Notes on New-Keynesian Models

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1 Smets & Wouters – Christiano, Eichenbaum, & Evans

1.1 Define the problem, FOCs, and equilibrium conditions

1.1.1 Final goods producers

The final good Y_t is a composite made of a continuum of goods:

$$Y_t = \left[\int_0^1 Y_t(i)^{\frac{1}{1+\lambda_{f,t}}} di \right]^{1+\lambda_{f,t}} \tag{1.1.1}$$

The **final goods producers** buy the intermediate goods on the market, package Y_t, and resell it to consumers. These firms maximize profits in a perfectly competitive environment. Their problem is:

$$
\begin{aligned}
&\max_{Y_t, Y_t(i)} P_t Y_t - \int_0^1 P_t(i) Y_t(i) di \\
&\text{s.t. } Y_t = \left[\int_0^1 Y_t(i)^{\frac{1}{1+\lambda_{f,t}}} di \right]^{1+\lambda_{f,t}} \quad (\mu_{f,t})
\end{aligned}
\tag{1.1.2}
$$

The FOCs are:

$$(\partial Y_t) \qquad\qquad P_t = \mu_{f,t} \qquad\qquad (1.1.3)$$

$$(\partial Y_t(i)) \quad -P_t(i) + \mu_{f,t}(1 + \lambda_{f,t})[\ldots]^{\lambda_{f,t}} Y_t(i)^{-\frac{\lambda_{f,t}}{1+\lambda_{f,t}}} = 0 \qquad (1.1.4)$$

Note that $[\ldots]^{\lambda_{f,t}} = Y_t^{\frac{\lambda_{f,t}}{1+\lambda_{f,t}}}$. From the FOCs one obtains:

$$Y_t(i) \;\; = \left(\frac{P_t(i)}{P_t}\right)^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} Y_t \qquad\qquad (1.1.5)$$

Combining this condition with the zero profit condition (these firms are competitive) one obtains an expression for the price of the composite good:

$$P_t = \left[\int_0^1 P_t(i)^{-\frac{1}{\lambda_{f,t}}} \, di\right]^{-\lambda_{f,t}} \qquad\qquad (1.1.6)$$

Note that the elasticity is $\frac{1+\lambda_{f,t}}{\lambda_{f,t}}$. $\lambda_{f,t} = 0$ corresponds to the linear case. $\lambda_{f,t} \to \infty$ corresponds to the Cobb-Douglas case. We will constrain $\lambda_{f,t} \in (0, \infty)$. $\lambda_{f,t}$ follows the exogenous process:

$$\ln \lambda_{f,t} = \ln \lambda_f + \epsilon_{\lambda,t}, \; \epsilon_{\lambda,t} \sim \ldots \qquad\qquad (1.1.7)$$

1.1.2   Intermediate goods producers

Intermediate goods producers i uses the following technology:

$$Y_t(i) = \max\{Z_t^{1-\alpha} K_t(i)^\alpha L_t(i)^{1-\alpha} - Z_t^* \Phi, 0\}, \qquad\qquad (1.1.8)$$

where

$$Z_t^* = Z_t \Upsilon^{\left(\frac{\alpha}{1-\alpha} t\right)}, \Upsilon > 1. \qquad\qquad (1.1.9)$$

Call $z_t = \log(Z_t/Z_{t-1})$. z_t follows the process:

$$(z_t - \gamma) = \rho_z(z_{t-1} - \gamma) + \epsilon_{z,t}, \; \epsilon_{z,t} \sim \ldots \qquad\qquad (1.1.10)$$

The firm's profit is given by:

$$P_t(i)Y_t(i) - W_t L_t(i) - R_t^k K_t(i).$$

Cost minimization subject to 1.1.8 yields the conditions:

$$(\partial L_t(i)) \quad \mathcal{V}_t(i)(1-\alpha)Z_t^{1-\alpha}K_t(i)^\alpha L_t(i)^{-\alpha} = W_t$$

$$(\partial K_t(i)) \quad \mathcal{V}_t(i)\alpha Z_t^{1-\alpha}K_t(i)^{\alpha-1}L_t(i)^{1-\alpha} = R_t^k$$

where $\mathcal{V}_t(i)$ is the Lagrange multiplier associated with 1.1.8. In turn, these conditions imply:

$$\frac{K_t(i)}{L_t(i)} = \frac{\alpha}{1-\alpha}\frac{W_t}{R_t^k}.$$

Note that if we integrate both sides of the equation wrt di and define $K_t = \int K_t(i)di$ and $L_t = \int L_t(i)di$ we obtain a relationship between aggregate labor and capital:

$$K_t = \frac{\alpha}{1-\alpha}\frac{W_t}{R_t^k}L_t. \tag{1.1.11}$$

Total variable cost is given by

$$
\begin{aligned}
\text{Variable Costs} \quad &= (W_t + R_t^k \tfrac{K_t(i)}{L_t(i)})L_t(i) \\
&= (W_t + R_t^k \tfrac{K_t(i)}{L_t(i)})\tilde{Y}_t(i)Z_t^{-(1-\alpha)}\left(\tfrac{K_t(i)}{L_t(i)}\right)^{-\alpha},
\end{aligned}
$$

where $\tilde{Y}_t(i) = Z_t^{1-\alpha}K_t(i)^\alpha L_t(i)^{1-\alpha}$ is the "variable" part of output. The marginal cost MC_t is the same for all firms and equal to:

$$
\begin{aligned}
MC_t \quad &= (W_t + R_t^k \tfrac{K_t(i)}{L_t(i)})Z_t^{-(1-\alpha)}\left(\tfrac{K_t(i)}{L_t(i)}\right)^{-\alpha} \\
&= \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}W_t^{1-\alpha}R_t^{k\ \alpha}Z_t^{-(1-\alpha)}.
\end{aligned} \tag{1.1.12}
$$

Profits can then be expressed as $(P_t(i) - MC_t)Y_t(i) - MC_t Z_t^*\Phi$. Note that since the last part of this expression does not depend on the firm's decision, it can be safely ignored. Prices are sticky as in Calvo (1983). Specifically, each firm can readjust prices with probability $1-\zeta_p$ in each period. We depart rfom Calvo (1983) in assuming that for those firms that cannot adjust prices, $P_t(i)$ will increase at the geometric weighted average (with weigths $1-\iota_p$ and ι_p, respectively) of the steady state rate of inflation π_* and of last period's inflation π_{t-1}. For those firms that can adjust prices, the problem is to choose a price level $\tilde{P}_t(i)$ that maximizes the expected present discounted value of

profits in all states of nature where the firm is stuck with that price in the future:

$$\max_{\tilde{P}_t(i)} \quad \Xi_t^p \left(\tilde{P}_t(i) - MC_t \right) Y_t(i)$$
$$+ E_t \sum_{s=1}^{\infty} \zeta_p^s \beta^s \Xi_{t+s}^p \left(\tilde{P}_t(i) \left(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p} \right) - MC_{t+s} \right) Y_{t+s}(i)$$
$$\text{s.t. } Y_{t+s}(i) = \left(\frac{\tilde{P}_t(i) \left(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p} \right)}{P_{t+s}} \right)^{-\frac{1+\lambda_{f,t+s}}{\lambda_{f,t+s}}} Y_{t+s},$$

$$(1.1.13)$$

where $\beta^s \Xi_{t+s}^p$ is today's value of a future dollar for the consumers (Ξ_{t+s}^p is the Lagrange multiplier associated with the consumer's nominal budget constraint - remember there are complete markets so $\beta^s \Xi_{t+s}^p$ is the same for all consumers). The FOC for the firm is:

$$\Xi_t^p \left(\frac{\tilde{P}_t(i)}{P_t} \right)^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}-1} \frac{1}{\lambda_{f,t} P_t} \left(\tilde{P}_t(i) - (1+\lambda_{f,t}) MC_t \right) Y_t(i) +$$
$$E_t \sum_{s=0}^{\infty} \zeta_p^s \beta^s \Xi_{t+s}^p \left(\frac{\tilde{P}_t(i) \left(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p} \right)}{P_{t+s}} \right)^{-\frac{1+\lambda_{f,t+s}}{\lambda_{f,t+s}}-1} \frac{\left(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p} \right)}{\lambda_{f,t+s} P_{t+s}} \quad (1.1.14)$$
$$\left(\tilde{P}_t(i) \left(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p} \right) - (1+\lambda_{f,t+s}) MC_{t+s} \right) Y_{t+s}(i) = 0$$

Note that all firms readjusting prices face an indentical problem. We will consider only the symmetric equilibrium in which all firms that can readjust prices will choose the same $\tilde{P}_t(i)$, so we can drop the i index from now on. From 1.1.6 it follows that:

$$P_t = [(1-\zeta_p)\tilde{P}_t^{-\frac{1}{\lambda_f}} + \zeta_p (\pi_{t-1}^{\iota_p} \pi_*^{1-\iota_p} P_{t-1})^{-\frac{1}{\lambda_f}}]^{-\lambda_f}. \qquad (1.1.15)$$

1.1.3 Households

The objective function for household j is given by:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s b_{t+s} \left[\log(C_{t+s}(j) - hC_{t+s-1}(j)) - \frac{\varphi_{t+s}}{1+\nu_l} L_{t+s}(j)^{1+\nu_l} + \frac{\chi_{t+s}}{1-\nu_m} \left(\frac{M_{t+s}(j)}{Z_{t+s}^* P_{t+s}} \right)^{1-\nu_m} \right]$$

$$(1.1.16)$$

where $C_t(j)$ is consumption, $L_t(j)$ is labor supply (total available hours are normalized to one), and $M_t(j)$ are money holdings. Note that the household is a "habit" guy for $h > 0$. φ_t affects the marginal utility of leisure: it is model as a stochastic preference shifter. Real money balances enter the utility function deflated by the (stochastic) trend growth of the economy, so to make real money demand stationary. χ_t is another stochastic

preference shifter that affects the marginal utility from real money balances. b_t is yet another stochastic preference shifter that scales the overall period utility. The preference shifters are exogenous processes (common to all households) that evolve as follows:

$$\ln \varphi_t = (1 - \rho_\varphi) \ln \varphi + \rho_\varphi \ln \varphi_{t-1} + \epsilon_{\varphi,t}, \ \epsilon_{\varphi,t} \sim \dots \quad (1.1.17)$$

$$\ln \chi_t = (1 - \rho_\chi) \ln \chi + \rho_\chi \ln \chi_{t-1} + \epsilon_{\chi,t}, \ \epsilon_{\chi,t} \sim \dots \quad (1.1.18)$$

$$\ln b_t = \rho_b \ln b_{t-1} + \epsilon_{b,t}, \ \epsilon_{b,t} \sim \dots \quad (1.1.19)$$

The household's budget constraint, written in nominal terms, is given by:

$$P_{t+s}C_{t+s}(j) + P_{t+s}I_{t+s}(j) + B_{t+s}(j) + M_{t+s}(j) \leq R_{t+s-1}B_{t+s-1}(j) + M_{t+s-1}(j)$$
$$+ \Pi_{t+s} + W_{t+s}(j)L_{t+s}(j) + \left(R^k_{t+s}u_{t+s}(j)\bar{K}_{t+s-1}(j) - P_{t+s}a(u_{t+s}(j))\Upsilon^{-t}\bar{K}_{t+s-1}(j)\right),$$
$$(1.1.20)$$

where $I_t(j)$ is investment, $B_t(j)$ is holdings of government bonds, R_t is the gross nominal interest rate paid on government bonds, Π_t is the per-capita profit the household gets from owning firms (assume household pool their firm shares, so that they all receive the same profit) $W_t(j)$ is the wage earned by household j. The term within parenthesis represents the return to owning $\bar{K}_t(j)$ units of capital. Households choose the utilization rate of their own capital, $u_t(j)$, and end up renting to firms in period t an amount of "effective" capital equal to:

$$K_t(j) = u_t(j)\bar{K}_{t-1}(j), \quad (1.1.21)$$

and getting $R^k_t u_t(j)\bar{K}_{t-1}(j)$ in return. They however have to pay a cost of utilization in terms of the consumption good which is equal to $a(u_t(j))\Upsilon^{-t}\bar{K}_{t-1}(j)$. Households accumulate capital according to the equation:

$$\bar{K}_t(j) = (1 - \delta)\bar{K}_{t-1}(j) + \Upsilon^t \mu_t \left(1 - S(\frac{I_t(j)}{I_{t-1}(j)})\right) I_t(j), \quad (1.1.22)$$

where δ is the rate of depreciation, and $S(\cdot)$ is the cost of adjusting investment, with $S'(\cdot) > 0, S''(\cdot) > 0$. The term μ_t is a stochastic disturbance to the price of investment relative to consumption, which follows the exogenous process:

$$\ln \mu_t = (1 - \rho_\mu) \ln \mu + \rho_\mu \ln \mu_{t-1} + \epsilon_{\mu,t}, \ \epsilon_{\mu,t} \sim \dots \quad (1.1.23)$$

Call $\Xi^p_t(j)$ the Lagrange multiplier associated with the budget constraint 1.1.20 (the marginal value of a dollar at time t). We assume there is a complete set of state contingent securities in nominal terms, although we do not explicitly write them in the household's budget constraint. This assumption implies that $\Xi^p_t(j)$ must be the same for all households in all periods and across all states of nature: $\Xi^p_t(j) = \Xi^p_t$ for all j and t. Although we so far kept the j index for all the appropriate variables, we will see that the assumption of complete markets implies that the index will drop out of most of these variables: In equilibrium households will make the same choice of consumption, money demand, investment and capital utilization. As we will see, wage rigidity á la Calvo implies that leisure and the wage will differ across households.

We first write the first order conditions for consumption and money demand. The FOCs for consumption, money holdings, and bonds are:

$$(\partial C_t(i)) \quad \frac{1}{P_t}\big(b_t(C_t(j) - hC_{t-1}(j))^{-1} \tag{1.1.24}$$

$$-\beta h \mathbb{E}_t[b_{t+1}(C_{t+1}(j) - hC_t(j))^{-1}]\big) = \Xi^p_t \tag{1.1.25}$$

$$(\partial M_t(i)) \quad \chi_t b_t \left(\frac{M_t(j)}{Z^*_t P_t}\right)^{-\nu_m} \frac{1}{Z^*_t P_t} = \Xi^p_t - \beta \mathbb{E}_t[\Xi^p_{t+1}] \tag{1.1.26}$$

$$(\partial B_t(i)) \quad \Xi^p_t = \beta R_t \mathbb{E}_t[\Xi^p_{t+1}] \tag{1.1.27}$$

The first FOC equates the marginal utility of consumption at time t, times the relative price of money in terms of the consumption good, to the marginal utility of one dollar at time t. FOCs 1.1.25 through 1.1.27 show that the quantity of consumption, money holdings, and bonds will also be the same apparent across households since the lagrange multiplier is the same, so that we can drop the j index. Separability in the utility function is key for this result: if the marginal utility of consumption depended on leisure, then equalling the marginal utility of consumption across households would not imply equal consumption, since leisure differs across j (depending on whether they can change their wage or not).

Now define $\Xi_t = P_t \Xi^p_t$. The FOCs for consumption, money and bonds can be rewrit-

ten as:

$$\Xi_t = b_t(C_t - hC_{t-1})^{-1} - \beta h E_t[b_{t+1}(C_{t+1} - hC_t)^{-1}] \tag{1.1.28}$$

$$\left(\frac{M_t}{Z_t^* P_t}\right)^{\nu_m} = \chi_t b_t \frac{R_t}{R_t - 1} \frac{1}{Z_t^* \Xi_t} \tag{1.1.29}$$

$$\Xi_t = \beta R_t E_t[\Xi_{t+1}\pi_{t+1}^{-1}]. \tag{1.1.30}$$

where inflation is defined as $\pi_t = P_t/P_{t-1}$.

Let us now address the capital accumulation/utilization problem. Call Ξ_t^k the Lagrange multiplier associated with constraint 1.1.22. The FOC with respect to investment, capital, and capital utilization are:

$$(\partial I_t) \qquad \Xi_t^k \Upsilon^t \mu_t \left(1 - S(\frac{I_t}{I_{t-1}}) - S'(\frac{I_t}{I_{t-1}})\frac{I_t}{I_{t-1}}\right)$$
$$+ \beta E_t[\Xi_{t+1}^k \Upsilon^{t+1} \mu_{t+1} S'(\frac{I_{t+1}}{I_t})(\frac{I_{t+1}}{I_t})^2] = \Xi_t \tag{1.1.31}$$

$$(\partial \bar{K}_t) \quad \Xi_t^k = \beta E_t[\Xi_{t+1}(\frac{R_{t+1}^k}{P_{t+1}}u_{t+1} - a(u_{t+1})\Upsilon^{-(t+1)}) + \Xi_{t+1}^k(1-\delta)] \tag{1.1.32}$$

$$(\partial u_t) \qquad\qquad \Upsilon^t \frac{R_t^k}{P_t} = a'(u_t) \tag{1.1.33}$$

The first FOC is the law of motion for the shadow value of capital. Note that if adjustment cost were absent, the FOC would simply say that $\Xi_t^k \Upsilon^t \mu_t$ is equal to the marginal utility of consumption. In other words, in absence of adjustment costs the shadow cost of taking resources away from consumption equals the shadow benefit (abstracting from $\Upsilon^t \mu_t$) of putting these resources into investment: Tobin's Q is equal to one. The second FOC says that if I buy a unit of capital today I have to pay its price in real terms, Ξ_t^k, but tomorrow I will get the proceeds from renting capital, plus I can sell back the capital that has not depreciated.

Now to the wage/leisure decision. Before going into the household's problem, more details on the labor market are needed. Labor used by the intermediate goods producers L_t is a composite:

$$L_t = \left[\int_0^1 L_t(j)^{\frac{1}{1+\lambda_{w,t}}} di\right]^{1+\lambda_{w,t}}.$$

There are **labor packers** who buy the labor from the households, package L_t, and resell it to the intermediate goods producers. Labor packers maximize profits in a perfectly

competitive environment. From the FOCs of the labor packers one obtains:

$$L_t(j) = \left(\frac{W_t(j)}{W_t} \right)^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} L_t \tag{1.1.34}$$

Combining this condition with the zero profit condition one obtains an expression for the wage:

$$W_t = \left[\int_0^1 W_t(j)^{\frac{1}{\lambda_{w,t}}} di \right]^{\lambda_{w,t}} \tag{1.1.35}$$

We will set $\lambda_{w,t} = \lambda_w \in (0, \infty)$. Given the structure of the labor market, the household has market power: she can choose her wage subject to 1.1.34. However, she is also subject to nominal rigidities á la Calvo. Specifically, households can readjust wages with probability $1 - \zeta_w$ in each period. For those that cannot adjust wages, $W_t(j)$ will increase at a geometrically weighted average of the steady state rate increase in wages (equal to steady state inflation π_* times the growth rate of the economy $e^\gamma \Upsilon^{\frac{\alpha}{1-\alpha}}$) and of last period's inflation times last period's productivity $(\pi_{t-1} e^{z^*_{t-1}})$. For those that can adjust, the problem is to choose a wage $\tilde{W}_t(j)$ that maximizes utility in all states of nature where the household is stuck with that wage in the future:

$$\max_{\tilde{W}_t(j)} \quad \mathbb{E}_t \sum_{s=0}^\infty (\zeta_w \beta)^s b_{t+s} \left[-\frac{\varphi_{t+s}}{\nu_l + 1} L_{t+s}(j)^{\nu_l+1} + \ldots \right]$$

$$\text{s.t.} \quad 1.1.20 \text{ and } 1.1.34 \text{ for } s = 0, \ldots, \infty, \text{ and}$$

$$W_{t+s}(j) = \left(\Pi_{l=1}^s (\pi_* e^\gamma \Upsilon^{\frac{\alpha}{1-\alpha}})^{1-\iota_w} (\pi_{t+l-1} e^{z^*_{t+l-1}})^{\iota_w} \right) \tilde{W}_t(j) \tag{1.1.36}$$

$$\text{for } s = 1, \ldots, \infty$$

where the ... indicate the terms in the utility function that are irrelevant for this problem. The FOC for this problem are:

$$(\partial \tilde{W}_t(j))$$

$$\frac{\Xi_t L_t}{\lambda_w W_t} \mathbb{E}_t \sum_{s=0}^\infty (\zeta_w \beta)^s \Xi_{t+s} L(j)_{t+s} \left[-\frac{\mathcal{X}_{t,s} \tilde{W}_t(j)}{P_{t+s}} + (1 + \lambda_w) \frac{b_{t+s} \varphi_{t+s} L_{t+s}(j)^{\nu_l}}{\Xi_{t+s}} \right] = 0. \tag{1.1.37}$$

where

$$\mathcal{X}_{t,s} = \begin{cases} 1 \text{ if } s = 0 \\ \Pi_{l=1}^s (\pi_* e^\gamma \Upsilon^{\frac{\alpha}{1-\alpha}})^{1-\iota_w} (\pi_{t+l-1} e^{z^*_{t+l-1}})^{\iota_w} \text{ otherwise.} \end{cases}$$

In absence of nominal rigidities this condition would amount to setting the real wage equal to ratio of the marginal utility of leisure over the marginal utility of consumption

times the markup $(1 + \lambda_w)$. All agents readjusting wages face an indentical problem. We will again consider only the symmetric equilibrium in which all agents that can readjust their wage will choose the same $\tilde{W}_t(j)$, so we can drop the i index from now on. From 1.1.35 it follows that:

$$W_t = [(1 - \zeta_w)\tilde{W}_t^{\frac{1}{\lambda_w}} + \zeta_w((\pi_* e^\gamma \Upsilon^{\frac{\alpha}{1-\alpha}})^{1-\iota_w}(\pi_{t-1} e^{z_{t-1}^*})^{\iota_w} W_{t-1})^{\frac{1}{\lambda_w}}]^{\lambda_w}. \tag{1.1.38}$$

1.1.4 Government Policies

The central bank follows a nominal interest rate rule by adjusting its instrument in response to deviations of inflation and output from their respective target levels:

$$\frac{R_t}{R^*} = \left(\frac{R_{t-1}}{R^*}\right)^{\rho_R}\left[\left(\frac{\pi_t}{\pi_*}\right)^{\psi_1}\left(\frac{Y_t}{Y_t^*}\right)^{\psi_2}\right]^{1-\rho_R} e^{\epsilon_{R,t}} \tag{1.1.39}$$

where R^* is the steady state nominal rate and Y_t^* is nominal output. The parameter ρ_R determines the degree of interest rate smoothing. The monetary policy shock $\epsilon_{R,t}$ is iid:

$$\epsilon_{R,t} \sim ... \tag{1.1.40}$$

The central bank supplies the money demanded by the household to support the desired nominal interest rate.

The government budget constraint is of the form

$$P_t G_t + R_{t-1} B_{t-1} + M_{t-1} = T_t + M_t + B_t, \tag{1.1.41}$$

where T_t are nominal lump-sum taxes (or subsidies) that also appear in household's budget constraint. Government spending is given by:

$$G_t = (1 - 1/g_t)Y_t \tag{1.1.42}$$

where g_t follows the process:

$$\ln g_t = (1 - \rho_g)\ln g + \rho_g \ln g_{t-1} + \epsilon_{g,t}, \, \epsilon_{g,t} \sim ... \tag{1.1.43}$$

1.1.5 Resource constraints

To obtain the market clearing condition for the final goods market first integrate the HH budget constraint across households, and combine it with the gvmt budget constraint:

$$P_t C_t + P_t I_t + P_t G_t \leq +\Pi_t + \int W_t(j) L_t(j) dj$$
$$+ R_t^k \int K_t(j) dj - P_t a(u_t) \Upsilon^{-t} \int \bar{K}_{t-1}(j) dj.$$

Next, realize that

$$\Pi_t = \int \Pi(i)_t di = \int P(i)_t Y(i)_t di - W_t L_t - R_t^k K_t,$$

where $L_t = \int L(i)_t di$ is total labor supplied by the labor packers (and demanded by the firms), and $K_t = \int K(i)_t di = \int K_t(j) dj$. Now replace the defintion of Π_t into the HH budget constraint, realize that by the labor and goods' packers' zero profit condition $W_t L_t = \int W_t(j) L_t(j) dj$, and $P_t Y_t = \int P(i)_t Y(i)_t di$ and obtain:

$$P_t C_t + P_t I_t + P_t G_t + P_t a(u_t) \Upsilon^{-t} \bar{K}_{t-1} = P_t Y_t,$$

or

$$C_t + I_t + a(u_t) \Upsilon^{-t} \bar{K}_{t-1} = \frac{1}{g_t} Y_t \tag{1.1.44}$$

where Y_t is defined by (1.1.1). The relationship between output and the aggregate inputs, labor anc capital, is:

$$
\begin{aligned}
\dot{Y}_t &= \int Z_t^{1-\alpha} K_t(i)^\alpha L_t(i)^{1-\alpha} di - Z_t^* \Phi \\
&= Z_t^{1-\alpha} \int (K/L)^\alpha L(i) di - Z_t^* \Phi \\
&= Z_t^{1-\alpha} K_t^\alpha L_t^{1-\alpha} - Z_t^* \Phi,
\end{aligned}
\tag{1.1.45}
$$

where I used the fact that the capital labor ratio is constant across firms (also, since $K(i) = (K/L)L(i)$ it must be the case that $\frac{\int K(i) di}{\int L(i) di} = K_t/L_t = (K/L)$). The problem with these resource constraints is that what we observe in the data is $\dot{Y}_t = \int Y_t(i) di$ and $\dot{L}_t = \int L_t(j) dj$, as opposed to Y_t and L_t. But note that from 1.1.5:

$$
\begin{aligned}
\dot{Y}_t &= Y_t P_t^{\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} \int P(i)_t^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} di \\
&= Y_t P_t^{\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} \dot{P}_t^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}},
\end{aligned}
$$

where $\dot{P}_t = \left(\int P_t(i)^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} di \right)^{-\frac{\lambda_{f,t}}{1+\lambda_{f,t}}}$, and

$$
\begin{aligned}
\dot{L}_t \quad &= \int L_t(j)dj \\
&= L_t W_t^{\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} \int W(j)_t^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} di \\
&= L_t W_t^{\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} \dot{W}_t^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} ,
\end{aligned}
$$

where $\dot{W}_t = \left(\int W(j)_t^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} dj \right)^{-\frac{\lambda_{w,t}}{1+\lambda_{w,t}}}$.

1.1.6 Exogenous Processes

The model is supposed to be fitted to data on output, consumption, investment, employment, wages, prices, nominal interest rates, and money.

- Technology process: let $z_t = \ln Z_t / Z_{t-1}$

$$(z_t - \gamma) = \rho_z(z_t - \gamma) + \epsilon_{z,t} \qquad (1.1.46)$$

  (We will probably restrict ρ_z to zero.)

- Preference for leisure:

$$\ln \varphi_t = (1 - \rho_\varphi) \ln \varphi + \rho_\varphi \ln \varphi_{t-1} + \epsilon_{\varphi,t} \qquad (1.1.47)$$

- Money Demand:

$$\ln \chi_t = (1 - \rho_\chi) \ln \chi + \rho_\chi \ln \chi_{t-1} + \epsilon_{\chi,t} \qquad (1.1.48)$$

- Price Mark-up shock:

$$\ln \lambda_{f,t} = \ln \lambda_f + \epsilon_{\lambda,t} \qquad (1.1.49)$$

- Capital adjustment cost process:

$$\ln \mu_t = (1 - \rho_\mu) \ln \mu + \rho_\mu \ln \mu_{t-1} + \epsilon_{\mu,t} \qquad (1.1.50)$$

- Intertemporal preference shifter:

$$\ln b_t = \rho_b \ln b_{t-1} + \epsilon_{b,t} \qquad (1.1.51)$$

- Government spending process:

$$\ln g_t = (1 - \rho_g) \ln g + \rho_g \ln g_{t-1} + \epsilon_{g,t} \qquad (1.1.52)$$

- Monetary Policy Shock $\epsilon_{R,t}$.

- Equation for $z_t^* = z_t + \frac{\alpha}{1-\alpha} \ln \Upsilon$

$$\left(z_t^* - \gamma - \frac{\alpha}{1 - \alpha} \ln \Upsilon\right) = \rho_z\left(z_t^* - \gamma - \frac{\alpha}{1 - \alpha} \ln \Upsilon\right) + \epsilon_{z,t} \qquad (1.1.53)$$

  (We will probably restrict ρ_z to zero.)

1.2 Detrending and steady state

We detrend the variables as in Altig et al. Lower case variables are all detrended variables – i.e., stationary stuff. Specifically:

$$c_t = \frac{C_t}{Z_t^*}, \; y_t = \frac{Y_t}{Z_t^*}, \; i_t = \frac{I_t}{Z_t^*}, \; k_t = \Upsilon^{-t}\frac{K_t}{Z_t^*}, \; \bar{k}_t = \Upsilon^{-t}\frac{\bar{K}_t}{Z_t^*},$$
$$r_t^k = \Upsilon^t \frac{R_t^k}{P_t}, \; w_t = \frac{W_t}{P_t Z_t^*}, \; m_t = \frac{M_t}{P_t Z_t^*}, \; \tilde{p}_t = \frac{\tilde{P}_t}{P_t}, \; \tilde{w}_t = \frac{\tilde{W}_t}{W_t}, \quad (1.2.1)$$
$$\xi_t = \Xi_t Z_t^*, \; \xi_t^k = \Xi_t^k Z_t^* \Upsilon^t, \; z_t^* = \log(Z_t^*/Z_{t-1}^*),$$

Denote with $_*$ the steady state values of the variables. Realize that at st.st. $z_* = \gamma$ and $z_*^* = \gamma + \frac{\alpha}{1-\alpha}\log\Upsilon$.

1.2.1 Intermediate goods producers

We start by expressing 1.1.12 in terms of detrended variables:

$$mc_t = \frac{MC_t}{P_t} = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}w_t^{1-\alpha}r_t^{k\,\alpha}. \quad (1.2.2)$$

Hence

$$mc_* = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}w_*^{1-\alpha}r_*^{k\,\alpha}. \quad (1.2.3)$$

Expression 8.8 becomes:

$$\frac{\xi_t}{\lambda_{f,t}}\tilde{p}_t^{-\frac{(1+\lambda_{f,t})}{\lambda_{f,t}}-1}\left(\tilde{p}_t - (1+\lambda_{f,t})mc_t\right)y_t(i)$$
$$+E_t\sum_{s=1}^{\infty}\zeta_p^s\beta^s\frac{\xi_{t+s}}{\lambda_{f,t+s}}\left(\frac{\tilde{p}_t}{\Pi_{l=1}^s\pi_{t+l}}\right)^{\frac{(1+\lambda_{f,t+s})}{\lambda_{f,t+s}}-1}\left(\Pi_{l=1}^s\pi_{t+l-1}^{\iota_p}\pi_*^{1-\iota_p}\right)^{\frac{(1+\lambda_{f,t+s})}{\lambda_{f,t+s}}} \quad (1.2.4)$$
$$\left(\tilde{p}_t\frac{\Pi_{l=1}^s\pi_{t+l-1}^{\iota_p}\pi_*^{1-\iota_p}}{\Pi_{l=1}^s\pi_{t+l}} - (1+\lambda_{f,t+s})mc_{t+s}\right)y_{t+s}(i) = 0$$

this implies that:

$$\tilde{p}_* = (1+\lambda_f)\alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}w_*^{1-\alpha}r_*^{k\,\alpha} \quad (1.2.5)$$

Expression 8.9 becomes:

$$1 = [(1-\zeta_p)\tilde{p}_t^{-\frac{1}{\lambda_{f,t}}} + \zeta_p(\pi_{t-1}^{\iota_p}\pi_*^{1-\iota_p}\pi_t^{-1})^{-\frac{1}{\lambda_{f,t}}}]^{-\lambda_{f,t}}. \quad (1.2.6)$$

which means that:

$$\tilde{p}_* = 1. \quad (1.2.7)$$

Equation 1.1.11 becomes:

$$k_t = \frac{\alpha}{1-\alpha} \frac{w_t}{r_t^k} L_t.$$

(1.2.8)

and at st.st.:

$$k_* = \frac{\alpha}{1-\alpha} \frac{w_*}{r_*^k} L_*.$$

(1.2.9)

Recall that aggregate profits are equal to:

$$\Pi_t = P_t Y_t - W_t L_t - R_t^k K_t.$$

In terms of detrended variables we then have :

$$
\begin{aligned}
\frac{\Pi_t}{P_t Z_t^*} &= y_t - w_t L_t - r_t^k k_t \\
&= k_t^\alpha L_t^{1-\alpha} - \Phi - w_t L_t - \frac{\alpha}{1-\alpha} w_t L_t \\
&= \left(\left(\frac{k_t}{L_t} \right)^\alpha - \frac{1}{1-\alpha} w_t \right) L_t - \Phi \\
&= \left(\left(\frac{\alpha}{1-\alpha} \right)^\alpha w_t^\alpha r_t^{k\ -\alpha} - \frac{1}{1-\alpha} w_t \right) L_t - \Phi
\end{aligned}
$$

At steady state we can use 1.2.5 to get that st. st. profits are:

$$\frac{\Pi_t}{P_t Z_t^*} = \frac{\lambda_f}{1-\alpha} w_* L_* - \Phi.$$

(1.2.10)

1.2.2 Households

Expression 1.1.28, 1.1.29, and 1.1.30 become:

$$\xi_t = b_t (c_t - h c_{t-1} e^{-z_t^*})^{-1} - \beta h \mathbb{E}_t [b_{t+1} (c_{t+1} e^{z_{t+1}^*} - h c_t)^{-1}],$$

(1.2.11)

$$m_t^{\nu_m} = \chi_t b_t \frac{R_t}{R_t - 1} \xi_t^{-1},$$

(1.2.12)

$$\xi_t = \beta R_t \mathbb{E}_t [\xi_{t+1} e^{-z_{t+1}^*} \pi_{t+1}^{-1}],$$

(1.2.13)

respectively. At steady state:

$$\xi_* = c_*^{-1} \left((1 - h e^{-\gamma} \Upsilon^{-\frac{\alpha}{1-\alpha}})^{-1} - \beta h (e^\gamma \Upsilon^{\frac{\alpha}{1-\alpha}} - h)^{-1} \right),$$

(1.2.14)

$$m_*^{\nu_m} = \chi \frac{R_*}{R_* - 1} \xi_*^{-1},$$

(1.2.15)

$$R_* = \beta^{-1} \pi_* e^\gamma \Upsilon^{\frac{\alpha}{1-\alpha}}.$$

(1.2.16)

Equation 1.1.21 and 1.1.22 become:

$$k_t = u_t \Upsilon^{-1} e^{-z_t^*} \bar{k}_{t-1}, \tag{1.2.17}$$

$$\bar{k}_t = (1-\delta)\Upsilon^{-1} e^{-z_t^*} \bar{k}_{t-1} + \mu_t \left(1 - S(\frac{i_t}{i_{t-1}} e^{z_t^*})\right) i_t. \tag{1.2.18}$$

which deliver the steady state relationships:

$$k_* = u_* e^{-\gamma} \Upsilon^{-\frac{1}{1-\alpha}} \bar{k}_*, \tag{1.2.19}$$

$$i_* = \mu \left(1 - (1-\delta)e^{-\gamma} \Upsilon^{-\frac{1}{1-\alpha}}\right) \bar{k}_*. \tag{1.2.20}$$

under the assumption that $S(e^{\gamma} \Upsilon^{\frac{\alpha}{1-\alpha}}) = 0$.

Equation 1.1.31, 1.1.32, and 1.1.33 become:

$$\xi_t^k \mu_t^\Upsilon \left(1 - S(\frac{i_t}{i_{t-1}} e^{z_t^*}) - S'(\frac{i_t}{i_{t-1}} e^{z_t^*})\frac{i_t}{i_{t-1}} e^{z_t^*}\right)$$
$$+ \beta \mathbb{E}_t [e^{-z_{t+1}^*} \xi_{t+1}^k \mu_{t+1} S'(\frac{i_{t+1}}{i_t} e^{z_{t+1}^*})(\frac{i_{t+1}}{i_t} e^{z_{t+1}^*})^2] = \xi_t \tag{1.2.21}$$

$$\xi_t^k = \beta \mathbb{E}_t \left[\Upsilon^{-1} e^{-z_{t+1}^*} \left(\xi_{t+1}(r_{t+1}^k u_{t+1} - a(u_{t+1})) + \xi_{t+1}^k(1-\delta)\right)\right] \tag{1.2.22}$$

$$r_t^k = a'(u_t) \tag{1.2.23}$$

which deliver the steady state relationships:

$$\xi_*^k \mu \left(1 - S(e^{\gamma} \Upsilon^{\frac{\alpha}{1-\alpha}}) - S'(e^{\gamma} \Upsilon^{\frac{\alpha}{1-\alpha}})e^{\gamma} \Upsilon^{\frac{\alpha}{1-\alpha}}\right)$$
$$+ \beta e^{-\gamma} \Upsilon^{-\frac{\alpha}{1-\alpha}} \xi_*^k \mu S'(e^{\gamma} \Upsilon^{\frac{\alpha}{1-\alpha}})(e^{\gamma} \Upsilon^{\frac{\alpha}{1-\alpha}})^2 = \xi_* \tag{1.2.24}$$

$$\xi_*^k = \beta e^{-\gamma} \Upsilon^{-\frac{1}{1-\alpha}} \left(\xi_*(r_*^k u_* - a(u_*)) + \xi_*^k(1-\delta)\right) \tag{1.2.25}$$

$$r_*^k = a'(u_*) \tag{1.2.26}$$

Under the assumptions that $S'(e^{\gamma} \Upsilon^{\frac{\alpha}{1-\alpha}}) = 0$, $u_* = 1$ and $a(u_*) = 0$, the above equations become:

$$\xi_*^k \mu = \xi_* \tag{1.2.27}$$

$$r_*^k = \mu^{-1} \left(\beta^{-1} e^{\gamma} \Upsilon^{\frac{1}{1-\alpha}} - (1-\delta)\right) \tag{1.2.28}$$

$$r_*^k = a'(u_*). \tag{1.2.29}$$

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Expressed in terms of detrended variables, equation 1.1.37 becomes:

$$\mathbb{E}_t \sum_{s=0}^{\infty} (\zeta_w \beta)^s L(j)_{t+s} \xi_{t+s} \left[- \tilde{\mathcal{X}}_{t,s} \tilde{w}_t w_t + (1 + \lambda_w) \frac{b_{t+s} \varphi_{t+s} L_{t+s}(j)^{\nu_l}}{\xi_{t+s}} \right] = 0, \quad (1.2.30)$$

where

$$\tilde{\mathcal{X}}_{t,s} = \begin{cases} 1 \quad \text{if } s = 0 \\ \dfrac{\Pi_{l=1}^s (\pi_* e^\gamma \Upsilon^{\frac{\alpha}{1-\alpha}})^{1-\iota_w} (\pi_{t+l-1} e^{z_{t+l-1}^*})^{\iota_w}}{\Pi_{l=1}^s \pi_{t+l} e^{z_{t+l}^*}} \quad \text{otherwise} \end{cases}$$

and

$$L_{t+s}(j) = \left(\tilde{w}_t w_t w_{t+s}^{-1} \tilde{\mathcal{X}}_{t,s} \right)^{-\frac{1+\lambda_w}{\lambda_w}} L_{t+s}.$$

Equation 1.1.38 becomes:

$$1 = [(1-\zeta_w) \tilde{w}_t^{\frac{1}{\lambda_w}} + \zeta_w ((\pi_* e^\gamma \Upsilon^{\frac{\alpha}{1-\alpha}})^{1-\iota_w} (\pi_{t-1} e^{z_{t-1}^*})^{\iota_w} \frac{w_{t-1}}{w_t} \pi_t^{-1} e^{-z_t^*})^{\frac{1}{\lambda_w}}]^{\lambda_w}. \quad (1.2.31)$$

which imply at steady state:

$$w_* = (1 + \lambda_w) \frac{\varphi L_*^{\nu_l}}{\xi_*}, \quad (1.2.32)$$

$$\tilde{w}_* = 1. \quad (1.2.33)$$

1.2.3 Resource constraints

The resource constraint(s) become:

$$g_t(c_t + i_t + a(u_t) \bar{e}^{-z_t^*} k_{t-1}) = y_t. \quad (1.2.34)$$

and

$$\dot{y}_t = k_t^\alpha L_t^{1-\alpha} - \Phi. \quad (1.2.35)$$

$$Y_t = \left(\frac{\dot{P}_t}{P_t} \right)^{\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} \dot{Y}_t$$

becomes

$$y_t = (\dot{p}_t)^{\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} \dot{y}_t \quad (1.2.36)$$

where

$$\begin{aligned}
\dot{p}_t &= \frac{\dot{P}_t}{P_t} \\
&= [(1-\zeta_p)(\frac{\tilde{P}_t}{P_t})^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} + \zeta_p (\pi_* \frac{P_{t-1}}{P_t})^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}}]^{-\frac{\lambda_{f,t}}{1+\lambda_{f,t}}} \quad (1.2.37) \\
&= [(1-\zeta_p) \tilde{p}_t^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} + \zeta_p (\pi_* \dot{p}_{t-1} \pi_t^{-1})^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}}]^{-\frac{\lambda_{f,t}}{1+\lambda_{f,t}}}
\end{aligned}$$

While

$$L_t = \left(\frac{\dot{W}_t}{W_t}\right)^{\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} \dot{L}_t$$

becomes

$$L_t = (\dot{w}_t)^{\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} \dot{L}_t \tag{1.2.38}$$

where

$$
\begin{aligned}
\dot{w}_t \quad &= \frac{\dot{W}_t}{W_t} \\
&= [(1-\zeta_w)(\frac{\tilde{W}_t}{W_t})^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} + \zeta_w(\pi_* e^\gamma \Upsilon^{\frac{\alpha}{1-\alpha}} \frac{\dot{W}_{t-1}}{W_t})^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}}]^{-\frac{\lambda_{w,t}}{1+\lambda_{w,t}}} \\
&= [(1-\zeta_w)\tilde{w}_t^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} + \zeta_w(\pi_* e^\gamma \Upsilon^{\frac{\alpha}{1-\alpha}} \pi_t^{-1} e^{-z_t^*} \frac{w_{t-1}}{w_t} \dot{w}_{t-1})^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}}]^{-\frac{\lambda_{w,t}}{1+\lambda_{w,t}}}
\end{aligned} \tag{1.2.39}
$$

At steady state we have:

$$g_*(c_* + i_*) = y_*. \tag{1.2.40}$$

and

$$y_* = k_*^\alpha L_*^{1-\alpha} - \Phi. \tag{1.2.41}$$

and

$$\dot{y}_* = y_*, \quad \dot{L}_* = L_*.$$

1.2.4 Government Policies

The Taylor rule 1.1.39 becomes:

$$\frac{R_t}{R^*} = \left(\frac{R_{t-1}}{R^*}\right)^{\rho_R} \left[\left(\frac{\pi_t}{\pi_*}\right)^{\psi_1} \left(\frac{y_t}{y_t^*}\right)^{\psi_2}\right]^{1-\rho_R} e^{\epsilon_{R,t}} \tag{1.2.42}$$

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1.3 Steady state

Define ϕ implicitly by defining L^* (note that you can only to consider policy changes that leave L^* unchanged). From 1.2.28 (if $\mu = 1$):

$$r_*^k = \beta^{-1} e^\gamma \Upsilon^{\frac{1}{1-\alpha}} - (1 - \delta). \tag{1.2.43}$$

From 1.2.5:

$$w_* = \left(\frac{1}{1 + \lambda_f} \alpha^\alpha (1 - \alpha)^{(1-\alpha)} r_*^{k \ -\alpha} \right)^{\frac{1}{1-\alpha}} \tag{1.2.44}$$

From 1.2.9

$$k_* = \frac{\alpha}{1 - \alpha} \frac{w_*}{r_*^k} L_*. \tag{1.2.45}$$

From 1.2.41:

$$y_* = k_*^\alpha L_*^{1-\alpha} - \Phi. \tag{1.2.46}$$

From 1.2.19 and 1.2.20:

$$\bar{k}_* = e^\gamma \Upsilon^{\frac{1}{1-\alpha}} k_*, \tag{1.2.47}$$

$$i_* = \left(1 - (1 - \delta) e^{-\gamma} \Upsilon^{-\frac{1}{1-\alpha}} \right) \bar{k}_*. \tag{1.2.48}$$

Hence it follows that:

$$\delta = 1 - e^\gamma \Upsilon^{\frac{1}{1-\alpha}} (1 - \frac{i_*}{\bar{k}_*}). \tag{1.2.49}$$

From 1.2.40:

$$c_* = \frac{y_*}{g_*} - i_*. \tag{1.2.50}$$

Given π_* (objective of central bank) and r^* (rreal interest rate) we have that 1.2.14, 1.2.16, and 1.2.15 deliver:

$$\xi_* = c_*^{-1} \left((1 - he^{-\gamma} \Upsilon^{-\frac{\alpha}{1-\alpha}})^{-1} - \beta h (e^\gamma \Upsilon^{\frac{\alpha}{1-\alpha}} - h)^{-1} \right),$$

$$= c_*^{-1} (e^{z_*^*} - h)^{-1} (e^{z_*^*} - h\beta) \tag{1.2.51}$$

$$R_* = r^* \pi_* \tag{1.2.52}$$

$$\beta = \frac{1}{r_*} e^\gamma \Upsilon^{\frac{\alpha}{1-\alpha}}. \tag{1.2.53}$$

$$m_*^{\nu_m} = \chi \frac{R_*}{R_* - 1} \xi_*^{-1}. \tag{1.2.54}$$

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From 1.2.27:

$$\xi_*^k = \xi_*. \tag{1.2.55}$$

From 1.2.32:

$$\varphi = \frac{w_*\xi_*}{(1+\lambda_w)L_*^{\nu_l}}. \tag{1.2.56}$$

The definition of the labor share LS_t is $LS_t = \frac{W_tL_t}{P_tY_t}$:

$$LS_t = \frac{W_tL_t}{P_tY_t} = \frac{w_tL_t}{y_t} \tag{1.2.57}$$

In absence of fixed costs, i.e. $\mathcal{F} = 0$, at steady state we have:

$$
\begin{aligned}
LS_* &= \frac{w_*L_*}{y_*} = \frac{w_*L_*}{k_*^\alpha L_*^{1-\alpha}} = w_*\left(\frac{L_*}{k_*}\right)^\alpha \\
&= \alpha^{-\alpha}(1-\alpha)^\alpha r_*^{k\ -\alpha}w_*^{1-\alpha} = (1-\alpha)mc_* \\
&= (1-\alpha)/(1+\lambda_f)
\end{aligned}
\tag{1.2.58}
$$

The following derivations are also obtained for $\mathcal{F} = 0$. We want to get the st.st. capital output ratio in this economy. Divide 1.2.9 by output (1.2.41), and obtain:

$$
\begin{aligned}
\frac{k_*}{y_*} &= \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha}\left(\frac{w_*}{r_*^k}\right)^{1-\alpha} \\
&= \frac{\alpha}{1+\lambda_f}r_*^{k\ -1} \\
&= \frac{\alpha}{1+\lambda_f}(\beta^{-1}e^\gamma\Upsilon^{\frac{1}{1-\alpha}} - (1-\delta))^{-1}
\end{aligned}
\tag{1.2.59}
$$

where we used the st.st. values of w_* and r_*^k computed above. Hence from the definition of $\bar{k}_*$:

$$
\begin{aligned}
\frac{\bar{k}_*}{y_*} &= e^\gamma\Upsilon^{\frac{1}{1-\alpha}}\frac{\alpha}{1+\lambda_f}(\beta^{-1}e^\gamma\Upsilon^{\frac{1}{1-\alpha}} - (1-\delta))^{-1} \\
&= \frac{\alpha}{1+\lambda_f}(\beta^{-1} - e^{-\gamma}\Upsilon^{-\frac{1}{1-\alpha}}(1-\delta))^{-1}
\end{aligned}
\tag{1.2.60}
$$

1.4 Log-linearized model

Eq. 1.2.2 becomes:

$$\widehat{mc}_t = (1 - \alpha)\,\widehat{w}_t + \alpha\,\widehat{r}_t^k. \tag{1.2.61}$$

- Eq. 1.2.6 becomes:

$$\widehat{\widetilde{p}}_t = \frac{\zeta_p}{1 - \zeta_p}(\,\widehat{\pi}_t - \,\iota_p\widehat{\pi}_{t-1}). \tag{1.2.62}$$

- Eq. 1.2.4 becomes (see appendix):

$$\begin{aligned}
\widehat{\widetilde{p}}_t &= (1 - \zeta_p\beta)(1 + \lambda_f)mc_*\widehat{mc}_t + (1 - \zeta_p\beta)\lambda_f mc_*\widehat{\lambda}_{f,t} - \iota_p\zeta_p\beta\widehat{\pi}_t \\
&\quad + \zeta_p\beta\,I\!E_t[\widehat{\pi}_{t+1}] + \zeta_p\beta\,I\!E_t[\widehat{\widetilde{p}}_{t+1}]
\end{aligned} \tag{1.2.63}$$

Combining 1.2.62 with 1.2.63 we obtain:

$$\begin{aligned}
\widehat{\pi}_t &= \frac{(1 - \zeta_p\beta)(1 - \zeta_p)}{(1 + \iota_p\beta)\zeta_p}\left[\widehat{mc}_t + \frac{\lambda_f}{1 + \lambda_f}\widehat{\lambda}_{f,t}\right] \\
&\quad + \frac{\iota_p}{1 + \iota_p\beta}\widehat{\pi}_{t-1} + \frac{\beta}{1 + \iota_p\beta}I\!E_t[\widehat{\pi}_{t+1}]
\end{aligned} \tag{1.2.64}$$

Eq. 1.2.8 becomes:

$$\widehat{k}_t = \widehat{w}_t - \widehat{r}_t^k + \widehat{L}_t. \tag{1.2.65}$$

Eq. 1.2.11 becomes:

$$\begin{aligned}
(e^{z^*_*} - h\beta)(e^{z^*_*} - h)\widehat{\xi}_t = {}& e^{z^*_*}(e^{z^*_*} - h)\widehat{b}_t - (e^{2z^*_*} + \beta h^2)\widehat{c}_t \\
&+ he^{z^*_*}\widehat{c}_{t-1} - he^{z^*_*}z_t^* - \beta h(e^{z^*_*} - h)I\!E_t[\widehat{b}_{t+1}] \\
&+ \beta he^{z^*_*}I\!E_t[\widehat{c}_{t+1}] + \beta he^{z^*_*}I\!E_t[z_{t+1}^*].
\end{aligned} \tag{1.2.66}$$

Eq. 1.2.12 becomes:

$$\nu_m\widehat{m}_t = \widehat{\chi}_t + \widehat{b}_t - \frac{1}{R_* - 1}\widehat{R}_t - \widehat{\xi}_t. \tag{1.2.67}$$

Eq. 1.2.13 becomes:

$$\widehat{\xi}_t = \widehat{R}_t + I\!E_t[\widehat{\xi}_{t+1}] - I\!E_t[z_{t+1}^*] - I\!E_t[\widehat{\pi}_{t+1}]. \tag{1.2.68}$$

Eq. 1.2.17 becomes:

$$\widehat{k}_t = \widehat{u}_t - z_t^* + \widehat{\overline{k}}_{t-1}. \tag{1.2.69}$$

Eq. 1.2.18 becomes:

$$\widehat{\bar{k}}_t = -(1 - \frac{i_*}{k_*})\, z_t^* + (1 - \frac{i_*}{k_*})\, \widehat{\bar{k}}_{t-1}$$
$$+ \frac{i_*}{k_*}\mu_t + \frac{i_*}{k_*}\widehat{i}_t. \tag{1.2.70}$$

Eq. 1.2.21 becomes:

$$\frac{1}{S''e^{2z_*^*}}\widehat{\xi}_t^k + \frac{1}{S''e^{2z_*^*}}\mu_t - \frac{1}{S''e^{2z_*^*}}\widehat{\xi}_t = z_t^* - \widehat{i}_{t-1} + (1+\beta)\widehat{i}_t$$
$$- \beta\, \mathbb{E}[z_{t+1}^*] - \beta\, \mathbb{E}[\widehat{i}_{t+1}]. \tag{1.2.71}$$

Eq. 1.2.22 becomes:

$$\widehat{\xi}_t^k = -\mathbb{E}_t[z_{t+1}^*] + \frac{r_*^k}{r_*^k + (1-\delta)}\mathbb{E}_t[\xi_{t+1}] + \frac{r_*^k}{r_*^k + (1-\delta)}\mathbb{E}_t[r_{t+1}^k]$$
$$+ \frac{1-\delta}{r_*^k + (1-\delta)}\mathbb{E}_t[\xi_{t+1}^k]. \tag{1.2.72}$$

Eq. 1.2.23 becomes:

$$r_*^k\widehat{r}_t^k = a''u_t. \tag{1.2.73}$$

Eq. 1.2.30 becomes:

$$(1 + \nu_l\tfrac{1+\lambda_w}{\lambda_w})\widehat{\bar{w}}_t + (1 + \zeta_w\beta\nu_l\tfrac{1+\lambda_w}{\lambda_w})\widehat{w}_t = (1 - \zeta_w\beta)(\,\widehat{b}_t + \widehat{\varphi}_t + \nu_l\widehat{L}_t - \widehat{\xi}_t)$$
$$- \zeta_w\beta(1 + \nu_l\tfrac{1+\lambda_w}{\lambda_w})\mathbb{E}_t[\iota_w\widehat{\pi}_t + \iota_w z_t^* - \widehat{\pi}_{t+1} - \widehat{z}_{t+1}^*] + \zeta_w\beta(1 + \nu_l\tfrac{1+\lambda_w}{\lambda_w})\mathbb{E}_t\left[\widehat{\bar{w}}_{t+1} + \widehat{w}_{t+1}\right] \tag{1.2.74}$$

Eq. 1.2.31 becomes:

$$\widehat{w}_t = \widehat{w}_{t-1} - \widehat{\pi}_t - z_t^* + \iota_w\widehat{\pi}_{t-1} + \iota_w z_{t-1}^* + \frac{1 - \zeta_w}{\zeta_w}\widehat{\bar{w}}_t. \tag{1.2.75}$$

Substituting $\widehat{\bar{w}}_t$ from 1.2.75 into 1.2.74 we obtain:

$$\widehat{w}_t - \widehat{w}_{t-1} + \widehat{\pi}_t + \widehat{z}_t^* - \iota_w\widehat{\pi}_{t-1} - \iota_w z_{t-1}^* = \frac{(1-\zeta_w)}{\zeta_w}\frac{1-\zeta_w\beta}{(1+\nu_l\frac{1+\lambda_w}{\lambda_w})}\left(\,\widehat{b}_t + \widehat{\varphi}_t + \nu_l\widehat{L}_t - \widehat{\xi}_t - \widehat{w}_t\right)$$
$$+ \beta\,\mathbb{E}_t\left[\widehat{w}_{t+1} - \widehat{w}_t + \widehat{\pi}_{t+1} + \widehat{z}_{t+1}^* - \iota_w\widehat{\pi}_t - \iota_w z_t^*\right] \tag{1.2.76}$$

where $\widehat{w}_t - \widehat{w}_{t-1} + \widehat{\pi}_t + \widehat{z}_t^*$ is nominal wage inflation. Eq. 1.2.34 becomes:

$$\widehat{y}_t = \widehat{g}_t + \frac{c_*}{c_* + i_*}\widehat{c}_t + \frac{i_*}{c_* + i_*}\widehat{i}_t + \frac{r_*^k k_*}{c_* + i_*}\widehat{u}_t. \tag{1.2.77}$$

Eq. 1.2.35 becomes (remember $\widehat{y}_t = \widehat{\bar{y}}_t$):

$$\widehat{y}_t = \alpha\frac{y_* + \Phi}{y_*}\widehat{k}_t + (1-\alpha)\frac{y_* + \Phi}{y_*}\widehat{L}_t \tag{1.2.78}$$

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Eq. 1.2.42 becomes:

$$\widehat{R}_t = \rho_R \widehat{R}_{t-1} + (1 - \rho_R)(\psi_1 \widehat{\pi}_t + \psi_2 \widehat{y}_t) + \epsilon_{R,t} \tag{1.2.79}$$

In absence of fixed costs, i.e. $\mathcal{F} = 0$, log-deviations the labor share equals marginal costs in terms of log deviations from steady state:

$$
\begin{aligned}
\widehat{LS}_t &= \widehat{w}_t + \widehat{L}_t - \widehat{y}_t \\
&= \widehat{w}_t + \widehat{L}_t - \alpha \widehat{k}_t - (1-\alpha)\widehat{L}_t \\
&= \widehat{w}_t - \alpha(\widehat{k}_t - \widehat{L}_t) \\
&= \widehat{w}_t + \alpha(\widehat{r}_{k,t} - \widehat{w}_t) \\
&= \widehat{mc}_t
\end{aligned} \tag{1.2.80}
$$

1.5 Measurement

Output growth (log differences, quarter-to-quarter, in %):

$$100 \times (\ln Y_t - \ln Y_{t-1}) = 100 \times (\ln y_t + \ln Z_t^* - \ln y_{t-1} - \ln Z_{t-1}^*)$$
$$= 100 \times (\hat{y}_t + \ln y^* - \hat{y}_{t-1} - \ln y^* + z_t^*) \qquad (1.2.81)$$
$$= 100 \times (\hat{y}_t - \hat{y}_{t-1} + \hat{z}_t^*) + 100\gamma + 100\frac{\alpha}{1-\alpha}\ln \Upsilon$$

where $\hat{z}_t^* = z_t^* - \gamma - \frac{\alpha}{1-\alpha}\ln \Upsilon$ and is modeled in the transition equation.

Consumption growth (log differences, quarter-to-quarter, in %):

$$100 \times (\ln C_t - \ln C_{t-1}) = 100 \times (\hat{c}_t - \hat{c}_{t-1} + \hat{z}_t^*) + 100\gamma + 100\frac{\alpha}{1-\alpha}\ln \Upsilon \qquad (1.2.82)$$

Investment growth (log differences, quarter-to-quarter, in %):

$$100 \times (\ln I_t - \ln I_{t-1}) = 100 \times (\hat{i}_t - \hat{i}_{t-1} + \hat{z}_t^*) + 100\gamma + 100\frac{\alpha}{1-\alpha}\ln \Upsilon \qquad (1.2.83)$$

Hours worked (log):
$$\ln L_t = \hat{L}_t + \ln L^* + \ln L^{adj} \qquad (1.2.84)$$

Nominal wage growth (log differences, quarter-to-quarter, in %):

$$100 \times (\ln W_t - \ln W_{t-1}) = 100 \times (\ln w_t + \ln w_{t-1} + z_t^* + \ln P_t - \ln P_{t-1})$$
$$= 100 \times (\hat{w}_t - \hat{w}_{t-1} + \hat{z}_t^* + \hat{\pi}_t) + 100 * \ln \pi_* + 100\gamma + 100\frac{\alpha}{1-\alpha}\ln \Upsilon$$
$$(1.2.85)$$

Inflation (quarter-to-quarter, in %):

$$100 \times (\ln P_t - \ln P_{t-1}) = 100 \ln \pi_t$$
$$= 100\hat{\pi}_t + 100 \ln \pi_*. \qquad (1.2.86)$$

Nominal M2 growth (log differences, quarter-to-quarter, in %):

$$100 \times (\ln M_t - \ln M_{t-1}) = 100 \times (\ln m_t + \ln m_{t-1} + z_t^* + \ln P_t - \ln P_{t-1})$$
$$= 100 \times (\hat{m}_t - \hat{m}_{t-1} + \hat{z}_t^* + \hat{\pi}_t) + 100 * \ln \pi_* + 100\gamma + 100\frac{\alpha}{1-\alpha}\ln \Upsilon$$
$$(1.2.87)$$

Nominal interest rate (annualized, in %):

$$400 \times (\ \ln R_t\ ) \quad = 4 \times 100 \hat{R}_t + 400 * \ln R^*. \tag{1.2.88}$$

Cointegrating relationships.

Log consumption - Log output (in %):

$$100 \times (\ \ln C_t - \ln Y_t\ ) \quad = 100 \times (\ \hat{c}_t - \hat{y}_t\ ) + 100(\ln c^* - \ln y^*) \tag{1.2.89}$$

Log investment - Log output (in %):

$$100 \times (\ \ln I_t - \ln Y_t\ ) \quad = 100 \times (\ \hat{i}_t - \hat{y}_t\ ) + 100(\ln i^* - \ln y^*) \tag{1.2.90}$$

Log nominal wage - Log output - Log Price (in %):

$$100 \times (\ \ln W_t - \ln Y_t - \ln P_t\ ) \quad = 100 \times (\ \hat{w}_t - \hat{y}_t\ ) + 100(\ln w^* - \ln y^*) \tag{1.2.91}$$

Log M2 - Log output - Log Price (in %):

$$100 \times (\ \ln M_t - \ln Y_t - \ln P_t\ ) \quad = 100 \times (\ \hat{m}_t - \hat{y}_t\ ) + 100(\ln m^* - \ln y^*) \tag{1.2.92}$$

Note that the transition equation has no constant. So we can rescale all theˆvariables by 100, and correspondingly make sure that the standard deviations of the exogenous shocks are measured in %.

2 Log-linearization of Eq. 1.2.4

Log-linearization of Eq. 1.2.4, which is reproduced here:

$$\xi_t \left(\tilde{p}_t - (1+\lambda_{f,t})mc_t\right) y_t(i)$$
$$+ \mathbb{E}_t \sum_{s=1}^{\infty} \zeta_p^s \beta^s \xi_{t+s} \left(\tilde{p}_t \frac{\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p}}{\Pi_{l=1}^s \pi_{t+l}} - (1+\lambda_{f,t+s})mc_{t+s} \right) y_{t+s}(i) \quad = 0 \tag{2.1}$$

Note that at st.st. the term within $(\dots)$ (namely $\tilde{p}_t \frac{\pi_*^s}{\Pi_{l=1}^s \pi_{t+l}} - (1+\lambda_{f,t+s})mc_{t+s}$) is equal to 0, so we need not bother with all the terms outside the parenthesis and we can set them to their st.st values. Call $d\ln x_t = \widehat{x}_t$. Now note that:

$$\frac{\partial \text{lhs}}{\partial \ln \tilde{p}_t} = \widehat{\tilde{p}}_t + \left(\sum_{s=1}^{\infty} \zeta_p^s \beta^s\right) \widehat{\tilde{p}}_t = \frac{1}{1-\zeta_p\beta}\widehat{\tilde{p}}_t$$

$$\frac{\partial \text{lhs}}{\partial \ln mc_t} = -(1+\lambda_f)mc_* \widehat{mc}_t$$

$$\frac{\partial \text{lhs}}{\Pi_{j=1,2,..} \partial \ln mc_{t+j}} = -(1+\lambda_f)mc_* \left(\sum_{s=1}^{\infty} \zeta_p^s \beta^s \mathbb{E}_t[\widehat{mc}_{t+s}]\right)$$

$$\frac{\partial \text{lhs}}{\partial \ln \lambda_{f,t}} = -\lambda_f mc_* \widehat{\lambda}_{f,t}$$

$$\frac{\partial \text{lhs}}{\Pi_{j=1,2,..} \partial \ln \lambda_{f,t+j}} = -\lambda_f mc_* \left(\sum_{s=1}^{\infty} \zeta_p^s \beta^s \mathbb{E}_t[\widehat{\lambda}_{f,t+s}]\right)$$

$$\frac{\partial \text{lhs}}{\Pi_{j=1,2,..} \partial \ln \pi_{t+j}} = -\left(\sum_{s=1}^{\infty} \zeta_p^s \beta^s \mathbb{E}_t[\sum_{l=1}^s (\widehat{\pi}_{t+l} - \iota_p \widehat{\pi}_{t+l-1})]\right)$$

Putting all together we get:

$$\widehat{\tilde{p}}_t = (1-\zeta_p\beta)(1+\lambda_f)mc_* \widehat{mc}_t + (1-\zeta_p\beta)\lambda_f mc_* \widehat{\lambda}_{f,t}$$
$$+ (1-\zeta_p\beta) \sum_{s=1}^{\infty} \zeta_p^s \beta^s \left[(1+\lambda_f)mc_* \mathbb{E}_t[\widehat{mc}_{t+s}] + \lambda_f mc_* \mathbb{E}_t[\widehat{\lambda}_{f,t+s}] + \mathbb{E}_t[\sum_{l=1}^s (\widehat{\pi}_{t+l} - \iota_p \widehat{\pi}_{t+l-1})]\right]$$
$$= (1-\zeta_p\beta)(1+\lambda_f)mc_* \widehat{mc}_t + (1-\zeta_p\beta)\lambda_f mc_* \widehat{\lambda}_{f,t} + \zeta_p\beta(\mathbb{E}_t[\widehat{\pi}_{t+1}] - \iota_p \widehat{\pi}_t)$$
$$+ \zeta_p\beta \mathbb{E}_t \Big[(1-\zeta_p\beta)(1+\lambda_f)mc_* \widehat{mc}_{t+1} + (1-\zeta_p\beta)\lambda_f mc_* \widehat{\lambda}_{f,t+1}$$
$$+ (1-\zeta_p\beta) \sum_{s=1}^{\infty} \zeta_p^s \beta^s \Big[(1+\lambda_f)\beta mc_* \mathbb{E}_{t+1}[\widehat{mc}_{t+1+s}] + \lambda_f mc_* \mathbb{E}_{t+1}[\widehat{\lambda}_{f,t+1+s}]$$
$$+ \mathbb{E}_{t+1}[\sum_{l=1}^s (\widehat{\pi}_{t+1+l} - \iota_p \widehat{\pi}_{t+l})] \Big] \Big]$$

or

$$\widehat{\tilde{p}}_t = (1-\zeta_p\beta)(1+\lambda_f)mc_* \widehat{mc}_t + (1-\zeta_p\beta)\lambda_f mc_* \widehat{\lambda}_{f,t}$$
$$+ \zeta_p\beta \mathbb{E}_t[\widehat{\pi}_{t+1}] + \zeta_p\beta \mathbb{E}_t[\widehat{\tilde{p}}_{t+1}] \tag{2.2}$$

3 Log-linearization of Eq. 1.2.30

Log-linearization of Eq. 1.2.30, which is reproduced here:

$$\mathbb{E}_t \sum_{s=0}^{\infty} (\zeta_w\beta)^s L_{t+s}(j)\xi_{t+s} \left[-\tilde{\mathcal{X}}_{t,s}\tilde{w}_t w_t + (1+\lambda_w)\frac{b_{t+s}\varphi_{t+s}L_{t+s}(j)^{\nu_l}}{\xi_{t+s}} \right] = 0, \tag{3.1}$$

where

$$\tilde{\mathcal{X}}_{t,s} = \begin{cases} 1 \ \text{ if } s = 0 \\ \dfrac{\Pi_{l=1}^{s}(\pi_* e^{\gamma}\Upsilon^{\frac{\alpha}{1-\alpha}})^{1-\iota_w}(\pi_{t+l-1}e^{z_{t+l-1}^*})^{\iota_w}}{\Pi_{l=1}^{s}\pi_{t+l}e^{z_{t+l}^*}} \ \text{ otherwise.} \end{cases}$$

At st.st. the term within $[\dots]$ (namely $-\tilde{\mathcal{X}}_{t,s}\tilde{w}_t w_t + (1+\lambda_w)\dfrac{b_{t+s}\varphi_{t+s}L_{t+s}(j)^{\nu_l}}{\xi_{t+s}}$) is equal to 0, so we need not bother with all the terms outside the parenthesis and we can set them to their st.st values. Loglinearizing:

$$\mathbb{E}_t \sum_{s=0}^{\infty}(\zeta_w\beta)^s \Big[-w_*\widehat{\tilde{w}}_t - w_*\widehat{w}_t - w_*\sum_{l=1}^{s}\big(\iota_w\widehat{\pi}_{t+l-1} + \iota_w\widehat{\pi}_{t+l-1} - \widehat{\pi}_{t+l} - \widehat{z}_{t+l}^*\big)$$
$$+ w_*\big(\widehat{b}_{t+s} + \widehat{\varphi}_{t+s} + \nu_l\widehat{L}_{t+s}(j) - \widehat{\xi}_{t+s}\big)\Big] = 0,$$

(3.2)

Realize that in terms of detrended variables:

$$L_{t+s}(j) = \Big(\tilde{w}_t w_t w_{t+s}^{-1}\tilde{\mathcal{X}}_{t,s}\Big)^{-\frac{1+\lambda_w}{\lambda_w}} L_{t+s},$$

hence

$$\widehat{L}_{t+s}(j) = -\frac{1+\lambda_w}{\lambda_w}\Big(\widehat{\tilde{w}}_t + \widehat{w}_t - \widehat{w}_{t+s} + \sum_{l=1}^{s}(\iota_w\widehat{\pi}_{t+l-1} + \iota_w\widehat{z}_{t+l-1}^* - \widehat{\pi}_{t+l} - \widehat{z}_{t+l}^*)\Big) + \widehat{L}_{t+s}.$$

Substituting in 3.2 we obtain:

$$\frac{1}{1-\zeta_w\beta}(1+\nu_l\frac{1+\lambda_w}{\lambda_w})(\widehat{\tilde{w}}_t + \widehat{w}_t) = \widehat{b}_t + \widehat{\varphi}_t + \nu_l\widehat{L}_t - \widehat{\xi}_t + \nu_l\frac{1+\lambda_w}{\lambda_w}\widehat{w}_t$$
$$+\mathbb{E}_t\sum_{s=1}^{\infty}(\zeta_w\beta)^s\Big[-(1+\nu_l\frac{1+\lambda_w}{\lambda_w})\sum_{l=1}^{s}\big(\iota_w\widehat{\pi}_{t+l-1} + \iota_w\widehat{z}_{t+l-1}^* - \widehat{\pi}_{t+l} - \widehat{z}_{t+l}^*\big)$$
$$+\widehat{b}_{t+s} + \widehat{\varphi}_{t+s} + \nu_l\widehat{L}_{t+s} - \widehat{\xi}_{t+s} + \nu_l\frac{1+\lambda_w}{\lambda_w}\widehat{w}_{t+s}\Big]$$
$$\frac{1}{1-\zeta_w\beta}(1+\nu_l\frac{1+\lambda_w}{\lambda_w})(\widehat{\tilde{w}}_t + \widehat{w}_t) = \widehat{b}_t + \widehat{\varphi}_t + \nu_l\widehat{L}_t - \widehat{\xi}_t + \nu_l\frac{1+\lambda_w}{\lambda_w}\widehat{w}_t$$
$$-\frac{\zeta_w\beta}{1-\zeta_w\beta}(1+\nu_l\frac{1+\lambda_w}{\lambda_w})\mathbb{E}_t[\iota_w\widehat{\pi}_t + \iota_w z_t^* - \widehat{\pi}_{t+1} - \widehat{z}_{t+1}^*]$$
$$+\zeta_w\beta\mathbb{E}_t\Big[\widehat{b}_{t+1} + \widehat{\varphi}_{t+1} + \nu_l\widehat{L}_{t+1} - \widehat{\xi}_{t+1}) + \nu_l\frac{1+\lambda_w}{\lambda_w}\widehat{w}_{t+1} + \sum_{s=1}^{\infty}(\zeta_w\beta)^s[-\sum_{l=1}^{s}\big(\iota_w\widehat{\pi}_{t+1+l-1} + \dots\big)$$
$$+\widehat{b}_{t+1+s} + \dots]\Big]$$

or

$$(1+\nu_l\frac{1+\lambda_w}{\lambda_w})\widehat{\tilde{w}}_t + (1+\zeta_w\beta\nu_l\frac{1+\lambda_w}{\lambda_w})\widehat{w}_t = (1-\zeta_w\beta)(\widehat{b}_t + \widehat{\varphi}_t + \nu_l\widehat{L}_t - \widehat{\xi}_t)$$
$$-\zeta_w\beta(1+\nu_l\frac{1+\lambda_w}{\lambda_w})\mathbb{E}_t[\iota_w\widehat{\pi}_t + \iota_w z_t^* - \widehat{\pi}_{t+1} - \widehat{z}_{t+1}^*] + \zeta_w\beta(1+\nu_l\frac{1+\lambda_w}{\lambda_w})\mathbb{E}_t\Big[\widehat{\tilde{w}}_{t+1} + \widehat{w}_{t+1}\Big].$$

(3.3)

This expression can be further simplified as:

$$\widehat{\tilde{w}}_t = \frac{(1-\zeta_w\beta)}{(1+\nu_l\frac{1+\lambda_w}{\lambda_w})}\big(\widehat{b}_t + \widehat{\varphi}_t + \nu_l\widehat{L}_t - \widehat{\xi}_t - \widehat{w}_t)$$
$$+\zeta_w\beta\mathbb{E}_t\Big[\widehat{\tilde{w}}_{t+1} + \widehat{w}_{t+1} - \widehat{w}_t + \widehat{\pi}_{t+1} + \widehat{z}_{t+1}^* - \iota_w\widehat{\pi}_t - \iota_w z_t^*\Big].$$

(3.4)

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4 The flexible price/wage version of the model

In the flexible price/wage version of the model $\zeta^p = \zeta^w = 0$.

- The price-setting problem of the intermediate good producer under flexible prices is:

$$\max_{\tilde{P}_t(i)} \left(\tilde{P}_t(i) - MC_t \right) Y_t(i)$$
$$\text{s.t. } Y_t(i) = \left(\frac{\tilde{P}_t(i)}{P_t} \right)^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} Y_t, \tag{4.1}$$

The FOC becomes:

$$\tilde{\Xi}_t^p \left(\frac{\tilde{P}_t(i)}{P_t} \right)^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}-1} \frac{1}{\lambda_{f,t}P_t} \left(\tilde{P}_t(i) - (1+\lambda_{f,t})MC_t \right) Y_t(i) = 0. \tag{4.2}$$

This affects the equilibrium conditions as follows. Equation 8.41 becomes:

$$\tilde{p}_t = (1 + \lambda_{f,t})mc_t, \tag{4.3}$$

and expression 1.2.6 becomes:

$$1 = \tilde{p}_t, \tag{4.4}$$

which implies:

$$1 = (1 + \lambda_{f,t})mc_t, \tag{4.5}$$

- The nominal interest rate and money need not be introduced, hence we can skip condition 1.2.13 .Same applies to 1.2.12.

- The wage-setting problem of the workers under flexible wages is:

$$\max_{\tilde{W}_t(j)} \left[-\frac{\varphi_t}{\nu_l+1}L_t(j)^{\nu_l+1} + \dots \right] \text{s.t. } 1.1.20 \text{ and } 1.1.34 \text{ for } s = 0. \tag{4.6}$$

The FOC becomes:

$$\tilde{w}_t w_t = (1 + \lambda_w)\frac{b_t\varphi_t L_t^{\nu_l}}{\xi_t}, \tag{4.7}$$

and equation 1.2.31 becomes:

$$1 = \tilde{w}_t, \tag{4.8}$$

which together imply:

$$w_t = (1 + \lambda_w)\frac{b_t\varphi_t L_t^{\nu_l}}{\xi_t}. \tag{4.9}$$

The steady state is unchanged. The log-linearized conditions are modified as follows:

- Eq. 1.2.64 drops out and is replaced by:

$$0 = (1 + \lambda_f)\widehat{mc}_t + \lambda_f \widehat{\lambda}_{f,t} \tag{4.10}$$

- Eq. 1.2.67, 1.2.68, and 1.2.79 drop out.

- Expressions 1.2.74 and 1.2.75 both drop out and are replaced by:

$$\widehat{w}_t = \widehat{b}_t + \widehat{\varphi}_t + \nu_l \widehat{L}_t - \widehat{\xi}_t. \tag{4.11}$$

5 Normalizations

We redefine the shocks as follows:

$$\tilde{\lambda}_{f,t} = \frac{(1 - \zeta_p)(1 - \beta\zeta_p)\lambda_f}{1 + \lambda_f}\widehat{\lambda}_{f,t} \tag{5.1}$$

$$\tilde{\mu}_t = \frac{1}{(1 + \beta)e^{2z_*^*}S''}\widehat{\mu}_t \tag{5.2}$$

$$\tilde{b}_t = \frac{(e^{z_*^*} - h)e^{z_*^*}}{(e^{2z_*^*} + h^2\beta)}\widehat{b}_t \tag{5.3}$$

$$\tilde{\varphi}_t = (1 - \zeta_w\beta)\widehat{\varphi}_t \tag{5.4}$$

$$\tilde{\chi}_t = \frac{1}{\nu_m}\widehat{\chi}_t \tag{5.5}$$

6 Introducing Capital Producers (decentralizing the investment decision)

In this section we decentralize the investment decision by introducing capital producers who buy goods, transform them into installed capital, and sell it back to the households at a price Q_t^k. We will see that Q_t^k is Tobin's Q – that is, the value of installed capital in terms of consumption, which previously was equal to Ξ_t^k/Ξ_t. The household's problem is the same as before except that now they do not decide about investment, but only on how much capital to buy from the capital goods producers.

The **household**'s budget constraint, written in nominal terms, is given by:

$$
\begin{aligned}
P_{t+s}C_{t+s}(j) + B_{t+s}(j) + M_{t+s}(j) &\leq R_{t+s}B_{t+s-1}(j) + M_{t+s-1}(j) \\
&+ \Pi_{t+s} + W_{t+s}(j)L_{t+s}(j) + \left(R^k_{t+s}u_{t+s}(j) - P_{t+s}a(u_{t+s}(j))\Upsilon^{-t}\right)\bar{K}_{t+s-1}(j) \\
&+ P_{t+s}Q^k_{t+s}\left((1-\delta)\bar{K}_{t+s-1}(j) - \bar{K}_{t+s}(j)\right),
\end{aligned}
\tag{6.1}
$$

where Q^k_t is the price of capital in terms of consumption goods. Note that households at the beginning of period t (but after the realization of the shocks) sell undepreciated capital from the previous period $((1-\delta)\bar{K}_{t-1}(j))$ to capital producers and at the end of the period purchase the new stock of capital $\bar{K}_t(j)$. Their FOC wrt $\bar{K}_{t+s}(j)$ are:

$$
(\partial\bar{K}_t) \quad \Xi_t Q^k_t = \beta E_t[\Xi_{t+1}(\frac{R^k_{t+1}}{P_{t+1}}u_{t+1} - a(u_{t+1})\Upsilon^{-(t+1)}) + \Xi_{t+1}Q^k_{t+1}(1-\delta)].
\tag{6.2}
$$

Note that this FOC is identical to 1.1.32 if we replace Q^k_t with Ξ^k_t/Ξ_t.

Capital Producers produce new capital by transforming general output, which they buy from final goods producers, into new capital via the technology:

$$
x' = x + \Upsilon^t\mu_t\left(1 - S(\frac{I_t}{I_{t-1}})\right)I_t.
\tag{6.3}
$$

where x is the initial capital purchased from households at the beginning of the period, and x' is the new stock of capital, which they sell back to households at the end of the period. Their period profits are therefore given by:

$$
\begin{aligned}
ll\Pi^k_t &= Q^k_t x' - Q^k_t x - I_t \\
&= Q^k_t \Upsilon^t\mu_t\left(1 - S(\frac{I_t}{I_{t-1}})\right)I_t - I_t.
\end{aligned}
\tag{6.4}
$$

Note that these profits do not depend on the initial level of capital x purchased, so effectively the only decision variable for capital producers is I_t. Since they discount profits using the households' discount rate $\beta^t\Xi_t$, their FOC wrt I_t are:

$$
\begin{aligned}
(\partial I_t) \quad & \Xi_t Q^k_t \Upsilon^t\mu_t\left(1 - S(\frac{I_t}{I_{t-1}}) - S'(\frac{I_t}{I_{t-1}})\frac{I_t}{I_{t-1}}\right) \\
& + \beta E_t[\Xi_{t+1}Q^k_{t+1}\Upsilon^{t+1}\mu_{t+1}S'(\frac{I_{t+1}}{I_t})(\frac{I_{t+1}}{I_t})^2] = \Xi_t
\end{aligned}
\tag{6.5}
$$

Note that this FOC is identical to 1.1.31 if we replace Q^k_t with Ξ^k_t/Ξ_t.

7 Adding BGG-type financial frictions as in Christiano, Motto, Rostagno

7.1 Households

The objective function for household j is unchanged (expression 1.1.16). The household's problem is different as households no longer hold the capital stock, and make investment and capital utilization decisions. Rather, they invest in deposits to the banking sector D_t (in addition to government bonds and money), which pay a gross nominal interest rate R_t^d. Household j's budget constraint is:

$$P_{t+s}C_{t+s}(j) + B_{t+s}(j) + D_{t+s}(j) + M_{t+s}(j) \leq R_{t+s}B_{t+s-1}(j) + R_{t+s}^d D_{t+s-1}(j) + M_{t+s-1}(j)$$
$$+ \Pi_{t+s} + W_{t+s}(j)L_{t+s}(j) + Tr_{t+s}, \tag{7.1}$$

where the term Tr_{t+s} represents transfers from the entrepreneurs, which we will discuss later. Households' first order conditions for consumption, money holdings, bonds, and wages, are unchanged. [deposits FOC?]

7.2 Capital Producers

There is a representative, competitive, capital producer who produces new capital by transforming general output – which is bought from final goods producers at the nominal price Q_t^k – into new capital via the technology:

$$x' = x + \Upsilon^t \mu_t \left(1 - S(\frac{I_t}{I_{t-1}})\right) I_t. \tag{7.2}$$

where x is the initial capital purchased from entrepreneurs in period t, and x' is the new stock of capital, which they sell back to entrepreneurs at the end of the same period. Their period profits, expressed in terms of consumption goods, are therefore given by:

$$\begin{aligned}
\Pi_t^k &= \frac{Q_t^k}{P_t}x' - \frac{Q_t^k}{P_t}x - I_t \\
&= \frac{Q_t^k}{P_t}\Upsilon^t \mu_t \left(1 - S(\frac{I_t}{I_{t-1}})\right) I_t - I_t.
\end{aligned} \tag{7.3}$$

Note that these profits do not depend on the initial level of capital x purchased, so effectively the only decision variable for capital producers is I_t. Since they discount profits using the households' discount rate $\beta^t \Xi_t$, their FOC wrt I_t are:

$$(\partial I_t) \qquad \Xi_t \frac{Q_t^k}{P_t} \Upsilon^t \mu_t \left(1 - S(\frac{I_t}{I_{t-1}}) - S'(\frac{I_t}{I_{t-1}})\frac{I_t}{I_{t-1}} \right)$$

$$+ \beta \mathbb{E}_t [\Xi_{t+1} \frac{Q_{t+1}^k}{P_{t+1}} \Upsilon^{t+1} \mu_{t+1} S'(\frac{I_{t+1}}{I_t})(\frac{I_{t+1}}{I_t})^2] = \Xi_t \qquad (7.4)$$

Note that this FOC is identical to 1.1.31 if we replace $\frac{Q_t^k}{P_t}$ with Ξ_t^k/Ξ_t.

7.3 Entrepreneurs

There is a continuum of entrepreneurs indexed by e. Each entrepreneur buys installed capital $\bar{K}_{t-1}(e)$ from the capital producers at the end of period $t-1$ using her own net worth $N_{t-1}(e)$ and a loan $B_{t-1}^d(e)$ from the banking sector:

$$Q_{t-1}^k \bar{K}_{t-1}(e) = B_{t-1}^d(e) + N_{t-1}(e)$$

where net worth is expressed in nominal terms. In the next period she rents capital out to firms, earning a rental rate R_t^k per unit of effective capital. In period t she is subject to an i.i.d. (across entrepreneurs and over time) shock $\omega(e)_t$ that increases or shrinks her capital, where $log\omega(e)_t \sim N(m_{\omega,t-1}, \sigma_{\omega,t-1}^2)$ where $m_{\omega,t-1}$ is such that $\mathbb{E}\omega(e)_t = 1$. Denote by $F_{t-1}(\omega)$ the cumulative distribution function of ω at time t, where the distribution needs to be known at time $t-1$. In addition, after observing the shock she can choose a level of utilization $u(e)_t$ by paying a cost in terms of general output equal to $a(u(e)_t)\Upsilon^{-t}$ per-unit-of-capital. At the end of period t the entrepreneurs sells undepreciated capital to the capital producers. Entrepreneurs' revenues in period t are therefore:

$$\left\{ R_t^k u(e)_t + (1-\delta)Q_t^k - P_t a(u(e)_t)\Upsilon^{-t} \right\} \omega(e)_t \bar{K}(e)_{t-1}$$

or equivalently

$$\omega(e)_t \tilde{R}^k(e)_t Q_{t-1}^k \bar{K}(e)_{t-1}$$

where

$$\tilde{R}^k(e)_t = \frac{R_t^k u(e)_t + (1-\delta)Q_t^k - P_t a(u(e)_t)\Upsilon^{-t}}{Q_{t-1}^k} \qquad (7.5)$$

is the gross nominal return to capital for entrepreneurs. From the profit function it is clear that the choice of the utilization rate is independent from the amount of capital purchased or the ω shock, and is given by the FOC:

$$\frac{R_t^k}{P_t} = a'(u(e)_t)\Upsilon^{-t}, \tag{7.6}$$

which is the same condition as 1.1.33. Consequently we can drop the index from the return $\tilde{R}_t^k$.

The debt contract undertaken by the entrepreneur in period $t-1$ consists of the triplet $(B^d(e)_{t-1}, R^d(e)_t, \bar{\omega}(e)_t)$ where $R_t^d(e)$ represents the contractual interest rate, and $\bar{\omega}(e)_t$ the theshold level of $\omega(e)_t$ below which the entrepreneur cannot pay back, which is therefore defined by the equation:

$$\bar{\omega}(e)_t \tilde{R}_t^k Q_{t-1}^k \bar{K}(e)_{t-1} = R^d(e)_t B^d(e)_{t-1}. \tag{7.7}$$

For $\omega(e)_t < \bar{\omega}(e)_t$ the bank monitors the entrepreneurs and extracts a fraction $(1-\mu_t^e)$ of its revenues $\tilde{R}_t^k Q_{t-1}^k \bar{K}(e)_{t-1}$, where μ_t^e represents exogenous bankrupcty costs. The bank's zero profit condition implies that [state by state?]:

$$[1 - F_{t-1}(\bar{\omega}(e)_t)] R^d(e)_t B^d(e)_{t-1} + (1-\mu_{t-1}^e) \int_0^{\bar{\omega}(e)_t} \omega dF_{t-1}(\omega) \tilde{R}_t^k Q_{t-1}^k \bar{K}(e)_{t-1} = R_{t-1} B^d(e)_{t-1}$$

where R_{t-1} is the rate paid by the bank to the depositors. If we define leverage as:

$$\varrho(e)_t \equiv \frac{B^d(e)_t}{N(e)_t},$$

use the definitions

$$\begin{aligned}\Gamma_{t-1}(\bar{\omega}_t) &\equiv \bar{\omega}_t \left[1 - F_{t-1}(\bar{\omega}_t)\right] + G_{t-1}(\bar{\omega}_t) \\ G_{t-1}(\bar{\omega}_t) &\equiv \int_0^{\bar{\omega}_t} \omega dF_{t-1}(\omega),\end{aligned}$$

as well as the definiton of $\bar{\omega}(e)_t$, the zero-profit condition can be rewritten as:

$$\left[\Gamma_{t-1}(\bar{\omega}(e)_t) - \mu_{t-1}^e G_{t-1}(\bar{\omega}(e)_t)\right] \frac{\tilde{R}_t^k}{R_{t-1}}(1 + \varrho(e)_{t-1}) = \varrho(e)_{t-1}. \tag{7.8}$$

Entrepreneurs' expected profits (before the realization of the shock ω_t) can be written as:

$$\begin{aligned}&\int_{\bar{\omega}(e)_t}^\infty \left[\omega(e)_t \tilde{R}^k(e)_t Q_{t-1}^k \bar{K}(e)_{t-1} - R^d(e)_t B^d(e)_{t-1}\right] dF_{t-1}(\omega(e)_t) = \\ &\left[\int_{\bar{\omega}(e)_t}^\infty \omega(e)_t dF_{t-1}(\omega(e)_t) - \bar{\omega}(e)_t[1 - F_{t-1}(\bar{\omega}(e)_t)]\right] \tilde{R}^k(e)_t Q_{t-1}^k \bar{K}(e)_{t-1} = \\ &[1 - \Gamma_{t-1}(\bar{\omega}(e)_t)] \frac{\tilde{R}_t^k}{R_{t-1}} [1 + \varrho(e)_{t-1}] R_{t-1} N(e)_{t-1}\end{aligned}$$

The contract that maximizes expected net worth for the entrepreneurs is given by:

$$\max_{\{\varrho(e)_{t-1},\bar{\omega}(e)_t\}} E_{t-1}\left[\begin{array}{c}[1-\Gamma_{t-1}(\bar{\omega}(e)_t)]\frac{\tilde{R}_t^k}{R_{t-1}}\left[1+\varrho(e)_{t-1}\right]R_{t-1}N(e)_{t-1}\\ +\eta_t\left\{\left[\Gamma_{t-1}(\bar{\omega}(e)_t)-\mu_{t-1}^e G_{t-1}(\bar{\omega}(e)_t)\right]\frac{\tilde{R}_t^k}{R_{t-1}}\left[1+\varrho(e)_{t-1}\right]-\varrho(e)_{t-1}\right\}\end{array}\right]$$

so that the FOCs are:

$$\begin{aligned}\varrho(e)_{t-1}:\quad 0 &= E_{t-1}\left[\left[1-\Gamma_{t-1}(\bar{\omega}(e)_t)\right]\frac{\tilde{R}_t^k}{R_{t-1}}\right]R_{t-1}N(e)_{t-1}\\ &\quad + E_{t-1}\left[\eta_t\left\{\left[\Gamma_{t-1}(\bar{\omega}(e)_t)-\mu_{t-1}^e G_{t-1}(\bar{\omega}(e)_t)\right]\frac{\tilde{R}_t^k}{R_{t-1}}-1\right\}\right]\\ \bar{\omega}(e)_t:\quad \eta_t &= \frac{\Gamma'_{t-1}(\bar{\omega}(e)_t)}{\Gamma'_{t-1}(\bar{\omega}(e)_t)-\mu_{t-1}^e G'_{t-1}(\bar{\omega}(e)_t)}R_{t-1}N(e)_{t-1}\end{aligned}$$

Substituting the second FOC into the first we obtain:

$$E_{t-1}\left[\left[1-\Gamma_{t-1}(\bar{\omega}_t)\right]\frac{\tilde{R}_t^k}{R_{t-1}}+\frac{\Gamma'_{t-1}(\bar{\omega}_t)}{\Gamma'_{t-1}(\bar{\omega}_t)-\mu_{t-1}^e G'_{t-1}(\bar{\omega}_t)}\left\{\left[\Gamma_{t-1}(\bar{\omega}_t)-\mu_{t-1}^e G_{t-1}(\bar{\omega}_t)\right]\frac{\tilde{R}_t^k}{R_{t-1}}-1\right\}\right]=0.$$
(7.9)

where we omit the the indicator (e) since the condition implies that $\bar{\omega}(e)_t$ only depends on aggregate variables and is the same across entrepreneurs. From the zero profits condition 7.8 this implies that leverage $\varrho(e)_{t-1}$ is also the same, hence we can rewrite 7.8 as a function of aggregate variables only:

$$\left[\Gamma_{t-1}(\bar{\omega}_t)-\mu_{t-1}^e G_{t-1}(\bar{\omega}_t)\right]\frac{\tilde{R}_t^k}{R_{t-1}}=\frac{Q_{t-1}^k\bar{K}_{t-1}-N_{t-1}}{Q_{t-1}^k\bar{K}_{t-1}}.$$
(7.10)

Aggregate entrepreneurs' equity evolves according to:

$$\begin{aligned}V_t &= \int_{\bar{\omega}_t}^{\infty}\omega_t\tilde{R}_t^k Q_{t-1}^k\bar{K}(e)_{t-1}dF_{t-1}(\omega_t)-[1-F_{t-1}(\bar{\omega}_t)]R^d(e)_t B^d(e)_{t-1}\\ &= \tilde{R}_t^k Q_{t-1}^k\bar{K}_{t-1}-\left[R_{t-1}+\mu_{t-1}^e G_{t-1}\left(\bar{\omega}_t\right)\tilde{R}_t^k\frac{Q_{t-1}^k\bar{K}_{t-1}}{Q_{t-1}^k\bar{K}_{t-1}-N_{t-1}}\right]\left(Q_{t-1}^k\bar{K}_{t-1}-N_{t-1}\right).\end{aligned}$$
(7.11)

A fraction $1-\gamma_t$ of entrepreneurs exits the economy and fraction γ_t survives to continue operating for another period. A fraction Θ of the total net worth owned by exiting entrepreneurs is consumed upon exit and the remaining fraction of their networth is transfered as a lump sum to the households. Each period new entrepreneurs enter and receive a net worth transfer W_t^e. Because W_t^e is small, this exit and entry process ensures that entrepreneurs do not accumulate enough net worth to escape the financial frictions. Aggregate entrepreneurs' net worth evolves accordingly as:

$$N_t = \gamma_t V_t + W_t^e.$$
(7.12)

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7.4 Detrending and steady state

We detrend the additional variables introduced by this extension as follows:

$$q_t^k = \frac{Q_t^k}{P_t}\Upsilon^t, \ n_t = \frac{N_t}{P_t Z_t^*}, \ v_t = \frac{V_t}{P_t Z_t^*}, \ w_t^e = \frac{W_t^e}{P_t Z_t^*}. \tag{7.13}$$

All other variables are detrended as in 8.38. Expressions 7.4, 7.5, 7.8, 7.11, and 7.12 become

$$\xi_t q_t^k \mu_t \left(1 - S(\frac{i_t}{i_{t-1}}e^{z_t^*}) - S'(\frac{i_t}{i_{t-1}}e^{z_t^*})\frac{i_t}{i_{t-1}}e^{z_t^*}\right)$$
$$+ \beta \mathbb{E}_t[e^{-z_{t+1}^*}\xi_{t+1}q_{t+1}^k \mu_{t+1} S'(\frac{i_{t+1}}{i_t}e^{z_{t+1}^*})(\frac{i_{t+1}}{i_t}e^{z_{t+1}^*})^2] = \xi_t \tag{7.14}$$

$$\tilde{R}_t^k = \frac{r_t^k u_t + (1-\delta)q_t^k - a(u_t)}{q_{t-1}^k \Upsilon}\pi_t \tag{7.15}$$

$$r_t^k = a'(u_t) \tag{7.16}$$

$$\bar{\omega}_t \tilde{R}_t^k = R_t^d \frac{q_{t-1}^k \bar{k}_{t-1} - n_{t-1}}{q_{t-1}^k \bar{k}_{t-1}} \tag{7.17}$$

$$\left[\Gamma_{t-1}(\bar{\omega}_t) - \mu_{t-1}^e G_{t-1}(\bar{\omega}_t)\right]\frac{\tilde{R}_t^k}{R_{t-1}} = \frac{q_{t-1}^k \bar{k}_{t-1} - n_{t-1}}{q_{t-1}^k \bar{k}_{t-1}} \tag{7.18}$$

$$v_t e^{z_t^*}\pi_t = \tilde{R}_t^k q_{t-1}^k \bar{k}_{t-1} - \left[R_{t-1} + \mu_{t-1}^e G_{t-1}(\bar{\omega}_t)\tilde{R}_t^k \frac{q_{t-1}^k \bar{k}_{t-1}}{q_{t-1}^k \bar{k}_{t-1} - n_{t-1}}\right](q_{t-1}^k \bar{k}_{t-1} - n_{t-1}) \tag{7.19}$$

$$n_t = \gamma_t v_t + w_t^e. \tag{7.20}$$

Expression 7.9 is already expressed in terms of detrended variables.

The steady state relationships are:

$$\xi_* q_*^k \mu \left(1 - S(e^\gamma \Upsilon^{\frac{\alpha}{1-\alpha}}) - S'(e^\gamma \Upsilon^{\frac{\alpha}{1-\alpha}})e^\gamma \Upsilon^{\frac{\alpha}{1-\alpha}}\right)$$
$$+ \beta e^{-\gamma}\Upsilon^{-\frac{\alpha}{1-\alpha}}\xi_* q_*^k \mu S'(e^\gamma \Upsilon^{\frac{\alpha}{1-\alpha}})(e^\gamma \Upsilon^{\frac{\alpha}{1-\alpha}})^2 = \xi_* \tag{7.21}$$

which implies since $S(.) = S'(.) = 0$ at steady state that $q_*^k = 1$. We also parameterize a(.) so that $u_* = 1$ and $a(u_*) = 0$. With this information, and after some simplification, we can rewrite the remaining steady state equations as

$$\frac{\tilde{R}_*^k}{\pi_*} = \frac{r_*^k + (1-\delta)}{\Upsilon} \tag{7.22}$$

$$\frac{\tilde{R}_*^k}{R_*} = \Psi(\bar{\omega}_*, \sigma_{\omega*}, \mu_*^e) \tag{7.23}$$

$$\frac{n_*}{\bar{k}_*} = 1 - [\Gamma_*(\bar{\omega}_*) - \mu_*^e G_*(\bar{\omega}_*)] \frac{\tilde{R}_*^k}{R_*} \tag{7.24}$$

$$\left(1 - \gamma_* \beta^{-1}\right) \frac{n_*}{\bar{k}_*} = \gamma_* \beta^{-1} \left\{ \frac{\tilde{R}_*^k}{R_*} [1 - \mu_*^e G_*(\bar{\omega}_*)] - 1 \right\} + \frac{w_*^e}{\bar{k}_*} \tag{7.25}$$

$$v_* = \gamma_*^{-1} (n_* - w_*^e). \tag{7.26}$$

with

$$
\begin{aligned}
\Psi\left(\bar{\omega}_*, \sigma_{\omega*}, \mu_*^e\right) &\equiv \frac{\Gamma_*'(\bar{\omega}_*)}{[1 - \Gamma_*(\bar{\omega}_*)][\Gamma_*'(\bar{\omega}_*) - \mu_*^e G_*'(\bar{\omega}_*)] + \Gamma_*'(\bar{\omega}_*)[\Gamma_*(\bar{\omega}_*) - \mu_*^e G_*(\bar{\omega}_*)]} \\
&= \frac{1}{1 - \mu_*^e \frac{G_*'(\bar{\omega}_*)}{\Gamma_*'(\bar{\omega}_*)} [1 - \Gamma_*(\bar{\omega}_*)] - \mu_*^e G_*(\bar{\omega}_*)}
\end{aligned}
\tag{7.27}
$$

Our strategy for computing the steady state is going to be the following: find a solution for the real return to capital $\frac{\tilde{R}_*^k}{\pi_*}$ and use 7.22 to find r_*^k:

$$r_*^k = \Upsilon \frac{\tilde{R}_*^k}{\pi_*} - (1 - \delta). \tag{7.28}$$

Once we have r_*^k we can proceed exactly as in section 1.3 to find the steady state for the other variables. Recall that from the Euler equation the steady state real rate is given by:

$$\frac{R_*}{\pi_*} = \beta^{-1} e^{z_*^*}.$$

In absence of financial friction $\frac{R_*}{\pi_*}$ and $\frac{\tilde{R}_*^k}{\pi_*}$ would be identical, but frictions induce a spread between the two, which we will compute subsequently as a function of the primitives in the economy ($\sigma_{\omega,*}^2$, μ_*^e, γ_*, w_*^e).

We solve for the steady state according to the following steps:

1. Set

$$F_*(\bar{\omega}_*) = \bar{F}_* \tag{7.29}$$

   and define

$$z_*^\omega \equiv \frac{\ln \bar{\omega}_* + \frac{1}{2}\sigma_{\omega*}^2}{\sigma_{\omega*}} = \Phi^{-1}(\bar{F}_*) \tag{7.30}$$

   which we can use to write

$$\bar{\omega}(\sigma_{\omega*}) = \exp\left\{ \sigma_{\omega*} z_*^\omega - \frac{1}{2}\sigma_{\omega*}^2 \right\} \tag{7.31}$$

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2. Given the value for the spread for debt contracts, R_*^d/R_*, we can use equation (7.17) to write

$$\frac{\tilde{R}_*^k}{R_*} = \frac{R_*^d}{R_*} \frac{1 - \frac{n_*}{q_*^k \bar{k}_*}}{\bar{\omega}_*} \tag{7.32}$$

(Note: this second step can be skipped if instead we calibrate/estimate $\tilde{R}_*^k/R_*$ directly.)

3. Given $\tilde{R}_*^k/R_*$, we can use (7.23) to write

$$
\begin{aligned}
\left(\frac{\tilde{R}_*^k}{R_*}\right)^{-1} &= 1 - \mu_*^e \frac{G_*'(\bar{\omega}_*)}{\Gamma_*'(\bar{\omega}_*)} \left[1 - \Gamma_*(\bar{\omega}_*)\right] - \mu_*^e G_*(\bar{\omega}_*) \\
&= 1 - \mu_*^e \left\{ \frac{G_*'(\bar{\omega}_*)}{\Gamma_*'(\bar{\omega}_*)} \left[1 - \Gamma_*(\bar{\omega}_*)\right] + G_*(\bar{\omega}_*) \right\}
\end{aligned}
$$

which we can use to set

$$\mu_*^e(\sigma_{\omega*}) = \frac{1 - \left(\frac{\tilde{R}_*^k}{R_*}\right)^{-1}}{\frac{G_*'(\bar{\omega}_*)}{\Gamma_*'(\bar{\omega}_*)} \left[1 - \Gamma_*(\bar{\omega}_*)\right] + G_*(\bar{\omega}_*)}$$

and plugging in the exact expressions we get

$$\mu_*^e(\sigma_\omega) = \frac{1 - \left(\frac{\tilde{R}_*^k}{R_*}\right)^{-1}}{\frac{1}{\sigma_{\omega*}} \frac{\phi(z_*^\omega)}{1 - \bar{F}_*} \left\{1 - \Phi(z_*^\omega - \sigma_{\omega*}) - \bar{\omega}_*\left(1 - \bar{F}_*\right)\right\} + \Phi(z_*^\omega - \sigma_{\omega*})} \tag{7.33}$$

4. Given the above and equation (7.24) we get

$$\frac{n_*}{\bar{k}_*}(\sigma_\omega) = 1 - \left\{\bar{\omega}_*\left[1 - \bar{F}_*\right] + (1 - \mu_*^e)\Phi(z_*^\omega - \sigma_{\omega*})\right\}\frac{\tilde{R}_*^k}{R_*} \tag{7.34}$$

5. Given the elasticity of the spread w.r.t. leverage, $\zeta_{sp,b}$, derived below in equation (7.42), we get the following expression

$$\frac{\frac{\frac{1 - \Phi(z_*^\omega - \sigma_{\omega*})}{1 - \bar{F}_*} - \bar{\omega}_*}{\bar{\omega}_* + (1 - \mu_*^e)\frac{\Phi(z_*^\omega - \sigma_{\omega*})}{1 - \bar{F}_*}}\left[1 - \frac{\mu_*^e}{\sigma_{\omega*}}\frac{\phi(z_*^\omega)}{1 - \bar{F}_*}\right] + 1}{\frac{\mu_*^e}{\bar{\omega}_*\sigma_{\omega*}^2}\frac{\frac{\phi(z_*^\omega)}{1 - \bar{F}_*} - z_*^\omega}{\left[1 - \frac{\mu_*^e}{\sigma_{\omega*}}\frac{\phi(z_*^\omega)}{1 - \bar{F}_*}\right]^2}\frac{\phi(z_*^\omega)}{\left(1 - \bar{F}_*\right)^2}}\frac{\frac{\tilde{R}_*^k}{R_*}}{\frac{n_*}{\bar{k}_*}} = 1 - \frac{\zeta_{sp,b}}{\left(\frac{n_*}{\bar{k}_*}\right)^{-1} - 1} \tag{7.35}$$

which we can solve for $\sigma_{\omega*}$. Once we find this value we can plug back into the previous expressions, that depend on $\sigma_{\omega*}$.

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6. Given γ_*, and using equation (7.25) we get

$$\frac{w_*^e}{\overline{k}_*} = \left(1 - \gamma_* \beta^{-1}\right) \frac{n_*}{\overline{k}_*} - \gamma_* \beta^{-1} \left\{ \frac{\tilde{R}_*^k}{R_*} \left[1 - \mu_*^e \Phi \left(z_*^\omega - \sigma_{\omega *}\right)\right] - 1 \right\} \tag{7.36}$$

and from equation (7.26)

$$\frac{v_*}{\overline{k}_*} = \gamma_*^{-1} \left(\frac{n_*}{\overline{k}_*} - \frac{w_*^e}{\overline{k}_*}\right) \tag{7.37}$$

7. We get r_*^k using equation (7.22) to write

$$r_*^k = \Upsilon \frac{\tilde{R}_*^k}{\pi_*} - (1 - \delta) \tag{7.38}$$

7.5   Log-linearization

Log-linearization of the FOC w.r.t. leverage (expression 7.9) yields:

$$0 = E_t \left(\widehat{\tilde{R}}_{t+1}^k - \hat{R}_t\right) + \zeta_{b,\bar{\omega}} E_t \widehat{\bar{\omega}}_{t+1} + \zeta_{b,\sigma_\omega} \hat{\sigma}_{\omega,t} + \zeta_{b,\mu^e} \hat{\mu}_t^e \tag{7.39}$$

with

$$\zeta_{b,x} \equiv \frac{\frac{\partial}{\partial x} \left[\left\{[1 - \Gamma(\bar{\omega})] + \frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu^e G'(\bar{\omega})} \left[\Gamma(\bar{\omega}) - \mu^e G(\bar{\omega})\right]\right\} \frac{\tilde{R}_*^k}{R_*} - \frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu^e G'(\bar{\omega})}\right]}{\left\{[1 - \Gamma_*(\bar{\omega}_*)] + \frac{\Gamma'_*(\bar{\omega}_*)}{\Gamma'_*(\bar{\omega}_*) - \mu_*^e G'_*(\bar{\omega}_*)} \left[\Gamma_*(\bar{\omega}_*) - \bar{\mu}^e G_*(\bar{\omega}_*)\right]\right\} \frac{\tilde{R}_*^k}{R_*}} \, x$$

defined for $x \in \{\bar{\omega}, \sigma_\omega^2, \mu^e\}$. Log-linearization of the zero profit condition (expression 7.18) yields:

$$\widehat{\tilde{R}}_t^k - \hat{R}_{t-1} + \zeta_{z,\bar{\omega}} \widehat{\bar{\omega}}_t + \zeta_{z,\sigma_\omega} \hat{\sigma}_{\omega,t-1} + \zeta_{z,\mu^e} \hat{\mu}_{t-1}^e = - \left(\varrho_*\right)^{-1} \left(\hat{n}_{t-1} - \hat{q}_{t-1}^k - \widehat{\overline{k}}_{t-1}\right) \tag{7.40}$$

with

$$\zeta_{z,x} \equiv \frac{\frac{\partial}{\partial x} \left[\Gamma(\bar{\omega}) - \mu^e G(\bar{\omega})\right]}{\Gamma_*(\bar{\omega}_*) - \mu_*^e G_*(\bar{\omega}_*)} \, x \tag{7.41}$$

defined for $x \in \{\bar{\omega}, \sigma_\omega^2, \mu^e\}$. We can further write

$$\widehat{\bar{\omega}}_t = -\frac{1}{\zeta_{z,\bar{\omega}} \varrho_*} \left(\hat{n}_{t-1} - \hat{q}_{t-1}^k - \widehat{\overline{k}}_{t-1}\right) - \frac{1}{\zeta_{z,\bar{\omega}}} \left(\widehat{\tilde{R}}_t^k - \hat{R}_{t-1} + \zeta_{z,\sigma_\omega} \hat{\sigma}_{\omega,t-1} + \zeta_{z,\mu^e} \hat{\mu}_{t-1}^e\right)$$

and plug this expression into 7.39 to obtain:

$$\begin{aligned}
0 &= E_t \left[\widehat{\tilde{R}}_{t+1}^k - \hat{R}_t\right] + \zeta_{b,\sigma_\omega} \hat{\sigma}_{\omega,t} + \zeta_{b,\mu^e} \hat{\mu}_t^e \\
&\quad - \frac{\zeta_{b,\bar{\omega}}}{\zeta_{z,\bar{\omega}}} \left[\frac{1}{\varrho_*} \left(\hat{n}_t - \hat{q}_t^k - \widehat{\overline{k}}_t\right) + E_t \left[\widehat{\tilde{R}}_{t+1}^k - \hat{R}_t\right] + \zeta_{z,\sigma_\omega} \hat{\sigma}_{\omega,t} + \zeta_{z,\mu^e} \hat{\mu}_t^e\right]
\end{aligned}$$

hence

$$E_t \left[\widehat{\tilde{R}}_{t+1}^k - \hat{R}_t \right] = \zeta_{sp,b} \left(\hat{q}_t^k + \widehat{\bar{k}}_t - \hat{n}_t \right) + \zeta_{sp,\sigma_\omega} \hat{\sigma}_{\omega,t} + \zeta_{sp,\mu^e} \hat{\mu}_t^e \tag{7.42}$$

where

$$\zeta_{sp,b} \equiv -\frac{\frac{\zeta_{b,\bar{\omega}}}{\zeta_{z,\bar{\omega}}}}{1 - \frac{\zeta_{b,\bar{\omega}}}{\zeta_{z,\bar{\omega}}}} \frac{1}{\varrho_*}$$

$$\zeta_{sp,\sigma_\omega} \equiv \frac{\frac{\zeta_{b,\bar{\omega}}}{\zeta_{z,\bar{\omega}}} \zeta_{z,\sigma_\omega} - \zeta_{b,\sigma_\omega}}{1 - \frac{\zeta_{b,\bar{\omega}}}{\zeta_{z,\bar{\omega}}}}$$

$$\zeta_{sp,\mu^e} \equiv \frac{\frac{\zeta_{b,\bar{\omega}}}{\zeta_{z,\bar{\omega}}} \zeta_{z,\mu^e} - \zeta_{b,\mu^e}}{1 - \frac{\zeta_{b,\bar{\omega}}}{\zeta_{z,\bar{\omega}}}}$$

Log-linearization of the expression 7.20, characterizing net worth, yields:

$$\hat{n}_t = \gamma_* \frac{v_*}{n_*} \left(\hat{\gamma}_t + \hat{v}_t \right) + \frac{w_*^e}{n_*} \hat{w}_t^e. \tag{7.43}$$

Log-linearization of the expression 7.19, characterizing the evolution of entrepreneurial equity, is

$$\begin{aligned}
\hat{v}_t &= -\hat{z}_t - \beta^{-1} \frac{\bar{k}_* - n_*}{v_*} \left(\hat{R}_{t-1} - \pi_t \right) + \frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}} \frac{\bar{k}_*}{v_*} (1 - \mu_*^e G_* (\bar{\omega}_*)) \left(\widehat{\tilde{R}}_t^k - \pi_t \right) + \beta^{-1} \frac{n_*}{v_*} \hat{n}_{t-1} \\
&\quad \left(\frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}} (1 - \mu_*^e G_* (\bar{\omega}_*)) - \beta^{-1} \right) \frac{\bar{k}_*}{v_*} \left(\hat{q}_{t-1}^k + \widehat{\bar{k}}_{t-1} \right) \\
&\quad - \mu_*^e G_* (\bar{\omega}_*) \frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}} \frac{\bar{k}_*}{v_*} \left[\hat{\mu}_{t-1}^e + \zeta_{G,\bar{\omega}} \widehat{\bar{\omega}}_t + \zeta_{G,\sigma_\omega} \hat{\sigma}_{\omega,t-1} \right].
\end{aligned} \tag{7.44}$$

Plugging in the expression for $\widehat{\bar{\omega}}_t$ we obtain

$$\begin{aligned}
\hat{v}_t &= -\hat{z}_t - \beta^{-1} \frac{\bar{k}_* - n_*}{v_*} \left(\hat{R}_{t-1} - \pi_t \right) + \frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}} \frac{\bar{k}_*}{v_*} (1 - \mu_*^e G_* (\bar{\omega}_*)) \left(\widehat{\tilde{R}}_t^k - \pi_t \right) + \beta^{-1} \frac{n_*}{v_*} \hat{n}_{t-1} \\
&\quad \left(\frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}} (1 - \mu_*^e G_* (\bar{\omega}_*)) - \beta^{-1} \right) \frac{\bar{k}_*}{v_*} \left(\hat{q}_{t-1}^k + \widehat{\bar{k}}_{t-1} \right) - \mu_*^e G_* (\bar{\omega}_*) \frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}} \frac{\bar{k}_*}{v_*} \left[\hat{\mu}_{t-1}^e + \zeta_{G,\sigma_\omega} \hat{\sigma}_{\omega,t-1} \right] \\
&\quad - \mu_*^e G_* (\bar{\omega}_*) \frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}} \frac{\bar{k}_*}{v_*} \zeta_{G,\bar{\omega}} \left[-\frac{1}{\zeta_{z,\bar{\omega}} \varrho_*} \left(\hat{n}_{t-1} - \hat{q}_{t-1}^k - \widehat{\bar{k}}_{t-1} \right) \right. \\
&\quad \left. -\frac{1}{\zeta_{z,\bar{\omega}}} \left(\widehat{\tilde{R}}_t^k - \hat{R}_{t-1} + \zeta_{z,\sigma_\omega} \hat{\sigma}_{\omega,t-1} + \zeta_{z,\mu^e} \hat{\mu}_{t-1}^e \right) \right]
\end{aligned}$$

Collecting terms yields

$$\begin{aligned}
\hat{v}_t &= -\hat{z}_t + \zeta_{v,\tilde{R}^k} \left(\widehat{\tilde{R}}_t^k - \pi_t \right) - \zeta_{v,R} \left(\hat{R}_{t-1} - \pi_t \right) + \zeta_{v,qK} \left(\hat{q}_{t-1}^k + \widehat{\bar{k}}_{t-1} \right) + \zeta_{v,n} \hat{n}_{t-1} \\
&\quad - \zeta_{v,\mu^e} \hat{\mu}_{t-1}^e - \zeta_{v,\sigma_\omega} \hat{\sigma}_{\omega,t-1}
\end{aligned} \tag{7.45}$$

with

$$\zeta_{v,\tilde{R}^k} \equiv \frac{\tilde{R}_*^k}{\pi_* e^{z_*}} \frac{\bar{k}_*}{v_*} \left[1 - \mu_*^e G_* \left(\bar{\omega}_* \right) \left(1 - \frac{\zeta_{G,\bar{\omega}}}{\zeta_{z,\bar{\omega}}} \right) \right]$$

$$\zeta_{v,R} \equiv \beta^{-1} \frac{\bar{k}_*}{v_*} \left[1 - \frac{n_*}{\bar{k}_*} + \mu_*^e G_* \left(\bar{\omega}_* \right) \frac{\tilde{R}_*^k}{R_*} \frac{\zeta_{G,\bar{\omega}}}{\zeta_{z,\bar{\omega}}} \right]$$

$$\zeta_{v,qK} \equiv \frac{\tilde{R}_*^k}{\pi_* e^{z_*}} \frac{\bar{k}_*}{v_*} \left[1 - \mu_*^e G_* \left(\bar{\omega}_* \right) \left(1 - \frac{\zeta_{G,\bar{\omega}}}{\zeta_{z,\bar{\omega}} \varrho_*} \right) \right] - \beta^{-1} \frac{\bar{k}_*}{v_*}$$

$$\zeta_{v,n} \equiv \beta^{-1} \frac{n_*}{v_*} + \frac{\tilde{R}_*^k}{\pi_* e^{z_*}} \frac{\bar{k}_*}{v_*} \mu_*^e G_* \left(\bar{\omega}_* \right) \frac{\zeta_{G,\bar{\omega}}}{\zeta_{z,\bar{\omega}} \varrho_*}$$

$$\zeta_{v,\mu^e} \equiv \mu_*^e G_* \left(\bar{\omega}_* \right) \frac{\tilde{R}_*^k}{\pi_* e^{z_*}} \frac{\bar{k}_*}{v_*} \left(1 - \zeta_{G,\bar{\omega}} \frac{\zeta_{z,\mu^e}}{\zeta_{z,\bar{\omega}}} \right)$$

$$\zeta_{v,\sigma_\omega} \equiv \mu_*^e G_* \left(\bar{\omega}_* \right) \frac{\tilde{R}_*^k}{\pi_* e^{z_*}} \frac{\bar{k}_*}{v_*} \zeta_{G,\bar{\omega}} \left(1 - \frac{\zeta_{z,\sigma_\omega}}{\zeta_{z,\bar{\omega}}} \right)$$

Finally, substituting this expression into 7.43 we get:

$$\begin{aligned}
\hat{n}_t &= \gamma_* \frac{v_*}{n_*} \hat{\gamma}_t + \frac{w_*^e}{n_*} \hat{w}_t^e - \gamma_* \frac{v_*}{n_*} \hat{z}_t \\
&+ \zeta_{n,\tilde{R}^k} \left(\widehat{\tilde{R}}_t^k - \pi_t \right) - \zeta_{n,R} \left(\hat{R}_{t-1} - \pi_t \right) + \zeta_{n,qK} \left(\hat{q}_{t-1}^k + \widehat{\bar{k}}_{t-1} \right) + \zeta_{n,n} \hat{n}_{t-1} \\
&- \zeta_{n,\mu^e} \hat{\mu}_{t-1}^e - \zeta_{n,\sigma_\omega} \hat{\sigma}_{\omega,t-1}
\end{aligned}$$

$$(7.46)$$

with

$$\zeta_{n,\tilde{R}^k} \equiv \gamma_* \frac{\tilde{R}_*^k}{\pi_* e^{z_*}} (1 + \varrho_*) \left[1 - \mu_*^e G_* \left(\bar{\omega}_* \right) \left(1 - \frac{\zeta_{G,\bar{\omega}}}{\zeta_{z,\bar{\omega}}} \right) \right]$$

$$\zeta_{n,R} \equiv \gamma_* \beta^{-1} (1 + \varrho_*) \left[1 - \frac{n_*}{\bar{k}_*} + \mu_*^e G_* \left(\bar{\omega}_* \right) \frac{\tilde{R}_*^k}{R_*} \frac{\zeta_{G,\bar{\omega}}}{\zeta_{z,\bar{\omega}}} \right]$$

$$\zeta_{n,qK} \equiv \gamma_* \frac{\tilde{R}_*^k}{\pi_* e^{z_*}} (1 + \varrho_*) \left[1 - \mu_*^e G_* \left(\bar{\omega}_* \right) \left(1 - \frac{\zeta_{G,\bar{\omega}}}{\zeta_{z,\bar{\omega}} \varrho_*} \right) \right] - \gamma_* \beta^{-1} (1 + \varrho_*)$$

$$\zeta_{n,n} \equiv \gamma_* \beta^{-1} + \gamma_* \frac{\tilde{R}_*^k}{\pi_* e^{z_*}} (1 + \varrho_*) \mu_*^e G_* \left(\bar{\omega}_* \right) \frac{\zeta_{G,\bar{\omega}}}{\zeta_{z,\bar{\omega}} \varrho_*}$$

$$\zeta_{n,\mu^e} \equiv \gamma_* \mu_*^e G_* \left(\bar{\omega}_* \right) \frac{\tilde{R}_*^k}{\pi_* e^{z_*}} (1 + \varrho_*) \left(1 - \zeta_{G,\bar{\omega}} \frac{\zeta_{z,\mu^e}}{\zeta_{z,\bar{\omega}}} \right)$$

$$\zeta_{n,\sigma_\omega} \equiv \gamma_* \mu_*^e G_* \left(\bar{\omega}_* \right) \frac{\tilde{R}_*^k}{\pi_* e^{z_*}} (1 + \varrho_*) \zeta_{G,\bar{\omega}} \left(1 - \frac{\zeta_{z,\sigma_\omega}}{\zeta_{z,\bar{\omega}}} \right)$$

Now normalize the shocks,

$$\tilde{\sigma}_{\omega,t} \equiv \zeta_{sp,\sigma_\omega} \hat{\sigma}_{\omega,t} \tag{7.47}$$

$$\tilde{\mu}_t^e \equiv \zeta_{sp,\mu^e} \hat{\mu}_t^e \tag{7.48}$$

$$\tilde{\gamma}_t \equiv \gamma_* \frac{v_*}{n_*} \hat{\gamma}_t \tag{7.49}$$

so that the relevant log-linear equations, (7.42) and (7.46), become:

$$E_t \left[\widehat{\tilde{R}}_{t+1}^k - \hat{R}_t \right] = \zeta_{sp,b} \left(\hat{q}_t^k + \widehat{\bar{k}}_t - \hat{n}_t \right) + \tilde{\sigma}_{\omega,t} + \tilde{\mu}_t^e \tag{7.50}$$

and

$$\begin{aligned}
\hat{n}_t &= \zeta_{n,\tilde{R}^k} \left(\widehat{\tilde{R}}_t^k - \pi_t \right) - \zeta_{n,R} \left(\hat{R}_{t-1} - \pi_t \right) + \zeta_{n,qK} \left(\hat{q}_{t-1}^k + \widehat{\bar{k}}_{t-1} \right) + \zeta_{n,n} \hat{n}_{t-1} \\
&+ \tilde{\gamma}_t + \frac{w_*^e}{n_*} \hat{w}_t^e - \gamma_* \frac{v_*}{n_*} \hat{z}_t - \frac{\zeta_{n,\mu^e}}{\zeta_{sp,\mu^e}} \tilde{\mu}_{t-1}^e - \frac{\zeta_{n,\sigma_\omega}}{\zeta_{sp,\sigma_\omega}} \tilde{\sigma}_{\omega,t-1}
\end{aligned} \tag{7.51}$$

Log-linearization of 7.15 and 7.14 yield:

$$\widehat{R}_t^k - \pi_t = \frac{r_*^k}{r_*^k + (1-\delta)}\hat{r}_t^k + \frac{(1-\delta)}{r_*^k + (1-\delta)}\hat{q}_t^k - \hat{q}_{t-1}^k, \tag{7.52}$$

and

$$\frac{1}{S''e^{2z_*}}\hat{q}_t^k + \frac{1}{S''e^{2z_*}}\mu_t = z_t^* - \widehat{i}_{t-1} + (1+\beta)\widehat{i}_t - \beta I\!E[z_{t+1}^*] - \beta I\!E[\widehat{i}_{t+1}]. \tag{7.53}$$

7.6 Log-linear distribution

Consider

$$\ln \omega \sim N\left(m_\omega, \sigma_\omega^2\right) \tag{7.54}$$

which has the properties

$$E\left[\omega\right] = e^{m_\omega + \frac{1}{2}\sigma_\omega^2} \tag{7.55}$$

In order to get $E\left[\omega\right] = 1$ we need to set

$$m_\omega = -\frac{1}{2}\sigma_\omega^2 \tag{7.56}$$

This implies that the pdf is

$$f\left(\omega\right) = \frac{1}{\omega\sigma_\omega\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln \omega + \frac{1}{2}\sigma_\omega^2}{\sigma_\omega}\right)^2} \tag{7.57}$$

The CDF is

$$F\left(\bar{\omega}\right) = \Phi\left(\frac{\ln \bar{\omega} + \frac{1}{2}\sigma_\omega^2}{\sigma_\omega}\right) \tag{7.58}$$

Further notice that

$$\phi\left(z\right) \equiv \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2} \tag{7.59}$$

$$\Phi\left(z\right) \equiv \int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx \tag{7.60}$$

for which we can use matlab functions `normpdf` and `normcdf`. We also need the following expression

$$z = \Phi^{-1}\left(\bar{F}\right) \tag{7.61}$$

for which we can use an inverse cdf function also available in matlab as `norminv`.

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The partial expectation obeys

$$E\left[\omega|\omega > \bar{\omega}\right] = \Phi\left(\frac{\frac{1}{2}\sigma_\omega^2 - \ln\bar{\omega}}{\sigma_\omega}\right) = 1 - \Phi\left(\frac{\ln\bar{\omega} - \frac{1}{2}\sigma_\omega^2}{\sigma_\omega}\right)$$

which implies that

$$
\begin{aligned}
G\left(\bar{\omega}\right) &\equiv \int_0^{\bar{\omega}} \omega f\left(\omega\right) d\omega = \int_0^\infty \omega f\left(\omega\right) d\omega - \int_{\bar{\omega}}^\infty \omega f\left(\omega\right) d\omega \\
&= \Phi\left(\frac{\ln\bar{\omega} - \frac{1}{2}\sigma_\omega^2}{\sigma_\omega}\right)
\end{aligned}
\tag{7.62}
$$

Finally we define

$$
\begin{aligned}
\Gamma\left(\bar{\omega}\right) &\equiv \int_0^{\bar{\omega}} \omega f\left(\omega\right) d\omega + \bar{\omega}\int_{\bar{\omega}}^\infty f\left(\omega\right) d\omega \\
&= \bar{\omega}\left[1 - \Phi\left(\frac{\ln\bar{\omega} + \frac{1}{2}\sigma_\omega^2}{\sigma_\omega}\right)\right] + G\left(\bar{\omega}\right)
\end{aligned}
\tag{7.63}
$$

If we define

$$z^\omega \equiv \frac{\ln\bar{\omega} + \frac{1}{2}\sigma_\omega^2}{\sigma_\omega} \tag{7.64}$$

then we get

$$G\left(\bar{\omega}\right) = \Phi\left(z^\omega - \sigma_\omega\right) \tag{7.65}$$

and

$$\Gamma\left(\bar{\omega}\right) = \bar{\omega}\left[1 - \Phi\left(z^\omega\right)\right] + \Phi\left(z^\omega - \sigma_\omega\right) \tag{7.66}$$

In order to compute the derivatives, first notice that we can write

$$\phi\left(z^\omega - \sigma_\omega\right) = \bar{\omega}\phi\left(z^\omega\right) \tag{7.67}$$

and

$$\phi'\left(z\right) = -z\phi\left(z\right), \ \forall z \tag{7.68}$$

Using this result we can write the derivatives as follows:

$$G'\left(\bar{\omega}\right) = \frac{1}{\sigma_\omega}\phi\left(z^\omega\right) \tag{7.69}$$

$$G''\left(\bar{\omega}\right) = -\frac{z^\omega}{\bar{\omega}\sigma_\omega}G'\left(\bar{\omega}\right) = -\frac{z^\omega}{\bar{\omega}\sigma_\omega^2}\phi\left(z^\omega\right) \tag{7.70}$$

$$\Gamma'(\bar{\omega}) = \frac{\Gamma(\bar{\omega}) - G(\bar{\omega})}{\bar{\omega}} = 1 - \Phi(z^{\omega}) \tag{7.71}$$

$$\Gamma''(\bar{\omega}) = -\frac{1}{\bar{\omega}}G'(\bar{\omega};\sigma_{\omega}) = -\frac{1}{\bar{\omega}\sigma_{\omega}}\phi(z^{\omega}) \tag{7.72}$$

and

$$\frac{\partial z^{\omega}}{\partial \sigma_{\omega}} = -\left(\frac{z^{\omega}}{\sigma_{\omega}} - 1\right) \tag{7.73}$$

$$G_{\sigma_{\omega}}(\bar{\omega}) = -\frac{z^{\omega}}{\sigma_{\omega}}\phi(z^{\omega} - \sigma_{\omega}) \tag{7.74}$$

$$G'_{\sigma_{\omega}}(\bar{\omega}) = -\frac{\phi(z^{\omega})}{\sigma_{\omega}^2}\left[1 - z^{\omega}(z^{\omega} - \sigma_{\omega})\right] \tag{7.75}$$

$$\Gamma_{\sigma_{\omega}}(\bar{\omega}) = -\phi(z^{\omega} - \sigma_{\omega}) \tag{7.76}$$

$$\Gamma'_{\sigma_{\omega}}(\bar{\omega}) = \left(\frac{z^{\omega}}{\sigma_{\omega}} - 1\right)\phi(z^{\omega}) \tag{7.77}$$

where we use notation $f'(\bar{\omega}) \equiv \partial f(\bar{\omega})/\partial\bar{\omega}$ and $f_{\sigma_{\omega}}(\bar{\omega}) \equiv \partial f(\bar{\omega})/\partial\sigma_{\omega}$, for $f \in \{G, \Gamma\}$.

7.7 Elasticities

First notice that we have several elasticities defined as

$$\zeta_{b,x} \equiv \frac{\frac{\partial}{\partial x}\left[\left\{1 - \Gamma(\bar{\omega}) + \frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu_*^e G'(\bar{\omega})}\left[\Gamma(\bar{\omega}) - \mu_*^e G(\bar{\omega})\right]\right\}\frac{\tilde{R}_*^k}{R_*} - \frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu_*^e G'(\bar{\omega})}\right]}{\left\{1 - \Gamma(\bar{\omega}) + \frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu_*^e G'(\bar{\omega})}\left[\Gamma(\bar{\omega}) - \mu_*^e G(\bar{\omega})\right]\right\}\frac{\tilde{R}_*^k}{R_*}} x$$

which we can rewrite as

$$\zeta_{b,x} \equiv \frac{\frac{\partial\tilde{\Psi}}{\partial x}x}{\left\{1 - \Gamma(\bar{\omega}) + \frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu_*^e G'(\bar{\omega})}\left[\Gamma(\bar{\omega}) - \mu_*^e G(\bar{\omega})\right]\right\}\frac{\tilde{R}_*^k}{R_*}}$$

with

$$\begin{aligned}
\tilde{\Psi} &\equiv \left\{1 - \Gamma(\bar{\omega}) + \frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu_*^e G'(\bar{\omega})}\left[\Gamma(\bar{\omega}) - \mu_*^e G(\bar{\omega})\right]\right\}\frac{\tilde{R}_*^k}{R_*} - \frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu_*^e G'(\bar{\omega})} \\
&= \left[1 - \Gamma(\bar{\omega})\right]\frac{\tilde{R}_*^k}{R_*} + \frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu_*^e G'(\bar{\omega})}\left[\left[\Gamma(\bar{\omega}) - \mu_*^e G(\bar{\omega})\right]\frac{\tilde{R}_*^k}{R_*} - 1\right] \tag{7.78}
\end{aligned}$$

Elasticities w.r.t. $\bar{\omega}$

First write

$$
\begin{aligned}
\frac{\partial \tilde{\Psi}}{\partial \bar{\omega}} =\ & -\Gamma'_* \left(\bar{\omega}_*\right) \frac{\tilde{R}^k_*}{R_*} \\
& + \left[\left[\Gamma_* \left(\bar{\omega}_*\right) - \mu^e_* G_* \left(\bar{\omega}_*\right)\right] \frac{\tilde{R}^k_*}{R_*} - 1 \right] \frac{\Gamma'' \left(\bar{\omega}\right) \left[\Gamma' \left(\bar{\omega}\right) - \mu^e_* G' \left(\bar{\omega}\right)\right] - \Gamma' \left(\bar{\omega}\right) \left[\Gamma'' \left(\bar{\omega}\right) - \mu^e_* G'' \left(\bar{\omega}\right)\right]}{\left[\Gamma' \left(\bar{\omega}\right) - \mu^e_* G' \left(\bar{\omega}\right)\right]^2} \\
& + \frac{\Gamma' \left(\bar{\omega}\right)}{\Gamma' \left(\bar{\omega}\right) - \mu^e_* G' \left(\bar{\omega}\right)} \left[\Gamma' \left(\bar{\omega}\right) - \mu^e_* G' \left(\bar{\omega}\right)\right] \frac{\tilde{R}^k_*}{R_*}
\end{aligned}
$$

and simplify to

$$
\begin{aligned}
\frac{\partial \tilde{\Psi}}{\partial \bar{\omega}} =\ & \mu^e_* \left\{ \left[\Gamma_* \left(\bar{\omega}_*\right) - \mu^e_* G_* \left(\bar{\omega}_*\right)\right] \frac{\tilde{R}^k_*}{R_*} - 1 \right\} \frac{G'' \left(\bar{\omega}\right) \Gamma' \left(\bar{\omega}\right) - G' \left(\bar{\omega}\right) \Gamma'' \left(\bar{\omega}\right)}{\left[\Gamma' \left(\bar{\omega}\right) - \mu^e_* G' \left(\bar{\omega}\right)\right]^2} \\
=\ & \mu^e_* \frac{n_*}{\bar{k}_*} \frac{\Gamma'' \left(\bar{\omega}\right) G' \left(\bar{\omega}\right) - G'' \left(\bar{\omega}\right) \Gamma' \left(\bar{\omega}\right)}{\left[\Gamma' \left(\bar{\omega}\right) - \mu^e_* G' \left(\bar{\omega}\right)\right]^2}
\end{aligned}
$$

which we can plug into the elasticity to get

$$
\zeta_{b,\bar{\omega}} = \frac{\mu^e_* \frac{n_*}{\bar{k}_*} \frac{\Gamma''_* \left(\bar{\omega}_*\right) G'_* \left(\bar{\omega}_*\right) - G''_* \left(\bar{\omega}_*\right) \Gamma'_* \left(\bar{\omega}_*\right)}{\left[\Gamma'_* \left(\bar{\omega}_*\right) - \mu^e_* G'_* \left(\bar{\omega}_*\right)\right]^2}}{\left\{ 1 - \Gamma_* \left(\bar{\omega}_*\right) + \Gamma'_* \left(\bar{\omega}_*\right) \frac{\Gamma_* \left(\bar{\omega}_*\right) - \mu^e_* G_* \left(\bar{\omega}_*\right)}{\Gamma'_* \left(\bar{\omega}_*\right) - \mu^e_* G'_* \left(\bar{\omega}_*\right)} \right\} \frac{\tilde{R}^k_*}{R_*}} \bar{\omega}_* \tag{7.79}
$$

We also have

$$
\zeta_{z,\bar{\omega}} \equiv \frac{\Gamma'_* \left(\bar{\omega}_*\right) - \mu^e_* G'_* \left(\bar{\omega}_*\right)}{\Gamma_* \left(\bar{\omega}_*\right) - \mu^e_* G_* \left(\bar{\omega}_*\right)} \bar{\omega}_* \tag{7.80}
$$

Notice that if we plug everything into

$$
\zeta_{sp,b} = -\frac{\frac{\zeta_{b,\bar{\omega}}}{\zeta_{z,\bar{\omega}}}}{1 - \frac{\zeta_{b,\bar{\omega}}}{\zeta_{z,\bar{\omega}}}} \frac{\frac{n_*}{k_*}}{1 - \frac{n_*}{\bar{k}_*}} \tag{7.81}
$$

which becomes

$$
\zeta_{sp,b} = \frac{\frac{\frac{n_*}{\bar{k}_*}}{1 - \frac{n_*}{\bar{k}_*}}}{1 - \frac{\left\{ \frac{\left[1 - \bar{\omega}_* \left[1 - \Phi\left(z^\omega_*\right)\right] - \Phi\left(z^\omega_* - \sigma_{\omega*}\right)\right]}{\bar{\omega}_* \left[1 - \Phi\left(z^\omega_*\right)\right] + \left(1 - \mu^e_*\right) \Phi\left(z^\omega_* - \sigma_{\omega*}\right)} \left[1 - \Phi\left(z^\omega_*\right) - \frac{\mu^e_*}{\sigma_{\omega*}} \phi\left(z^\omega_*\right)\right] + 1 - \Phi\left(z^\omega_*\right) \right\} \frac{\tilde{R}^k_*}{R_*}}{\frac{\mu^e_* \phi\left(z^\omega_*\right)}{\bar{\omega}_* \sigma^2_{\omega*}} \frac{\phi\left(z^\omega_*\right) - z^\omega_* \left[1 - \Phi\left(z^\omega_*\right)\right]}{\left[1 - \Phi\left(z^\omega_*\right) - \frac{\mu^e_*}{\sigma_{\omega*}} \phi\left(z^\omega_*\right)\right]^2} \frac{n_*}{\bar{k}_*}}} \tag{7.82}
$$

Elasticity of w.r.t. σ_ω

First we compute the derivative

$$
\begin{aligned}
\frac{\partial \tilde{\Psi}}{\partial \sigma_\omega} =\ & -\Gamma_{\sigma_\omega} \left(\bar{\omega}\right) \frac{\tilde{R}^k_*}{R_*} + \frac{\Gamma' \left(\bar{\omega}\right)}{\Gamma' \left(\bar{\omega}\right) - \mu^e_* G' \left(\bar{\omega}\right)} \left[\Gamma_{\sigma_\omega} \left(\bar{\omega}\right) - \mu^e_* G_{\sigma_\omega} \left(\bar{\omega}\right)\right] \frac{\tilde{R}^k_*}{R_*} \\
& + \frac{\Gamma'_{\sigma_\omega} \left(\bar{\omega}\right) \left[\Gamma' \left(\bar{\omega}\right) - \mu^e_* G' \left(\bar{\omega}\right)\right] - \Gamma' \left(\bar{\omega}\right) \left[\Gamma'_{\sigma_\omega} \left(\bar{\omega}\right) - \mu^e_* G'_{\sigma_\omega} \left(\bar{\omega}\right)\right]}{\left[\Gamma' \left(\bar{\omega}\right) - \mu^e_* G' \left(\bar{\omega}\right)\right]^2} \left[\left[\Gamma \left(\bar{\omega}\right) - \mu^e_* G \left(\bar{\omega}\right)\right] \frac{\tilde{R}^k_*}{R_*} - 1 \right]
\end{aligned}
$$

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hence

$$\frac{\partial \tilde{\Psi}}{\partial \sigma_\omega} = \left(\frac{1 - \mu_*^e \frac{G_{\sigma_\omega}(\bar{\omega})}{\Gamma_{\sigma_\omega}(\bar{\omega})}}{1 - \mu_*^e \frac{G'(\bar{\omega})}{\Gamma'(\bar{\omega})}} - 1 \right) \Gamma_{\sigma_\omega}(\bar{\omega}) \frac{\tilde{R}_*^k}{R_*} + \mu_*^e \frac{n_*}{\bar{k}_*} \frac{G'(\bar{\omega})\Gamma'_{\sigma_\omega}(\bar{\omega}) - \Gamma'(\bar{\omega})G'_{\sigma_\omega}(\bar{\omega})}{[\Gamma'(\bar{\omega}) - \mu_*^e G'(\bar{\omega})]^2}$$

so that

$$\zeta_{b,\sigma_\omega} = \frac{\left(\frac{1 - \mu_*^e \frac{G_{\sigma_\omega *}(\bar{\omega}_*)}{\Gamma_{\sigma_\omega *}(\bar{\omega}_*)}}{1 - \mu_*^e \frac{G'_*(\bar{\omega}_*)}{\Gamma'_*(\bar{\omega}_*)}} - 1 \right) \Gamma_{\sigma_\omega *}(\bar{\omega}_*) \frac{\tilde{R}_*^k}{R_*} + \mu_*^e \frac{n_*}{\bar{k}_*} \frac{G'_*(\bar{\omega}_*)\Gamma'_{\sigma_\omega *}(\bar{\omega}_*) - \Gamma'_*(\bar{\omega}_*)G'_{\sigma_\omega *}(\bar{\omega}_*)}{[\Gamma'_*(\bar{\omega}_*) - \mu_*^e G'_*(\bar{\omega}_*)]^2}}{[1 - \Gamma_*(\bar{\omega}_*)] \frac{\tilde{R}_*^k}{R_*} + \frac{\Gamma'_*(\bar{\omega}_*)}{\Gamma'_*(\bar{\omega}_*) - \mu_*^e G'_*(\bar{\omega}_*)} \left(1 - \frac{n_*}{k_*}\right)} \sigma_{\omega *} \tag{7.83}$$

We also have

$$\zeta_{z,\sigma_\omega} = \frac{\Gamma_{\sigma_\omega *}(\bar{\omega}_*) - \mu_*^e G_{\sigma_\omega *}(\bar{\omega}_*)}{\Gamma_*(\bar{\omega}_*) - \mu_*^e G_*(\bar{\omega}_*)} \sigma_{\omega *} \tag{7.84}$$

and finally we can write

$$\zeta_{sp,\sigma_\omega} = \frac{\frac{\zeta_{b,\bar{\omega}}}{\zeta_{z,\bar{\omega}}} \zeta_{z,\sigma_\omega} - \zeta_{b,\sigma_\omega}}{1 - \frac{\zeta_{b,\bar{\omega}}}{\zeta_{z,\bar{\omega}}}} \tag{7.85}$$

Elasticity of w.r.t. μ^e

First solve

$$\frac{\partial \tilde{\Psi}}{\partial \mu^e} = -\frac{\Gamma'(\bar{\omega})G'(\bar{\omega})}{[\Gamma'(\bar{\omega}) - \mu_*^e G'(\bar{\omega})]^2} \frac{n_*}{\bar{k}_*} - \frac{\Gamma'(\bar{\omega})G(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu_*^e G'(\bar{\omega})} \frac{\tilde{R}_*^k}{R_*}$$

so that

$$\zeta_{b,x} \equiv -\frac{\frac{\Gamma'_*(\bar{\omega}_*)G'_*(\bar{\omega}_*)}{\Gamma'_*(\bar{\omega}_*) - \mu_*^e G'_*(\bar{\omega}_*)} \frac{n_*}{\bar{k}_*} + \Gamma'_*(\bar{\omega}_*) G_*(\bar{\omega}_*) \frac{\tilde{R}_*^k}{R_*}}{[1 - \Gamma_*(\bar{\omega}_*)] [\Gamma'_*(\bar{\omega}_*) - \mu_*^e G'_*(\bar{\omega}_*)] \frac{\tilde{R}_*^k}{R_*} + \Gamma'_*(\bar{\omega}_*) \left(1 - \frac{n_*}{k_*}\right)} \mu_*^e \tag{7.86}$$

We also have

$$\zeta_{z,\mu^e} = -\frac{G_*(\bar{\omega}_*)}{\Gamma_*(\bar{\omega}_*) - \mu_*^e G_*(\bar{\omega}_*)} \mu_*^e \tag{7.87}$$

Finally we write

$$\zeta_{sp,\mu^e} = \frac{\frac{\zeta_{b,\bar{\omega}}}{\zeta_{z,\bar{\omega}}} \zeta_{z,\mu^e} - \zeta_{b,\mu^e}}{1 - \frac{\zeta_{b,\bar{\omega}}}{\zeta_{z,\bar{\omega}}}} \tag{7.88}$$

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8 SW original model

In this section we describe in detail the Smets and Wouters (2007) model, henceforth SW), and emphasize the differences with the model presented in Section 1 of these notes.

8.1 Model

8.1.1 Intermediate firms

We follow SW and assume the production function to be:

$$Y_t(i) = \max\{e^{\tilde{z}_t} K_t(i)^\alpha \left(L_t(i)e^{\gamma t}\right)^{1-\alpha} - \Phi e^{(\gamma + \frac{\alpha}{1-\alpha}\log\Upsilon)t}, 0\}, \tag{8.1}$$

where

$$\tilde{z}_t = \rho_z \tilde{z}_{t-1} + \sigma_z \epsilon_{z,t}, \; \epsilon_{z,t} \sim N(0,1) \tag{8.2}$$

(Note that what SW call "γ" in our notation is e^γ, and that they assume $\Upsilon = 1$.) SW assume that productivity $\tilde{z}_t$ is stationary. Define Z_t as follows:

$$\ln Z_t = \frac{1}{1-\alpha}\tilde{z}_t. \tag{8.3}$$

For $\rho_z \in (0,1)$ the process $\ln Z_t$ is stationary, as in SW. For $\rho_z = 1$ it follows a random walk. This specification accomodates both. Note that we can rewrite the production function as:

$$Y_t(i) = \max\{K_t(i)^\alpha \left(L_t(i)Z_t\right)^{1-\alpha} - \Phi e^{-\frac{1}{1-\alpha}\tilde{z}_t} Z_t e^{(\gamma + \frac{\alpha}{1-\alpha}\log\Upsilon)t}, 0\}. \tag{8.4}$$

Cost minimization subject to 8.4 yields the conditions:

$$(\partial L_t(i)) \quad \mathcal{V}_t(i)(1-\alpha)Z_t^{1-\alpha}K_t(i)^\alpha L_t(i)^{-\alpha} = W_t$$
$$(\partial K_t(i)) \quad \mathcal{V}_t(i)\alpha Z_t^{1-\alpha}K_t(i)^{\alpha-1}L_t(i)^{1-\alpha} = R_t^k$$

where $\mathcal{V}_t(i)$ is the Lagrange multiplier associated with 1.1.8. In turn, these conditions imply:

$$\frac{K_t(i)}{L_t(i)} = \frac{\alpha}{1-\alpha}\frac{W_t}{R_t^k}.$$

Note that if we integrate both sides of the equation wrt di and define $K_t = \int K_t(i)di$ and $L_t = \int L_t(i)di$ we obtain a relationship between aggregate labor and capital:

$$K_t = \frac{\alpha}{1-\alpha} \frac{W_t}{R_t^k} L_t. \tag{8.5}$$

Total variable cost is given by

$$
\begin{aligned}
\text{Variable Costs} \quad &= (W_t + R_t^k \tfrac{K_t(i)}{L_t(i)}) L_t(i) \\
&= (W_t + R_t^k \tfrac{K_t(i)}{L_t(i)}) \tilde{Y}_t(i) Z_t^{-(1-\alpha)} \left(\tfrac{K_t(i)}{L_t(i)}\right)^{-\alpha},
\end{aligned}
$$

where $\tilde{Y}_t(i) = Z_t^{1-\alpha} K_t(i)^\alpha L_t(i)^{1-\alpha}$ is the "variable" part of output. The marginal cost MC_t is the same for all firms and equal to:

$$
\begin{aligned}
MC_t \quad &= (W_t + R_t^k \tfrac{K_t(i)}{L_t(i)}) Z_t^{-(1-\alpha)} \left(\tfrac{K_t(i)}{L_t(i)}\right)^{-\alpha} \\
&= \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)} W_t^{1-\alpha} R_t^{k\,\alpha} Z_t^{-(1-\alpha)}.
\end{aligned} \tag{8.6}
$$

[TO DO WITH KIMBALL] Prices are sticky as in Calvo (1983). Specifically, each firm can readjust prices with probability $1 - \zeta_p$ in each period. We depart rfom Calvo (1983) in assuming that for those firms that cannot adjust prices, $P_t(i)$ will increase at the geometric weighted average (with weigths $1 - \iota_p$ and ι_p, respectively) of the steady state rate of inflation π_* and of last period's inflation π_{t-1}. For those firms that can adjust prices, the problem is to choose a price level $\tilde{P}_t(i)$ that maximizes the expected present discounted value of profits in all states of nature where the firm is stuck with that price in the future:

$$
\begin{aligned}
\max_{\tilde{P}_t(i)} \quad & \Xi_t^p \left(\tilde{P}_t(i) - MC_t \right) Y_t(i) \\
& + E_t \sum_{s=1}^\infty \zeta_p^s \beta^s \Xi_{t+s}^p \left(\tilde{P}_t(i) \left(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p} \right) - MC_{t+s} \right) Y_{t+s}(i) \\
\text{s.t. } Y_{t+s}(i) = & \left(\frac{\tilde{P}_t(i) \left(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p} \right)}{P_{t+s}} \right)^{-\frac{1+\lambda_{f,t+s}}{\lambda_{f,t+s}}} Y_{t+s},
\end{aligned} \tag{8.7}
$$

where $\beta^s \Xi_{t+s}^p$ is today's value of a future dollar for the consumers (Ξ_{t+s}^p is the Lagrange multiplier associated with the consumer's nominal budget constraint - remember there

are complete markets so $\beta^s \Xi^p_{t+s}$ is the same for all consumers). The FOC for the firm is:

$$\Xi^p_t \left(\frac{\tilde{P}_t(i)}{P_t}\right)^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}-1} \frac{1}{\lambda_{f,t}P_t}\left(\tilde{P}_t(i) - (1+\lambda_{f,t})MC_t\right)Y_t(i)+$$

$$E_t \sum_{s=0}^{\infty} \zeta_p^s \beta^s \Xi^p_{t+s}\left(\frac{\tilde{P}_t(i)\left(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p}\pi_*^{1-\iota_p}\right)}{P_{t+s}}\right)^{-\frac{1+\lambda_{f,t+s}}{\lambda_{f,t+s}}-1}\frac{\left(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p}\pi_*^{1-\iota_p}\right)}{\lambda_{f,t+s}P_{t+s}} \tag{8.8}$$

$$\left(\tilde{P}_t(i)\left(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p}\pi_*^{1-\iota_p}\right) - (1+\lambda_{f,t+s})MC_{t+s}\right)Y_{t+s}(i) = 0$$

Note that all firms readjusting prices face an indentical problem. We will consider only the symmetric equilibrium in which all firms that can readjust prices will choose the same $\tilde{P}_t(i)$, so we can drop the i index from now on. From 1.1.6 it follows that:

$$P_t = [(1-\zeta_p)\tilde{P}_t^{-\frac{1}{\lambda_f}} + \zeta_p(\pi_{t-1}^{\iota_p}\pi_*^{1-\iota_p}P_{t-1})^{-\frac{1}{\lambda_f}}]^{-\lambda_f}. \tag{8.9}$$

8.1.2 Households

Household j's utility is (as opposed to 1.1.16):

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left[\frac{1}{1-\sigma_c}(C_{t+s}(j) - hC_{t+s-1})^{1-\sigma_c}\right]\exp\left(\frac{\sigma_c-1}{1+\nu_l}L_{t+s}(j)^{1+\nu_l}\right) \tag{8.10}$$

where $C_t(j)$ is consumption, $L_t(j)$ is labor supply. Three observations are in order regarding this utility function. First, utility is increasing in consumption and leisure regardless of the value of σ_c. Second, there are no "discount rate" or "leisure" shocks in the utility function. Third, SW have external (as opposed to internal) habit.

The household's budget constraint, written in real terms, is given by:

$$C_{t+s}(j) + I_{t+s}(j) + \frac{B_{t+s}(j)}{b_{t+s}R_{t+s}P_{t+s}} \leq \frac{B_{t+s-1}(j)}{P_{t+s}}$$
$$+ \frac{W_{t+s}^h}{P_{t+s}}L_{t+s}(j) + \left(\frac{R_{t+s}^k}{P_{t+s}}u_{t+s}(j)\bar{K}_{t+s-1}(j) - a(u_{t+s}(j))\Upsilon^{-t}\bar{K}_{t+s-1}(j)\right) + \Pi_{t+s} - T_{t+s},$$
$$\tag{8.11}$$

where $I_t(j)$ is investment, $B_t(j)$ is holdings of government bonds, R_t is the gross nominal interest rate paid on government bonds, Π_t is the per-capita profit the household gets from owning firms (assume household pool their firm shares, T_{t+s} is lump-sum taxes, so that they all receive the same profit) $W_t^h(j)$ is the wage earned by household j. b_t is a "risk premium shock". The term within parenthesis represents the return to owning

$\bar{K}_t(j)$ units of capital. Households choose the utilization rate of their own capital, $u_t(j)$, and end up renting to firms in period t an amount of "effective" capital equal to:

$$K_t(j) = u_t(j)\bar{K}_{t-1}(j), \tag{8.12}$$

and getting $R_t^k u_t(j)\bar{K}_{t-1}(j)$ in return. They however have to pay a cost of utilization in terms of the consumption good which is equal to $a(u_t(j))\Upsilon^{-t}\bar{K}_{t-1}(j)$. Households accumulate capital according to the equation:

$$\bar{K}_t(j) = (1-\delta)\bar{K}_{t-1}(j) + \Upsilon^t \mu_t \left(1 - S(\frac{I_t(j)}{I_{t-1}(j)})\right) I_t(j), \tag{8.13}$$

where δ is the rate of depreciation, and $S(\cdot)$ is the cost of adjusting investment, with $S'(\cdot) > 0, S''(\cdot) > 0$. The term μ_t is a stochastic disturbance to the price of investment relative to consumption

Households are all identical, so the j subscript is pretty redundant except for the fact that we have external habits. We will drop the j subsequently.

The FOCs for consumption, bonds, and labor are:

$$(\partial C_t(j)) \qquad (C_t - hC_{t-1})^{-\sigma_c} \exp\left(\frac{\sigma_c - 1}{1 + \nu_l} L_t^{1+\nu_l}\right) = \Xi_t \tag{8.14}$$

$$(\partial B_t(j)) \qquad \Xi_t = \beta R_t b_t \mathbb{E}_t[\frac{\Xi_{t+1}}{\pi_{t+1}}] \tag{8.15}$$

$$(\partial L_t(j)) \qquad (C_t - hC_{t-1})^{1-\sigma_c} \exp\left(\frac{\sigma_c - 1}{1 + \nu_l} L_t^{1+\nu_l}\right) L_t^{\nu_l} = \Xi_t \frac{W_t^h}{P_t}. \tag{8.16}$$

Note that households take W_t^h as given and maximize with respect to L_t. The wage stickiness part will be discussed below. Using 8.14 we can rewrite 8.16 as:

$$(C_t - hC_{t-1}) L_t^{\nu_l} = \frac{W_t^h}{P_t}. \tag{8.17}$$

Let us now address the capital accumulation/utilization problem. Call Ξ_t^k the Lagrange multiplier associated with constraint 8.13. The FOC with respect to investment, capital, and capital utilization are:

$$(\partial I_t) \qquad \Xi_t^k \Upsilon^t \mu_t \left(1 - S(\frac{I_t}{I_{t-1}}) - S'(\frac{I_t}{I_{t-1}})\frac{I_t}{I_{t-1}}\right)$$
$$+ \beta \mathbb{E}_t[\Xi_{t+1}^k \Upsilon^{t+1} \mu_{t+1} S'(\frac{I_{t+1}}{I_t})(\frac{I_{t+1}}{I_t})^2] = \Xi_t \tag{8.18}$$

$$(\partial \bar{K}_t) \quad \Xi_t^k = \beta \mathbb{E}_t[\Xi_{t+1}(\frac{R_{t+1}^k}{P_{t+1}} u_{t+1} - a(u_{t+1})\Upsilon^{-(t+1)}) + \Xi_{t+1}^k(1-\delta)] \tag{8.19}$$

$$(\partial u_t) \qquad \Upsilon^t \frac{R_t^k}{P_t} = a'(u_t) \tag{8.20}$$

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The first FOC is the law of motion for the shadow value of capital. Note that if adjustment cost were absent, the FOC would simply say that $\Xi_t^k \Upsilon^t \mu_t$ is equal to the marginal utility of consumption. In other words, in absence of adjustment costs the shadow cost of taking resources away from consumption equals the shadow benefit (abstracting from $\Upsilon^t \mu_t$) of putting these resources into investment: Tobin's Q is equal to one. The second FOC says that if I buy a unit of capital today I have to pay its price in real terms, Ξ_t^k, but tomorrow I will get the proceeds from renting capital, plus I can sell back the capital that has not depreciated. Define $Q_t^k = \frac{\Xi_t^k}{\Xi_t}$. Q_t^k has the interpretation of the value of installed capital relative to consumption goods (i.e., Tobin's Q). Then condition 8.19 can be rewritten as:

$$Q_t^k = \beta \mathbb{E}_t \left[\frac{\Xi_{t+1}}{\Xi_t} \left(\frac{R_{t+1}^k}{P_{t+1}} u_{t+1} - a(u_{t+1}) \Upsilon^{-(t+1)} + Q_{t+1}^k (1-\delta) \right) \right]. \tag{8.21}$$

8.1.3 Government Policies

The central bank follows a nominal interest rate rule by adjusting its instrument in response to deviations of inflation and output from their respective target levels:

$$\frac{R_t}{R^*} = \left(\frac{R_{t-1}}{R^*} \right)^{\rho_R} \left[\left(\frac{\pi_t}{\pi_*} \right)^{\psi_1} \left(\frac{Y_t}{Y_t^f} \right)^{\psi_2} \right]^{1-\rho_R} \left(\frac{Y_t}{Y_{t-1}} \frac{Y_{t-1}^f}{Y_t^f} \right)^{\psi_3} e^{r_t^m} \tag{8.22}$$

where the parameter ρ_R determines the degree of interest rate smoothing, R^* is the steady state nominal rate and Y_t^f is output under flexible/prices and wages. Note that policy reacts to both level differences between Y_t and Y_t^f ($\psi_2(1-\rho_R)$ coefficient) as well as growth differences (ψ_2 coefficient). Note also that the exogenours part of monetary policy is captured by the process r_t^m, which follows an autoregressive process. The central bank supplies the money demanded by the household to support the desired nominal interest rate.

The government budget constraint is of the form

$$P_t G_t + B_{t-1} = P_t T_t + \frac{B_t}{b_t R_t}, \tag{8.23}$$

where T_t are nominal lump-sum taxes (or subsidies) that also appear in household's budget constraint. SW, who assume technology is stationary, express government spending relative to the *deterministic* trend in output:

$$G_t = \tilde{g}_t y_* e^{z_*^* t} \tag{8.24}$$

where y_* is the steady state of detrended output. Since we detrend everything (see below) by Z_t^*, we need to be careful. Define

$$g_t = \frac{G_t}{y_* Z_t^*} = \tilde{g}_t e^{-\frac{1}{1-\alpha} \tilde{z}_t}. \tag{8.25}$$

At steady state $\tilde{g}_* = g_*$. Note the difference with DSSW, where $g_t = \frac{Y_t}{Y_t - G_t}$ and $g_* = \frac{y_*}{c_* + i_*} > 1$. In SW $g_* \in (0, 1)$.

8.1.4 Resource constraints

To obtain the market clearing condition for the final goods market first integrate the HH budge constraint across households, and combine it with the gvmt budget constraint:

$$P_t C_t + P_t I_t + P_t G_t \leq +\Pi_t + \int W_t(j) L_t(j) dj$$
$$+ R_t^k \int K_t(j) dj - P_t a(u_t) \Upsilon^{-t} \int \bar{K}_{t-1}(j) dj.$$

Next, realize that

$$\Pi_t = \int \Pi(i)_t di = \int P(i)_t Y(i)_t di - W_t L_t - R_t^k K_t,$$

where $L_t = \int L(i)_t di$ is total labor supplied by the labor packers (and demanded by the firms), and $K_t = \int K(i)_t di = \int K_t(j) dj$. Now replace the defintion of Π_t into the HH budget constraint, realize that by the labor and goods' packers' zero profit condition $W_t L_t = \int W_t(j) L_t(j) dj$, and $P_t Y_t = \int P(i)_t Y(i)_t di$ and obtain:

$$C_t + I_t + a(u_t) \Upsilon^{-t} \bar{K}_{t-1} + G_t = Y_t \tag{8.26}$$

where Y_t is defined by (1.1.1). The relationship between output and the aggregate inputs, labor anc capital, is:

$$\begin{aligned} \dot{Y}_t &= \int Z_t^{1-\alpha} K_t(i)^\alpha L_t(i)^{1-\alpha} di - Z_t^* \Phi \\ &= Z_t^{1-\alpha} \int (K/L)^\alpha L(i) di - Z_t^* \Phi \\ &= Z_t^{1-\alpha} K_t^\alpha L_t^{1-\alpha} - Z_t^* \Phi, \end{aligned} \tag{8.27}$$

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where I used the fact that the capital labor ratio is constant across firms (also, since $K(i) = (K/L)L(i)$ it must be the case that $\frac{\int K(i)di}{\int L(i)di} = K_t/L_t = (K/L)$). The problem with these resource constraints is that what we observe in the data is $\dot{Y}_t = \int Y_t(i)di$ and $\dot{L}_t = \int L_t(j)dj$, as opposed to Y_t and L_t. But note that from 1.1.5:

$$
\begin{aligned}
\dot{Y}_t &= Y_t P_t^{\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} \int P(i)_t^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} di \\
&= Y_t P_t^{\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} \dot{P}_t^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}},
\end{aligned}
$$

where $\dot{P}_t = \left(\int P_t(i)^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} di \right)^{-\frac{\lambda_{f,t}}{1+\lambda_{f,t}}}$, and

$$
\begin{aligned}
\dot{L}_t &= \int L_t(j)dj \\
&= L_t W_t^{\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} \int W(j)_t^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} di \\
&= L_t W_t^{\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} \dot{W}_t^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}},
\end{aligned}
$$

where $\dot{W}_t = \left(\int W(j)_t^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} dj \right)^{-\frac{\lambda_{w,t}}{1+\lambda_{w,t}}}$.

8.1.5 Exogenous Processes

When technology is stationary or has a unit root, its process is given by 8.3, which we report here:

$$
\tilde{z}_t = \rho_z \tilde{z}_{t-1} + \sigma_z \varepsilon_{z,t}.
$$

We now discuss the process for g_t. SW assume a stationary process for $\hat{\tilde{g}}_t = log(\frac{\tilde{g}_t}{g_*})$, which is correlated with shocks in technology:

$$
\hat{\tilde{g}}_t = \rho_g \hat{\tilde{g}}_{t-1} + \sigma_g \varepsilon_{g,t} + \eta_{gz} \sigma_z \varepsilon_{z,t}. \tag{8.28}
$$

If technology is not stationary, this process does not make sense since $\hat{\tilde{g}}_t$ is non stationary. Hence we replace it by the assumption that $\hat{g}_t = log(\frac{g_t}{g_*})$ is stationary

$$
\hat{g}_t = \rho_g \hat{g}_{t-1} + \sigma_g \varepsilon_{g,t} + \eta_{gz} \sigma_z \varepsilon_{z,t}. \tag{8.29}
$$

We express all remaining processes in log deviations from their steady state value, which is assumed to be 1:

$$\hat{b}_t = \rho_b \hat{b}_{t-1} + \sigma_b \varepsilon_{b,t}, \tag{8.30}$$

$$\hat{\mu}_t = \rho_\mu \hat{\mu}_{t-1} + \sigma_\mu \varepsilon_{\mu,t}, \tag{8.31}$$

$$\hat{r}_t^m = \rho_{r^m} \hat{r}_{t-1}^m + \sigma_r \varepsilon_{r^m,t}, \tag{8.32}$$

The mark-up shocks follow ARMA(1,1) processes:

$$\hat{\lambda}_{f,t} = \rho_{\lambda_f} \hat{\lambda}_{f,t-1} + \sigma_{\lambda_f} \varepsilon_{\lambda_f,t} + \eta_{\lambda_f} \sigma_{\lambda_f} \varepsilon_{\lambda_f,t-1}, \tag{8.33}$$

$$\hat{\lambda}_{w,t} = \rho_{\lambda_w} \hat{\lambda}_{w,t-1} + \sigma_{\lambda_w} \varepsilon_{\lambda_w,t} + \eta_{\lambda_w} \sigma_{\lambda_w} \varepsilon_{\lambda_w,t-1}, \tag{8.34}$$

8.2 Detrending

SW detrend the variables by the deterministic trend $e^{\gamma t}$ (or by $e^{(\gamma + \frac{\alpha}{1-\alpha} \log \Upsilon)t}$ if there is a trend in the relative price of capital). We detrend by

$$Z_t^* = Z_t e^{(\gamma + \frac{\alpha}{1-\alpha} \log \Upsilon)t}, \Upsilon > 1. \tag{8.35}$$

Define $z_t^* = \log(Z_t^*/Z_{t-1}^*)$. Denote with $_*$ the steady state values of the variables, and realize that at st.st. $z_*^* = \gamma + \frac{\alpha}{1-\alpha} \log \Upsilon$. From 8.2 and 8.3 we see that

$$\hat{z}_t^* = z_t^* - z_*^* = \frac{1}{1-\alpha}(\rho_z - 1)\tilde{z}_{t-1} + \frac{1}{1-\alpha}\sigma_z \epsilon_{z,t}, \tag{8.36}$$

and

$$E_t[\hat{z}_{t+1}^*] = \frac{1}{1-\alpha}(\rho_z - 1)\tilde{z}_t. \tag{8.37}$$

Note that for $\rho_z = 1$ $\tilde{z}_t$ has no impact on $\hat{z}_t$.

Specifically:

$$\begin{aligned}
&c_t = \frac{C_t}{Z_t^*}, \ y_t = \frac{Y_t}{Z_t^*}, \ i_t = \frac{I_t}{Z_t^*}, \ k_t = \Upsilon^{-t}\frac{K_t}{Z_t^*}, \ \bar{k}_t = \Upsilon^{-t}\frac{\bar{K}_t}{Z_t^*}, \\
&r_t^k = \Upsilon^t\frac{R_t^k}{P_t}, \ w_t = \frac{W_t}{P_t Z_t^*}, \ w_t^h = \frac{W_t^h}{P_t Z_t^*}, \ \tilde{p}_t = \frac{\tilde{P}_t}{P_t}, \ \tilde{w}_t = \frac{\tilde{W}_t}{W_t}, \\
&\xi_t = \Xi_t Z_t^{*\sigma_c}, \ \xi_t^k = \Xi_t^k Z_t^{*\sigma_c}\Upsilon^t, \ q_t^k = Q_t^k \Upsilon^t.
\end{aligned} \tag{8.38}$$

Note that this implies that some of the equilibrium conditions will look different from SW.

Intermediate goods producers

We start by expressing 8.6 in terms of detrended variables:

$$mc_t = \frac{MC_t}{P_t} = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}w_t^{1-\alpha}r_t^{k\ \alpha}. \tag{8.39}$$

Hence

$$mc_* = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}w_*^{1-\alpha}r_*^{k\ \alpha}. \tag{8.40}$$

***************** (TO BE DONE) ******************

Expression 8.8 becomes:

$$\begin{aligned}
&\frac{\xi_t}{\lambda_{f,t}}\tilde{p}_t^{-\frac{(1+\lambda_{f,t})}{\lambda_{f,t}}-1}(\tilde{p}_t - (1+\lambda_{f,t})mc_t)\,y_t(i) \\
&+ E_t \sum_{s=1}^{\infty} \zeta_p^s \beta^s \frac{\xi_{t+s}}{\lambda_{f,t+s}}\left(\frac{\tilde{p}_t}{\Pi_{l=1}^s \pi_{t+l}}\right)^{-\frac{(1+\lambda_{f,t+s})}{\lambda_{f,t+s}}-1}\left(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p}\pi_*^{1-\iota_p}\right)^{\frac{(1+\lambda_{f,t+s})}{\lambda_{f,t+s}}} \\
&\left(\tilde{p}_t \frac{\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p}\pi_*^{1-\iota_p}}{\Pi_{l=1}^s \pi_{t+l}} - (1+\lambda_{f,t+s})mc_{t+s}\right)y_{t+s}(i) = 0
\end{aligned} \tag{8.41}$$

this implies that:

$$\tilde{p}_* = (1 + \lambda_f)\alpha^{-\alpha}(1 - \alpha)^{-(1-\alpha)}w_*^{1-\alpha}r_*^{k\ \alpha} \tag{8.42}$$

Expression **??** becomes:

$$1 = [(1 - \zeta_p)\tilde{p}_t^{-\frac{1}{\lambda_{f,t}}} + \zeta_p(\pi_{t-1}^{\iota_p}\pi_*^{1-\iota_p}\pi_t^{-1})^{-\frac{1}{\lambda_{f,t}}}]^{-\lambda_{f,t}}. \tag{8.43}$$

which means that:

$$\tilde{p}_* = 1. \tag{8.44}$$

Recall that aggregate profits are equal to:

$$\Pi_t = P_t Y_t - W_t L_t - R_t^k K_t.$$

In terms of detrended variables we then have :

$$
\begin{aligned}
\frac{\Pi_t}{P_t Z_t^*} &= y_t - w_t L_t - r_t^k k_t \\
&= k_t^\alpha L_t^{1-\alpha} - \Phi - w_t L_t - \frac{\alpha}{1-\alpha}w_t L_t \\
&= \left((\frac{k_t}{L_t}^\alpha - \frac{1}{1-\alpha}w_t\right)L_t - \Phi \\
&= \left((\frac{\alpha}{1-\alpha})^\alpha w_t^\alpha r_t^{k\ -\alpha} - \frac{1}{1-\alpha}w_t\right)L_t - \Phi
\end{aligned}
$$

At steady state we can use 8.42 to get that st. st. profits are:

$$\frac{\Pi_t}{P_t Z_t^*} = \frac{\lambda_f}{1-\alpha}w_* L_* - \Phi. \tag{8.45}$$

****

Equation 8.5 becomes:

$$k_t = \frac{\alpha}{1-\alpha}\frac{w_t}{r_t^k}L_t. \tag{8.46}$$

and at st.st.:

$$k_* = \frac{\alpha}{1-\alpha}\frac{w_*}{r_*^k}L_*. \tag{8.47}$$

Households

Expressions 8.14, 8.15, and 8.17 become:

$$\xi_t = \left(c_t - hc_{t-1}e^{-z_t^*}\right)^{-\sigma_c}\exp\left(\frac{\sigma_c - 1}{1+\nu_l}L_t^{1+\nu_l}\right), \tag{8.48}$$

$$\xi_t = \beta R_t b_t E_t[\xi_{t+1}e^{-\sigma_c z_{t+1}^*}\pi_{t+1}^{-1}], \tag{8.49}$$

$$\left(c_t - hc_{t-1}e^{-z_t^*}\right)L_t^{\nu_l} = w_t^h, \tag{8.50}$$

respectively. At steady state:

$$\xi_* = c_*^{-\sigma_c}(1 - he^{-z_*^*})^{-\sigma_c} \exp\left(\frac{\sigma_c - 1}{1 + \nu_l}L_*^{1+\nu_l}\right), \tag{8.51}$$

$$R_* = \beta^{-1}\pi_* e^{\sigma_c z_*^*}, \tag{8.52}$$

$$c_*\left(1 - he^{-z_*^*}\right)L_*^{\nu_l} = w_*^h. \tag{8.53}$$

Equation 8.12 and 8.13 become:

$$k_t = u_t \Upsilon^{-1} e^{-z_t^*}\bar{k}_{t-1}, \tag{8.54}$$

$$\bar{k}_t = (1 - \delta)\Upsilon^{-1}e^{-z_t^*}\bar{k}_{t-1} + \mu_t\left(1 - S(\frac{i_t}{i_{t-1}}e^{z_t^*})\right)i_t. \tag{8.55}$$

which deliver the steady state relationships:

$$k_* = e^{-\gamma}\Upsilon^{-\frac{1}{1-\alpha}}\bar{k}_*, \tag{8.56}$$

$$i_* = \mu\left(1 - (1 - \delta)e^{-\gamma}\Upsilon^{-\frac{1}{1-\alpha}}\right)\bar{k}_*. \tag{8.57}$$

under the assumption that $S(e^\gamma \Upsilon^{\frac{\alpha}{1-\alpha}}) = 0$.

Equation 8.18, 8.21, and 8.20 become:

$$\xi_t^k \mu_t\left(1 - S(\frac{i_t}{i_{t-1}}e^{z_t^*}) - S'(\frac{i_t}{i_{t-1}}e^{z_t^*})\frac{i_t}{i_{t-1}}e^{z_t^*}\right)$$
$$+ \beta E_t[e^{-\sigma_c z_{t+1}^*}\xi_{t+1}^k\mu_{t+1}S'(\frac{i_{t+1}}{i_t}e^{z_{t+1}^*})(\frac{i_{t+1}}{i_t}e^{z_{t+1}^*})^2] = \xi_t \tag{8.58}$$

$$q_t^k = \beta E_t\left[\Upsilon^{-1}e^{-\sigma_c z_{t+1}^*}\frac{\xi_{t+1}}{\xi_t}\left(r_{t+1}^k u_{t+1} - a(u_{t+1}) + q_{t+1}^k(1 - \delta)\right)\right] \tag{8.59}$$

$$r_t^k = a'(u_t). \tag{8.60}$$

Under the assumptions that $S'(e^\gamma \Upsilon^{\frac{\alpha}{1-\alpha}}) = 0$, $u_* = 1$ and $a(u_*) = 0$, the above equations at steady state imply

$$\xi_*^k = \xi_* \tag{8.61}$$

$$r_*^k = \beta^{-1}e^{\sigma_c z_*^*}\Upsilon - (1 - \delta) \tag{8.62}$$

$$r_*^k = a'(u_*). \tag{8.63}$$

where 8.61 implies $q_*^k = 1$ (note the $a(.)$ function can be normalized so to make $a'(1)$ be whatever the steady state r_*^k is).

Expressed in terms of detrended variables, equation 1.1.37 becomes:

$$\mathbb{E}_t \sum_{s=0}^{\infty} (\zeta_w \beta)^s L(j)_{t+s} \xi_{t+s} \left[-\tilde{\mathcal{X}}_{t,s} \tilde{w}_t w_t + (1 + \lambda_w) \frac{b_{t+s} \varphi_{t+s} L_{t+s}(j)^{\nu_l}}{\xi_{t+s}} \right] = 0, \quad (8.64)$$

where

$$\tilde{\mathcal{X}}_{t,s} = \begin{cases} 1 & \text{if } s = 0 \\ \dfrac{\Pi_{l=1}^{s} (\pi_* e^{\gamma} \Upsilon^{\frac{\alpha}{1-\alpha}})^{1-\iota_w} (\pi_{t+l-1} e^{z^*_{t+l-1}})^{\iota_w}}{\Pi_{l=1}^{s} \pi_{t+l} e^{z^*_{t+l}}} & \text{otherwise} \end{cases}$$

and

$$L_{t+s}(j) = \left(\tilde{w}_t w_t w_{t+s}^{-1} \tilde{\mathcal{X}}_{t,s} \right)^{-\frac{1+\lambda_w}{\lambda_w}} L_{t+s}.$$

Equation 1.1.38 becomes:

$$1 = [(1 - \zeta_w) \tilde{w}_t^{\frac{1}{\lambda_w}} + \zeta_w ((\pi_* e^{\gamma} \Upsilon^{\frac{\alpha}{1-\alpha}})^{1-\iota_w} (\pi_{t-1} e^{z^*_{t-1}})^{\iota_w} \frac{w_{t-1}}{w_t} \pi_t^{-1} e^{-z^*_t})^{\frac{1}{\lambda_w}}]^{\lambda_w}. \quad (8.65)$$

which imply at steady state:

$$w_* = (1 + \lambda_w) \frac{\varphi L_*^{\nu_l}}{\xi_*}, \quad (8.66)$$

$$\tilde{w}_* = 1. \quad (8.67)$$

Resource constraints

If the technology process is stationary, the resource constraint become:

$$y_* \tilde{g}_t e^{-\frac{1}{1-\alpha} \tilde{z}_t} + c_t + i_t + a(u_t) \bar{e}^{-z^*_t} k_{t-1} = y_t, \quad (8.68)$$

otherwise it becomes:

$$y_* g_t + c_t + i_t + a(u_t) \bar{e}^{-z^*_t} k_{t-1} = y_t. \quad (8.69)$$

Detrended output is also given as a function of inputs by:

$$\dot{y}_t = k_t^{\alpha} L_t^{1-\alpha} - \Phi e^{-\frac{1}{1-\alpha} \tilde{z}_t}. \quad (8.70)$$

$$Y_t = \left(\frac{\dot{P}_t}{P_t} \right)^{\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} \dot{Y}_t$$

becomes

$$y_t = (\dot{p}_t)^{\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} \dot{y}_t \quad (8.71)$$

where

$$\dot{p}_t = \frac{\dot{P}_t}{P_t}$$

$$= [(1-\zeta_p)(\frac{\tilde{P}_t}{P_t})^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} + \zeta_p(\pi_*\frac{\dot{P}_{t-1}}{P_t})^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}}]^{-\frac{\lambda_{f,t}}{1+\lambda_{f,t}}} \tag{8.72}$$

$$= [(1-\zeta_p)\tilde{p}_t^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} + \zeta_p(\pi_*\dot{p}_{t-1}\pi_t^{-1})^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}}]^{-\frac{\lambda_{f,t}}{1+\lambda_{f,t}}}$$

While

$$L_t = \left(\frac{\dot{W}_t}{W_t}\right)^{\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} \dot{L}_t$$

becomes

$$L_t = (\dot{w}_t)^{\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} \dot{L}_t \tag{8.73}$$

where

$$\dot{w}_t = \frac{\dot{W}_t}{W_t}$$

$$= [(1-\zeta_w)(\frac{\tilde{W}_t}{W_t})^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} + \zeta_w(\pi_*e^\gamma\Upsilon^{\frac{\alpha}{1-\alpha}}\frac{\dot{W}_{t-1}}{W_t})^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}}]^{-\frac{\lambda_{w,t}}{1+\lambda_{w,t}}} \tag{8.74}$$

$$= [(1-\zeta_w)\tilde{w}_t^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} + \zeta_w(\pi_*e^\gamma\Upsilon^{\frac{\alpha}{1-\alpha}}\pi_t^{-1}e^{-z_t^*}\frac{w_{t-1}}{w_t}\dot{w}_{t-1})^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}}]^{-\frac{\lambda_{w,t}}{1+\lambda_{w,t}}}$$

At steady state we have:

$$\frac{1}{1-g_*}(c_* + i_*) = y_*. \tag{8.75}$$

and

$$y_* = k_*^\alpha L_*^{1-\alpha} - \Phi. \tag{8.76}$$

and

$$\dot{y}_* = y_*, \quad \dot{L}_* = L_*.$$

8.3 Steady State

For now treat L^* as a parameter (we will see that the real variables are all defined as a ratio to L_*, so L_* is just a normalization constant). Define the real rate

$$r_* = \frac{R_*}{\pi_*}, \tag{8.77}$$

then from 8.52 we have:

$$r_* = \beta^{-1}e^{\sigma_c z_*^*}. \tag{8.78}$$

From 8.62:

$$r_*^k = r_* \Upsilon - (1 - \delta). \tag{8.79}$$

From 8.42:

$$w_* = \left(\frac{1}{1 + \lambda_f} \alpha^\alpha (1 - \alpha)^{(1-\alpha)} r_*^{k \ -\alpha} \right)^{\frac{1}{1-\alpha}} \tag{8.80}$$

From 8.47

$$k_* = \frac{\alpha}{1 - \alpha} \frac{w_*}{r_*^k} L_*. \tag{8.81}$$

From 8.56 and 8.57:

$$\bar{k}_* = e^\gamma \Upsilon^{\frac{1}{1-\alpha}} k_*, \tag{8.82}$$

$$i_* = \left(1 - (1 - \delta) e^{-\gamma} \Upsilon^{-\frac{1}{1-\alpha}} \right) \bar{k}_*. \tag{8.83}$$

From 8.76:

$$y_* = k_*^\alpha L_*^{1-\alpha} - \Phi. \tag{8.84}$$

SW use the reparameterization $\Phi_p = \frac{y_* + \Phi}{y_*}$, implying that steady state output is given by:

$$y_* = \frac{k_*^\alpha L_*^{1-\alpha}}{\Phi_p}. \tag{8.85}$$

From 8.75:

$$c_* = (1 - g_*)y_* - i_*, \tag{8.86}$$

(as opposed to $c_* = \frac{y_*}{g_*} - i_*$ in DSSW). (Aside: note that 8.53 implies

$$c_*(1 - he^{-z_*^*})L_*^{\nu_l} = w_*.$$

Since we already have c_* and w_* it would seem L_* is given. In fact, this is because SW do not use the parameter φ, which would make 8.53 hold for any L_*.)

8.4 Log-linear

1. If technology is stationary, eq. 8.68 becomes:

$$\hat{y}_t = \hat{\tilde{g}}_t - \frac{1}{1 - \alpha} \tilde{z}_t + \frac{c_*}{y_*} \hat{c}_t + \frac{i_*}{y_*} \hat{i}_t + \frac{r_*^k k_*}{y_*} \hat{u}_t, \tag{8.87}$$

If technology has a unit root, eq. 8.69 becomes:

$$\hat{y}_t = \hat{g}_t + \frac{c_*}{y_*}\hat{c}_t + \frac{i_*}{y_*}\hat{i}_t + \frac{r_*^k k_*}{y_*}\hat{u}_t, \tag{8.88}$$

This is one of the two equilibrium conditions for which we need to write two different versions for the stationary and non-stationary case (the other being the production function). The difference with DSSW (eq. 1.2.77) are due to a different definition of the government spending process which in SW is given by $g_t = \frac{G_t}{y_* Z_t^*}$, with $g_* = 1 - \frac{c_* + i_*}{y_*}$ (in our case, $g_t = \frac{Y_t}{Y_t - G_t}$ and $g_* = \frac{y_*}{c_* + i_*}$). Note that in SW $g_* \in (0, 1)$. This is eq. (1) in SW using the reparameterizations

$$c_y = \frac{c_*}{y_*}, \quad i_y = \frac{i*}{y_*}, \quad z_y = r_*^k \frac{k_*}{y_*}.$$

2. Eq. 8.49 becomes:

$$\hat{\xi}_t = \hat{R}_t + \hat{b}_t + \mathbb{E}_t[\hat{\xi}_{t+1}] - \mathbb{E}_t[\hat{\pi}_{t+1}] - \sigma_c \mathbb{E}_t[\hat{z}_{t+1}], \tag{8.89}$$

and eq. 8.48 becomes:

$$\hat{\xi}_t = -\sigma_c(1 - he^{-z_*^*})^{-1}\left(\hat{c}_t - he^{-z_*^*}\hat{c}_{t-1} + he^{-z_*^*}z_t\right) + (\sigma_c - 1)L_*^{1+\nu_l}\hat{L}_t,$$

which becomes using 8.53:

$$\frac{(1 - he^{-z_*^*})}{\sigma_c}\hat{\xi}_t = -\left(\hat{c}_t - he^{-z_*^*}\hat{c}_{t-1} + he^{-z_*^*}z_t\right) + \frac{(\sigma_c - 1)}{\sigma_c}\frac{w_* L_*}{c_*}\hat{L}_t. \tag{8.90}$$

Putting 8.89 and 8.90 two together we obtain:

$$\hat{c}_t = -\frac{(1 - he^{-z_*^*})}{\sigma_c(1 + he^{-z_*^*})}\left(\hat{R}_t - \mathbb{E}_t[\hat{\pi}_{t+1}] + \hat{b}_t\right) + \frac{he^{-z_*^*}}{(1 + he^{-z_*^*})}\left(\hat{c}_{t-1} - \hat{z}_t\right)$$
$$+ \frac{1}{(1 + he^{-z_*^*})}\mathbb{E}_t\left[\hat{c}_{t+1} + \hat{z}_{t+1}\right] + \frac{(\sigma_c - 1)}{\sigma_c(1 + he^{-z_*^*})}\frac{w_* L_*}{c_*}\left(\hat{L}_t - \mathbb{E}_t[\hat{L}_{t+1}]\right). \tag{8.91}$$

This corresponds to eq. (2) in SW, and to the combination of eqs 1.2.68 and 1.2.66 in DSSW. In the code we follow SW's code and use the normalization:

$$\hat{\tilde{b}}_t = -\frac{(1 - he^{-z_*^*})}{\sigma_c(1 + he^{-z_*^*})}\hat{b}_t. \tag{8.92}$$

3. Eq. 8.58 becomes:

$$\hat{i}_t = \frac{1}{S'' e^{2z_*^*}(1 + \beta e^{(1-\sigma_c)z_*^*})} \hat{q}_t^k + \frac{1}{1 + \beta e^{(1-\sigma_c)z_*^*}} \left(\hat{i}_{t-1} - \hat{z}_t\right)$$
$$+ \frac{\beta e^{(1-\sigma_c)z_*^*}}{1 + \beta e^{(1-\sigma_c)z_*^*}} \mathbb{E}_t\left[\hat{i}_{t+1} + \hat{z}_{t+1}\right] + \hat{\mu}_t, \quad (8.93)$$

where we follows SW and renormalize the process $\hat{\mu}_t$ by dividing it for $S'' e^{2z_*^*}(1 + \beta e^{(1-\sigma_c)z_*^*})$. This is eq. (3) in SW, and corresponds to eq. 1.2.71 in DSSW (which was expressed in terms of ξ_t^k). The equation can be expressed, perhaps more intuitively, in terms of $\hat{q}_t^k$:

$$\hat{q}_t^k = S'' e^{2z_*^*}(1 + \beta e^{(1-\sigma_c)z_*^*}) \left(\hat{i}_t - \frac{1}{1 + \beta e^{(1-\sigma_c)z_*^*}} \left(\hat{i}_{t-1} - \hat{z}_t\right)\right.$$
$$\left. - \frac{\beta e^{(1-\sigma_c)z_*^*}}{1 + \beta e^{(1-\sigma_c)z_*^*}} \mathbb{E}_t\left[\hat{i}_{t+1} + \hat{z}_{t+1}\right] - \hat{\mu}_t\right), \quad (8.94)$$

4. Eq. 8.59 becomes

$$\frac{r_*^k}{r_*^k + (1-\delta)} \mathbb{E}_t[r_{t+1}^k] + \frac{1-\delta}{r_*^k + (1-\delta)} \mathbb{E}_t[q_{t+1}^k] - \hat{q}_t^k = \hat{R}_t + \hat{b}_t - \mathbb{E}_t[\hat{\pi}_{t+1}] \quad (8.95)$$

where we used 8.89. This is eq. (4) in SW (using the value of r_*^k one can see they correspond) and is the same as eq. 1.2.72 in DSSW, except that this was expressed in terms of ξ_t^k. In the code we use the normalization 8.92, consistently with 8.91.

5. Eq. 1.2.78 becomes:

$$\hat{y}_t = \Phi_p\left(\alpha\hat{k}_t + (1-\alpha)\hat{L}_t\right) + (\Phi_p - 1)\frac{1}{1-\alpha}\tilde{z}_t. \quad (8.96)$$

This is eq. (5) in SW. Note that the last term in 8.96 is non-stationary if $\rho_z = 1$. So in this case it needs to be dropped (which amounts to assuming the fixed costs are proportional to Z_t as opposed to just $e^{\gamma t}$) and the eq. becomes

$$\hat{y}_t = \Phi_p\left(\alpha\hat{k}_t + (1-\alpha)\hat{L}_t\right). \quad (8.97)$$

6. Eq. 1.2.69 remains the same:

$$\hat{k}_t = \hat{u}_t - \hat{z}_t + \hat{\bar{k}}_{t-1}. \quad (8.98)$$

This is eq. (6) in SW.

7. Equations 8.99 becomes:

$$\frac{1-\psi}{\psi}\hat{r}_t^k = u_t. \qquad (8.99)$$

where $\frac{1-\psi}{\psi}$ is simply a reparameterization of the ratio $\frac{r_*^k}{a''}$ that appears in 1.2.73. This is eq. (7) in SW with $z_1 = \frac{1-\psi}{\psi}$, and eq. 1.2.73 in DSSW.

8. Eq. 8.55 becomes:

$$\hat{\bar{k}}_t = (1 - \frac{i_*}{\bar{k}_*})\left(\hat{\bar{k}}_{t-1} - \hat{z}_t\right) + \frac{i_*}{\bar{k}_*}\hat{i}_t + \frac{i_*}{\bar{k}_*}S'' e^{2z_*^*}(1 + \beta e^{(1-\sigma_c)z_*^*})\hat{\mu}_t. \qquad (8.100)$$

This is eq. (8) in SW, and corresponds to eq. 1.2.70 in DSSW, except for the renormalization of the exogenous process μ_t. Note that in SW's code the term $(1 + \beta e^{(1-\sigma_c)z_*^*})$ is erroneously omitted from the coefficient multiplying $\hat{mu}_t$.

9. Eq. 1.2.61 remains the same as in DSSW:

$$\hat{mc}_t = (1-\alpha)\,\hat{w}_t + \alpha\,\hat{r}_t^k. \qquad (8.101)$$

This is eq. (9) in SW, where $\hat{\mu}_t^p = -\hat{mc}_t$ and where they used (8.104) to substitute for $\hat{r}_t^k$. That is actually what we also do in the code, obtaining:

$$\hat{mc}_t = \hat{w}_t + \alpha\hat{L}_t - \alpha\hat{k}_t. \qquad (8.102)$$

10. Eq. (TO DO) becomes:

$$\hat{\pi}_t = \frac{(1 - \zeta_p\beta e^{(1-\sigma_c)z_*^*})(1 - \zeta_p)}{(1 + \iota_p\beta e^{(1-\sigma_c)z_*^*})\zeta_p((\Phi_p - 1)\epsilon_p + 1)}\hat{mc}_t$$
$$+ \frac{\iota_p}{1 + \iota_p\beta e^{(1-\sigma_c)z_*^*}}\hat{\pi}_{t-1} + \frac{\beta e^{(1-\sigma_c)z_*^*}}{1 + \iota_p\beta e^{(1-\sigma_c)z_*^*}}\mathbb{E}_t[\hat{\pi}_{t+1}] + \hat{\lambda}_{f,t} \qquad (8.103)$$

This is eq. (10) in SW.

11. Eq. 1.2.65 remains the same:

$$\hat{k}_t = \hat{w}_t - \hat{r}_t^k + \hat{L}_t. \qquad (8.104)$$

This is eq. (11) in SW.

12. Eq. 8.50, which essentially defines the household's marginal rate of subsitution between consumption and labor,

$$\frac{1}{1 - he^{-z_*^*}} \left(\hat{c}_t - he^{-z_*^*} \hat{c}_{t-1} + he^{-z_*^*} \hat{z}_t \right) + \nu_l \hat{L}_t = \hat{w}_t^h. \qquad (8.105)$$

This corresponds to eq. (12) in SW, except that they express it in terms of the markup $\hat{\mu}_t^w = \hat{w}_t - \hat{w}_t^h$, which is what we also do in our code. DSSW did not have this equation as we plugged it the wage Phillips curve directly.

13. Eq. (TO DO) becomes:

$$\hat{w}_t = \frac{(1 - \zeta_w \beta e^{(1-\sigma_c)z_*^*})(1 - \zeta_w)}{(1 + \beta e^{(1-\sigma_c)z_*^*})\zeta_w((\lambda_w - 1)\epsilon_w + 1)} \left(\hat{w}_t^h - \hat{w}_t \right)$$
$$- \frac{1 + \iota_w \beta e^{(1-\sigma_c)z_*^*}}{1 + \beta e^{(1-\sigma_c)z_*^*}} \hat{\pi}_t + \frac{1}{1 + \beta e^{(1-\sigma_c)z_*^*}} \left(\hat{w}_{t-1} - \hat{z}_t - \iota_w \hat{\pi}_{t-1} \right)$$
$$+ \frac{\beta e^{(1-\sigma_c)z_*^*}}{1 + \beta e^{(1-\sigma_c)z_*^*}} E_t \left[\hat{w}_{t+1} + \hat{z}_{t+1} + \hat{\pi}_{t+1} \right] + \hat{\lambda}_{w,t} \quad (8.106)$$

This is eq. (13) in SW. In the code we follow SW and replace $\hat{w}_t^h - \hat{w}_t$ with $-\hat{\mu}_t^w$.

14. Eq. 1.2.79 becomes:

$$\hat{R}_t = \rho_R \hat{R}_{t-1} + (1 - \rho_R) \left(\psi_1 \hat{\pi}_t + \psi_2 (\hat{y}_t - \hat{y}_t^f) \right)$$
$$+ \psi_3 \left((\hat{y}_t - \hat{y}_t^f) - (\hat{y}_{t-1} - \hat{y}_{t-1}^f) \right) + \hat{r}_t^m \quad (8.107)$$

where the differences are (1) the use of flexible price/wage output to measure the output gap, (2) the addition of the term $\psi_3 \left((\hat{y}_t - \hat{y}_t^f) - (\hat{y}_{t-1} - \hat{y}_{t-1}^f) \right)$; (3) the fact that the residual $\hat{r}_t^m$ is autocorrelated.

15. Additional flexible price/wages equations, where we replace the real rate r_t^f for $\hat{R}_t - \mathbb{E}_t[\hat{\pi}_{t+1}]$:

$$\hat{y}_t^f = \hat{g}_t + \frac{c_*}{y_*}\hat{c}_t^f + \frac{i_*}{y_*}\hat{i}_t^f + \frac{r_*^k k_*}{y_*}\hat{u}_t^f, \tag{8.108}$$

$$\hat{c}_t^f = -\frac{(1-he^{-z_*^*})}{\sigma_c(1+he^{-z_*^*})}\left(\hat{r}_t^f + \hat{b}_t\right) + \frac{he^{-z_*^*}}{(1+he^{-z_*^*})}\left(\hat{c}_{t-1}^f - \hat{z}_t\right) + \frac{1}{(1+he^{-z_*^*})}\mathbb{E}_t[\hat{c}_{t+1}^f + \hat{z}_{t+1}]$$

$$+ \frac{(\sigma_c - 1)}{\sigma_c(1+he^{-z_*^*})}\frac{w_* L_*}{c_*}\left(\hat{L}_t^f - \mathbb{E}_t[\hat{L}_{t+1}^f]\right), \tag{8.109}$$

$$\hat{q}_t^{kf} = S''e^{2z_*^*}(1+\beta e^{(1-\sigma_c)z_*^*})\left(\hat{i}_t^f - \frac{1}{1+\beta e^{(1-\sigma_c)z_*^*}}\left(\hat{i}_{t-1}^f - \hat{z}_t\right)\right.$$

$$\left. - \frac{\beta e^{(1-\sigma_c)z_*^*}}{1+\beta e^{(1-\sigma_c)z_*^*}}\mathbb{E}_t\left[\hat{i}_{t+1}^f + \hat{z}_{t+1}\right] - \hat{\mu}_t\right), \tag{8.110}$$

$$\hat{r}_t^f = \frac{r_*^k}{r_*^k + (1-\delta)}\mathbb{E}_t[r_{t+1}^{kf}] + \frac{1-\delta}{r_*^k + (1-\delta)}\mathbb{E}_t[q_{t+1}^{kf}] - \hat{q}_t^{kf} - \hat{b}_t \tag{8.111}$$

$$\hat{y}_t^f = \Phi_p\left(\alpha\hat{k}_t^f + (1-\alpha)\hat{L}_t^f\right) + (\Phi_p - 1)\frac{1}{1-\alpha}\tilde{z}_t, \tag{8.112}$$

$$\hat{k}_t^f = \hat{u}_t^f - \hat{z}_t + \hat{\bar{k}}_{t-1}^f, \tag{8.113}$$

$$u_t^f = \frac{1-\psi}{\psi}\hat{r}_t^{kf}, \tag{8.114}$$

$$\hat{\bar{k}}_t^f = (1 - \frac{i_*}{\bar{k}_*})\left(\hat{\bar{k}}_{t-1}^f - \hat{z}_t\right) + \frac{i_*}{\bar{k}_*}\hat{i}_t^f + \frac{i_*}{\bar{k}_*}S''e^{2z_*^*}(1+\beta e^{(1-\sigma_c)z_*^*})\hat{\mu}_t. \tag{8.115}$$

$$0 = (1-\alpha)\,\hat{w}_t^f + \alpha\,\hat{r}_t^{kf}. \tag{8.116}$$

$$\hat{k}_t^f = \hat{w}_t^f - \hat{r}_t^{kf} + \hat{L}_t^f, \tag{8.117}$$

$$\hat{w}_t^f = \frac{1}{1-he^{-z_*^*}}\left(\hat{c}_t^f - he^{-z_*^*}\hat{c}_{t-1}^f + he^{-z_*^*}\hat{z}_t\right) + \nu_l\hat{L}_t^f. \tag{8.118}$$

16. The exogenous processes are described in section 8.1.5.

8.5 Adding BGG financial frictions to SW

Amounts to replacing 8.95 with conditions 7.50 and 7.52 (see section 7), which we repeat here for convenience:

$$E_t\left[\widehat{\tilde{R}}_{t+1}^k - \hat{R}_t\right] = -\hat{b}_t\zeta_{sp,b}\left(\hat{q}_t^k + \widehat{\bar{k}}_t - \hat{n}_t\right) + \tilde{\sigma}_{\omega,t} + \tilde{\mu}_t^e \tag{8.119}$$

$$\widehat{\tilde{R}}_t^k - \pi_t = \frac{r_*^k}{r_*^k + (1-\delta)}\hat{r}_t^k + \frac{(1-\delta)}{r_*^k + (1-\delta)}\hat{q}_t^k - \hat{q}_{t-1}^k, \tag{8.120}$$

and adding the eq. condition 7.51 describing the evolution of entrepreneurial net worth

$$
\begin{aligned}
\hat{n}_t &= \zeta_{n,\tilde{R}^k}\left(\widehat{\tilde{R}}_t^k - \pi_t\right) - \zeta_{n,R}\left(\hat{R}_{t-1} - \pi_t\right) + \zeta_{n,qK}\left(\hat{q}_{t-1}^k + \widehat{\tilde{k}}_{t-1}\right) + \zeta_{n,n}\hat{n}_{t-1} \\
&\quad + \tilde{\gamma}_t + \frac{w_*^e}{n_*}\hat{w}_t^e - \gamma_* \frac{v_*}{n_*}\hat{z}_t - \frac{\zeta_{n,\mu^e}}{\zeta_{sp,\mu^e}}\tilde{\mu}_{t-1}^e - \frac{\zeta_{n,\sigma_\omega}}{\zeta_{sp,\sigma_\omega}}\tilde{\sigma}_{\omega,t-1}
\end{aligned}
\tag{8.121}
$$

Note that if $\zeta_{sp,b} = 0$ and the financial friction shocks are zero, 8.95 coincides with 7.50 plus 7.52. In particular, we stick to SW's assumption that returns to deposit are not subject to the same "intermediation cost" shock b_t as government bonds. This assumption mirrors SW's assumption that capital investment was not subject to that transaction cost.

8.6 Anticipated policy shocks

We modify the policy rule (8.107) so to incorporate anticipated policy shocks. In order to do so we add the anticipated shocks to the exogenous component of monetary policy as follows:

$$
\hat{r}_t^m = \rho_{r^m}\hat{r}_{t-1}^m + \sigma_r\varepsilon_{r^m,t} + \sum_{k=1}^{K}\sigma_{k,r}\varepsilon_{k,t-k}^R,
\tag{8.122}
$$

where $\varepsilon_{R,t}$ is the usual contemporaneous policy shock and $\varepsilon_{k,t-k}^R$ is a policy shock that is known to agents at time $t-k$, but affects the policy rule k periods later, that is, at time t. We assume as usual that $\varepsilon_{k,t-k}^R \sim N(0,1),\ i.i.d..$

In order to solve the model we need to express the anticipated shocks in recursive form. For this purpose, we augment the state vector s_t with K additional states $\nu_t^R,\ldots,\nu_{t-K}^R$ whose law of motion is as follows:

$$
\begin{aligned}
\nu_{1,t}^R &= \nu_{2,t-1}^R + \sigma_{1,r}\varepsilon_{1,t}^R \\
\nu_{2,t}^R &= \nu_{3,t-1}^R + \sigma_{2,r}\varepsilon_{2,t}^R \\
&\ \ \vdots \\
\nu_{K,t}^R &= \sigma_{K,r}\varepsilon_{K,t}^R
\end{aligned}
$$

and rewrite expression (8.123) as

$$
\hat{r}_t^m = \rho_{r^m}\hat{r}_{t-1}^m + \sigma_r\varepsilon_{r^m,t} + \nu_{1,t-1}^R,
\tag{8.123}
$$

It is easy to verify that $\nu_{1,t-1}^R = \sum_{k=1}^K \sigma_{k,r}\varepsilon_{k,t-k}^R$, that is, $\nu_{1,t-1}^R$ is a "bin" that collects all anticipated shocks that affect the policy rule in period t. In the implementation, we assume that these shocks have the same standard deviation as the contemporaneous shock: $\sigma_{k,r} = \sigma_r$.

8.7 Adding long run changes in productivity

We add long run changes in productivity. Specifically we assume that the production function is:

$$Y_t(i) = \max\{e^{\tilde{z}_t} K_t(i)^\alpha \left(L_t(i)e^{\gamma t}Z_t^p\right)^{1-\alpha} - \Phi Z_t^*, 0\}, \tag{8.124}$$

where $\tilde{z}_t$ and $z_t^p = log(Z_t^p/Z_{t-1}^p)$ follow AR(1) processes:

$$\tilde{z}_t = \rho_z\tilde{z}_{t-1} + \sigma_z\epsilon_{z,t}, \ \epsilon_{z,t} \sim N(0,1), \tag{8.125}$$

$$z_t^p = \rho_{z^p}z_{t-1}^p + \sigma_{z^p}\epsilon_{z^p,t}, \ \epsilon_{z^p,t} \sim N(0,1), \tag{8.126}$$

and

$$Z_t^* = Z_tZ_t^p e^{\left(\gamma+\frac{\alpha}{1-\alpha}\log\Upsilon\right)t}, \ Z_t = e^{\frac{1}{1-\alpha}\tilde{z}_t}. \tag{8.127}$$

We detrend by Z_t^* as in section 8.2. Define $z_t^* = \log(Z_t^*/Z_{t-1}^*)$, with $z_*^* = \gamma+\frac{\alpha}{1-\alpha}\log\Upsilon$ at st.state. From 8.126, 8.125, and 8.127 we see that

$$\hat{z}_t^* = z_t^* - z_*^* = \frac{1}{1-\alpha}(\rho_z - 1)\tilde{z}_{t-1} + \frac{1}{1-\alpha}\sigma_z\epsilon_{z,t} + z_t^p, \tag{8.128}$$

and

$$E_t[\hat{z}_{t+1}^*] = \frac{1}{1-\alpha}(\rho_z - 1)\tilde{z}_t + \rho_z z_t^p. \tag{8.129}$$

Note that we can accommodate both cases where $\tilde{z}_t$ is stationary and random walk ($\rho_z = 1$). Regardless, there is a stochastic trend in growth so the resource constraint and the production function need to be written as in eqs 8.88 and 8.97 in section 8.4.