t}\}_{t=1}^{\infty}$ is monotonically increasing. Thus, Lemma 13 states that condition (273) is sufficient for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 0$.

2. Lemma 14 states that $\frac{K_{t+1}}{A_{t+1}} < \frac{K_t}{A_t}$ for all $t \geq 1$ if

$$\frac{A_{t+1}}{A_t} > 1 + \hat{\delta} \left[\left(\frac{K^*/A^*}{K_t/A_t} \right)^{1-\theta} - 1 \right] \text{ at all } t \geq 1. \quad (279)$$

In Lemma 15, we establish an upper bound on $\hat{k}_t := \frac{K^*/A^*}{K_t/A_t}$. Therefore, we have:

$$\left(\frac{K^*/A^*}{K_t/A_t} \right)^{1-\theta} > \left[1 - \frac{A_t}{A_1} \left(\frac{A_0 - A_1}{A_0} \right) [1 - \hat{\delta}(1 - \theta)]^{t-1} - \sum_{u=1}^{t-1} \frac{A_t}{A_{t-u}} \left(\frac{A_{t-u}}{A_{t-u+1}} - 1 \right) [1 - \hat{\delta}(1 - \theta)]^{u-1} \right]^{1-\theta} \quad (280)$$

Hence, the following condition is sufficient for condition (279):

$$\frac{A_{t+1}}{A_t} > 1 + \hat{\delta} \left[\left[1 - \frac{A_t}{A_1} \left(\frac{A_0 - A_1}{A_0} \right) \left[1 - \hat{\delta}(1 - \theta) \right]^{t-1} - \sum_{u=1}^{t-1} \frac{A_t}{A_{t-u}} \left(\frac{A_{t-u}}{A_{t-u+1}} - 1 \right) \left[1 - \hat{\delta}(1 - \theta) \right]^{u-1} \right]^{1-\theta} - 1 \right] \quad \forall t \geq 1.$$

Under this condition, $\{\frac{K_t}{A_t}\}_{t=1}^{\infty}$ is monotonically decreasing. Thus, Lemma 13 states that condition (275) is sufficient for $\beta R_{t+1} < \frac{w_{t+1}}{w_t}$ at all $t \geq 0$. \square

B.1.2 Further Discussion on Corollary 2

As we see in Section 4.2.2, the Euler equation between $t = 0$ and $t = 1$ for low-income agents holds at the time of MIT shock ($t = 1$), and households do not respond to future anticipated productivity shocks. This is not generally the case in a limited-commitment model with CRRA utility and in a neoclassical growth model with logarithmic utility.

CRRA utility We discuss what happens under a CRRA utility function ($\frac{c^{1-\sigma}-1}{1-\sigma}$ with $\sigma \neq 1$). Under the no-savings condition for high-income states (Assumption 3), households finance their consumption in future low-income states with their state-contingent assets, so the budget constraint is given by:

$$R_t \mathbf{a}_{s,t} = \mathbf{c}_{s,t} + \frac{1-\nu}{R_{t+1}} \mathbf{c}_{s+1,t+1} + \frac{1-\nu}{R_{t+1}} \frac{1-\nu}{R_{t+2}} \mathbf{c}_{s+2,t+2} + \dots \quad (281)$$

Suppose a deterministic sequence of future interest rates $\{R_\tau\}_{\tau \geq t+1}$ is given. Then, the optimal consumption in all future low-income states is determined by the Euler equation:

$$\mathbf{c}_{s+1,\tau+1} = (\beta R_{\tau+1})^{\frac{1}{\sigma}} \mathbf{c}_{s,\tau} \quad \text{for all } s \geq 0, \tau \geq t.$$

The budget constraint is written as follows:

$$R_t \mathbf{a}_{s,t} = \mathbf{c}_{s,t} \left[1 + (1-\nu) \beta^{\frac{1}{\sigma}} (R_{t+1})^{\frac{1-\sigma}{\sigma}} + (1-\nu)^2 \beta^{\frac{2}{\sigma}} (R_{t+1} R_{t+2})^{\frac{1-\sigma}{\sigma}} + \dots \right], \quad (282)$$

implying that today's consumption depends on future interest rates $\{R_\tau\}_{\tau \geq t+1}$:

$$\mathbf{c}_{s,t} = \frac{R_t \mathbf{a}_{s,t}}{\left[1 + (1-\nu) \beta^{\frac{1}{\sigma}} (R_{t+1})^{\frac{1-\sigma}{\sigma}} + (1-\nu)^2 \beta^{\frac{2}{\sigma}} (R_{t+1} R_{t+2})^{\frac{1-\sigma}{\sigma}} + \dots \right]}. \quad (283)$$

Note that if $\sigma = 1$ (logarithmic utility), the denominator becomes a constant, implying that consumption is proportional to today's interest rate. This leads to the previous result that the Euler equation between t and $t + 1$ holds despite an unexpected shock at $t + 1$. However, under $\sigma \neq 1$, since consumption today depends on future interest rates, the expectation about future productivity affects today's consumption decisions.

Suppose that a future path of interest rates unexpectedly changes at time $t + 1$. The optimal consumption at $t + 1$ is chosen according to equation (283), which is generally

different from the consumption anticipated at time t .⁴⁵ Hence, the Euler equation between t and $t + 1$ no longer holds. This implies that MIT shocks and anticipated shocks have different implications for households' consumption and the law of motion of capital in the case of CRRA utility with $\sigma \neq 1$.

Comparison with a Neoclassical Growth Model The two results (the Euler equation at the time of the shock and the indifference between MIT shocks and anticipated shocks) are not true in a standard neoclassical growth model unless the capital fully depreciates ($\delta = 1$).

Suppose the economy is in a steady state at $t = -5$ with aggregate productivity A_0 . Agents realize the news at $t = -4$ that productivity increases from A_0 to A_1 at $t = 1$ permanently:

$$A_t = \begin{cases} A_0 & \text{if } t \leq 0 \\ A_1 & \text{if } t \geq 1 \end{cases}$$

Transitional dynamics of a standard neoclassical growth model is described by an Euler equation and a resource constraint. Then, the law of motion of capital is derived as:

$$K_{t+2} = A_{t+1}^{1-\theta} K_{t+1}^\theta + (1 - \delta)K_{t+1} - \beta \underbrace{(\theta A_{t+1}^{1-\theta} K_{t+1}^{\theta-1} + 1 - \delta)}_{=R_{t+1}} \underbrace{[A_t^{1-\theta} K_t^\theta + (1 - \delta)K_t - K_{t+1}]}_{C_t}. \quad (284)$$

Here we continue to assume the logarithmic utility. Capital in the initial steady state (K_{-5}) and capital in the new steady state (K_∞) allow us to numerically compute the sequence of capital. Since choice variables at time t (C_t, K_{t+1}) depend on future capital (K_{t+2}), the agents respond to future anticipated shocks (Figure 6). Intuitively, higher productivity from $t = 1$ onwards allows agents to consume more. Since agents prefer a smooth consumption profile, they start to consume a little more when they realize the news at $t = -4$.

The interest rate does not change at $t = -4$ as $A_{-5} = A_{-4} = A_0$ and $K_{-5} = K_{-4}$. The response of C_{-4} to the future shock implies that the Euler equation between $t = -5$ and $t = -4$ does not hold. An optimality condition requires that the Euler equation between $t = 0$ and $t = 1$ holds.⁴⁶

⁴⁵ $c_{s,t+1}$ is still proportional to an interest rate R_{t+1} if $\{R_\tau\}_{\tau \geq t+2}$ does not change from the anticipated one. However, after an unanticipated TFP shock at $t + 1$, which affects the law of motion of capital at $t + 1$, a new sequence of interest rates $\{R_\tau\}_{\tau \geq t+2}$ generally differs from the anticipated one.

⁴⁶The Euler equation between $t = 0$ and $t = 1$ does not generally hold if an MIT shock is realized at $t = 1$.

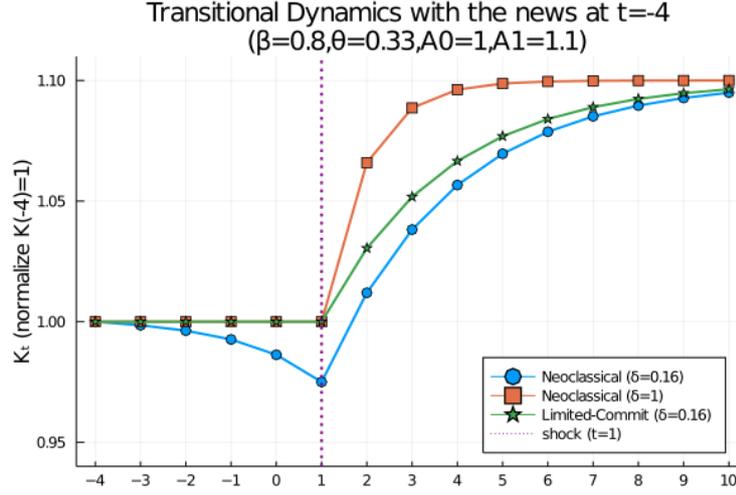


Figure 6: Transitional Dynamics of K_t after an Anticipated Shock

B.1.3 Consumption Inequality along the Transition

We characterize the forces determining the evolution of consumption inequality along the transition to the new steady state. In period $t = 1$ capital was predetermined and thus the impact on capital- and labor income was exclusively determined by the exogenous shock itself, a fact that enabled us sharply characterize the consumption distribution at that date. From period $t = 2$ on aggregate capital adjusts, following the law of motion (39), and with it wages w_t and rates of returns R_t . Since consumption follows a ladder structure, consumption inequality at each date can be characterized by the gap between consumption of high income individuals, $c_{0,t}$ and of low-income individuals who last had high income s periods ago, $c_{s,t}$. The evolution of this consumption gap $-\log(c_{s,t}/c_{0,t})$ is characterized in Proposition 17. We first derive the consumption gap $\frac{c_{s,t}}{c_{s,0}}$ in the following Lemma.

Lemma 16. *Suppose an economy is in a stationary equilibrium at $t = 0$, and a productivity shock is realized at $t = 1$. Suppose that Assumptions 2, 3, and 4 hold. The evolution of deflated consumption for low-income agents is characterized as:*

$$\frac{c_{s,t}}{c_{s,0}} = \begin{cases} \left(\prod_{u=1}^t \frac{R_u}{R_0} \right) \frac{w_0}{w_t} & \text{if } s \geq t \\ \left(\prod_{u=t-s+1}^t \frac{R_u}{R_0} \right) \frac{w_{t-s}}{w_t} & \text{if } 1 \leq s < t \end{cases} \quad (285)$$

Notice that by denoting $R_u = R_0$ for $u \leq 0$ and $w_{t-s} = w_0$ for $s \geq t$, the latter expression includes the former as a special case.

Proof. The ratio of deflated consumption between time t and time 0 for agents s with

$s \geq t$, $\frac{c_{s,t}}{c_{s,0}}$, is derived using the Euler equation:

$$\begin{aligned} w_t c_{s,t} &= \beta R_t w_{t-1} c_{s-1,t-1} \\ &= (\beta R_t)(\beta R_{t-1}) \cdots (\beta R_1) w_0 c_{s-t,0} \\ c_{s,0} &= (\beta R_0)^t c_{s-t,0} \\ \therefore \frac{c_{s,t}}{c_{s,0}} &= \left(\frac{R_1}{R_0}\right) \cdots \left(\frac{R_t}{R_0}\right) \frac{w_0}{w_t} \end{aligned}$$

This equation can be expressed in a sequential way:

$$\begin{aligned} \frac{c_{s,t}}{c_{s,0}} &= \frac{c_{s,t-1}}{c_{s,0}} \left(\frac{R_t}{R_0}\right) \left(\frac{w_{t-1}}{w_t}\right) \\ &= \prod_{u=1}^t \left(\frac{R_u}{R_0}\right) \left(\frac{w_{u-1}}{w_u}\right) = \left(\prod_{u=1}^t \frac{R_u}{R_0}\right) \frac{w_0}{w_t}. \end{aligned}$$

If $s < t$, the low-income agents have experienced high income after the shock. Hence, wage at the time of last high income is w_{t-s} with $w_{t-s} > w_0$. The ratio of deflated consumption between time t and time 0 is given by:

$$\begin{aligned} w_t c_{s,t} &= \beta R_t w_{t-1} c_{s-1,t-1} \\ &= (\beta R_t) \cdots (\beta R_{t-s+1}) w_{t-s} c_{h,t-s} \\ c_{s,0} &= (\beta R_0)^s c_{h,0} \\ \therefore \frac{c_{s,t}}{c_{s,0}} &= \left(\frac{R_{t-s+1}}{R_0}\right) \cdots \left(\frac{R_t}{R_0}\right) \frac{w_{t-s}}{w_t} = \left(\prod_{u=t-s+1}^t \frac{R_u}{R_0}\right) \frac{w_{t-s}}{w_t} \end{aligned}$$

□

Proposition 17. *Suppose an economy is in a stationary equilibrium at $t = 0$ and is hit by a permanent productivity shock at $t = 1$. Let Assumption 3 be satisfied and $0 < \delta < 1$. The evolution of the consumption gap between high-income and low-income agents relative to the steady state is characterized by:*

$$-\log\left(\frac{c_{s,t}}{c_{0,t}}\right) + \log\left(\frac{c_{s,0}}{c_{0,0}}\right) = \begin{cases} \log\left(\frac{w_1 R_0}{w_0 R_1}\right) & \text{if } s \geq 1 \text{ if } t = 1 \\ \log\left(\frac{w_1 R_0}{w_0 R_1}\right) + \sum_{u=1}^{t-1} \log\left(\frac{w_{u+1} R_0}{w_u R_{u+1}}\right) & \text{if } s \geq t \text{ and } t \geq 2 \\ \sum_{u=t-s}^{t-1} \log\left(\frac{w_{u+1} R_0}{w_u R_{u+1}}\right) & \text{if } s < t \text{ and } t \geq 2 \end{cases} \quad (286)$$

The consumption gap expands at time 1 since $\log\left(\frac{w_1 R_0}{w_0 R_1}\right) > 0$. From time 1 until time $s \geq 2$, the consumption gap continues to be higher than in the stationary equilibrium if $\log\left(\frac{w_1 R_0}{w_0 R_1}\right) + \sum_{u=1}^{t-1} \log\left(\frac{w_{u+1} R_0}{w_u R_{u+1}}\right) > 0$. From time $s + 1$ onward, the consumption gap is smaller than the stationary equilibrium if $\sum_{u=t-s}^{t-1} \log\left(\frac{w_{u+1} R_0}{w_u R_{u+1}}\right) < 0$.

Proof. From equation (285) in Lemma 16, we know:

$$\begin{aligned}\frac{c_{s,t}}{c_{s,0}} &= \left(\frac{R_{t-s+1}}{R_0}\right) \cdots \left(\frac{R_t}{R_0}\right) \frac{w_{t-s}}{w_t}, \\ \frac{c_{s,t-1}}{c_{s,0}} &= \left(\frac{R_{t-s}}{R_0}\right) \cdots \left(\frac{R_{t-1}}{R_0}\right) \frac{w_{t-1-s}}{w_{t-1}}.\end{aligned}$$

Dividing the former equation by the latter gives:

$$\frac{c_{s,t}}{c_{s,t-1}} = \left(\frac{R_t}{R_{t-s}}\right) \left(\frac{w_{t-s}}{w_t}\right) \left(\frac{w_{t-1}}{w_{t-1-s}}\right).$$

This allows us to express $\frac{c_{s,t}}{c_{0,t}}$ in a sequential way, where we use $c_{0,t} = c_{0,t-1} = c_h$:

$$\begin{aligned}\frac{c_{s,t}}{c_{0,t}} &= \left(\frac{R_t}{R_{t-s}}\right) \left(\frac{w_{t-s}}{w_t}\right) \left(\frac{w_{t-1}}{w_{t-1-s}}\right) \frac{c_{s,t-1}}{c_{0,t-1}} \\ &= \left[\prod_{u=1}^t \frac{R_u}{R_{u-s}} \frac{w_{u-s}}{w_u} \frac{w_{u-1}}{w_{u-1-s}} \right] \frac{c_{s,0}}{c_{0,0}} \text{ for } s \geq 1.\end{aligned}\quad (287)$$

By taking a log:

$$\begin{aligned}-\log\left(\frac{c_{s,t}}{c_{0,t}}\right) &= \sum_{u=1}^t \left[\underbrace{\log\left(\frac{w_u}{w_{u-1}}\right)}_{\text{wage increase for high income}} - \underbrace{\log\left(\frac{w_{u-s}}{w_{u-s-1}}\right)}_{\text{wage increase for } s \text{ agent}} - \underbrace{\log\left(\frac{R_u}{R_{u-s}}\right)}_{\text{higher interest rate}} \right] - \underbrace{\log\left(\frac{c_{s,0}}{c_{0,0}}\right)}_{\text{initial gap}} \\ &= \log\left(\frac{w_t}{w_0}\right) - \log\left(\frac{w_{t-s}}{w_0}\right) - \sum_{u=t-s+1}^t \log\left(\frac{R_u}{R_0}\right) - \log\left(\frac{c_{s,0}}{c_{0,0}}\right).\end{aligned}\quad (288)$$

This equation holds for any $s \geq 1$. If $s \geq t$, we use the facts that:

$$\begin{aligned}\log\left(\frac{w_{t-s}}{w_0}\right) &= \log\left(\frac{w_0}{w_0}\right) = 0 \\ \sum_{u=t-s+1}^0 \log\left(\frac{R_u}{R_0}\right) &= \sum_{u=t-s+1}^0 \log\left(\frac{R_0}{R_0}\right) = 0 \text{ for } s > t.\end{aligned}$$

Then, (288) is written as follows:

$$\begin{aligned}-\log\left(\frac{c_{s,t}}{c_{0,t}}\right) + \log\left(\frac{c_{s,0}}{c_{0,0}}\right) &= \log\left(\frac{w_t}{w_0}\right) - \sum_{u=1}^t \log\left(\frac{R_u}{R_0}\right) \text{ for } s \geq t \\ &= \sum_{u=1}^t \log\left(\frac{w_u}{w_{u-1}} \frac{R_0}{R_u}\right) \\ &= \begin{cases} \log\left(\frac{w_1}{w_0} \frac{R_0}{R_1}\right) + \sum_{u=1}^{t-1} \log\left(\frac{w_{u+1}}{w_u} \frac{R_0}{R_{u+1}}\right) & \text{if } t \geq 2 \\ \log\left(\frac{w_1}{w_0} \frac{R_0}{R_1}\right) & \text{if } t = 1 \end{cases}\end{aligned}\quad (289)$$

Combined with the case of $s < t$, we have:

$$-\log\left(\frac{c_{s,t}}{c_{0,t}}\right) + \log\left(\frac{c_{s,0}}{c_{0,0}}\right) = \begin{cases} \log\left(\frac{w_1 R_0}{w_0 R_1}\right) & \text{for all } s \geq 1 \text{ if } t = 1 \\ \log\left(\frac{w_1 R_0}{w_0 R_1}\right) + \sum_{u=1}^{t-1} \log\left(\frac{w_{u+1} R_0}{w_u R_{u+1}}\right) & \text{if } s \geq t \text{ and } t \geq 2 \\ \sum_{u=t-s}^{t-1} \log\left(\frac{w_{u+1} R_0}{w_u R_{u+1}}\right) & \text{if } s < t \text{ and } t \geq 2 \end{cases}$$

As we saw before,

$$\frac{w_1 R_0}{w_0 R_1} = \frac{\theta A_1^{1-\theta} K_0^{\theta-1} + (1-\delta) \left(\frac{A_1}{A_0}\right)^{1-\theta}}{\theta A_1^{1-\theta} K_0^{\theta-1} + 1 - \delta} > 1 \text{ if } \delta < 1. \quad (290)$$

This confirms that consumption gap between high-income agents and s -th low-income agents expands at time 1 if $\delta < 1$. Therefore, the productivity shock increases consumption of high-income agents more than low-income agents at time 1. \square

We provide an intuition for the proposition. Since low-income agents ($s \geq 1$) have no labor income and consume a fraction of savings accumulated in the last high-income period, their consumption gap depends on whether the last high income realization materialized before or after the time 0 productivity shock. Low-income agents with $s \geq t$ have not yet experienced high income after the shock, and the wage in their last high-income state was w_0 . In contrast, low-income agents with $s < t$ already realized high income after the shock, and wages in the last high-income period are $w_{t-s} > w_0$.

We plot the evolution over time of the consumption gaps in Figure 7 for the parametric example used above. As the proposition shows, the gap is determined by three factors, which we display separately in Figure 8: (i) a higher wage at time t compared to time 0 benefits high-income agents, which expands the consumption gap; (ii) a higher wage at time $t - s$ compared to time 0 benefits type s low-income agents; and (iii) a higher interest rate from time $t - s + 1$ to time t benefits type s low-income agents because of the higher return on savings.

Because wages increase monotonically after the positive productivity shock, the wage effect $\log\left(\frac{w_t}{w_0}\right) - \log\left(\frac{w_{t-s}}{w_0}\right) = \log\left(\frac{w_t}{w_{t-s}}\right)$ expands the consumption gap (see the first three panels of Figure 8). On the other hand, because the interest rate is higher than in the stationary equilibrium, the interest rate effect $\sum_{u=t-s+1}^t \log\left(\frac{R_u}{R_0}\right)$ shrinks the gap. The overall impact depends on the relative magnitude of the two effects, as Figure 7 shows. Here, the consumption expands in period $t = 1$, then starts to shrink at time $t = 2$ and undershoots the original gap before converging to the new stationary equilibrium.

B.1.4 Inequality after a Negative Shock

This subsection summarizes symmetric results for an unexpected negative productivity shock at time 1. Figure 9 illustrates that the transition path of consumption gaps after a

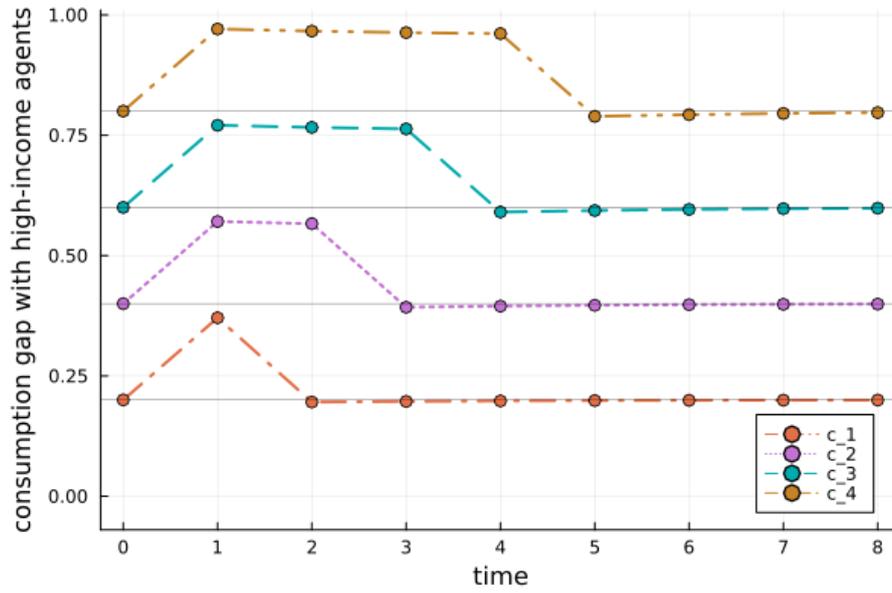


Figure 7: Evolution of the Consumption Gap. The figure plots $-\log(c_{s,t}/c_{0,t})$ against time t along the transition. We use the same parameter values as in Figure 3.

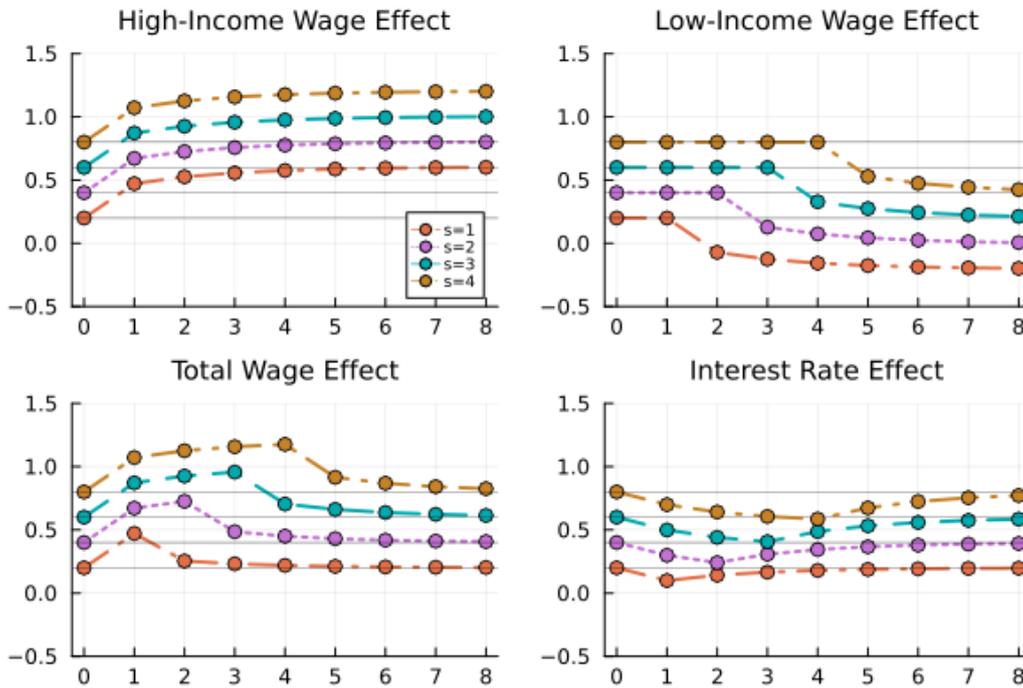
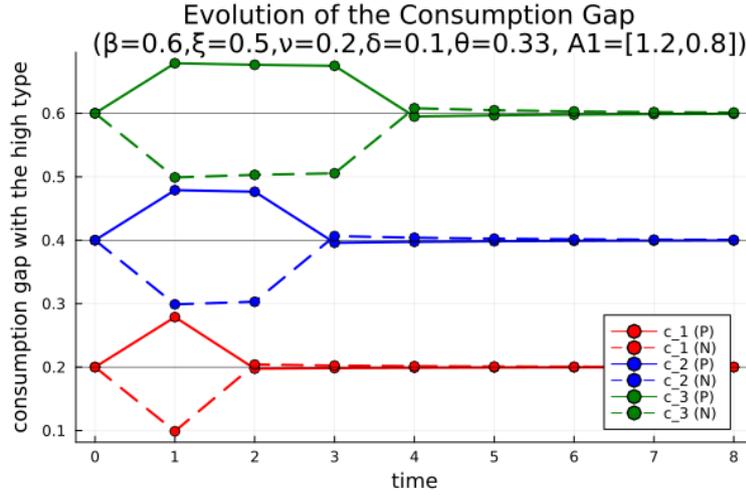


Figure 8: Decomposition of the Consumption Gap. First panel: wage effect for high income agents. Second panel: wage effect for low-income agents. Third panel: combined wage effect. Last panel: interest rate effect. We use the same parameter values as in Figure 3.

Figure 9: Evolution of the Consumption Gap (Positive & Negative Productivity Shock)



negative productivity shock is symmetric with a transition path after a positive shock.

Corollary 4 (Long-Run Consumption Gap after a Negative Shock). *Consider a transition path after a negative productivity shock at time 1. For sufficiently large $t \geq s + 1$, the consumption gap converges to the initial level for all $s \geq 1$. This means that the deflated consumption distribution in the new stationary equilibrium is the same as the initial stationary equilibrium.*

Proof. It follows directly from Proposition 8 □

Corollary 5 (Negative Productivity Shock with Full Depreciation). *Consider a transition path after a negative productivity shock.*

$$\text{If } \delta = 1, \frac{w_{t+1} R_0}{w_t R_{t+1}} = 1 \text{ for all } t \geq 0.$$

This implies that with full depreciation of capital ($\delta = 1$), the deflated consumption distribution is constant along the transition.

Proof. It follows directly from Proposition 9 □

Corollary 6 (Transition of Consumption Gap after a Negative Shock). *Consider a transition path after a negative productivity shock at time 1. The evolution of the consumption gap between high-income agents and s -th low-income agents is given by (286):*

$$-\log \left(\frac{c_{s,t}}{c_{0,t}} \right) + \log \left(\frac{c_{s,0}}{c_{0,0}} \right) = \begin{cases} \log \left(\frac{w_1 R_0}{w_0 R_1} \right) & \text{if } t = 1 \\ \log \left(\frac{w_1 R_0}{w_0 R_1} \right) + \sum_{u=1}^{t-1} \log \left(\frac{w_{u+1} R_0}{w_u R_{u+1}} \right) & \text{if } 2 \leq t \leq s \text{ and } s \geq 2 \\ \sum_{u=t-s}^{t-1} \log \left(\frac{w_{u+1} R_0}{w_u R_{u+1}} \right) & \text{if } t \geq s + 1 \end{cases}$$

Assume $0 < \delta < 1$. Then, the consumption gap shrinks at time 1, since $\log\left(\frac{w_1 R_0}{w_0 R_1}\right) < 0$. From time 1 until time $s \geq 2$, the consumption gap continues to be lower than the stationary equilibrium if $\log\left(\frac{w_1 R_0}{w_0 R_1}\right) + \sum_{u=1}^{s-1} \log\left(\frac{w_{u+1} R_0}{w_u R_{u+1}}\right) < 0$. From time $s + 1$ onwards, the consumption gap is higher than the stationary equilibrium if $\sum_{u=t-s}^{t-1} \log\left(\frac{w_{u+1} R_0}{w_u R_{u+1}}\right) > 0$.

Proof. It follows directly from Propositions 10 and 17. \square

B.2 Additional Discussions about Asset Pricing

B.2.1 Comparison of $(K_0, \{A_t\}_{t=0}^\infty)$ in the two models

Saving Rate and Capital Accumulation in the LC model We can rewrite \hat{s}^{LC} as:

$$\hat{s}^{LC} = \beta\theta + (1 - \theta)\nu \left[\frac{\xi}{\nu\left(\frac{1}{\beta} - 1 + \xi + \nu\right)} - \frac{\theta}{(1 - \theta)\frac{1}{\beta}} \right] \quad (291)$$

With $\delta = 1$, the second term is strictly positive if and only if Assumption 5 holds.

With $\delta = 1$, we also derive K_t in closed form. Since the economy has a constant saving rate,

$$K_{t+1} = sA_t^{1-\theta}K_t^\theta, \text{ where } s \in \{\hat{s}^{LC}, s^{Rep} := \beta\theta\}$$

This implies:

$$\log K_t = (1 - \theta) \left[\sum_{\tau=1}^{t-1} \theta^{t-1-\tau} \log A_\tau \right] + \frac{1 - \theta^{t-1}}{1 - \theta} \log \hat{s} + \theta^{t-1} \log K_0 \quad (292)$$

Comparison given $(K_0, \{A_t\}_{t=0}^\infty)$ The expression of K_{t+1} above implies that for any given K_t and A_t , the limited-commitment model always accumulates more capital:

$$\log K_{t+1}^{LC} - \log K_{t+1}^{Rep} = \underbrace{\log \hat{s}^{LC} - \log s^{Rep}}_{>0 \text{ under Assumption 5}} + \theta \left(\log K_t^{LC} - \log K_t^{Rep} \right). \quad (293)$$

Starting from the same initial capital K_0 , the difference in K_t is expressed as:

$$\log K_t^{LC} - \log K_t^{Rep} = (1 + \theta + \theta^2 + \dots + \theta^t)(\log \hat{s}^{LC} - \log s^{Rep}), \quad (294)$$

where $(\log \hat{s}^{LC} - \log s^{Rep}) = \log \left[\frac{\xi}{1 - (1 - \nu - \xi)\beta} \frac{1 - \theta}{\theta} + 1 - \nu \right] > 0$ under Assumption 5.

In the long-run, this will converge to:

$$\lim_{t \rightarrow \infty} (\log K_t^{LC} - \log K_t^{Rep}) = \frac{1}{1 - \theta} \log \left[\frac{\hat{s}}{\beta\theta} \right], \quad (295)$$

which is consistent with the capital ratio in the steady state:

$$\frac{K^{*LC}}{K^{*Rep}} = \left(\frac{\hat{s}}{\beta\theta} \right)^{\frac{1}{1-\theta}}. \quad (296)$$

We make three remarks here: (i) Given $(K_0, \{A_t\}_{t=0}^{\infty})$, K_t^{LC} is always larger than K_t^{Rep} , (ii) For any sequence of $\{A_t\}_{t \geq 0}$, $\frac{K_t^{LC}}{K_t^{Rep}}$ monotonically converges to the ratio in the steady state $\frac{K^{*LC}}{K^{*Rep}}$, (iii) As ξ and ν approach zero ($\xi \rightarrow 0$ and $\nu \rightarrow 0$), the law of motion of capital in the limited-commitment model also approaches to the one in the representative-agent model:

$$\lim_{\xi \rightarrow 0, \nu \rightarrow 0} K_t^{LC} = K_t^{Rep} \text{ for any } t \geq 1, \text{ given } K_0. \quad (297)$$

B.2.2 Subsection 5.2.4: Quantitative Exploration of Asset Pricing

The risk premium in the limited-commitment model and the representative-agent model are stated in Lemmas 1 and 2. In order to quantitatively assess the risk premium, it is expressed as a function of productivity-adjusted capital \tilde{K}_t , aggregate productivity growth rate $g_{A,t+1} = \frac{A_{t+1}}{A_t}$, and depreciation rate δ_{t+1} .

Asset Pricing in a Limited-Commitment Model Given the law of motion of aggregate capital, risk premium in the limited-commitment model is expressed as:⁴⁷

$$1 + \lambda_t^{LC}(\Omega^t) = \mathbb{E}_t \left[\theta g_{A,t+1}(\Omega_{t+1})^{1-\theta} \left[\hat{s} \tilde{K}_t(\Omega^t)^\theta + (1 - \hat{\delta}_t(\Omega_t)) \tilde{K}_t(\Omega^t) \right]^{\theta-1} + 1 - \delta_{t+1}(\Omega_{t+1}) \right] \\ \times \mathbb{E}_t \left[\frac{1}{\theta g_{A,t+1}(\Omega_{t+1})^{1-\theta} \left[\hat{s} \tilde{K}_t(\Omega^t)^\theta + (1 - \hat{\delta}_t(\Omega_t)) \tilde{K}_t(\Omega^t) \right]^{\theta-1} + 1 - \delta_{t+1}(\Omega_{t+1})} \right] \quad (300)$$

Stochastic Depreciation in a Representative-Agent Model In a representative-agent model with shocks to aggregate productivity $A_t(\Omega^t)$ and depreciation rate $\delta_t(\Omega^t)$, aggregate consumption $C_t(\Omega^t)$ and capital $K_{t+1}(\Omega^t)$ are chosen to satisfy the Euler equation and the resource constraint:

$$\frac{1}{C_t(\Omega^t)} = \beta \mathbb{E}_t \left[R_{t+1}(\Omega^{t+1}) \frac{1}{C_{t+1}(\Omega^{t+1})} \right], \quad (301)$$

$$K_{t+1} = K_t^\theta A_t^{1-\theta} + (1 - \delta_t) K_t - C_t. \quad (302)$$

⁴⁷The interest rate and aggregate capital are written as:

$$R_{t+1}(\tilde{K}_{t+1}, \Omega^{t+1}) = \theta \tilde{K}_{t+1}^{\theta-1} + 1 - \delta_{t+1}(\Omega_{t+1}) \quad (298)$$

$$\tilde{K}_{t+1}(\tilde{K}_t, \Omega^{t+1}) := \frac{K_{t+1}}{A_{t+1}} = \frac{1}{g_A(\Omega_{t+1})} \left[\hat{s} \tilde{K}_t^\theta + (1 - \hat{\delta}_t(\Omega_t)) \tilde{K}_t \right] \quad (299)$$

We rewrite the Euler equation (301) and the resource constraint (302) in terms of $\tilde{K}_t := \frac{K_t}{A_t}$ and $\tilde{C}_t := \frac{C_t}{A_t}$:

$$\frac{1}{\tilde{C}_t(\Omega^t)} = \beta \mathbb{E}_t \left[\frac{A_t}{A_{t+1}} \left\{ \theta \tilde{K}_{t+1}^{\theta-1} + 1 - \delta_{t+1} \right\} \frac{1}{\tilde{C}_{t+1}(\Omega^{t+1})} \right], \quad (303)$$

$$\tilde{K}_{t+1} = \frac{A_t}{A_{t+1}} \left[\tilde{K}_t^\theta + (1 - \delta_t) \tilde{K}_t - \tilde{C}_t \right]. \quad (304)$$

Assuming that both productivity growth rates ($g_{A,t+1} := \frac{A_{t+1}}{A_t}$) and depreciation rate (δ_{t+1}) follow *iid* (they do not depend on realization of past shocks), we conjecture that \tilde{C}_t is a function of \tilde{K}_t and δ_t :

$$\tilde{C}_t = \mathcal{C}(\tilde{K}_t, \delta_t) \quad (305)$$

Under the conjecture, the Euler equation is written as follows, after substituting \tilde{K}_{t+1} using equation (304):

$$\begin{aligned} \frac{1}{\mathcal{C}(\tilde{K}, \delta)} &= \beta \sum_{g_A} \sum_{\delta'} \pi(g_A) \pi(\delta') \frac{1}{g_A} \left\{ \theta \left(\frac{1}{g_A} \left[\tilde{K}^\theta + (1 - \delta) \tilde{K} - \mathcal{C}(\tilde{K}, \delta) \right] \right)^{\theta-1} + 1 - \delta' \right\} \\ &\quad \times \frac{1}{\mathcal{C} \left(\frac{1}{g_A} \left[\tilde{K}^\theta + (1 - \delta) \tilde{K} - \mathcal{C}(\tilde{K}, \delta) \right], \delta' \right)}. \end{aligned} \quad (306)$$

We solve for $\mathcal{C}(\tilde{K}, \delta)$ with a policy function iteration over the Euler equation (306). Given the consumption rule, the risk premium is:

$$\begin{aligned} 1 + \lambda^{\text{Rep}}(\Omega^t) &= \mathbb{E}_t [R_{t+1}(\Omega^{t+1})] \mathbb{E}_t \left[\beta \frac{C_t(\Omega^t)}{C_{t+1}(\Omega^{t+1})} \right] \\ &= \mathbb{E}_t \left[\theta \left(\tilde{K}_{t+1}(\Omega^{t+1}) \right)^{\theta-1} + 1 - \delta_{t+1}(\Omega_{t+1}) \right] \mathbb{E}_t \left[\beta \frac{1}{g_{A,t+1}(\Omega_{t+1})} \frac{\mathcal{C}(\tilde{K}_t(\Omega^t), \delta_t(\Omega_t))}{\mathcal{C}(\tilde{K}_{t+1}(\Omega^{t+1}), \delta_{t+1}(\Omega_{t+1}))} \right]. \end{aligned} \quad (307)$$

Equivalent Formulation of Risk Premium Figure 5 and Table 2 in Section 5.2.4 quantitatively explore the size of risk premium, given the formulae for the risk premium in the limited-commitment model (300) and in the representative-agent model (307). We describe the derivations of relevant expressions in the section.

The risk premium in equation (96) is derived using the definition of covariance and the Euler equation:

$$\begin{aligned} 1 &= \mathbb{E}_t \left[R_{t+1} \beta \frac{c_t}{c_{t+1}} \right] \\ &= \mathbb{E}_t [R_{t+1}] \mathbb{E}_t \left[\beta \frac{c_t}{c_{t+1}} \right] + \text{cov} \left(R_{t+1}, \beta \frac{c_t}{c_{t+1}} \right), \end{aligned}$$

where c_t is the consumption of unconstrained agents. This gives the following expression:

$$1 + \lambda_t := \mathbb{E}_t [R_{t+1}] \mathbb{E}_t \left[\beta \frac{c_t}{c_{t+1}} \right] \quad (308)$$

$$= 1 - cov \left(R_{t+1}, \beta \frac{c_t}{c_{t+1}} \right) \quad (309)$$

Second Moments of Idiosyncratic Income Process: Our calibration targets the auto-correlation of the idiosyncratic income process. Remember that the Markov transition matrix is given by:

$$\pi(z_{t+1}|z_t) = \begin{bmatrix} 1 - \nu & \nu \\ \xi & 1 - \xi \end{bmatrix},$$

where ξ is the probability of switching from high-income ($z_t = \zeta$) to low-income ($z_t = 0$) state. It implies the stationary income distribution:

$$(\psi_l, \psi_h) = \left(\frac{\xi}{\xi + \nu}, \frac{\nu}{\xi + \nu} \right). \quad (310)$$

Average labor productivity is normalized to one:

$$\frac{\nu}{\xi + \nu} \zeta = 1.$$

The variance of z_t is given by:

$$\begin{aligned} Var(z_t) &= \mathbb{E} [z_t^2] - (\mathbb{E}[z_t])^2 \\ &= \frac{\nu}{\xi + \nu} \zeta^2 - 1 \\ &= \frac{\xi + \nu}{\nu} - 1 = \frac{\xi}{\nu}. \end{aligned}$$

The covariance is:

$$\begin{aligned} Cov(z_t, z_{t+1}) &= \mathbb{E} [z_t z_{t+1}] - \mathbb{E}[z_t] \mathbb{E}[z_{t+1}] \\ &= \frac{\nu}{\xi + \nu} (1 - \xi) \zeta^2 - 1 \\ &= \frac{\xi}{\nu} (1 - \xi - \nu). \end{aligned}$$

Therefore, the autocorrelation, $corr(z_t, z_{t+1})$, is given by $1 - \xi - \nu$.

Calibration The fraction of the working population is around 2/3. We set the autocorrelation of income to be 0.9. This pins down (ξ, ν) as follows:

$$\begin{aligned} \frac{\nu}{\xi + \nu} &= \frac{2}{3} \text{ and } 1 - \xi - \nu = 0.9 \\ \rightarrow (\xi, \nu) &= (0.066, 0.033) \end{aligned}$$

B.3 Additional Discussions about Section 6 (Literature)

B.3.1 Speed of Convergence

As mentioned in Subsection 4.2, the effective depreciation rate ($\hat{\delta} = \nu + \rho$) is increasing in ν (i.e., the probability from low-income state to high-income state). In Figure 10, aggregate capital in a limited-commitment model converges to a new stationary equilibrium faster or slower than the neoclassical growth model, depending on parameter values.⁴⁸ The numerical result shows that ξ does not affect the speed of convergence.

Compared to a Solow growth model with a save saving rate \hat{s} and a depreciation rate δ , a limited-commitment model converges faster. This is because the effective depreciation rate in a limited-commitment model, $\hat{\delta}$, is higher (since $1 - \hat{\delta} = (1 - \nu)\beta(1 - \delta) < 1 - \delta$), and thus the capital persists less than a Solow model.

Linear approximation of the transitional dynamics gives the approximated aggregate capital at time t . Aggregate capital at t is given by:

$$K_t \approx \left[1 - (1 - \theta)\hat{\delta}\right]^{t-1} (K_1 - K^*) + K^*,$$

where $\hat{\delta} > \delta$. This equation shows that the speed of convergence is given by $(1 - \theta)\hat{\delta}$. That is, the aggregate capital converges to the new steady state, K^* , faster if $(1 - \theta)\hat{\delta}$ is closer to 1. The speed of convergence depends on the share of capital in a Cobb-Douglas production function, θ , and the effective depreciation rate, $\hat{\delta}$, but not on the saving rate.⁴⁹ Since $\hat{\delta} > \delta$, a limited-commitment model converges faster than a Solow growth model. See Barro and Sala-i Martin (2004) for a discussion about speeds of convergence in a continuous-time model.

B.3.2 An economy without idiosyncratic shocks ($\xi = \nu = 0$)

This subsection describes the case with $\xi = \nu = 0$. Since the transition probability of idiosyncratic shocks is zero, idiosyncratic productivity is a permanent type. The upshot is that if agents can freely borrow, the aggregate law of motion in this economy is the same as the standard neoclassical growth model. If we assume that agents face a limited-commitment constraint, high-income agents have zero initial assets, and Assumption 3 is satisfied, then the law of motion is the same as the two-agent model in Section 6.1,

⁴⁸We normalize the initial capital, K_0 , to one in order to compare the speed of convergence. The level of capital in a stationary equilibrium depends on ν and ξ .

⁴⁹The growth rate of capital depends on $\hat{\delta}$ and θ but not on the saving rate:

$$g_{t,t+1} := \frac{K_{t+1} - K_t}{K_t} = \left[\left(\frac{K^*}{K_t} \right)^{1-\theta} - 1 \right] [1 - (1 - \nu)\beta(1 - \delta)].$$

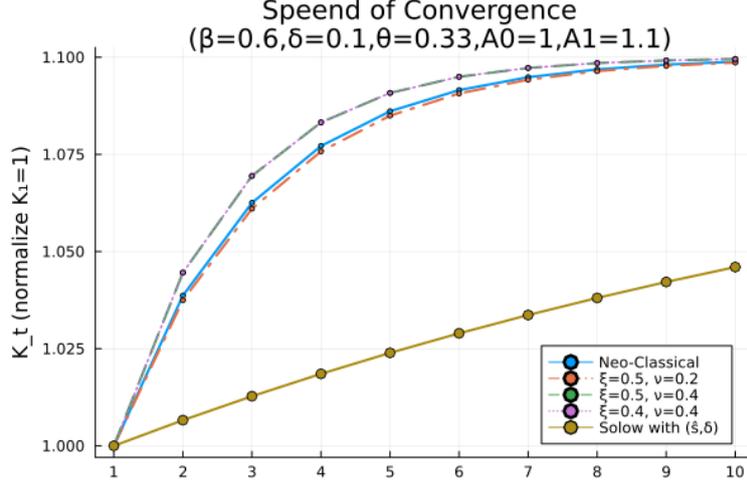


Figure 10: Speed of Convergence

which coincides with the limited-commitment model with $\nu \rightarrow 0$ and $\xi \rightarrow 0$. In either case, aggregate capital in the steady state is the same as the standard neoclassical growth model (and the two-agent model).

We first briefly review the law of motion in the two cases: (a) the standard neoclassical growth model and (b) a limited-commitment model with $\xi \rightarrow 0$ and $\nu \rightarrow 0$ (but $\xi > 0$, $\nu > 0$, and Assumption 3 are satisfied). In case (b), (ν, ξ) satisfies $\kappa := \xi/\nu \in (0, \infty)$.

The law of motion is given as follows. In case (a), aggregate capital follows the standard Euler equation:

$$\frac{1}{K_t^\theta + (1 - \delta)K_t - K_{t+1}} = \frac{\beta[\theta K_{t+1}^{\theta-1} + 1 - \delta]}{K_{t+1}^\theta + (1 - \delta)K_{t+1} - K_{t+2}}. \quad (311)$$

In case (b), only low-income agents save, so the law of motion follows:

$$K_{t+1} = \beta[\theta K_{t+1}^\theta + (1 - \delta)K_t]. \quad (312)$$

Note that we maintain Assumption 3 so that the economy is in a partial insurance equilibrium. In a stationary equilibrium, $\beta R < 1$ (and $\lim_{\nu, \xi \rightarrow 0} \beta R = 1$) is satisfied with Assumption 5. In both cases, capital in the steady state is given by:

$$K = \left(\frac{\theta}{\frac{1}{\beta} - 1 + \delta} \right)^{\frac{1}{1-\theta}} \quad (313)$$

and satisfies $\beta R = 1$.

Now we analyze the case with $\xi = \nu = 0$. If agents can freely borrow, the Euler equation is satisfied for both agents:

$$\frac{1}{c_t^i} = \beta R_{t+1} \frac{1}{c_{t+1}^i} \text{ for } i \in \{\text{high, low}\}.$$

The fraction of each type of household is given by the “stationary distribution” of income shocks by assuming the ratio of ξ and ν as $\kappa := \xi/\nu$:

$$\begin{aligned} (\psi_l, \psi_h) &:= \left(\frac{\xi}{\xi + \nu}, \frac{\nu}{\xi + \nu} \right) \\ &= \left(\frac{\kappa}{\kappa + 1}, \frac{1}{\kappa + 1} \right). \end{aligned} \quad (314)$$

Aggregation leads to the Euler equation of aggregate consumption:

$$\underbrace{\sum_i \psi_i \mathbf{c}_{t+1}^i}_{=C_{t+1}} = \beta R_{t+1} \underbrace{\sum_i \psi_i \mathbf{c}_t^i}_{=C_t}.$$

Together with the resource constraint, the aggregate law of motion is described by two equations:

$$C_{t+1} = \beta R_{t+1} C_t \quad (315)$$

$$C_t + K_{t+1} = Y_t + (1 - \delta)K_t. \quad (316)$$

The two equations give the law of motion in the neoclassical growth model in equation (311).

Consider that agents face a limited-commitment constraint, implying $\mathbf{a}_{t+1} \geq 0$ at any $t \geq 0$. Given that low-income agents will never switch to a high-income state and have the logarithmic utility function, their consumption and saving rule is given by:

$$\mathbf{c}_t = (1 - \beta)R_t \mathbf{a}_t \quad (317)$$

$$\mathbf{a}_{t+1} = \beta R_t \mathbf{a}_t. \quad (318)$$

The optimal consumption and saving decision of high-income agents follows the Kuhn-Tucker condition:

$$\frac{1}{\mathbf{c}_t} = \beta R_{t+1} \frac{1}{\mathbf{c}_{t+1}} \text{ if } \mathbf{a}_{t+1} > 0 \quad (319)$$

$$\frac{1}{\mathbf{c}_t} \geq \beta R_{t+1} \frac{1}{\mathbf{c}_{t+1}} \text{ if } \mathbf{a}_{t+1} = 0 \quad (320)$$

Given Assumption 3 and zero initial assets, $\mathbf{c}_t = \zeta w_t$ and $\mathbf{a}_{t+1} = 0$ for all $t \geq 0$ satisfy the condition (320). Given those decision rules, the aggregate law of motion is the same as the two-agent model and the limited-commitment model with $\nu \rightarrow 0$ and $\xi \rightarrow 0$ in equation (312). Note that Assumption 5 implies that Assumption 3 is satisfied in the steady state. As long as the initial aggregate capital K_0 satisfies $K_0 \geq K^*$ (equal to or above the capital in the steady state), Assumption 3 is satisfied along the transition.