Defying Gödel: Let's Prove Consistency of PA

A Crash Course on Proof Theory

Jin Wei

Sep 08, 2023

Table of Contents

Why Consistency? Why PA?

Problem with ZFC

Set theory/ZFC forms the *de facto* foundation of modern mathematics. Namely, all mathematical objects and structures, with a few exceptions, are sets.

Problem with ZFC

Set theory/ZFC forms the *de facto* foundation of modern mathematics. Namely, all mathematical objects and structures, with a few exceptions, are sets.

Back to early 20th century, people are skeptical about ZFC. Besides its being non-intuitive, ZFC seems to lack two key features of an ideal foundation of mathematics:

Problem with ZFC

Set theory/ZFC forms the *de facto* foundation of modern mathematics. Namely, all mathematical objects and structures, with a few exceptions, are sets.

Back to early 20th century, people are skeptical about ZFC. Besides its being non-intuitive, ZFC seems to lack two key features of an ideal foundation of mathematics:

- **completeness**: any mathematical statement is either provably true or provably false under this foundation.

Problem with ZFC

Set theory/ZFC forms the *de facto* foundation of modern mathematics. Namely, all mathematical objects and structures, with a few exceptions, are sets.

Back to early 20th century, people are skeptical about ZFC. Besides its being non-intuitive, ZFC seems to lack two key features of an ideal foundation of mathematics:

- **completeness**: any mathematical statement is either provably true or provably false under this foundation.
- **consistency**: no contradiction would arise from this foundation.

Problem with ZFC

Set theory/ZFC forms the *de facto* foundation of modern mathematics. Namely, all mathematical objects and structures, with a few exceptions, are sets.

Back to early 20th century, people are skeptical about ZFC. Besides its being non-intuitive, ZFC seems to lack two key features of an ideal foundation of mathematics:

- **completeness**: any mathematical statement is either provably true or provably false under this foundation.
- **consistency**: no contradiction would arise from this foundation.

ZFC does not look satisfactory from this standard:

Problem with ZFC

Set theory/ZFC forms the *de facto* foundation of modern mathematics. Namely, all mathematical objects and structures, with a few exceptions, are sets.

Back to early 20th century, people are skeptical about ZFC. Besides its being non-intuitive, ZFC seems to lack two key features of an ideal foundation of mathematics:

- **completeness**: any mathematical statement is either provably true or provably false under this foundation.
- **consistency**: no contradiction would arise from this foundation.

ZFC does not look satisfactory from this standard:

- People failed using ZFC to prove or disprove some important mathematical principles, especially the **Continuum hypothesis**.

Problem with ZFC

Set theory/ZFC forms the *de facto* foundation of modern mathematics. Namely, all mathematical objects and structures, with a few exceptions, are sets.

Back to early 20th century, people are skeptical about ZFC. Besides its being non-intuitive, ZFC seems to lack two key features of an ideal foundation of mathematics:

- **completeness**: any mathematical statement is either provably true or provably false under this foundation.
- **consistency**: no contradiction would arise from this foundation.

ZFC does not look satisfactory from this standard:

- People failed using ZFC to prove or disprove some important mathematical principles, especially the **Continuum hypothesis**.
- After the **Russell's paradox**, people are afraid that a similar paradox might arise from ZFC again.

Hilbert's Program and Peano Arithmetic

Hilbert puts hope in his grand program: to reduce all (hopefully) mathematics to **basic arithmetic**, which he believes to be solid and ideal. During this program, Hilbert coined the term *beweistheorie*/**proof theory** as a study on the nature of proofs. His work is the precursor of **reverse mathematics**/subsystesms of second-order arithmetic.

Hilbert's Program and Peano Arithmetic

Hilbert puts hope in his grand program: to reduce all (hopefully) mathematics to **basic arithmetic**, which he believes to be solid and ideal. During this program, Hilbert coined the term *beweistheorie*/**proof theory** as a study on the nature of proofs. His work is the precursor of **reverse mathematics**/subsystesms of second-order arithmetic.

Definition (Peano Arithmetic)

Peano Arithmetic/PA is a theory of natural numbers:

Hilbert's Program and Peano Arithmetic

Hilbert puts hope in his grand program: to reduce all (hopefully) mathematics to **basic arithmetic**, which he believes to be solid and ideal. During this program, Hilbert coined the term *beweistheorie*/**proof theory** as a study on the nature of proofs. His work is the precursor of **reverse mathematics**/subsystesms of second-order arithmetic.

Definition (Peano Arithmetic)

Peano Arithmetic/PA is a theory of natural numbers:

- Logic: **classical first order logic** using $\{\neg, \vee, \wedge, \rightarrow, \exists, \forall, =\}$.

Hilbert's Program and Peano Arithmetic

Hilbert puts hope in his grand program: to reduce all (hopefully) mathematics to **basic arithmetic**, which he believes to be solid and ideal. During this program, Hilbert coined the term *beweistheorie*/**proof theory** as a study on the nature of proofs. His work is the precursor of **reverse mathematics**/subsystesms of second-order arithmetic.

Definition (Peano Arithmetic)

Peano Arithmetic/PA is a theory of natural numbers:

- Logic: **classical first order logic** using $\{\neg, \vee, \wedge, \rightarrow, \exists, \forall, =\}$.
- Language: 0, Succ, $+$, $\times$, $<$.

Hilbert's Program and Peano Arithmetic

Hilbert puts hope in his grand program: to reduce all (hopefully) mathematics to **basic arithmetic**, which he believes to be solid and ideal. During this program, Hilbert coined the term *beweistheorie*/**proof theory** as a study on the nature of proofs. His work is the precursor of **reverse mathematics**/subsystesms of second-order arithmetic.

Definition (Peano Arithmetic)

Peano Arithmetic/PA is a theory of natural numbers:

- Logic: **classical first order logic** using $\{\neg, \vee, \wedge, \to, \exists, \forall, =\}$.
- Language: 0, Succ, $+$, $\times$, $<$.
- Characteristic axioms of above symbols.

Hilbert's Program and Peano Arithmetic

Hilbert puts hope in his grand program: to reduce all (hopefully) mathematics to **basic arithmetic**, which he believes to be solid and ideal. During this program, Hilbert coined the term *beweistheorie*/**proof theory** as a study on the nature of proofs. His work is the precursor of **reverse mathematics**/subsystesms of second-order arithmetic.

Definition (Peano Arithmetic)

Peano Arithmetic/PA is a theory of natural numbers:

- Logic: **classical first order logic** using $\{\neg, \vee, \wedge, \rightarrow, \exists, \forall, =\}$.
- Language: 0, Succ, $+$, $\times$, $<$.
- Characteristic axioms of above symbols.
- Induction: for any first-order formula $\varphi(x)$:

$$(\varphi(0) \wedge \forall n\ (\varphi(n) \rightarrow \varphi(\mathrm{Succ}(n))) \rightarrow \forall n\ \varphi(n)$$

Con(PA)

Definition

Con(PA) $:= \neg(\exists x$ "x is a code for a proof of $0 = 1$ from axioms of PA")

Con(PA)

Definition

Con(PA) := $\neg(\exists x$ "x is a code for a proof of $0 = 1$ from axioms of PA")

Con(PA), **representing** consistency of PA, is a statement expressible in PA itself. In particular, if PA is indeed complete, then PA ought to prove Con(PA) because it better not proves $\neg$Con(PA) (why?).

Con(PA)

Definition

$\text{Con}(PA) := \neg(\exists x$ "x is a code for a proof of $0 = 1$ from axioms of PA")

Con(PA), **representing** consistency of PA, is a statement expressible in PA itself. In particular, if PA is indeed complete, then PA ought to prove Con(PA) because it better not proves ¬Con(PA) (why?).

Con(PA) v.s. Consistency of PA

Con(PA) is **NOT** exactly the consistency of PA. The consistency will not be established by a mere proof of Con(PA) in PA, as the following example shows:

Con(PA)

Definition

Con(PA) $:= \neg(\exists x$ "x is a code for a proof of $0 = 1$ from axioms of PA")

Con(PA), **representing** consistency of PA, is a statement expressible in PA itself. In particular, if PA is indeed complete, then PA ought to prove Con(PA) because it better not proves $\neg$Con(PA) (why?).

Con(PA) v.s. Consistency of PA

Con(PA) is **NOT** exactly the consistency of PA. The consistency will not be established by a mere proof of Con(PA) in PA, as the following example shows:

The theory **PA+"0=1"** proves Con(PA+"0=1") but it is inconsistent.

THE Theorem

Gödel proves PA is **incomplete** and furthermore kills any hope for a complete theory of arithmetic by his second incompleteness theorem.

THE Theorem

Gödel proves PA is **incomplete** and furthermore kills any hope for a complete theory of arithmetic by his second incompleteness theorem.

Theorem (Gödel's Second Incompleteness)

Assuming PA is consistent, PA does not prove Con(PA). Namely any proof of Con(PA) will result in a proof of "0=1" in PA.

THE Theorem

Gödel proves PA is **incomplete** and furthermore kills any hope for a complete theory of arithmetic by his second incompleteness theorem.

Theorem (Gödel's Second Incompleteness)

Assuming PA is consistent, PA does not prove Con(PA). Namely any proof of Con(PA) will result in a proof of "0=1" in PA.

It also extends to any reasonable theory that is strong enough to interpret PA, such as ZFC. Similarly, ZFC does not prove Con(ZFC).

THE Theorem

Gödel proves PA is **incomplete** and furthermore kills any hope for a complete theory of arithmetic by his second incompleteness theorem.

Theorem (Gödel's Second Incompleteness)

Assuming PA is consistent, PA does not prove Con(PA). Namely any proof of Con(PA) will result in a proof of "0=1" in PA.

It also extends to any reasonable theory that is strong enough to interpret PA, such as ZFC. Similarly, ZFC does not prove Con(ZFC).

The true consistency of PA, however, does not belong to this realm. Con(-) is just a particular type of sentences Gödel utilizes to show a general incompleteness phenomenon.

Consistency of PA

Now let's forget about Con(PA) and actually prove the consistency of PA. The proof should be a form of reasoning that turns inconsistency of PA (a proof of $0 = 1$ for instance) into inconsistency of our meta-logic system.

Consistency of PA

Now let's forget about Con(PA) and actually prove the consistency of PA. The proof should be a form of reasoning that turns inconsistency of PA (a proof of $0 = 1$ for instance) into inconsistency of our meta-logic system.

"... in recognizing that such conditions exist and must be respected we find ourselves in agreement with the philosophers, especially with Kant. Kant already taught - and indeed it is part and parcel of his doctrine - that mathematics has at its disposal a content secured independently of all logic and hence can never be provided with a foundation by means of logic alone"

Consistency of PA

Now let's forget about Con(PA) and actually prove the consistency of PA. The proof should be a form of reasoning that turns inconsistency of PA (a proof of $0 = 1$ for instance) into inconsistency of our meta-logic system.

"... in recognizing that such conditions exist and must be respected we find ourselves in agreement with the philosophers, especially with Kant. Kant already taught - and indeed it is part and parcel of his doctrine - that mathematics has at its disposal a content secured independently of all logic and hence can never be provided with a foundation by means of logic alone"

–David Hilbert, *On the Infinite*

Primitive Recursive Arithmetic

To reason about consistency of PA, we need a meta-logic system as weak as possible. We will use **PRA** in this talk, a weak and **finitistic** theory on natural numbers. Every theorem in this talk is implicitly reasoned in PRA with two exceptions.

Primitive Recursive Arithmetic

To reason about consistency of PA, we need a meta-logic system as weak as possible. We will use **PRA** in this talk, a weak and **finitistic** theory on natural numbers. Every theorem in this talk is implicitly reasoned in PRA with two exceptions.

Definition (Primitive Recursive Arithmetic)

Primitive Recursive Arithmetic

To reason about consistency of PA, we need a meta-logic system as weak as possible. We will use **PRA** in this talk, a weak and **finitistic** theory on natural numbers. Every theorem in this talk is implicitly reasoned in PRA with two exceptions.

Definition (Primitive Recursive Arithmetic)

- Logic: **classical quantifier-free logic** using $\{\neg, \vee, \wedge, \rightarrow, =\}$ (in fact, intuitionistic logic suffices).

Primitive Recursive Arithmetic

To reason about consistency of PA, we need a meta-logic system as weak as possible. We will use **PRA** in this talk, a weak and **finitistic** theory on natural numbers. Every theorem in this talk is implicitly reasoned in PRA with two exceptions.

Definition (Primitive Recursive Arithmetic)

- Logic: **classical quantifier-free logic** using $\{\neg, \vee, \wedge, \rightarrow, =\}$ (in fact, intuitionistic logic suffices).
- Language: 0, Succ, and all functions definable from Succ using compositions, projections, and **primitive recursions**.

Primitive Recursive Arithmetic

To reason about consistency of PA, we need a meta-logic system as weak as possible. We will use **PRA** in this talk, a weak and **finitistic** theory on natural numbers. Every theorem in this talk is implicitly reasoned in PRA with two exceptions.

Definition (Primitive Recursive Arithmetic)

- Logic: **classical quantifier-free logic** using $\{\neg, \vee, \wedge, \rightarrow, =\}$ (in fact, intuitionistic logic suffices).
- Language: 0, Succ, and all functions definable from Succ using compositions, projections, and **primitive recursions**.
- Characteristic axioms of above symbols.

Primitive Recursive Arithmetic

To reason about consistency of PA, we need a meta-logic system as weak as possible. We will use **PRA** in this talk, a weak and **finitistic** theory on natural numbers. Every theorem in this talk is implicitly reasoned in PRA with two exceptions.

Definition (Primitive Recursive Arithmetic)

- Logic: **classical quantifier-free logic** using $\{\neg, \vee, \wedge, \rightarrow, =\}$ (in fact, intuitionistic logic suffices).
- Language: 0, Succ, and all functions definable from Succ using compositions, projections, and **primitive recursions**.
- Characteristic axioms of above symbols.
- Induction: for any quantifier-free formula φ:
  from $\varphi(0)$ and $\varphi(x) \rightarrow \varphi(x+1)$, deduce $\varphi(t)$ for any term t.

Gentzen's Proof

Axiomatization of Logic

To understand Gentzen's proof, we need to shift our focus to pure logic for a bit. In some sense, we will **reason about logic using arithmetic** (PRA).

Axiomatization of Logic

To understand Gentzen's proof, we need to shift our focus to pure logic for a bit. In some sense, we will **reason about logic using arithmetic** (PRA).

A **proof system** is an axiomatization of logic, including axioms and inference rules.

Axiomatization of Logic

To understand Gentzen's proof, we need to shift our focus to pure logic for a bit. In some sense, we will **reason about logic using arithmetic** (PRA).

A **proof system** is an axiomatization of logic, including axioms and inference rules.

Definition (Hilbert-Style Proof System)

Axiomatization of Logic

To understand Gentzen's proof, we need to shift our focus to pure logic for a bit. In some sense, we will **reason about logic using arithmetic** (PRA).

A **proof system** is an axiomatization of logic, including axioms and inference rules.

Definition (Hilbert-Style Proof System)

- Defining axioms for each of connectives $\neg, \to, \wedge, \vee, \forall, \exists$

Axiomatization of Logic

To understand Gentzen's proof, we need to shift our focus to pure logic for a bit. In some sense, we will **reason about logic using arithmetic** (PRA).

A **proof system** is an axiomatization of logic, including axioms and inference rules.

Definition (Hilbert-Style Proof System)

- Defining axioms for each of connectives $\neg, \rightarrow, \wedge, \vee, \forall, \exists$
- Modus Ponens:

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash A \rightarrow B}{\Gamma \vdash B} \text{ Modus Ponens}$$

Axiomatization of Logic

To understand Gentzen's proof, we need to shift our focus to pure logic for a bit. In some sense, we will **reason about logic using arithmetic** (PRA).

A **proof system** is an axiomatization of logic, including axioms and inference rules.

Definition (Hilbert-Style Proof System)

- Defining axioms for each of connectives $\neg, \rightarrow, \wedge, \vee, \forall, \exists$
- Modus Ponens:

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash A \rightarrow B}{\Gamma \vdash B} \text{ Modus Ponens}$$

You cannot prove much about this proof system and it is hard to reason about what can be proven and what cannot. Gentzen provides two alternative proof systems, **natural deduction** and **sequent calculus**, and we will focus on the latter.

Sequent Calculus

Definition (Gentzen-style Proof System for Intuitionistic Logic)

Sequent Calculus

Definition (Gentzen-style Proof System for Intuitionistic Logic)

$$\frac{}{A \vdash A}\ \text{Id} \qquad \frac{}{\bot \vdash A}\ \text{Ex Falso}$$

Sequent Calculus

Definition (Gentzen-style Proof System for Intuitionistic Logic)

$$\frac{}{A \vdash A}\,\text{Id} \qquad \frac{}{\bot \vdash A}\,\text{Ex Falso}$$

$$\frac{\Gamma \vdash A}{\Gamma, B \vdash A}\,\text{WEK}$$

Sequent Calculus

Definition (Gentzen-style Proof System for Intuitionistic Logic)

$$\frac{}{A \vdash A}\,\text{Id} \qquad \frac{}{\bot \vdash A}\,\text{Ex Falso}$$

$$\frac{\Gamma \vdash A}{\Gamma, B \vdash A}\,\text{WEK}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}\,\text{L}\!\to \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\,\text{R}\!\to$$

Sequent Calculus

Definition (Gentzen-style Proof System for Intuitionistic Logic)

$$\frac{}{A \vdash A}\text{Id} \qquad \frac{}{\bot \vdash A}\text{Ex Falso}$$

$$\frac{\Gamma \vdash A}{\Gamma, B \vdash A}\text{WEK}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}\text{L}\to \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\text{R}\to$$

$$\frac{\Gamma, A_i \vdash C}{\Gamma, A_0 \wedge A_1 \vdash C}\text{L}\wedge \qquad \frac{\Gamma \vdash A_0 \qquad \Gamma \vdash A_1}{\Gamma \vdash A_0 \wedge A_1}\text{R}\wedge$$

Sequent Calculus

Definition (Gentzen-style Proof System for Intuitionistic Logic)

$$\frac{}{A \vdash A}\ \text{Id} \qquad \frac{}{\bot \vdash A}\ \text{Ex Falso}$$

$$\frac{\Gamma \vdash A}{\Gamma, B \vdash A}\ \text{WEK}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}\ \text{L}\!\to \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ \text{R}\!\to$$

$$\frac{\Gamma, A_i \vdash C}{\Gamma, A_0 \wedge A_1 \vdash C}\ \text{L}\wedge \qquad \frac{\Gamma \vdash A_0 \qquad \Gamma \vdash A_1}{\Gamma \vdash A_0 \wedge A_1}\ \text{R}\wedge$$

$$\frac{\Gamma, A_0 \vdash C \qquad \Gamma, A_1 \vdash C}{\Gamma, A_0 \vee A_1 \vdash C}\ \text{L}\vee \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_0 \vee A_1}\ \text{R}\vee$$

Sequent Calculus

Definition (Gentzen-style Proof System for Intuitionistic Logic)

$$\dfrac{}{A \vdash A}\ \text{Id} \qquad \dfrac{}{\bot \vdash A}\ \text{Ex Falso}$$

$$\dfrac{\Gamma \vdash A}{\Gamma, B \vdash A}\ \text{WEK}$$

$$\dfrac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}\ \text{L}\!\to \qquad \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ \text{R}\!\to$$

$$\dfrac{\Gamma, A_i \vdash C}{\Gamma, A_0 \wedge A_1 \vdash C}\ \text{L}\wedge \qquad \dfrac{\Gamma \vdash A_0 \qquad \Gamma \vdash A_1}{\Gamma \vdash A_0 \wedge A_1}\ \text{R}\wedge$$

$$\dfrac{\Gamma, A_0 \vdash C \qquad \Gamma, A_1 \vdash C}{\Gamma, A_0 \vee A_1 \vdash C}\ \text{L}\vee \qquad \dfrac{\Gamma \vdash A_i}{\Gamma \vdash A_0 \vee A_1}\ \text{R}\vee$$

$$\dfrac{\Gamma \vdash A \qquad \Gamma, A \vdash B}{\Gamma \vdash B}\ \text{CUT}$$

Subformula Property

Seqent calculus has an extremely good structural property:

Subformula Property

Seqent calculus has an extremely good structural property:

Definition (Subformula Property)

For any inference rule except CUT, all formulas in the premises of a deduction appear as some subformulas in the conclusion.

Subformula Property

Seqent calculus has an extremely good structural property:

Definition (Subformula Property)

For any inference rule except CUT, all formulas in the premises of a deduction appear as some subformulas in the conclusion.

For proofs without cut, we have a good sense of what can be proven and what cannot. Even more, we can have a concrete algorithm to search for a proof for given proposition: just enumerate all possible subformulas of that proposition and test them from bottom up. It would be so nice if our proof system does not have the cut rule, right?

Cut Elimination/Hauptsatz

Theorem (Hauptsatz)

The Cut rule is redundant and thus the above system has full subformula property.

Cut Elimination/Hauptsatz

Theorem (Hauptsatz)

The Cut rule is redundant and thus the above system has full subformula property.

Corollary (Consistency of Intuitionistic logic)

Intuitionistic logic is consistent. Namely, $\perp$/contradiction is not derivable.

Cut Elimination/Hauptsatz

Theorem (Hauptsatz)

The Cut rule is redundant and thus the above system has full subformula property.

Corollary (Consistency of Intuitionistic logic)

Intuitionistic logic is consistent. Namely, $\bot$/contradiction is not derivable.

This proof reduces consistency of LJ to consistency of PRA: a finitistic algorithm transforms any potential proof of $\bot$ in LJ into a proof of $0 = 1$ in PRA. Since we believe in consistency of PRA, there will not be such a proof of $\bot$ in LJ.

Cut Elimination/Hauptsatz

Theorem (Hauptsatz)

The Cut rule is redundant and thus the above system has full subformula property.

Corollary (Consistency of Intuitionistic logic)

Intuitionistic logic is consistent. Namely, $\bot$/contradiction is not derivable.

This proof reduces consistency of LJ to consistency of PRA: a finitistic algorithm transforms any potential proof of $\bot$ in LJ into a proof of $0 = 1$ in PRA. Since we believe in consistency of PRA, there will not be such a proof of $\bot$ in LJ.

Theorem

Classical logic and first-order logic are consistent.

Move to PA

Such a fancy proof does not really justify logic, because it is circular. Nevertheless, we can apply this idea to prove consistency of theories that stronger than PRA, such as PA, which is exactly Gentzen's original plan:

Move to PA

Such a fancy proof does not really justify logic, because it is circular. Nevertheless, we can apply this idea to prove consistency of theories that stronger than PRA, such as PA, which is exactly Gentzen's original plan:

Theorem

PA equipped with sequent calculus admits cut elimination. As the consequence, no proof of $0 = 1$ *is derivable from PA and PA is consistent.*

Move to PA

Such a fancy proof does not really justify logic, because it is circular. Nevertheless, we can apply this idea to prove consistency of theories that stronger than PRA, such as PA, which is exactly Gentzen's original plan:

Theorem

PA equipped with sequent calculus admits cut elimination. As the consequence, no proof of $0 = 1$ *is derivable from PA and PA is consistent.*

Unfortunately, this theorem cannot be proved in PRA without contradicting with Gödel's incompleteness: PA interprets PRA and thus would prove Con(PA) under this theorem.

Move to PA

Such a fancy proof does not really justify logic, because it is circular. Nevertheless, we can apply this idea to prove consistency of theories that stronger than PRA, such as PA, which is exactly Gentzen's original plan:

Theorem

PA equipped with sequent calculus admits cut elimination. As the consequence, no proof of $0 = 1$ *is derivable from PA and PA is consistent.*

Unfortunately, this theorem cannot be proved in PRA without contradicting with Gödel's incompleteness: PA interprets PRA and thus would prove Con(PA) under this theorem.

Beside PRA, Gentzen needs an additional assumption rejecting the existence of infinitely descending chains of length up to ϵ_0. Thus, Gentzen reduces consistency of PA to PRA + well-foundedness of ϵ_0.

Mysterious ϵ_0

A quick introduction to ordinals:

Mysterious ϵ_0

A quick introduction to ordinals:

- $\omega := \lim\{0, 1, 2, \cdots\}$

Mysterious ϵ_0

A quick introduction to ordinals:

- $\omega := \lim\{0, 1, 2, \cdots\}$
- $\omega + \omega := \lim\{\omega, \omega + 1, \omega + 2, \cdots\}$

Mysterious ϵ_0

A quick introduction to ordinals:

- $\omega := \lim\{0, 1, 2, \cdots\}$
- $\omega + \omega := \lim\{\omega, \omega + 1, \omega + 2, \cdots\}$
- $\omega \cdot \omega := \lim\{\omega, \omega + \omega, \omega + \omega + \omega, \cdots\}$

Mysterious ϵ_0

A quick introduction to ordinals:

- $\omega := \lim\{0, 1, 2, \cdots\}$
- $\omega + \omega := \lim\{\omega, \omega + 1, \omega + 2, \cdots\}$
- $\omega \cdot \omega := \lim\{\omega, \omega + \omega, \omega + \omega + \omega, \cdots\}$
- $\omega^\omega := \lim\{\omega, \omega \cdot \omega, \omega \cdot \omega \cdot \omega, \cdots\}$

Mysterious ϵ_0

A quick introduction to ordinals:

- $\omega := \lim\{0, 1, 2, \cdots\}$
- $\omega + \omega := \lim\{\omega, \omega + 1, \omega + 2, \cdots\}$
- $\omega \cdot \omega := \lim\{\omega, \omega + \omega, \omega + \omega + \omega, \cdots\}$
- $\omega^\omega := \lim\{\omega, \omega \cdot \omega, \omega \cdot \omega \cdot \omega, \cdots\}$
- $\epsilon_0 := \lim\{\omega, \omega^\omega, \omega^{\omega^\omega}, \cdots\}$

Mysterious ϵ_0

A quick introduction to ordinals:

- $\omega := \lim\{0, 1, 2, \cdots\}$
- $\omega + \omega := \lim\{\omega, \omega + 1, \omega + 2, \cdots\}$
- $\omega \cdot \omega := \lim\{\omega, \omega + \omega, \omega + \omega + \omega, \cdots\}$
- $\omega^\omega := \lim\{\omega, \omega \cdot \omega, \omega \cdot \omega \cdot \omega, \cdots\}$
- $\epsilon_0 := \lim\{\omega, \omega^\omega, \omega^{\omega^\omega}, \cdots\}$

Our daily induction on natural numbers is really the well-foundedness of ω.

Ordinal Analysis

It turns out this assumption of ϵ_0 is optimal in the following sense:

Ordinal Analysis

It turns out this assumption of ϵ_0 is optimal in the following sense:

Theorem (Ordinal Strength of PA)

For any ordinal $\alpha < \epsilon_0$, PA proves well-foundedness of α. To be consistent with Gödel's incompleteness, ϵ_0 is the least upper bound of such ordinals.

Ordinal Analysis

It turns out this assumption of ϵ_0 is optimal in the following sense:

Theorem (Ordinal Strength of PA)

For any ordinal $\alpha < \epsilon_0$, PA proves well-foundedness of α. To be consistent with Gödel's incompleteness, ϵ_0 is the least upper bound of such ordinals.

For various theories of arithmetic, we can compute the corresponding ordinal strength, which represents how strong the theory is/how hard to prove its consistency. This idea gives birth to **ordinal analysis**.

Ordinal Analysis

It turns out this assumption of ϵ_0 is optimal in the following sense:

Theorem (Ordinal Strength of PA)

For any ordinal $\alpha < \epsilon_0$, PA proves well-foundedness of α. To be consistent with Gödel's incompleteness, ϵ_0 is the least upper bound of such ordinals.

For various theories of arithmetic, we can compute the corresponding ordinal strength, which represents how strong the theory is/how hard to prove its consistency. This idea gives birth to **ordinal analysis**.

The ordinal strength of PRA is ω^{ω}. Hence, PRA is weaker than PA but well-foundedness of ϵ_0 is stronger. Together, PRA + welledness of ϵ_0 is neither stronger nor weaker than PA. That's how we avoid Gödel's incompleteness.

Gödel's Proof

Gödel's Approach and Heyting Arithmetic

Gödel is not satisfied with Gentzen's proof but agrees with his philosophy. He later presents his own proof of consistency of PA based on PRA, via constructive mathematics (note: Gödel is not a constructivist).

Gödel's Approach and Heyting Arithmetic

Gödel is not satisfied with Gentzen's proof but agrees with his philosophy. He later presents his own proof of consistency of PA based on PRA, via constructive mathematics (note: Gödel is not a constructivist).

Definition (Heyting Arithemtic)

Heyting arithmetic/HA is the intuitionistic version of Peano arithmetic: PA = HA + LEM where LEM is Law of Excluded Middle.

HA fits in the idea of constructive mathematics: each proof conduced in HA has a **realizer** that captures the constructive information that the conclusion demands.

Gödel-Genzten Translation

Although intuitionistic logic is a weaker logic, classical logic can be embeded into intuitionistic logic via **double-negation translation**.

Gödel-Genzten Translation

Although intuitionistic logic is a weaker logic, classical logic can be embeded into intuitionistic logic via **double-negation translation**.

Definition (Double-negation Translation)

For any formula φ and ψ, we inductively define φ^N:

Gödel-Genzten Translation

Although intuitionistic logic is a weaker logic, classical logic can be embeded into intuitionistic logic via **double-negation translation**.

Definition (Double-negation Translation)

For any formula φ and ψ, we inductively define φ^N:

- $\varphi^N := \neg\neg\varphi$ if φ is atomic
- $(\neg\varphi)^N := \neg\varphi^N$
- $(\varphi \to \psi)^N := \varphi^N \to \psi^N$
- $(\varphi \wedge \psi)^N := \varphi^N \wedge \psi^N$
- $(\varphi \vee \psi)^N := \neg(\neg\varphi^N \wedge \neg\psi^N)$
- $(\forall x\ \varphi)^N := \forall x\ (\varphi^N)$
- $(\exists x\ \varphi)^N := \neg(\forall\ \neg\varphi^N)$

Reduce Consistency of PA to HA

Proofs in PA can be simulated/**interpreted** in HA in the following sense:

Reduce Consistency of PA to HA

Proofs in PA can be simulated/**interpreted** in HA in the following sense:

Theorem

PA proves φ if and only if HA proves φ^N.

Reduce Consistency of PA to HA

Proofs in PA can be simulated/**interpreted** in HA in the following sense:

Theorem

PA proves φ if and only if HA proves φ^N.

In particular, PA proves $0 = 1$ if and only if HA proves $(0 = 1)^N = \neg\neg(0 = 1)$, which is equivalent to HA proving $\bot$.

Reduce Consistency of PA to HA

Proofs in PA can be simulated/**interpreted** in HA in the following sense:

Theorem

PA proves φ if and only if HA proves φ^N.

In particular, PA proves $0 = 1$ if and only if HA proves $(0 = 1)^N = \neg\neg(0 = 1)$, which is equivalent to HA proving $\bot$.

Theorem

PA is consistent if and only if HA is consistent.

Reduce Consistency of PA to HA

Proofs in PA can be simulated/**interpreted** in HA in the following sense:

Theorem

PA proves φ if and only if HA proves φ^N.

In particular, PA proves $0 = 1$ if and only if HA proves $(0 = 1)^N = \neg\neg(0 = 1)$, which is equivalent to HA proving $\bot$.

Theorem

PA is consistent if and only if HA is consistent.

Now we will try to interpret proofs of HA in PRA and thus hopefully reduce consistency of PA to PRA.

Reducing Use of Quantifiers

To interpret proofs of HA in PRA, we need to eliminate the use of quantifiers in HA in a way that still preserves (some) meaning of orginal formulas. We make use of three non-constructive principles:

Reducing Use of Quantifiers

To interpret proofs of HA in PRA, we need to eliminate the use of quantifiers in HA in a way that still preserves (some) meaning of orginal formulas. We make use of three non-constructive principles:

Definition

- Axiom of Choice: $\forall x\ \exists y\ \varphi(x, y) \rightarrow \exists Y\ \forall x\ \varphi(x, Y(x))$.

Reducing Use of Quantifiers

To interpret proofs of HA in PRA, we need to eliminate the use of quantifiers in HA in a way that still preserves (some) meaning of orginal formulas. We make use of three non-constructive principles:

Definition

- Axiom of Choice: $\forall x\ \exists y\ \varphi(x, y) \to \exists Y\ \forall x\ \varphi(x, Y(x))$.
- Independence of Premise: $(\varphi \to \exists x\ \psi) \to \exists x\ (\varphi \to \psi)$.

Reducing Use of Quantifiers

To interpret proofs of HA in PRA, we need to eliminate the use of quantifiers in HA in a way that still preserves (some) meaning of orginal formulas. We make use of three non-constructive principles:

Definition

- Axiom of Choice: $\forall x \, \exists y \; \varphi(x, y) \to \exists Y \; \forall x \; \varphi(x, Y(x))$.
- Independence of Premise: $(\varphi \to \exists x \; \psi) \to \exists x \; (\varphi \to \psi)$.
- Markov's Principle: $(\neg \forall x \; \varphi(x)) \to \exists x \; \neg\varphi(x)$ where φ is quantifier free.

Reducing Use of Quantifiers

To interpret proofs of HA in PRA, we need to eliminate the use of quantifiers in HA in a way that still preserves (some) meaning of orginal formulas. We make use of three non-constructive principles:

Definition

- Axiom of Choice: $\forall x\ \exists y\ \varphi(x, y) \to \exists Y\ \forall x\ \varphi(x, Y(x))$.
- Independence of Premise: $(\varphi \to \exists x\ \psi) \to \exists x\ (\varphi \to \psi)$.
- Markov's Principle: $(\neg\forall x\ \varphi(x)) \to \exists x\ \neg\varphi(x)$ where φ is quantifier free.

They are all valid in PA but none of them is true in HA from constructive point of view.

Dialectica Interpretation

Denote $\varphi^D = \exists x\ \forall y\ \varphi_D$ and $\psi^D = \exists u\ \forall v\ \psi_D$ where φ_D and ψ_D are quantifier-free.

Dialectica Interpretation

Denote $\varphi^D = \exists x\ \forall y\ \varphi_D$ and $\psi^D = \exists u\ \forall v\ \psi_D$ where φ_D and ψ_D are quantifier-free.

Definition (Dialectica Interpretation)

- $\varphi^D = \varphi_D := \varphi$ if φ is atomic.
- $(\neg\varphi)^D := \exists Y\ \forall x\ \neg\varphi_D(x, Y(x))$
- $(\varphi \to \psi)^D := \exists U, Y\ \forall x, v\ (\varphi_D(x, Y(x, v)) \to \psi_D(U(x), v))$
- $(\varphi \wedge \psi)^D := \exists x, u\ \forall y, v\ (\varphi_D \wedge \psi_D)$
- $(\varphi \vee \psi)^D := \exists z, x, u\ \forall y, v\ ((z = 0 \wedge \varphi_D) \vee (z = 1 \wedge \psi_D))$
- $(\forall z\ \varphi)^D := \exists X\ \forall z, y\ \varphi_D(X(z), y, z)$
- $(\exists z\ \varphi)^D := \exists z, x\ \forall y\ \varphi_D(x, y, z)$

Dialectica Interpretation

Denote $\varphi^D = \exists x\ \forall y\ \varphi_D$ and $\psi^D = \exists u\ \forall v\ \psi_D$ where φ_D and ψ_D are quantifier-free.

Definition (Dialectica Interpretation)

- $\varphi^D = \varphi_D := \varphi$ if φ is atomic.
- $(\neg\varphi)^D := \exists Y\ \forall x\ \neg\varphi_D(x, Y(x))$
- $(\varphi \rightarrow \psi)^D := \exists U, Y\ \forall x, v\ (\varphi_D(x, Y(x, v)) \rightarrow \psi_D(U(x), v))$
- $(\varphi \wedge \psi)^D := \exists x, u\ \forall y, v\ (\varphi_D \wedge \psi_D)$
- $(\varphi \vee \psi)^D := \exists z, x, u\ \forall y, v\ ((z = 0 \wedge \varphi_D) \vee (z = 1 \wedge \psi_D))$
- $(\forall z\ \varphi)^D := \exists X\ \forall z, y\ \varphi_D(X(z), y, z)$
- $(\exists z\ \varphi)^D := \exists z, x\ \forall y\ \varphi_D(x, y, z)$

The quantifier-free version φ_D inherits constructive information of φ. The problem is φ_D is not really a sentence in PRA since it might contains terms of higher types, i.e., functions of natural numbers, and functions of functions, and so on.

Primitive Recursive Functionals of Finite Type

Gödel define an extension of PRA containing **primitive recursive functionals of finite type**, which he names **system T**. It is a typed system and a precursor of modern type theory.

Primitive Recursive Functionals of Finite Type

Gödel define an extension of PRA containing **primitive recursive functionals of finite type**, which he names **system T**. It is a typed system and a precursor of modern type theory.

Definition (System T)

- $\mathbb{N}$ as the basic natural number type

Primitive Recursive Functionals of Finite Type

Gödel define an extension of PRA containing **primitive recursive functionals of finite type**, which he names **system T**. It is a typed system and a precursor of modern type theory.

Definition (System T)

- $\mathbb{N}$ as the basic natural number type
- If σ and τ are types, so is $\sigma \to \tau$ and $\sigma \times \tau$.

Primitive Recursive Functionals of Finite Type

Gödel define an extension of PRA containing **primitive recursive functionals of finite type**, which he names **system T**. It is a typed system and a precursor of modern type theory.

Definition (System T)

- $\mathbb{N}$ as the basic natural number type
- If σ and τ are types, so is $\sigma \to \tau$ and $\sigma \times \tau$.
- Every variables are associated with a unique type.

Primitive Recursive Functionals of Finite Type

Gödel define an extension of PRA containing **primitive recursive functionals of finite type**, which he names **system T**. It is a typed system and a precursor of modern type theory.

Definition (System T)

- $\mathbb{N}$ as the basic natural number type
- If σ and τ are types, so is $\sigma \to \tau$ and $\sigma \times \tau$.
- Every variables are associated with a unique type.
- Constant $0 : \mathbb{N}$ and function $\text{Succ} : \mathbb{N} \to \mathbb{N}$.

Primitive Recursive Functionals of Finite Type

Gödel define an extension of PRA containing **primitive recursive functionals of finite type**, which he names **system T**. It is a typed system and a precursor of modern type theory.

Definition (System T)

- $\mathbb{N}$ as the basic natural number type
- If σ and τ are types, so is $\sigma \to \tau$ and $\sigma \times \tau$.
- Every variables are associated with a unique type.
- Constant $0 : \mathbb{N}$ and function $\text{Succ} : \mathbb{N} \to \mathbb{N}$.
- Combinators $K_{\sigma,\tau} : \sigma \to \tau \to \sigma$ and $S_{\rho,\sigma,\tau} : (\rho \to \sigma \to \tau) \to (\rho \to \sigma) \to (\rho \to \tau)$ and defining equations.

Primitive Recursive Functionals of Finite Type

Gödel define an extension of PRA containing **primitive recursive functionals of finite type**, which he names **system T**. It is a typed system and a precursor of modern type theory.

Definition (System T)

- $\mathbb{N}$ as the basic natural number type
- If σ and τ are types, so is $\sigma \to \tau$ and $\sigma \times \tau$.
- Every variables are associated with a unique type.
- Constant $0 : \mathbb{N}$ and function $\mathrm{Succ} : \mathbb{N} \to \mathbb{N}$.
- Combinators $K_{\sigma,\tau} : \sigma \to \tau \to \sigma$ and $S_{\rho,\sigma,\tau} : (\rho \to \sigma \to \tau) \to (\rho \to \sigma) \to (\rho \to \tau)$ and defining equations.
- Recursor $R_\sigma : \sigma \to (\mathbb{N} \to \sigma \to \sigma) \to \mathbb{N} \to \sigma$ and its defining equations (primitive recursion).

Primitive Recursive Functionals of Finite Type

Gödel define an extension of PRA containing **primitive recursive functionals of finite type**, which he names **system T**. It is a typed system and a precursor of modern type theory.

Definition (System T)

- $\mathbb{N}$ as the basic natural number type
- If σ and τ are types, so is $\sigma \to \tau$ and $\sigma \times \tau$.
- Every variables are associated with a unique type.
- Constant $0 : \mathbb{N}$ and function $\mathrm{Succ} : \mathbb{N} \to \mathbb{N}$.
- Combinators $K_{\sigma,\tau} : \sigma \to \tau \to \sigma$ and $S_{\rho,\sigma,\tau} : (\rho \to \sigma \to \tau) \to (\rho \to \sigma) \to (\rho \to \tau)$ and defining equations.
- Recursor $R_\sigma : \sigma \to (\mathbb{N} \to \sigma \to \sigma) \to \mathbb{N} \to \sigma$ and its defining equations (primitive recursion).
- Induction principle.

Soundness Theorem

Theorem

Suppose that HA proves φ, then there exist terms $\vec{t}$ such that T proves $\varphi_D(\vec{t}, \vec{y})$ for all $\vec{y}$.

Soundness Theorem

Theorem

Suppose that HA proves φ, then there exist terms $\vec{t}$ such that T proves $\varphi_D(\vec{t}, \vec{y})$ for all $\vec{y}$.

Corollary

Suppose that HA proves a Π^0_2-formula, i.e., of the form $\forall x \; \exists y \; \varphi(x, y)$ where φ is quantifier-free. Then there exists a function term f in T such that T proves $\varphi(x, f(x))$.

Soundness Theorem

Theorem

Suppose that HA proves φ, then there exist terms $\vec{t}$ such that T proves $\varphi_D(\vec{t}, \vec{y})$ for all $\vec{y}$.

Corollary

Suppose that HA proves a Π^0_2-formula, i.e., of the form $\forall x \; \exists y \; \varphi(x, y)$ where φ is quantifier-free. Then there exists a function term f in T such that T proves $\varphi(x, f(x))$.

In other words, T extract constructive information from all Π^0_2 theorems in HA. Since $0 = 1$ is trivially a Π^0_2-formula, it follows:

Soundness Theorem

Theorem

Suppose that HA proves φ, then there exist terms $\vec{t}$ such that T proves $\varphi_D(\vec{t}, \vec{y})$ for all $\vec{y}$.

Corollary

Suppose that HA proves a Π^0_2-formula, i.e., of the form $\forall x\ \exists y\ \varphi(x, y)$ where φ is quantifier-free. Then there exists a function term f in T such that T proves $\varphi(x, f(x))$.

In other words, T extract constructive information from all Π^0_2 theorems in HA. Since $0 = 1$ is trivially a Π^0_2-formula, it follows:

Corollary

If HA proves $0 = 1$, then T proves $0 = 1$. Thus, consistency of T implies consistency of HA and consequentially PA.

ϵ_0 Hits Us Again

Now we reduce consistency of PA to consistency of T. Gödel argues that T is a natural extension of PRA and essentially finitistic. Hence, consistency of PA is secured.

ϵ_0 Hits Us Again

Now we reduce consistency of PA to consistency of T. Gödel argues that T is a natural extension of PRA and essentially finitistic. Hence, consistency of PA is secured.

One actual proof of consistency of T is to provide a type system allowing infinite term length and prove strong normalization result (cut-elimination in disguise) for terms with length up to ϵ_0. Therefore, the ordinal strength of T is also ϵ_0 and thus Gödel's result is the same as Gentzen's in essence.

ϵ_0 Hits Us Again

Now we reduce consistency of PA to consistency of T. Gödel argues that T is a natural extension of PRA and essentially finitistic. Hence, consistency of PA is secured.

One actual proof of consistency of T is to provide a type system allowing infinite term length and prove strong normalization result (cut-elimination in disguise) for terms with length up to ϵ_0. Therefore, the ordinal strength of T is also ϵ_0 and thus Gödel's result is the same as Gentzen's in essence.

To summarize, consistency of PA from a finitistic point of view is exactly equivalent to the belief in well-foundedness of ϵ_0. How much faith in ϵ_0? Well, up to you!

Applied Proof Theory

A Non-Constructive Mathematical Proof

Theorem (Jackson's Theorem)

Let f be a continuous function on $[0, 1]$. Then for any n, there is a unique best L_1-approximation of f in $\mathcal{P}_n$, polynomials of degree up to n.

A Non-Constructive Mathematical Proof

Theorem (Jackson's Theorem)

Let f be a continuous function on $[0,1]$. Then for any n, there is a unique best L_1-approximation of f in $\mathcal{P}_n$, polynomials of degree up to n.

The proof is non-constructive and we know nothing about the unique best approximation other than its pure existence and uniqueness.

A Non-Constructive Mathematical Proof

Theorem (Jackson's Theorem)

Let f be a continuous function on $[0,1]$. Then for any n, there is a unique best L_1-approximation of f in $\mathcal{P}_n$, polynomials of degree up to n.

The proof is non-constructive and we know nothing about the unique best approximation other than its pure existence and uniqueness.

Theorem (A Better Theorem)

Say f is a function on $[0,1]$ with modulus of continuity ω_f. Then for any n,

$$\Phi_f(\epsilon) := \frac{\zeta}{2}\min\{\frac{1}{10(n+2)^2}, \omega_f(\zeta/12), \frac{\zeta}{24n^2(n+1)^2\|f\|_1}\}$$

is a modulus of uniqueness for approximation by $\mathcal{Q}_n$ where $\mathcal{Q}_n$ is a refinement of $\mathcal{P}_n$ of polynomials with norm $\leqslant 2\|f\|_1$, $\zeta = \epsilon/(4(n+1)20^n(n+1)^{2n}n^n$.

Proof Mining

How do we obtain such a better theorem other than randomly search for it? Observe that the original statement is equivalent to the following:

Proof Mining

How do we obtain such a better theorem other than randomly search for it? Observe that the original statement is equivalent to the following:

$$\forall f \in \mathbb{R}^{\mathbb{Q}\cap[0,1]}\forall\omega_f \in (\mathbb{Q}^+)^{\mathbb{Q}^+}\forall M\forall\epsilon \in \mathbb{Q}^+[(\mathrm{ucont}(f,\omega_f) \wedge \int |f|dx < M) \to \exists\delta \in \mathbb{Q}^+\mathrm{approxQ}(f,\epsilon,\delta)]$$

where $[-]$ only quantifies over effectively compact domains.

Proof Mining

How do we obtain such a better theorem other than randomly search for it? Observe that the original statement is equivalent to the following:

$$\forall f \in \mathbb{R}^{\mathbb{Q}\cap[0,1]} \forall \omega_f \in (\mathbb{Q}^+)^{\mathbb{Q}^+} \forall M \forall \epsilon \in \mathbb{Q}^+ [(\text{ucont}(f, \omega_f) \wedge \int |f| dx < M) \rightarrow \exists \delta \in \mathbb{Q}^+ \text{approxQ}(f, \epsilon, \delta)]$$

where $[-]$ only quantifies over effectively compact domains. Under this version, Jackson's theorem is interpretable in some reasonable extension of PA (say PA*) and it is Π_2-statement!

Proof Mining

How do we obtain such a better theorem other than randomly search for it? Observe that the original statement is equivalent to the following:

$$\forall f \in \mathbb{R}^{\mathbb{Q}\cap[0,1]}\forall\omega_f \in (\mathbb{Q}^+)^{\mathbb{Q}^+}\forall M\forall\epsilon \in \mathbb{Q}^+[(\mathsf{ucont}(f,\omega_f) \wedge \int |f|dx < M) \rightarrow \exists\delta \in \mathbb{Q}^+\mathsf{approxQ}(f,\epsilon,\delta)]$$

where $[-]$ only quantifies over effectively compact domains. Under this version, Jackson's theorem is interpretable in some reasonable extension of PA (say PA*) and it is Π_2-statement!

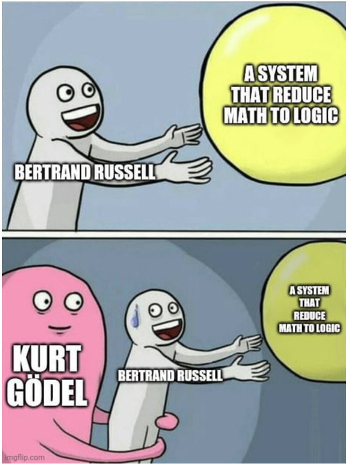
We can interpret this ZFC proof in PA*, in HA* via Gödel-Gentzen, lastly in T* via Dialectica. After a modification of original proof to satisfy the Π_2-restriction, we extract a term in T* that spits out the constructive proof. This method is called **Functional Interpretation/Proof Mining**.

The End

Thank you and enjoy the pizza!

The End

Thank you and enjoy the pizza!



References

[1] J. Avigad, *Mathematical logic and computation*, Cambridge, 2022.
[2] J. Avigad, *Gödel's Functional ("Dialectica") Interpretation*, Elsevier, 1998.
[3] U. Kohlenbach, *Applied Proof Theory : Proof Interpretations And Their Use in Mathematics*, Springer, 2008.
[4] H. Towsner, "A Worked Example of the Functional Interpretation", 2015.