Introduction to Algebraic Geometry

Molly Bradley

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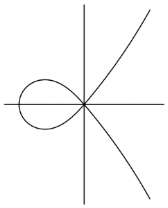
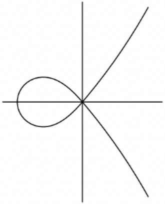


Overview

Algebraic Geometry

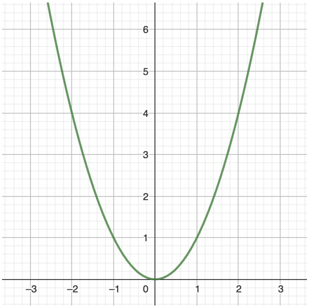
Study of geometric structures defined by polynomials



Our Talk

We will explore this relationship through learning about affine algebraic sets, affine varieties, and the coordinate ring.

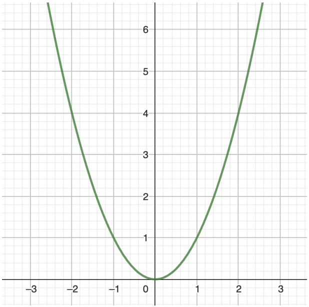
Affine Algebraic Sets



A Simple Equation

$$y = x^2$$

Affine Algebraic Sets

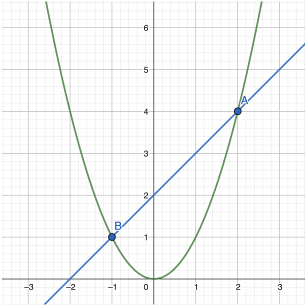


A New Perspective

In the **polynomial ring** $\mathbb{R}[x, y]$, we write $f(x, y) = y - x^2$

Given a set of polynomials $S \subset \mathbb{R}[x, y]$, an **affine algebraic set** is a set of points $(x, y) \in \mathbb{R}^2$ such that $f(x, y) = 0$ for all $f \in S$.

Affine Algebraic Sets



A New Example

If $S = \{y - x^2, y - x - 2\}$, then $V(S) = \{(-1, 1), (2, 4)\}$

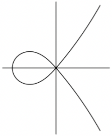
Definition

A **variety** is an irreducible algebraic set ($V(S)$ is not a variety)

Affine Varieties and Prime Ideals

Important Notation

- k: algebraically closed field
- I: ideal in the ring of polynomials $k[x_1, x_2, ..., x_n]$
- V: affine variety in affine n-space $\mathbb{A}^n$ (we view this as k^n)



Important Constructions

- If $I = (S)$ is the ideal generated by S, then $V(S) = V(I)$.
- $I(V) = \{f \in k[x_1, x_2, ..., x_n] : f(x) = 0 \text{ for all } x \in V\}$
- THM: An affine set V is a variety if and only if $I(V)$ is prime.

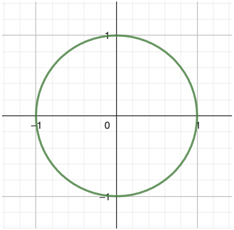
Coordinate Ring

Definition

Given variety V, we define the coordinate ring $K[V] = k[x_1, ..., x_n]/I(V)$.

Building Intuition: An Example

We consider $k = \mathbb{C}$ and view the real part of $V(x^2 + y^2 - 1)$



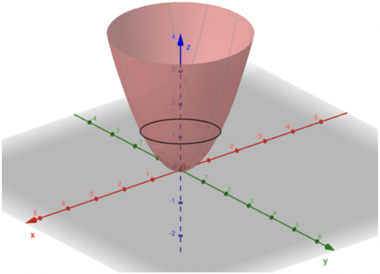
Question

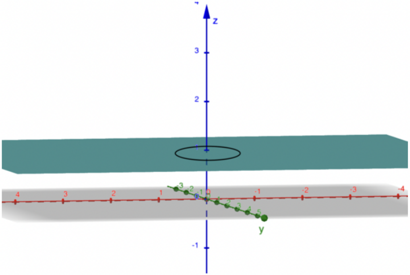
What are the different polynomial functions from $V(x^2 + y^2 - 1)$ to $\mathbb{C}$?

More on the Coordinate Ring

Polynomial Functions $\mathbb{C}[x, y] \to \mathbb{C}$

We consider $f(x, y) = x^2 + y^2$ and $f(x, y) = 1$





Observation

Restricting our attention to $V(x^2 + y^2 - 1)$, these functions are the same!

More on the Coordinate Ring

Example

We recall the coordinate ring $K[V] = k[x_1, ..., x_n]/I(V)$.

In this case, $I(V) = (x^2 + y^2 - 1)$, so $K[V] = k[x_1, ..., x_n]/(x^2 + y^2 - 1)$

$x^2 + y^2 + (x^2 + y^2 - 1) = 1 + (x^2 + y^2 - 1)$

Construction

Two functions f and g are equivalent on V if $f(x) = g(x)$ for all $x \in V$

$f(x) - g(x) = 0 \implies (f - g)(x) = 0 \implies f - g \in I(V) \implies$
$f + I(V) = g + I(V)$.

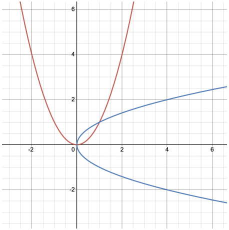
Algebra Meets Geometry

Theorem

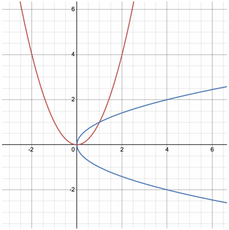
There is a natural bijection between polynomial maps between varieties $\varphi : V \to W$ and homomorphisms $\hat{\varphi} \colon K[W] \to K[V]$

Example

We consider $V = V(y - x^2 - 1)$ and $W = V(x - y^2 - 1)$.



Algebra Meets Geometry



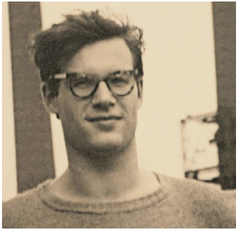
Example Continued

We consider $V = V(y - x^2 - 1)$ and $W = V(x - y^2 - 1)$.

We define $\varphi\colon V \to W$ by $(x, y) \mapsto (y, x)$.

We see that $\hat{\varphi} = f \circ \varphi$, so $\hat{\varphi}\colon K[W] \to K[V]$ by $f(x, y) \mapsto f(y, x)$.

Foundations of Modern Algebraic Geometry



Alexander Grothendieck

All commutative rings can be viewed as functions on some geometric space

Basis of scheme theory!

Acknowledgements

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Source: Algebraic Curves: An Introduction to Algebraic Geometry by William Fulton.