(Reduced-Form) Vector Autoregressions

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Model Specification

- A VAR(p) is a multivariate generalization of the AR(p) model:

$$y_t = \Phi_1 y_{t-1} + \ldots + \Phi_p y_{t-p} + u_t$$

- y_t is a $n \times 1$ random vector that takes values in $\mathbb{R}^n$.
- $u_t \sim iid(0, \Sigma)$ is a vector of reduced-form innovations.

Goals

- Derivation of likelihood function.
- Derivation of posterior for multivariate regression with unknown variance.
- Use prior distribution to cope with high-dimensional parameter space (regularization).

Likelihood Function

- Define the $(np+1) \times 1$ vector x_t as

$$x_t = [y'_{t-1}, \ldots, y'_{t-p}, 1]'$$

- Moreover, define the matrixes

$$Y = \begin{bmatrix} y'_1 \\ \vdots \\ y'_T \end{bmatrix}, \quad X = \begin{bmatrix} x'_1 \\ \vdots \\ x'_T \end{bmatrix}, \quad \Phi = [\Phi_1, \ldots, \Phi_p, \Phi_c]'$$

- The conditional density of y_t:

$$p(y_t | Y_{1:t-1}, Y_{1-p:0}, \Phi, \Sigma) \propto |\Sigma|^{-1/2} \exp\left\{ -\frac{1}{2}(y'_t - x'_t\Phi)\Sigma^{-1}(y'_t - x'_t\Phi)' \right\},$$

where $Y_{t_0:t_1} = [y_{t_0}, \ldots y_{t_1}]$.

Likelihood Function

- Take the product of the conditional densities of $y_1, \ldots, y_T$ to obtain the joint density.
- Let $Y_{1-p:0}$ be a vector with initial observations

$$\begin{aligned} p(Y_{1:T}|Y_{1-p:0}, \Phi, \Sigma) &= \prod_{t=1}^{T} p(y_t|Y_{1:t-1}, Y_{1-p:0}, \Phi, \Sigma) \\ &\propto |\Sigma|^{-T/2} \exp\left\{-\frac{1}{2}\sum_{t=1}^{T}(y_t' - x_t'\Phi)\Sigma^{-1}(y_t' - x_t'\Phi)'\right\} \end{aligned}$$

Some Matrix Algebra

- **Lemma:** Let A and B be two $n \times n$ matrices, then

$$\text{tr}[A + B] = \text{tr}[A] + \text{tr}[B] \quad \square$$

- **Lemma:** Let a be a $n \times 1$ vector, B be a symmetric positive definite $n \times n$ matrix, and tr the trace operator that sums the diagonal elements of a matrix. Then

$$a'Ba = \text{tr}[Baa'] \quad \square$$

- **Deduce**:

$$\sum_{t=1}^{T}(y_t' - x_t'\Phi)\Sigma^{-1}(y_t' - x_t'\Phi)' = \text{tr}\left[\Sigma^{-1}\sum_{t=1}^{T}(y_t' - x_t'\Phi)'(y_t' - x_t'\Phi)\right].$$

Likelihood Function

$$
\begin{aligned}
& p(Y_{1:T}|Y_{1-p:0}, \Phi, \Sigma) \\
& \quad \propto \quad |\Sigma|^{-T/2} \exp\left\{-\frac{1}{2}\mathrm{tr}\left[\Sigma^{-1}\sum_{t=1}^{T}(y_t' - x_t'\Phi)'(y_t' - x_t'\Phi)\right]\right\} \\
& \quad \propto \quad |\Sigma|^{-T/2} \exp\left\{-\frac{1}{2}\mathrm{tr}[\Sigma^{-1}(Y - X\Phi)'(Y - X\Phi)]\right\}
\end{aligned}
$$

Likelihood Function

- Define the "OLS" estimator

$$\hat{\Phi} = (X'X)^{-1}X'Y.$$

- Define the sum of squared OLS residual matrix

$$\hat{S} = (Y - X\hat{\Phi})'(Y - X\hat{\Phi}) = Y'Y - Y'X(X'X)^{-1}X'Y.$$

- It can be verified that

$$(Y - X\Phi)'(Y - X\Phi) = (\Phi - \hat{\Phi})'X'X(\Phi - \hat{\Phi}) + \hat{S}$$

Likelihood Function

- This leads to the following representation of the likelihood function

$$p(Y|\Phi,\Sigma) \propto |\Sigma|^{-T/2}\exp\left\{-\frac{1}{2}tr[\Sigma^{-1}\hat{S}]\right\} \times \exp\left\{-\frac{1}{2}tr[\Sigma^{-1}(\Phi-\hat{\Phi})'X'X(\Phi-\hat{\Phi})]\right\}.$$

- Let $\beta = vec(\Phi)$ and $\hat{\beta} = vec(\hat{\Phi})$. It can be verified that

$$tr[\Sigma^{-1}(\Phi-\hat{\Phi})'X'X(\Phi-\hat{\Phi})] = (\beta-\hat{\beta})'[\Sigma\otimes(X'X)^{-1}]^{-1}(\beta-\hat{\beta}).$$

- The likelihood function has the alternative representation

$$p(Y|\Phi,\Sigma) \propto |\Sigma|^{-T/2}\exp\left\{-\frac{1}{2}tr[\Sigma^{-1}\hat{S}]\right\} \times \exp\left\{-\frac{1}{2}(\beta-\hat{\beta})'[\Sigma\otimes(X'X)^{-1}]^{-1}(\beta-\hat{\beta})\right\}.$$

"Inverting" the Likelihood Function

- Suppose that

$$p(\Phi, \Sigma) \propto c.$$

- Then

$$p(\Phi, \Sigma | Y) \propto p(Y | \Phi, \Sigma).$$

Background: Matricvariate Normal Distribution

- Suppose that the random matrix Φ has density

$$p(\Phi|\Sigma, X'X) \propto |\Sigma \otimes (X'X)^{-1}|^{-1/2} \exp\left\{-\frac{1}{2} tr[\Sigma^{-1}(\Phi - \hat{\Phi})'X'X(\Phi - \hat{\Phi})]\right\}$$

  then $\Phi|(\Sigma, X'X)$ is matricvariate normal

$$MN(\hat{\Phi}, \Sigma \otimes (X'X)^{-1}).$$

- Let $\beta = vec(\Phi)$ and $\hat{\beta} = vec(\hat{\Phi})$. Then

$$\beta|\Sigma, X'X \sim N\left(\hat{\beta}, \Sigma \otimes (X'X)^{-1}\right).$$

- To generate a draw Z from a multivariate $N(\mu, \Sigma)$, decompose $\Sigma = CC'$. E.g., C could be the lower triangular Cholesky factor. Then let $Z = \mu + C \cdot N(0, I)$.

Background: Inverted Wishart Distribution

- Let Σ be a $n \times n$ positive definite random matrix. Σ has the Inverted Wishart $IW(S, \nu)$ distribution if its density is of the form

$$p(\Sigma|S, \nu) \propto |S|^{\nu/2}|\Sigma|^{-(\nu+n+1)/2} \exp\left\{-\frac{1}{2} tr[\Sigma^{-1}S]\right\}$$

- To sample a Σ from an Inverted Wishart $IW(S, \nu)$ distribution, draw $n \times 1$ vectors $Z_1, \ldots, Z_\nu$ from a multivariate normal $N(0, S^{-1})$ and let

$$\Sigma = \left[\sum_{i=1}^{\nu} Z_i Z_i'\right]^{-1}$$

Likelihood Function Interpreted as PDF for (Φ, Σ) – Step 1: $p(\Phi|\Sigma, Y)$

- Interpret the likelihood as density for (Φ, Σ):

$$
\begin{aligned}
& p(\Phi, \Sigma|Y) \\
& \propto |\Sigma|^{-T/2} \exp\left\{-\frac{1}{2} tr[\Sigma^{-1}\hat{S}]\right\} \\
& \quad \times \exp\left\{-\frac{1}{2} tr[\Sigma^{-1}(\Phi - \hat{\Phi})' X' X (\Phi - \hat{\Phi})]\right\} \\
& \propto |\Sigma|^{-T/2} |\Sigma \otimes (X'X)^{-1}|^{1/2} \exp\left\{-\frac{1}{2} tr[\Sigma^{-1}\hat{S}]\right\} \\
& \quad \times (2\pi)^{-nk/2} |\Sigma \otimes (X'X)^{-1}|^{-1/2} \\
& \quad \times \exp\left\{-\frac{1}{2} tr[\Sigma^{-1}(\Phi - \hat{\Phi})' X' X (\Phi - \hat{\Phi})]\right\}.
\end{aligned}
$$

- Thus,

$$\Phi|(\Sigma, Y) \sim MN(\hat{\phi}, \Sigma \otimes (X'X)^{-1}).$$

Likelihood Function Interpreted as PDF for (Φ, Σ) – Step 2: $p(\Sigma|Y)$

- Compute $p(\Sigma|Y) \propto \int p(Y|\Phi, \Sigma)d\Phi$...
- Note that:

$$|\Sigma \otimes (X'X)^{-1}|^{1/2} = |\Sigma|^{k/2}|X'X|^{-n/2}.$$

- Therefore,

$$p(\Sigma|Y) \propto |\Sigma|^{-(T-k)/2}|X'X|^{-n/2} \exp\left\{-\frac{1}{2}tr[\Sigma^{-1}\hat{S}]\right\}.$$

- Deduce

$$\Sigma|Y \sim IW(\hat{S}, T-k-n-1), \quad \Phi|(\Sigma, Y) \sim MN(\hat{\phi}, \Sigma \otimes (X'X)^{-1}).$$

- Write as

$$(\Phi, \Sigma)|Y \sim MNIW\left(\hat{\Phi}, (X'X)^{-1}, \hat{S}, T-k-n-1\right).$$

Bayesian Analysis with Improper Prior

- Replace (improper) prior $p(\Phi, \Sigma) \propto c$ by (improper) prior:

$$p(\Phi, \Sigma) \propto |\Sigma|^{-(n+1)/2}.$$

- Then, the posterior is obtained from

$$p(\Phi, \Sigma | Y) \propto p(Y | \Phi, \Sigma) p(\Phi, \Sigma).$$

- Our previous analysis implies that

$$(\Phi, \Sigma) | Y \sim MNIW\left(\hat{\Phi}, (X'X)^{-1}, \hat{S}, T - k\right).$$

Algorithm: Direct Sampling of VAR Parameters

For $s = 1, \ldots, n_{sim}$:

1. Draw $\Sigma^{(s)}$ from an $IW(\hat{S}, T-k)$ distribution.
2. Draw $\Phi^{(s)}$ from the conditional distribution $MN(\hat{\Phi}, \Sigma^{(s)} \otimes (X'X)^{-1})$. $\square$

The Use of Priors for VARs

- Priors are used to "regularize" the VAR likelihood and cope with the dimensionality problem: the number of free parameters is often large relative to the number of observations.
- Priors add information to the estimation problem.

Prior with Kronecker Structure

- Consider the prior:

$$\Sigma \sim IW(\underline{\nu}, \underline{S}), \quad \Phi|\Sigma \sim MN(\underline{\mu}_\Phi, \Sigma \otimes \underline{P}_\Phi^{-1}), \quad .$$

- Prior density:

$$\begin{aligned} p(\Phi, \Sigma) &= (2\pi)^{-nk/2}|\Sigma|^{-k/2}|\underline{P}_\Phi|^{n/2} \exp\left\{-\frac{1}{2}\mathrm{tr}\left[\Sigma^{-1}(\Phi - \underline{\mu}_\Phi)'\underline{P}_\Phi(\Phi - \underline{\mu}_\Phi)\right]\right\} \\ &\quad \times \underline{C}_{IW}|\Sigma|^{-(\underline{\nu}+n+1)/2} \exp\left\{-\frac{1}{2}\mathrm{tr}\left[\Sigma^{-1}\underline{S}\right]\right\}. \end{aligned}$$

- $\underline{C}_{IW}$ is the normalization constant of the IW prior.

Posterior

$$
\begin{aligned}
& p(Y|\Phi,\Sigma)p(\phi,\Sigma) \\
& \quad \propto \quad (2\pi)^{-nk/2}|\Sigma|^{-k/2}|\underline{P}_\Phi|^{n/2}\underline{C}_{IW}|\Sigma|^{-(\underline{\nu}+n+1)/2}(2\pi)^{-nT/2}|\Sigma|^{-T/2} \\
& \qquad \times \exp\left\{-\frac{1}{2}\mathrm{tr}\left[\Sigma^{-1}(\Phi-\underline{\mu}_\Phi)'\underline{P}_\Phi(\Phi-\underline{\mu}_\Phi)\right]\right\}\exp\left\{-\frac{1}{2}\mathrm{tr}\left[\Sigma^{-1}(\Phi-\hat{\Phi})X'X(\Phi-\hat{\Phi})\right]\right\} \\
& \qquad \times \exp\left\{-\frac{1}{2}\mathrm{tr}\left[\Sigma^{-1}\underline{S}\right]\right\}\exp\left\{-\frac{1}{2}\mathrm{tr}\left[\Sigma^{-1}\hat{S}\right]\right\}
\end{aligned}
$$

Posterior

- Define

$$\bar{P}_\Phi = \underline{P}_\Phi + X'X, \quad \bar{\mu}_\Phi = \bar{P}_\Phi^{-1}\left[\underline{P}_\Phi \underline{\mu}_\Phi + X'X\hat{\Phi}\right].$$

- Write

$$\begin{aligned} & p(Y|\Phi,\Sigma)p(\phi,\Sigma) \\ = \; & (2\pi)^{-nk/2}|\Sigma|^{-k/2}|\underline{P}_\Phi|^{n/2}\underline{C}_{IW}|\Sigma|^{-(\underline{\nu}+n+1)/2}(2\pi)^{-nT/2}|\Sigma|^{-T/2}|\bar{P}_\phi|^{n/2}|\bar{P}_\phi|^{-n/2} \\ & \times \exp\left\{-\frac{1}{2}\mathrm{tr}\left[\Sigma^{-1}(\Phi-\bar{\mu}_\Phi)\bar{P}_\phi(\Phi-\bar{\mu}_\Phi)\right]\right\} \\ & \times \exp\left\{-\frac{1}{2}\mathrm{tr}\left[\Sigma^{-1}\left(\underline{S}+\underline{\mu}_\Phi'\underline{P}_\Phi\underline{\mu}_\Phi + Y'Y - \bar{\mu}_\Phi'\bar{P}_\Phi\bar{\mu}_\Phi\right)\right]\right\} \end{aligned}$$

Posterior

- Define

$$\bar{S} = \underline{S} + \underline{\mu}_\Phi' \underline{P}_\Phi \underline{\mu}_\Phi + Y'Y - \bar{\mu}_\Phi' \bar{P}_\Phi \bar{\mu}_\Phi, \quad \bar{\nu} = \underline{\nu} + T$$

- Deduce that

$$\Sigma | Y \sim IW(\bar{S}, \bar{\nu}), \quad \Phi | (\Sigma, Y) \sim MN(\bar{\mu}_\Phi, \Sigma \otimes \bar{P}^{-1}).$$

- Draws from the posterior can be generated by direct sampling.

Marginal Data Density

- Let $\bar{C}_{IW}$ be the normalization constant of the posterior IW distribution. Then,

$$\begin{aligned} p(Y) &= \int\int p(Y|\Phi,\Sigma)p(\phi,\Sigma)d\Phi d\Sigma \\ &= (2\pi)^{-nT/2}\int |\underline{P}_\Phi|^{k/2}|\bar{P}_\phi|^{-k/2}\underline{C}_{IW}|\Sigma|^{-(\bar{\nu}+n+1)/2}\exp\left\{-\frac{1}{2}\mathrm{tr}\left[\Sigma^{-1}\bar{S}\right]\right\}d\Sigma \\ &= (2\pi)^{-nT/2}\frac{|\underline{P}_\Phi|^{n/2}}{|\bar{P}_\Phi|^{n/2}}\frac{\underline{C}_{IW}}{\bar{C}_{IW}}, \end{aligned}$$

- where

$$\frac{\underline{C}_{IW}}{\bar{C}_{IW}} = \frac{|\underline{S}|^{\underline{\nu}/2}}{|\bar{S}|^{\bar{\nu}/2}}\frac{2^{n_y\bar{\nu}/2}\prod_{i=1}^{n_y}\Gamma((\bar{\nu}+1-i)/2)}{2^{n_y\underline{\nu}/2}\prod_{i=1}^{n_y}\Gamma((\underline{\nu}+1-i)/2)}.$$

Hierarchical Models and Hyperparameter Selection

- "Performance" of Bayesian VAR is sensitive to prior variance.
- Choose prior variance in a data-driven way.
- Hierarchical model

$$p(Y|\Phi,\Sigma)p(\Phi,\Sigma|\lambda)p(\lambda),$$

where λ controls features of the prior.

Selection vs. Averaging

- **Selection:**
  - Compute

$$p(Y|\lambda) = \int p(Y|\Phi, \Sigma)p(\Phi, \Sigma|\lambda)d(\Phi, \Sigma).$$

  - Define: $\hat{\lambda} = \text{argmax}\, p(Y|\lambda)$.
  - Work with $p(\Phi, \Sigma|Y, \hat{\lambda})$.
- **Averaging:**
  - Use prior $p(\lambda)$
  - Factorize posterior as

$$p(\Phi, \Sigma, \lambda|Y) = p(\Phi, \Sigma|Y, \lambda)p(\lambda|Y),$$

    where $p(\lambda|Y) \propto p(Y|\lambda)p(\lambda)$.

Illustration: Marginal Likelihood of λ

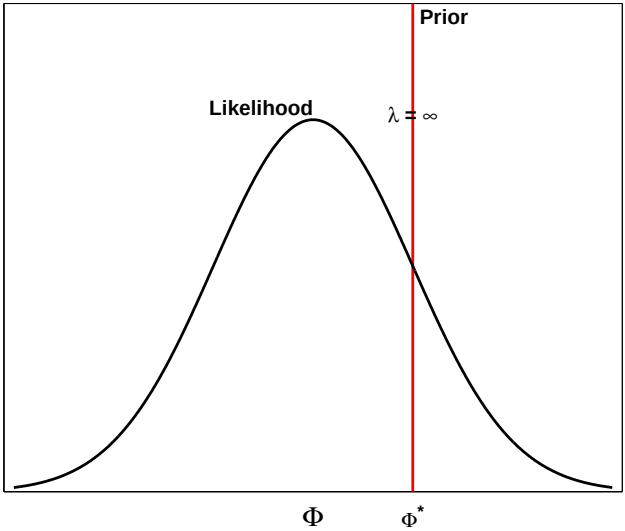
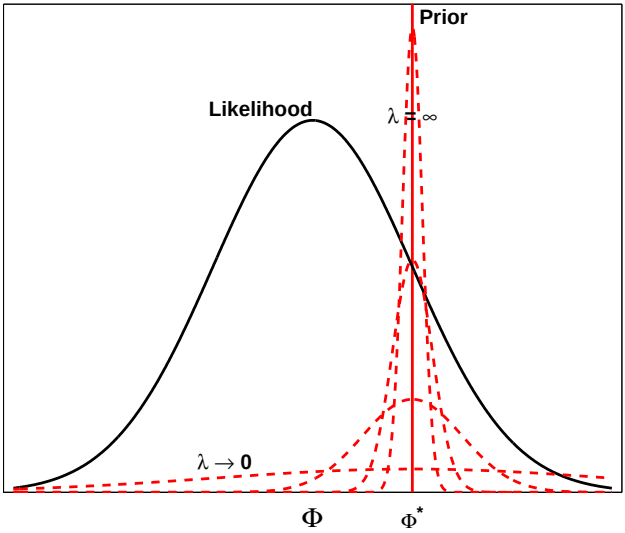


Illustration: Marginal Likelihood of λ



Example: Marginal Likelihood of λ

- Suppose the VAR takes the special form of an AR(1) model:

$$y_t = \phi y_{t-1} + u_t, \quad u_t \sim iidN(0,1)$$

- Suppose the prior takes the form

$$\phi \sim N\left(\phi^*, \frac{1}{\lambda T \gamma_0}\right).$$

  where $\gamma_0 = 1/(1-\phi_*^2)$

- Define $\gamma_1 = \phi_* \gamma_0$ and denote the sample autocovariances by

$$\hat{\gamma}_0 = \frac{1}{T}\sum y_t^2, \quad \hat{\gamma}_1 = \frac{1}{T}\sum y_t y_{t-1}.$$

- For convenience, we standardized the prior variance by T.

Example: Marginal Likelihood of λ

- After some algebra it can be shown that marginal likelihood of λ takes the following form

$$\ln p(Y|\lambda, \phi^*) = -T/2 \ln(2\pi) - \frac{T}{2}\tilde{\sigma}^2(\lambda, \phi^*) - \frac{1}{2}c(\lambda, \phi^*).$$

- The term $\tilde{\sigma}^2(\lambda, \phi^*)$ measures the in-sample one-step-ahead forecast error:

$$\begin{aligned}\lim_{\lambda \longrightarrow 0} \tilde{\sigma}^2(\lambda, \phi^*) &= \frac{1}{T}\sum(y_t - \hat{\phi} y_{t-1})^2 \\ \lim_{\lambda \longrightarrow \infty} \tilde{\sigma}^2(\lambda, \phi^*) &= \frac{1}{T}\sum(y_t - \phi^* y_{t-1})^2.\end{aligned}$$

- The third term above can be interpreted as a penalty for model complexity and is of the form

$$c(\lambda, \phi^*) = \ln\left(1 + \frac{\hat{\gamma}_0}{\lambda \gamma_0}\right).$$

- As λ approaches zero, the marginal log likelihood function tends to minus infinity.

Example: Marginal Likelihood of λ

- Recall that marginal likelihood of λ takes the following form

$$\ln p(Y|\lambda, \phi^*) = -T/2 \ln(2\pi) - \frac{T}{2}\tilde{\sigma}^2(\lambda, \phi^*) - \frac{1}{2}c(\lambda, \phi^*).$$

- If an interior maximum of marginal likelihood exists, it is given by

$$\hat{\lambda} = \frac{\gamma_0 \hat{\gamma}_0^2}{T(\hat{\gamma}_0\gamma_1 - \gamma_0\hat{\gamma}_1)^2 - (\gamma_0)^2\hat{\gamma}_0}.$$

Example: Minnesota Prior

- Reference: Doan, Litterman, and Sims (1984), Sims and Zha (1998).
- Consider the following Gaussian bivariate VAR(2).

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} y_{1,t-2} \\ y_{2,t-2} \end{bmatrix} + \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix}$$

- Minnesota Prior is centered at:

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_{1,t-2} \\ y_{2,t-2} \end{bmatrix} + \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix}$$

Considerations

- It's relatively easy to think about variances of marginal prior distributions of parameters, but hard to think of a full covariance matrix.
- In high-dimensional parameter spaces, independence of parameters may lead to a lot of probability mass in regions of the parameter space in which the model behaves unreasonable.
- VAR: want plausible long-run properties, e.g., co-trending, steady states, etc.
- Constructing priors from artifical (dummy) observations can help introducing reasonable correlations.

Constructing a Prior from "Dummy Observations"

- Suppose we have T^* dummy observations (Y^*, X^*), plug the dummy observations into the likelihood function and multiply the likelihood function by the (initial) improper prior:

$$p(\Phi, \Sigma) \propto |\Sigma|^{-(n+1)/2} p(Y^*|\Phi, \Sigma).$$

- Define

$$\Phi^* = (X^{*'}X^*)^{-1}X^{*'}Y^*, \quad S^* = (Y^* - X^*\Phi^*)'(Y^* - X^*\Phi^*).$$

- This leads to a prior

$$\Sigma \sim IW(\underline{\nu}, \underline{S}), \quad \Phi|\Sigma \sim MN(\underline{\mu}_\Phi, \Sigma \otimes \underline{P}_\Phi^{-1}), \quad .$$

with

$$\underline{\nu} = T^* - k, \quad \underline{S} = S^*, \quad \underline{\mu}_\Phi = \Phi^*, \quad \underline{P}_\Phi = X^{*'}X^*.$$

Posterior with Dummy Observation Prior

- Posterior is proportional to

$$p(\Phi, \Sigma, Y) \propto p(Y|\Phi, \Sigma)p(Y^*|\Phi, \Sigma)|\Sigma|^{-(n+1)/2}$$

- Define $\bar{T} = T^* + T$ and

$$\begin{aligned}\bar{\Phi} &= (X^{*'}X^* + X'X)^{-1}(X^{*'}Y^* + X'Y) \\ \bar{S} &= \left[Y^{*'}Y^* + Y'Y) - (X^{*'}Y^* + X'Y)'(X^{*'}X^* + X'X)^{-1}(X^{*'}Y^* + X'Y)\right].\end{aligned}$$

- Then, let $\bar{X} = [X^{*'}, X']'$ and deduce:
- Deduce that

$$\Sigma|Y \sim IW(\bar{S}, \bar{T} - k), \quad \Phi|(\Sigma, Y) \sim MN(\bar{\Phi}, \Sigma \otimes (\bar{X}'\bar{X})^{-1}).$$

Example: Minnesota Prior

- Dummies for the β coefficients:

$$
\begin{aligned}
Y^* &= X^*\Phi + U \\
\begin{bmatrix} \lambda_1 \underline{s}_1 & 0 \\ 0 & \lambda_1 \underline{s}_2 \end{bmatrix} &= \begin{bmatrix} \lambda_1 \underline{s}_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 \underline{s}_2 & 0 & 0 & 0 \end{bmatrix} \Phi + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}
\end{aligned}
$$

The first observation implies, for instance, that

$$
\begin{aligned}
\lambda_1 \underline{s}_1 = \lambda_1 \underline{s}_1 \beta_{11} + u_{11} \quad &\Longrightarrow \quad \beta_{11} = 1 - \frac{u_{11}}{\lambda_1 \underline{s}_1} \\
&\Longrightarrow \quad \beta_{11} \sim \mathcal{N}\left(1, \frac{\Sigma_{11}}{\lambda_1^2 \underline{s}_1^2}\right) \\
0 = \lambda_1 \underline{s}_1 \beta_{21} + u_{12} \quad &\Longrightarrow \quad \beta_{21} = -\frac{u_{12}}{\lambda_1 \underline{s}_1} \\
&\Longrightarrow \quad \beta_{21} \sim \mathcal{N}\left(0, \frac{\Sigma_{22}}{\lambda_1^2 \underline{s}_1^2}\right)
\end{aligned}
$$

Example: Minnesota Prior

- Dummies for the γ coefficients:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \lambda_1 \underline{s}_1 2^{\lambda_2} & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 \underline{s}_2 2^{\lambda_2} & 0 \end{bmatrix} \Phi + U$$

- For lags of order p the entry above would be $\lambda_1 \underline{s}_i p_2^{\lambda}$.
- The prior for the covariance matrix is implemented by λ_3 replications of

$$\begin{bmatrix} \underline{s}_1 & 0 \\ 0 & \underline{s}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Phi + U$$

Example: Minnesota Prior

- Sums-of-coefficients dummy observations, reflecting the belief that when y_i has been stable at its initial level, it will tend to persist at that level, regardless of the value of other variables:

$$\begin{bmatrix} \lambda_4 \underline{y}_1 & 0 \\ 0 & \lambda_4 \underline{y}_2 \end{bmatrix} = \begin{bmatrix} \lambda_4 \underline{y}_1 & 0 & \lambda_4 \underline{y}_1 & 0 & 0 \\ 0 & \lambda_4 \underline{y}_2 & 0 & \lambda_4 \underline{y}_2 & 0 \end{bmatrix} \Phi + U$$

- Co-persistence prior dummy observations, reflecting the belief that when data on all y's are stable at their initial levels, thy will tend to persist at that level:

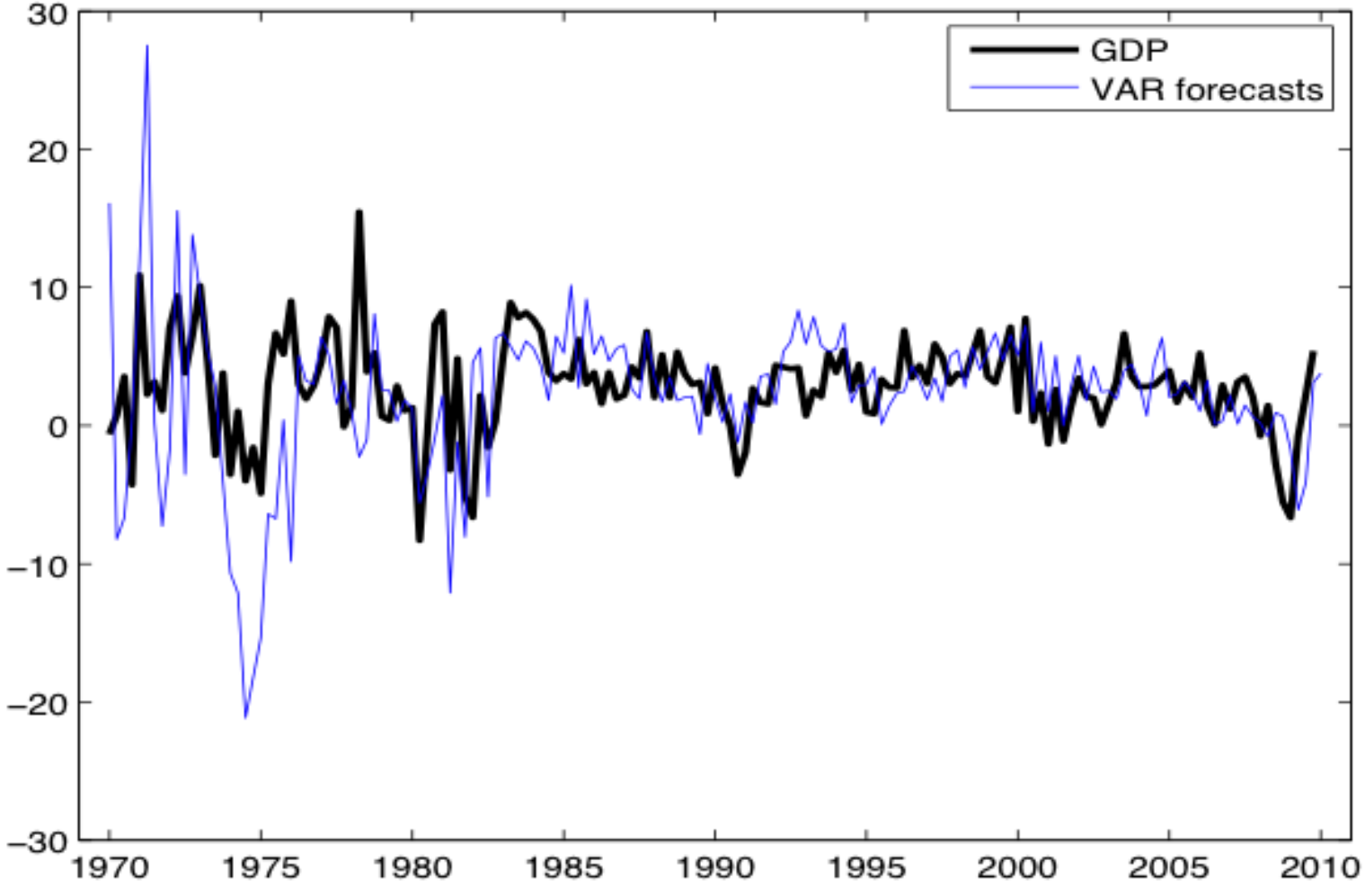
$$\begin{bmatrix} \lambda_5 \underline{y}_1 & \lambda_5 \underline{y}_2 \end{bmatrix} = \begin{bmatrix} \lambda_5 \underline{y}_1 & \lambda_5 \underline{y}_2 & \lambda_5 \underline{y}_1 & \lambda_5 \underline{y}_2 & \lambda_5 \end{bmatrix} \Phi + U$$

Example: Minnesota Prior - Hyperparameters

- λ_1 is the overall tightness of the prior. Large values imply a small prior covariance matrix.
- λ_2: the variance for the coefficients of lag h is scaled down by the factor $(l^{-\lambda_2})^2$.
- λ_3: determines the weight for the prior on Σ. Suppose that $Z_i = \mathcal{N}(0, \sigma^2)$. Then an estimator for σ^2 is $\hat{\sigma}^2 = \frac{1}{\lambda_3} \sum_{i=1}^{\lambda_3} Z_i^2$. The larger λ_3, the more informative the estimator, and in the context of the VAR, the tighter the prior.
- λ_4 and λ_5: tuning parameters for sums-of-coefficients and co-persistence dummies.
- In addition: $\underline{s} = \text{std}(Y_{-\tau,0})$ and $\underline{y} = \text{mean}(Y_{-\tau,0})$, where $Y_{-\tau,0}$ is a short presample.

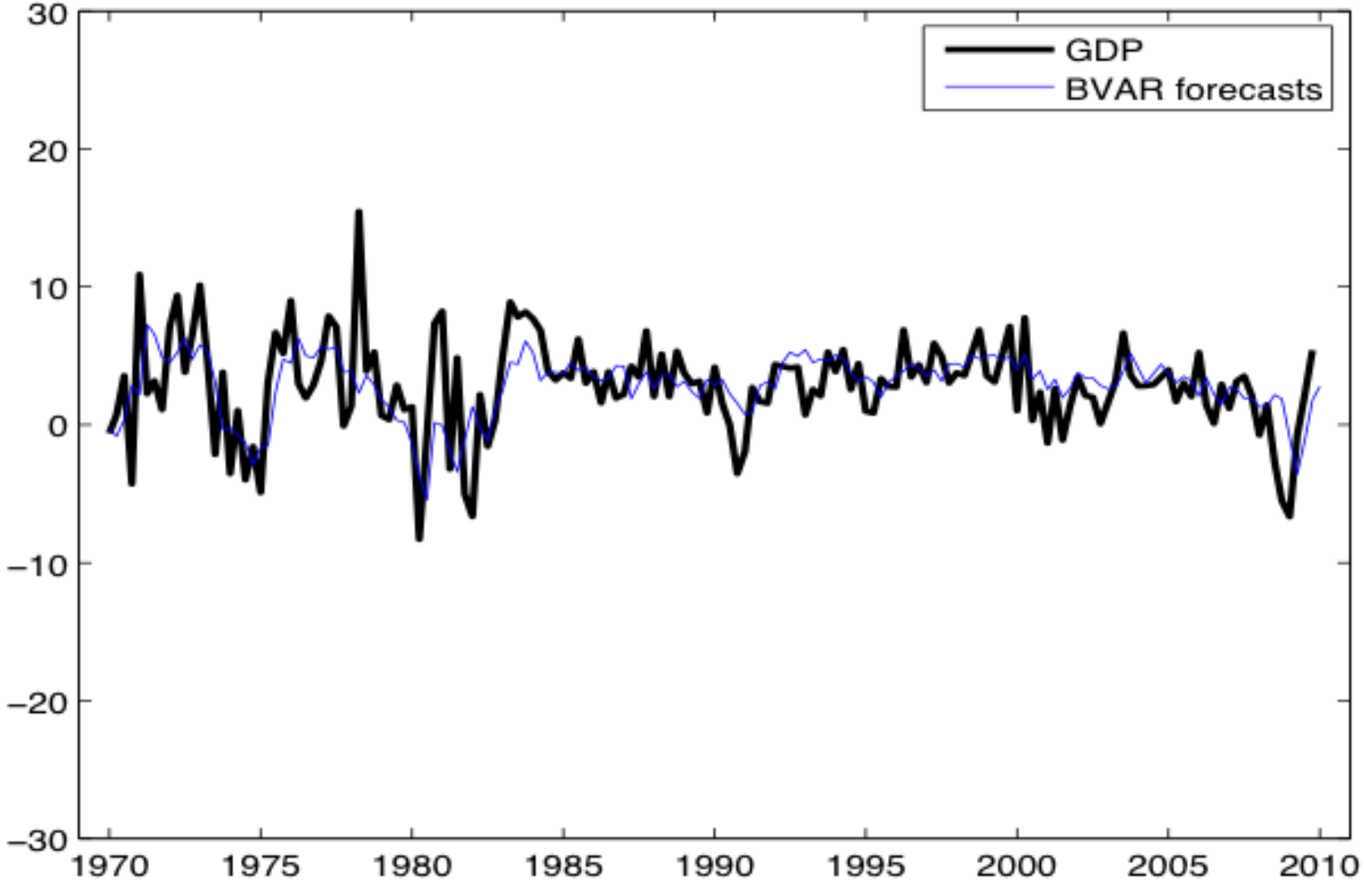
US GDP growth and VAR forecast (1-step ahead)

Flat-prior VAR



US GDP growth and BVAR forecast (1-step ahead)

BVAR (MN+SOC+DIO priors + hyperparameter selection)



BVARs: Forecasting performance

Mean Squared Forecast Errors

		7-variable VAR		
		Flat-prior	BVAR with MN prior (λ=0.2)	BVAR with MN+SOC+DIO
1 Quarter Ahead	Real GDP	19.18	9.61	7.97
	GDP Deflator	2.27	1.53	1.35
	Federal Funds Rate	1.83	1.08	1.03
1 Year Ahead	Real GDP	11.90	5.48	3.42
	GDP Deflator	2.22	1.85	1.58
	Federal Funds Rate	0.56	0.40	0.31

Example: Minnesota Prior w/ Dummy Observations

- The marginal likelihood can be calculated from the normalization constants of the MNIW distribution (see Zellner (1971, Appendix)):

$$p(Y|\lambda) = (2\pi)^{-nT/2} \frac{|\bar{X}'\bar{X}|^{-\frac{n}{2}}|\bar{S}|^{-\frac{\bar{T}-k}{2}}}{|X^{*\prime}X^{*}|^{-\frac{n}{2}}|S^{*}|^{-\frac{T^{*}-k}{2}}} \frac{2^{\frac{n(\bar{T}-k)}{2}}\prod_{i=1}^{n}\Gamma[(\bar{T}-k+1-i)/2]}{2^{\frac{n(T^{*}-k)}{2}}\prod_{i=1}^{n}\Gamma[(T^{*}-k+1-i)/2]}.$$

- The hyperparameters $(\bar{y}, \bar{s}, \lambda)$ enter through the dummy observations X^{*} and Y^{*}.

Extension 1: Alternative Priors

- Giannone, Lenza, Primiceri (2018): "Priors for the Long-Run," *Journal of American Statistical Association*, forthcoming.
- Estimation is typically based on conditional likelihood functions that ignore the likelihood of the initial observations.
- Example:

$$y_t = c + \phi y_{t-1} + u_t = \underbrace{\phi^{t-1} y_1 + c \sum_{s=0}^{t-2} \phi^s}_{DC_t} + \underbrace{\sum_{s=0}^{t-2} \phi^s u_{t-j}}_{SC_t}$$

- Write

$$DC_t = \begin{cases} y_1 + (t-1)c & \text{if } \phi = 1 \\ \frac{c}{1-\phi} + \phi^{t-1}(y_1 - \frac{c}{1-\phi}) & \text{if} |\phi| < 1 \end{cases}$$

- Deterministic component may absorb too much low frequency variation of the time series.

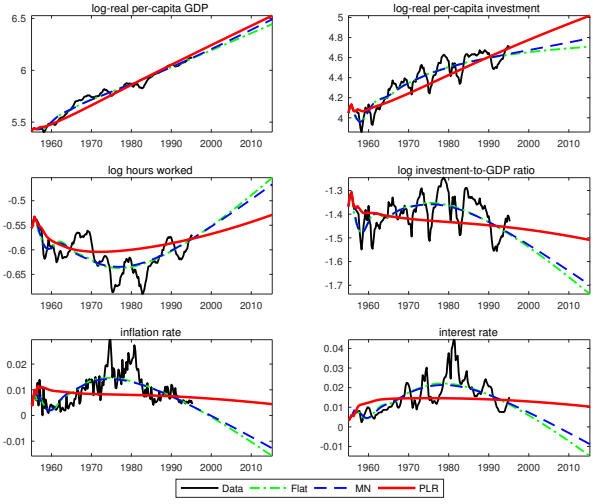


FIGURE 2.1. Deterministic component for selected variables implied by various 7-variable VARs. Flat: BVAR with a flat prior; MN: BVAR with the Minnesota prior; PLR: BVAR with the prior for the long run.

Extension 1: Alternative Priors – The Basic Idea

- Write VAR in VECM form:

$$\Delta y_t = \Pi_0 + \Pi_* y_{t-1} + \sum_{j=1}^{p-1} \Pi_i \Delta y_{t-j} + u_t$$

  where $\Pi_* = \alpha\beta'$.

- Reasonable prior for columns of α will depend on the rows of β':
  - if i'th row of β' corresponds to a linear combination that is stationary, then it makes sense to choose a prior for i'th column of α with mass away form zero.
  - if i'th row of β' corresponds to a linear combination that is non-stationary, then it makes sense to choose a prior for i'th column of α with mass away form zero.

- See paper for details on how to implement this.

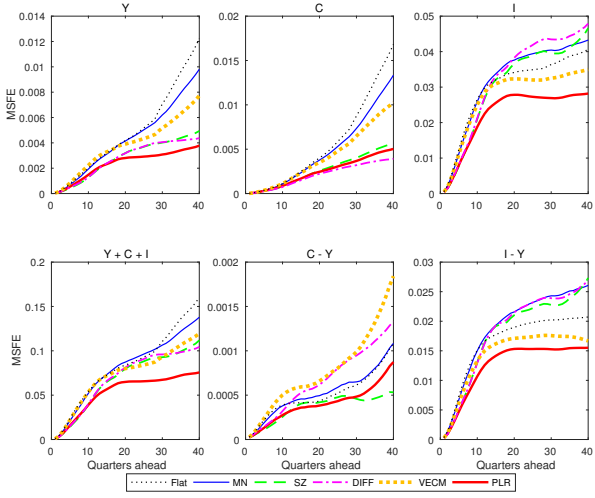


FIGURE 5.1. Mean squared forecast errors in models with three variables. Flat: BVAR with a flat prior; MN: BVAR with the Minnesota prior; SZ: BVAR with the Minnesota and sum-of-coefficient priors; DIFF: VAR with variables in first differences; VECM: vector error-correction model that imposes the existence of a common stochastic trend for Y, C and I, without any additional prior information; PLR: BVAR with the Minnesota prior and the prior for the long run.

Extension 2: Sparse versus Dense Models

- Giannone, Lenza, Primiceri (2018): "Economic Prediction With Big Data: The Illusion of Sparsity," *Manuscript*, FRB New York, ECB, and Northwestern University.
- Sparse models: only a few predictors are relevant.
- Dense models: many predictors are relevant but only have small individual effects.
- Model:

$$y_t = x_t'\phi + z_t'\beta + u_t.$$

  Here x_t's are included in all specifications (low dimensional), z_t's are optional (high dimensional).

- Prior – part 1:

$$p(\sigma^2) \propto \frac{1}{\sigma^2}, \quad \phi \propto c.$$

Extension 2: Sparse versus Dense Models

- Prior – part 2: "spike and slab"

$$\beta_i|(\sigma^2,\gamma^2,q) \sim \begin{cases} N(0,\sigma^2\gamma^2) & \text{with prob. } q \\ 0 & \text{with prob. } 1-q \end{cases}$$

- For $q = 1$ we obtain our "standard" prior ("Ridge Regression")

- Rewrite prior as

$$\beta_i|(\sigma^2,\gamma^2,\nu_i) \sim N(0,\sigma^2\gamma^2,\nu_i), \quad \nu_i \sim \text{Bernoulli}(q).$$

- By changing the mixing distribution, we can generate a wide variety of priors, including a Bayesian version of LASSO.

Extension 2: Sparse versus Dense Models

- In problems of this form it is often good to standardize and orthogonalize the regressors x_t prior to the estimation.
- To specify a prior on the hyperparameters (q, γ^2) they suggest to define

$$R^2(\gamma^2, q) = \frac{qk\gamma^2\bar{\sigma}_z^2}{qk\gamma^2\bar{\sigma}_z^2 + 1}$$

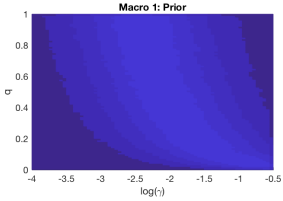
  where k is the number of regressors z and $\bar{\sigma}_z^2$ is the average sample variance of the z_j's.
- The prior takes the form

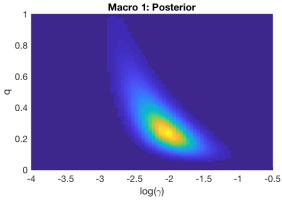
$$q \sim \text{Beta}(a, b), \quad R^2 \sim \text{Beta}(A, B).$$

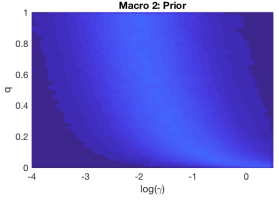
- The paper works out the posterior.

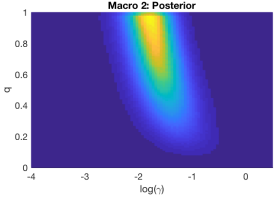
TABLE 1. Description of the datasets.

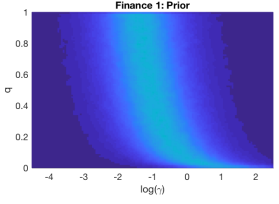
	Dependent variable	**Possible predictors**	**Sample**
Macro 1	Monthly growth rate of US industrial production	130 lagged macroeconomic indicators	659 monthly time-series observations, from February 1960 to December 2014
Macro 2	Average growth rate of GDP over the sample 1960-1985	60 socio-economic, institutional and geographical characteristics, measured at pre-60s value	90 cross-sectional country observations
Finance 1	US equity premium (S&P 500)	16 lagged financial and macroeconomic indicators	58 annual time-series observations, from 1948 to 2015
Finance 2	Stock returns of US firms	144 dummies classifying stock as very low, low, high or very high in terms of 36 lagged characteristics	1400k panel observations for an average of 2250 stocks over a span of 624 months, from July 1963 to June 2015
Micro 1	Per-capita crime (murder) rates	Effective abortion rate and 284 controls including possible covariate of crime and their transformations	576 panel observations for 48 US states over a span of 144 months, from January 1986 to December 1997
Micro 2	Number of pro-plaintiff eminent domain decisions in a specific circuit and in a specific year	Characteristics of judicial panels capturing aspects related to gender, race, religion, political affiliation, education and professional history of the judges, together with some interactions among the latter, for a total of 138 regressors	312 panel circuit/year observations, from 1975 to 2008

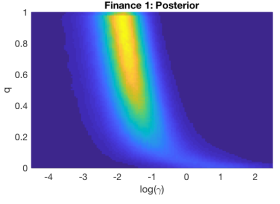

Macro 1: Prior
q
log(γ)

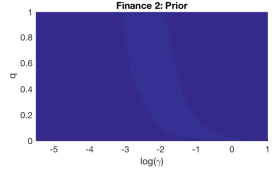

Macro 1: Posterior
q
log(γ)


Macro 2: Prior
q
log(γ)


Macro 2: Posterior
q
log(γ)


Finance 1: Prior
q
log(γ)


Finance 1: Posterior
q
log(γ)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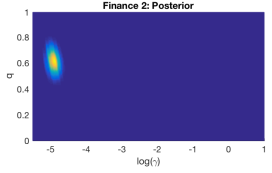
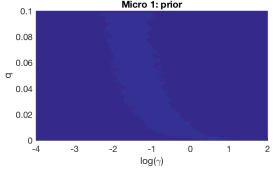
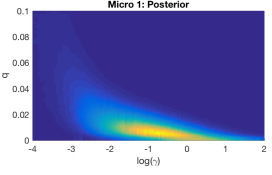
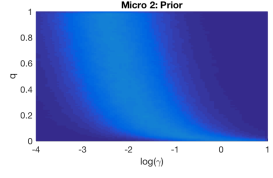
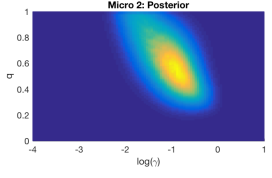
q
log(γ)

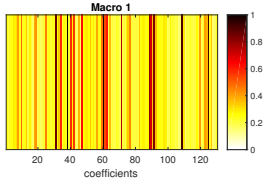

Finance 2: Posterior
q
log(γ)


Micro 1: prior
q
log(γ)


Micro 1: Posterior
q
log(γ)


Micro 2: Prior
q
log(γ)


Micro 2: Posterior
q
log(γ)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coefficients
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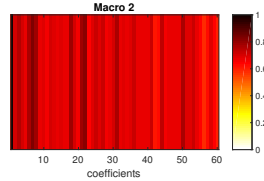
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coefficients
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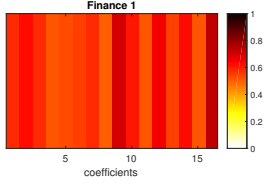
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coefficients
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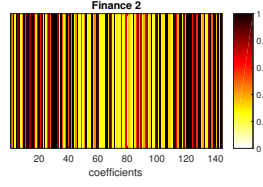
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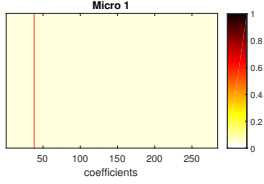
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coefficients
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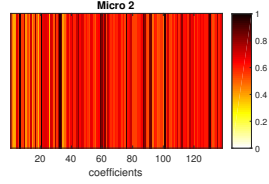
Micro 2
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coefficients
1 0.8 0.6 0.4 0.2 0