Proof Theory of Continuous Logic

A search for Gentzen-style proof system

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Ultraproduct, despite its extreme usefulness, has it limit. Ultraproducts preserve **first-order properties** by Łos's theorem, but almost always violate other properties by saturation.

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Note: it is a natural and interesting question to ask what size the Dedekind completion of $^*\mathbb{R}$ has.

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Note: it is a natural and interesting question to ask what size the Dedekind completion of $^*\mathbb{R}$ has.

Spoiler alert: it may not be what you think it should be.

Special Ultraproducts for Complete Metric Spaces

It is a bad news to operator algebraists for obvious reasons. Nevertheless, they manage to construct an exotic "ultraproduct" that works for their purpose. In doing so, they assume a logic of *positive bounded formulas with an approximate semantics*.

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Remark

Continuous logic is actually not new. Keisler and Chang wrote a book in 1966 called *Continuous Model Theory*. Their theory, however, is far too general to work with.

Introducing Continuous Logic

In continuous logic, we are replacing the truth values $\{0, 1\}$ with $[0, 1]$.

Remark

We consider 0 as truth, instead of 1. It might look strange but is a very natural choice, which will be clear in a few lines.

A model of continuous logic is (M, d) where $d : M \times M \to [0, 1]$ is a complete metric. Here are the interpretation of symbols:

- The equality in our model is just d. In particular, two elements are equal if the equality predicate is evaluated at 0.
- A n-ary relation R is an **uniformly continuous** function $R^{\mathfrak{M}} : M^n \to [0, 1]$ (uniformliness is needed in order to have a unique syntax, i.e., model-independence).
- A n-ary function f is an **uniformly continuous** function $f^{\mathfrak{M}} : M^n \to M$.

Continuous Logic Continued

We inductively build formulas of higher complexity:

- $\perp^{\mathfrak{M}} := 1$
- $(A \to B)^{\mathfrak{M}} := B^{\mathfrak{M}} \dot{-} A^{\mathfrak{M}}$
- $\top^{\mathfrak{M}} : (\perp \to \perp)^{\mathfrak{M}} = 0$
- $(\neg A)^{\mathfrak{M}} := (A \to \perp)^{\mathfrak{M}} = 1 \dot{-} A^{\mathfrak{M}}$
- $(A \wedge B)^{\mathfrak{M}} := \max(A^{\mathfrak{M}}, B^{\mathfrak{M}})$
- $(A \vee B)^{\mathfrak{M}} := \min(A^{\mathfrak{M}}, B^{\mathfrak{M}})$
- $(\forall x\ A(x))^{\mathfrak{M}} := \sup_{x \in M} A^{\mathfrak{M}}(x)$
- $(\exists x\ A(x))^{\mathfrak{M}} := \inf_{x \in M} A^{\mathfrak{M}}(x)$

Definition

We say $\mathfrak{M} \models \varphi$ if $\varphi^{\mathfrak{M}} = 0$.

Approximation of Truth

It turns out our regular set of logic connectives $\{\bot, \top, \to, \wedge, \vee, \forall, \exists\}$ are insufficient. Any uniformly continuous function $[0,1]^n \to [0,1]$ is in a way a legitimate logic connective and there are continuum many such. We naturally add a new modal operator $\frac{1}{2}$:

- $(\frac{1}{2}A)^{\mathfrak{M}} := \frac{1}{2}A^{\mathfrak{M}}$

We obtain a finite set of connectives dense in all logic connectives. Now we need to relax the notion of entailment to complete our logic:

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Definition

We say $\mathfrak{M} \models \varphi$ iff $\varphi^{\mathfrak{M}} < \frac{1}{n}$ for all n, or equivalently $\varphi^{\mathfrak{M}}$ converges to 0. We say $\Gamma \models \varphi$ iff for all $\mathfrak{M}$, if every formula in Γ converges to zero in $\mathfrak{M}$, then $\varphi^{\mathfrak{M}}$ converges to zero.

Notice it is incorporated well with approximated version of entailment since we demand all logic symbols to be uniformly continuous.

Internalize That Special Ultraproduct

Recall the first-order ultraproduct $\mathfrak{M}^{\mathcal{U}}$ where two elements $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ in M^I are equivalent iff

$$\{i \in I \mid x_i = y_i\} \in \mathcal{U}$$

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By approximation, we relax the condition into

$$\text{For all } n,\ \{i \in I \mid d(x_i, y_i) < \frac{1}{n}\} \in \mathcal{U}$$

Correct Logic for Metric Structures

This notion of ultraproduct recovers the exact ultraprodcut invented by operator algraists eariler. In particular, continuous logic preserves completeness of metric structures.

Theorem

If M is a complete bounded metric space, then $M^{\mathcal{U}}$ under the new definition is still a complete bounded metric space.

Furthermore, you can axiomatize Hilbert spaces, C^*-algebras, probability spaces, and various metric structures in continuous logic. To some extent, it is the correct logic to study metric structures.

What a Nice Logic

Quote From Yaacov and Pedersen

"...continuous first-order logic satisfies suitably phrased forms of the compactness theorem, the Löwenheim-Skolem theorems, the diagram arguments, Craig's interpolation theorem, Beth's definability theorem, characterizations of quantifier elimination and model completeness, the existence of saturated and homogeneous models results, the omitting types theorem, fundamental results of stability theory, and nearly all other results of elementary model theory."

For its famous application to the Connes Embedding problem, you can check on Isaac Goldbring's website.

Syntax of Continuous Logic

A Demand for Proof System

Continuous logic was reinvented according to semantic needs. It is natural to ask for a syntactic counterpart, i.e., a proof system such that is sound and complete with respect to the semantics. There is one based on Łukasiewicz (propositional) logic.

A Demand for Proof System

Continuous logic was reinvented according to semantic needs. It is natural to ask for a syntactic counterpart, i.e., a proof system such that is sound and complete with respect to the semantics. There is one based on Łukasiewicz (propositional) logic.

Remark

Łukasiewicz logic is the strongest among fuzzy logics and it is sound and complete with respect to MV-algebras on $[0,1]$ or any linearly ordered MV-algbras.

A Hilbert-Style Proof System

As usual, we has *modus ponens* as the only inference rule:

$$\frac{A \qquad A \rightarrow B}{B}$$

Together with three groups of axiom schemata.

Łukasiewicz axioms

- (Ł1): $A \rightarrow (B \rightarrow A)$
- (Ł2): $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
- (Ł3): $((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A)$
- (Ł4): $(\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$

Continuous axioms

- (A5): $\frac{1}{2}A \rightarrow (\frac{1}{2}A \rightarrow A)$
- (A6): $(\frac{1}{2}A \rightarrow A) \rightarrow \frac{1}{2}A$

Proof System Continued

Qualifier axioms

- (A7): $\forall x(A \to B) \to (\forall x\ A \to \forall x\ B)$
- (A8): $\forall x\ A \to A[t/x]$
- (A8): $A \to \forall x\ A$

Metric Structure Axioms for Given Relation R and Function f

- d is a metric
- R is uniformly continuous
- f is uniformly continuous

Theorem

The proof system $\vdash_{Ł}$ induced from above is sound and complete with respect to to the semantics $\models$.

Defining More Connectives

Notice that we only axiomatize three connectives $\{\to, \bot, \frac{1}{2}\}$ and define others accordingly:

- $A \vee B := (A \to B) \to B$ (weak disjunction) is semantically $\min(A, B)$
- $A \wedge B := \neg(\neg A \vee \neg B)$ (weak conjunction) is semantically $\max(A, B)$.
- $A \odot B := \neg A \to B$ (strong disjunction) is semantically $\max(A + B - 1, 0)$.
- $A \oplus B := \neg(A \to \neg B)$ (strong conjunction) is semantically $\min(A + B, 1)$.

Remark

Strong/weak naming unfortunately does not match their semantics well. For instance, weak disjunction is stronger than strong disjunction:

$$A \vee \neg A \text{ is never true while } A \odot \neg A \text{ is provably true}$$

Hilbert System Is the Worst

The above proof system works. You cannot do, however, much proof theory because it is a Hilbert system and everyone knows Hilbert-style systems are terrible. We want a Gentzen-style one. Unfortunately, Łukasiewicz logic, the one continuous logic is based on, is famously known to not have good sequent calculi.

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Failure of the Deduction Theorem

- $P, P \to (P \to Q) \vdash_{Ł} Q$ by applying *modus ponens* twice.
- $P \to (P \to Q) \nvdash_{Ł} (P \to Q)$. Evaluate P at $\frac{1}{2}$ and Q at 1. We have the premise $P \to (P \to Q)$ to be zero, i.e., true; but the the conclusion $(P \to Q)$ is $\frac{1}{2}$ and not true.

Attempts Joint with Eben Blaisdell

Affine Logic Approach

Even though the deduction theorem fails, a weaker form still holds:

Theorem

$\Gamma, A \vdash_{\L} B \iff \Gamma \vdash_{\L} (A \to (A \to \cdots (A \to B)))$ *for some copies of A*

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These features exactly point towards an affine logic setting.

Real Valued Logic Approach

Affine logic alone is not strong enough to prove Axiom Ł3, nor an important consequence: $(A \to B) \vee (B \to A)$ (linearity of truth value).

We later got inspiration from a paper by D. Diaconescu et al. on real valued modal logic. We start to consider a logic allowing mutliple formulas on both sides but interpreted conjunctively on both sides.

In particular, we hope to have the following simple $\to$ introduction rules, which are natural in $\mathbb{R}$-truth value:

$$\frac{\Gamma, B \vdash A; \Delta}{\Gamma, B - A \vdash \Delta} \qquad \frac{\Gamma, A \vdash B; \Delta}{\Gamma \vdash B - A; \Delta}$$

They do not make sense in $[0, 1]$-truth values, but not far. We view our $A \to B$ as $(B - A) \wedge 0$ in disguise where $-$ is the arrow in real valued logic and $\wedge$ is additive.

$[0,1]$-Valued Sequent Calculus

$$\dfrac{}{A \vdash A}\ \text{Identity} \qquad \dfrac{}{\bot \vdash A}\ \text{Ex Falso}$$

$$\dfrac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ \text{WK} \qquad \dfrac{\Gamma_0 \vdash \Delta_0 \qquad \Gamma_1 \vdash \Delta_1}{\Gamma_0, \Gamma_1 \vdash \Delta_0; \Delta_1}\ \text{MIX}$$

$$\dfrac{\Gamma, B \vdash A; \Delta}{\Gamma, A \to B \vdash \Delta}\ \text{L}\!\to \qquad \dfrac{\Gamma, A \vdash B; \Delta \qquad \Gamma \vdash \Delta}{\Gamma \vdash A \to B; \Delta}\ \text{R}\!\to$$

$$\dfrac{\Gamma_0, A \vdash \Delta_0 \qquad \Gamma_1 \vdash A; \Delta_1}{\Gamma_0, \Gamma_1 \vdash \Delta_0; \Delta_1}\ \text{CUT}$$

(The above Γ and Δ are assumed to be multisets and the intended meaning of $\Gamma \vdash \Delta$ is that $\sum \Gamma \geqslant \sum \Delta$)

Soundness and Completeness

We denote this provability relation as $\vdash_{[0,1]}$. We then have soundness and completeness over $\models$, with language restricted to $\{\perp, \rightarrow\}$, over tautologies:

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Soundess and Completeness over Tautologies

$\vdash_{[0,1]} A$ if and only if $\models A$

We can extend the result to sequents modulo some conversion between sets and multisets. Say Γ is a multiset and we use Γ' to denote its extracted set in the obvious way.

Soundness

If $\Gamma \vdash_{[0,1]} A$, then $\Gamma' \models A$.

Completeness

If $\Gamma \models A$, then there exists some mutliset Δ with $\Delta \vdash_{[0,1]} A$ and $\Delta' = \Gamma$.

Proof of Soundness and Completeness

Proof.

Soundness is clear. For completeness, we make use of the Hilbert system and derive all its axioms. For instance, here is the derivation of (Ł3):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \to A, A \vdash A}}{A \vdash (B \to A) \to A}}{B, A \vdash B; (B \to A) \to A} \qquad \dfrac{\dfrac{\dfrac{}{A, B \vdash A; B}}{B, B \to A \vdash A}}{B \vdash (B \to A) \to A}}{\dfrac{B \vdash A \to B; (B \to A) \to A}{(A \to B) \to B \vdash (B \to A) \to A}}}{\vdash ((A \to B) \to B) \to ((B \to A) \to A)}
$$

$\square$

Proof of Soundness and Completeness

Proof.

Soundness is clear. For completeness, we make use of the Hilbert system and derive all its axioms. For instance, here is the derivation of (Ł3):

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$B \to A, A \vdash A$}
\UnaryInfC{$A \vdash (B \to A) \to A$}
\UnaryInfC{$B, A \vdash B; (B \to A) \to A$}
\AxiomC{}
\UnaryInfC{$A, B \vdash A; B$}
\UnaryInfC{$B, B \to A \vdash A$}
\UnaryInfC{$B \vdash (B \to A) \to A$}
\BinaryInfC{$B \vdash A \to B; (B \to A) \to A$}
\UnaryInfC{$(A \to B) \to B \vdash (B \to A) \to A$}
\UnaryInfC{$\vdash ((A \to B) \to B) \to ((B \to A) \to A)$}
\end{prooftree}
$$

□

Notice that we need the CUT rule to admit *modus ponens*, which is standard.

Example

This sequent calculus is very interesting and fun to play with. It makes doing syntactic proofs much much easier. For instance, we have an elegant and straightforward proof of linearity while the same proof the in Hilbert system takes pages and is a pain.

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Examples

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \to A, B, A \vdash B, A} \qquad \dfrac{\dfrac{}{A, B \vdash A, B}}{B \to A, B \vdash A}}{B \to A, B \vdash A \to B, A}}{(A \to B) \to (B \to A), B \vdash A}}{\vdash ((A \to B) \to (B \to A)) \to (B \to A)}}{\vdash (A \to B) \vee (B \to A)}$$

This sequent calculus also extends naturally to first order continuous logic by adding rules for $\frac{1}{2}$ and quantifiers.

CUT Elimination

It is far from perfect yet. We need to eliminate the CUT rule, in other words, to show the CUT rule is admissible.

$$\frac{\Gamma_0, A \vdash \Delta_0 \qquad \Gamma_1 \vdash A; \Delta_1}{\Gamma_0, \Gamma_1 \vdash \Delta_0, \Delta_1}\ \mathrm{CUT}$$

A cut-free Gentzen system is the most ideal proof system: we will have the **subformula property**, which allows proof searching, and syntactic proof of unprovability and so on. Here is a quote from Andre Scedrov, a professor at Penn:

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“If you don’t have a cut-free proof system, you don’t have a proof system.”

Trouble in Proving CUT Elimination

Our simple $\to$ introduction rules cause trouble for us to perform the standard CUT elimination procedure, where you recursively reduce complexity of CUT formulas. In the following case:

$$\dfrac{\dfrac{\Gamma_0, B \vdash A; \Delta_0}{\Gamma_0, A \to B \vdash \Delta_0} \qquad \dfrac{\Gamma_1, A \vdash B; \Delta_1 \qquad \Gamma_1 \vdash \Delta_1}{\Gamma_1 \vdash A \to B; \Delta_1}}{\Gamma_0, \Gamma_1 \vdash \Delta_0; \Delta_1}$$

You can perform a lower-degree CUT:

$$\dfrac{\Gamma_0, B \vdash A; \Delta_0 \qquad \Gamma_1, A \vdash B; \Delta_1}{\Gamma_0, \Gamma_1, A \vdash A; \Delta_0; \Delta_1}$$

But then you will never be able to CUT off A.

CAN Elimination

This suggests us to consider an alternative rule:

$$\frac{\Gamma, A \vdash A; \Delta}{\Gamma \vdash \Delta}\ \text{CAN}$$

CAN is equivalent to CUT but more natural in this setting.

To eliminate CAN, however, we have to give up the idea of a local-scale elimination. Instead, we take an instance of CAN, trace all the way back to atomic levels, and cancel all its subformulas there. In doing so, we realize an absent rule:

$$\frac{\Gamma, \Gamma \vdash \Delta; \Delta}{\Gamma \vdash \Delta}\ \text{RED}$$

Trouble in Proving CAN Elimination

There is another trouble due to local-irreversibility of left $\to$:

Examples

We have a proof of

$$A, A \to B \vdash B; B \to A$$

but neither $A, B \vdash A; B; B \to A$ nor $A \vdash B; B \to A$ is provable.

In the worst scenario, the proof tree splits into two structually non-equivalent subtrees for every left $\to$ rules, which makes the analysis grows exponentially.

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In the worst scenario, the proof tree splits into two structually non-equivalent subtrees for every left $\rightarrow$ rules, which makes the analysis grows exponentially.

We also tried to work in real valued logic instead, which has an easy CUT elimination proof. We interpret $[0, 1]$-provability in $\mathbb{R}$-provability and prove a conservativity result. However, we need to add additive $\vee$ and $\wedge$ to real valued logic, which causes its own problem in CUT elimination.

The Solution And ...

Hypersequent Approach

Encouraged by my advisor Henry Townser, we consider a hypersequent calculus setting, which is very natural due to disparity between the left and right $\rightarrow$ rules. Two premises are needed to introduce $\rightarrow$ on the right:

$$\frac{\Gamma, A \vdash B; \Delta \qquad \Gamma \vdash \Delta}{\Gamma \vdash A \rightarrow B; \Delta}\ \mathrm{R}{\rightarrow}$$

and there are two ways to introduce $\rightarrow$ on the left:

$$\frac{\Gamma, B \vdash A; \Delta}{\Gamma, A \rightarrow B \vdash \Delta}\ \mathrm{L}{\rightarrow} \qquad\qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta}\ \mathrm{WK}$$

It suggests the following left $\rightarrow$ rule in a hypersequent seting:

$$\frac{\Gamma, B \vdash A; \Delta \mid \Gamma \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta}$$

$[0,1]$-Hypersequent Calculus

Hypersequents are of the form $\Gamma_0 \vdash \Delta_0 \mid \cdots \mid \Gamma_n \vdash \Delta_n$. The intended meaning is that a hypersequent is true if one of the component $\Gamma_i \vdash \Delta_i$ is true. We use G, H to denote hypersequents.

$$\frac{}{A \vdash A}\ \text{Identity} \qquad \frac{}{\bot \vdash A}\ \text{Ex Falso}$$

$$\frac{G \mid \Gamma_0 \vdash \Delta_0}{G \mid \Gamma_0 \vdash \Delta_0 \mid \Gamma_1 \vdash \Delta_1}\ \text{EW} \qquad \frac{G \mid \Gamma \vdash \Delta \mid \Gamma \vdash \Delta}{G \mid \Gamma \vdash \Delta}\ \text{EC}$$

$$\frac{G \mid \Gamma_0, \Gamma_1 \vdash \Delta_0; \Delta_1}{G \mid \Gamma_0 \vdash \Delta_0 \mid \Gamma_1 \vdash \Delta_1}\ \text{SEP} \qquad \frac{G \mid \Gamma_0 \vdash \Delta_0 \qquad G \mid \Gamma_1 \vdash \Delta_1}{G \mid \Gamma_0, \Gamma_1 \vdash \Delta_0; \Delta_1}\ \text{MIX}$$

$$\frac{G \mid \Gamma, B \vdash A; \Delta \mid \Gamma \vdash \Delta}{G \mid \Gamma, A \to B \vdash \Delta}\ \text{L}\to \qquad \frac{G \mid \Gamma, A \vdash B; \Delta \qquad G \mid \Gamma \vdash \Delta}{G \mid \Gamma \vdash A \to B; \Delta}\ \text{R}\to$$

$$\frac{G \mid \Gamma, A \vdash A; \Delta}{G \mid \Gamma \vdash \Delta}\ \text{CAN}$$

CAN Elimination but then...

This hypersequent calculus is an extension of the sequent calculus.
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What is tragic is that we soon found out that our logic system is not new. In a paper in 2005, George Metcalfe et al. construct exactly the same hypersequent calculus for Łukasiewicz logic. All discoveries we have so far constitute exactly the main content of this paper.

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Moral of This Story

Check the literature carefully before you work on a problem!

Wrapping Up

For certain reasons, I believe the original **sequent calculus** is actually CAN-free; or I would very love to see a counterexample to that. There are also many other projects left for proof theory of continuous logic, such as incorporating $\frac{1}{2}$ and quantifiers into this system, another approach to hypersequent calculus as an extension of finitely multi-valued logics, and eventually proof mining. But that will be the story for the future I guess, hopefully.

The End

Thank you all for having me!

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