

Probability and the Infinite Monkey Theorem

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- How can we be sure?



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 - 1 A set Ω called the sample space
 - 2 A σ -algebra \mathcal{F} of subsets of Ω , meaning $\Omega \in \mathcal{F}$ and \mathcal{F} is closed under complements and countable unions
 - 3 A probability measure $P : \mathcal{F} \rightarrow [0; 1]$ that satisfies
 - 1 $P(\emptyset) = 0$
 - 2 $P(\Omega) = 1$
 - 3 For pairwise disjoint $A_1; A_2; A_3; \dots \in \mathcal{F}$, we have

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

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Basic Results

- Need two basic properties of probability spaces for this proof
 - 1 Subadditivity: if $A \subseteq \bigcup_{n=1}^{\infty} A_n$, then $P(A) \leq \sum_{n=1}^{\infty} P(A_n)$
 - 2 Continuity from above: if $A_1 \supseteq A_2 \supseteq \dots$ with $\bigcap_{n=1}^{\infty} A_n = A$, then $\lim_{n \rightarrow \infty} P(A_n) = P(A)$

Borel-Cantelli Lemma

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- In other words, if the sum of the probabilities is finite, then the probability of an infinite number of these events occurring is 0.

Borel-Cantelli Proof

- Proof:

- Observe

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$$\forall \epsilon > 0 \quad \sum_{k=n}^{\infty} P(A_k) < \epsilon$$

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$$\bigcap_{n=1}^{\infty} A_n \quad \bigcap_{n=2}^{\infty} A_n \quad \forall \bigcap_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \limsup_{n \rightarrow \infty} A_n$$

- Continuity from above

$$P \left(\limsup_{n \rightarrow \infty} A_n \right) = \lim_{N \rightarrow \infty} P \left(\bigcap_{n=N}^{\infty} A_n \right)$$

Borel-Cantelli Proof (Contd.)

- Proof:
 - Subadditivity

$$P\left[\bigcup_{n=N}^{\infty} A_n\right] \leq \sum_{n=N}^{\infty} P(A_n) < 1$$

Borel-Cantelli Proof (Contd.)

- Proof:

- Subadditivity

$$P\left(\bigcup_{n=N}^{\infty} A_n\right) \leq \sum_{n=N}^{\infty} P(A_n) < 1$$

- Since $\sum_{n=1}^{\infty} P(A_n) < 1$

$$\lim_{N \rightarrow \infty} \sum_{n=N}^{\infty} P(A_n) = 0$$

$$\Rightarrow P(\limsup A_n) = \lim_{N \rightarrow \infty} P\left(\bigcup_{n=N}^{\infty} A_n\right) = 0$$

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$$\begin{array}{ccc} \underbrace{a_1^1 a_1^2 \dots a_1^n}_{\text{1st trial}} & \underbrace{a_2^1 a_2^2 \dots a_2^n}_{\text{2nd trial}} & \underbrace{a_i^1 a_i^2 \dots a_i^n}_{\text{ith trial}} \end{array}$$

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- The probability of typing a given text correctly at any trial is greater than zero, say $p = \frac{1}{k^n}$ where k is the number of characters on the typewriter and n is the length of the text.
- Therefore, $P(A_i) = (1 - p)^i$ since the trials are independent.



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- This is a geometric series, and its sum is $\frac{1-p}{1-(1-p)} = \frac{1-p}{p}$.

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- Therefore, the probability that the monkey never types any given text correctly is zero, i.e., the probability that they do type the text is 1.
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- Hence, the monkey will almost surely type the text at some point, proving the Infinite Monkey Theorem.

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- In other words, it would take on average $3.43 \times 10^{183,946}$ trials before the monkey starts typing *Hamlet*.

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- Suppose every atom were a “monkey” typing at 2000 characters per minute, which translates to a billion characters in a year. Chances increase to only 1 in

$$3:43 \quad 10^{183,946} = (10^{80} \quad 10^{20} \quad 10^9) = 3:43 \quad 10^{183,819}$$

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- This theorem, while captivating, is more of a thought experiment than a practical reality
- Interesting to explore the limits of probability and the boundaries of possibility.

References

- Durrett, R. (2019). *Probability: Theory and Examples* (5th ed.). Cambridge University Press.

Thank you!