Extreme Value Theorem and Mean Value Theorem

Theorem 1. *(Extreme Value Theorem) Let* $f : [a, b] \to \mathbb{R}$ *be continuous. Then* f *is bounded, and there exists* x^* *and* x_* *such that* $f(x^*) = \sup\{f(x) : x \in [a, b]\}$ *and* $f(x_*) = \inf\{f(x) : x \in [a, b]\}$*.*

We will first show that if f is continuous then it is bounded above. Assume that it is not bounded above. Then for every natural number n, we can find $x_n \in [a, b]$ with $f(x_n) > n$. Why is x_n bounded? What can we conclude if it is bounded?

Recall that since f is continuous, it is also sequentially continuous. What does this tell us?

Recall that convergent sequences are bounded. How does this give us a contradiction?

Therefore, we conclude that f is bounded above. Since $\{f(x) : x \in [a, b]\}$ is bounded above and nonempty, the set has a supremum which we will call M. Our goal is to find x^* such that $f(x^*) = M$ and $x^* \in [a, b]$. For every natural number, n, there exists $y_n \in [a, b]$ with $M - 1/n < f(x_n) \leq M$. Why is this true?

Why does y_n have a convergent subsequence, y_{n_j} that converges to y?

We can say $f(y_{n_j}) \to f(y)$ and $f(y_{n_j}) \to M$. Why can we say both of these statements and why does this mean $M = f(y)$?

Finally, is $y \in [a, b]$?

Definition 1. *(Limit of function) Let $f : A \to \mathbb{R}$ where $A \subset \mathbb{R}$. We say that $f(x) \to L$ as $x \to c$ in A also denoted*

$$\lim_{x \to c} f(x) = L$$

if the following holds:

$$\forall \epsilon > 0, \exists \delta > 0, \text{ s.t. } x \in A \text{ and } 0 < |x - c| < \delta \implies |f(x) - L| < \epsilon.$$

If a function is continuous at c, what is L?

In our definition for continuous, we did not say $0 < |x - c|$. Why can't $|x - c| = 0$ in the definition given here

Definition 2. *(Right Limit of function) Let $f : A \to \mathbb{R}$ where $A \subset \mathbb{R}$. We say that $f(x) \to L$ as $x \to c$ in A from above or from the right or "as x decreases to c" also denoted*

$$\lim_{x \to c^+} f(x) = L \text{ or } \lim_{x \downarrow c} f(x) = L$$

if the following holds:

$$\forall \epsilon > 0, \exists \delta > 0, \text{ s.t. } x \in A \text{ and } 0 < x - c < \delta \implies |f(x) - L| < \epsilon.$$

Definition 3. *(Left Limit of function) Let $f : A \to \mathbb{R}$ where $A \subset \mathbb{R}$. We say that $f(x) \to L$ as $x \to c$ in A from below or from the left or "as x increases to c" also denoted*

$$\lim_{x \to c^-} f(x) = L \text{ or } \lim_{x \uparrow c} f(x) = L$$

if the following holds:

$$\forall \epsilon > 0, \exists \delta > 0, \text{ s.t. } x \in A \text{ and } 0 < c - x < \delta \implies |f(x) - L| < \epsilon.$$

Write down a function that is not continuous at c. Write what L would be for each of the above definitions, if it exists.

Definition 4. *Let $f : A \to \mathbb{R}$ where $A \subset \mathbb{R}$ that is not bounded from above. We say that $f(x) \to L$ as $x \to \infty$ also denoted*

$$\lim_{x\to\infty} f(x) = L$$

if the following holds:

$$\forall \epsilon > 0, \exists R > 0, \text{ s.t. } x \in A \text{ and } x > R \implies |f(x) - L| < \epsilon.$$

Write down what the definition would be for the limit as $x \to -\infty$.

Definition 5. *Let $f : A \to \mathbb{R}$ where $A \subset \mathbb{R}$ that is not bounded from above. We say that $f(x) \to L$ as $x \to \infty$ also denoted*

$$\lim_{x\to\infty} f(x) = L$$

if the following holds:

$$\forall \epsilon > 0, \exists R > 0, \text{ s.t. } x \in A \text{ and } x > R \implies |f(x) - L| < \epsilon.$$

Definition 6. *Let $f : A \to \mathbb{R}$ where $A \subset \mathbb{R}$. We say that $f(x) \to \infty$ as $x \to c$ if*

$$\forall M > 0, \exists \delta > 0 \text{ s.t. } x \in A \text{ and } 0 < |x - c| < \delta \implies f(x) > M.$$

Using the previous two definitions, write down what you think the definition would be to say $f(x) \to \infty$ as $x \to \infty$.

Properties of limits:

- If the limit exists, then the right limit and left limit will also exist and will both be equivalent to the limit.
- Limits of sums and products of functions are the sums and products of the limits when the limits are defined.
- If $f(x)/g(x)$ is defined and the limit of $g(x)$ is defined then the limit of $f(x)/g(x)$ is the quotient of the limits.
- If $f(x) \leq g(x)$ and $f(x) \to \alpha$, $g(x) \to \beta$ as $x \to c$, then $\alpha \leq \beta$.

Definition 7. *We say $f : (a,b) \to \mathbb{R}$ is differentiable at $x_0 \in (a,b)$ if*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. In this case we denote it $f'(x_0)$ and call it the derivative of f at x_0.

Remark. *When the derivative of f exists for all $x_0 \in A$, we consider the derivative to be a function $f' : A \to \mathbb{R}$.*

Remark.

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Show that if $(-f)'(x_0) = 0$, then $f'(x_0) = 0$.

Definition 8. *Let $f : A \to \mathbb{R}$ and let $x_0 \in A$. We say x_0 is a local/global maxima/minima for f if:*

- *Local maxima: $\exists r > 0$ s.t. $x \in A$ and $|x - x_0| < r \implies$*
- *Local mimima:*
- *Global maxima: $x \in A \implies$*
- *Global minima: $x \in A \implies$*

If f has a minima at c, show that $-f$ has a maxima at c.

Theorem 2. *(Fermat's Principle) Let $f : (a, b) \to \mathbb{R}$ be differentiable and suppose it has a maxima or minima at some $c \in (a, b)$. Then $f'(c) = 0$.*

Let $f : (a, b) \to \mathbb{R}$ have a maxima at c and be differentiable. Then

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}.$$

Show that $\lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} \geq 0$ and $\lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} \leq 0$.

We conclude that $f'(c) = 0$. If $f : (a, b) \to \mathbb{R}$ has a minima at c and is differentiable, then $-f(c)$ is a maxima. Using the above argument, $-f'(c) = 0$. As shown above, this shows $f'(c) = 0$.

Theorem 3. *(Rolle's Theorem) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose $f(a) = f(b)$. Then there exists a $c \in (a, b)$ with $f'(c) = 0$*

Show this is true if f is constant.

Apply the extreme value theorem. We know that either [$x_* \neq a$ and $x_* \neq b$] or [$x^* \neq a$ and $x_* \neq b$]. Why?

How would you use Fermat's Principle to complete the proof?

Theorem 4. *(Mean Value Theorem). Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then $\exists c \in (a, b)$ with*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Set $k = \frac{f(b)-f(a)}{b-a}$. Notice that k is a constant. Now let $g(x) = f(x) - kx$. Since g is continuous on $[a, b]$ and differentiable on (a, b) and $-kx$ is continuous on $[a, b]$ and differentiable on (a, b), we have that g satisfies the assumptions of Rolle's Theorem. What does this tell us? How can we finish the proof?