Probability and the Infinite Monkey Theorem

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December 12, 2023

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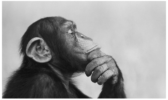
- How can we be sure?



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  1. A set Ω called the sample space
  2. A σ-algebra $\mathcal{F}$ of subsets of Ω, meaning $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under complements and countable unions
  3. A probability measure $P : \mathcal{F} \to [0, 1]$ that satisfies
     1. $P(\emptyset) = 0$
     2. $P(\Omega) = 1$
     3. For pairwise disjoint $A_1, A_2, A_3, \ldots \in \mathcal{F}$, we have

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

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  1. Subadditivity: if $A \subseteq \cup_{n=1}^{\infty} A_n$, then $P(A) \leq \sum_{n=1}^{\infty} P(A_n)$
  2. Continuity from above: if $A_1 \supset A_2 \supset \cdots$ with $\bigcap_{n=1}^{\infty} A_n = A$, then $\lim_{n\to\infty} P(A_n) = P(A)$

Borel-Cantelli Lemma

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- In other words, if the sum of the probabilities is finite, then the probability of an infinite number of these events occuring is 0.

Borel-Cantelli Proof

- Proof:
  - Observe

$$\bigcup_{n=1}^{\infty} A_n \supseteq \bigcup_{n=2}^{\infty} A_n \supseteq \cdots \supseteq \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \limsup_{n \to \infty} A_n$$

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  - Continuity from above

$$P\left(\limsup_{n\to\infty} A_n\right) = \lim_{N\to\infty} P\left(\bigcup_{n=N}^{\infty} A_n\right)$$

Borel-Cantelli Proof (Contd.)

- Proof:
  - Subadditivity

$$P\left(\bigcup_{n=N}^{\infty} A_n\right) \leq \sum_{n=N}^{\infty} P(A_n) < \infty$$

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- Proof:
  - Subadditivity

$$P\left(\bigcup_{n=N}^{\infty} A_n\right) \leq \sum_{n=N}^{\infty} P(A_n) < \infty$$

  - Since $\sum_{n=1}^{\infty} P(A_n) < \infty$

$$\lim_{N\to\infty} \sum_{n=N}^{\infty} P(A_n) = 0$$

$$\implies P(\limsup_{n\to\infty} A_n) = \lim_{N\to\infty} P\left(\bigcup_{n=N}^{\infty} A_n\right) \leq \lim_{N\to\infty} \sum_{n=N}^{\infty} P(A_n) = 0$$

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$$\underbrace{a_1^1 a_1^2 \cdots a_1^n}_{\text{1st trial}} \underbrace{a_2^1 a_2^2 \cdots a_2^n}_{\text{2nd trial}} \cdots \underbrace{a_i^1 a_i^2 \cdots a_i^n}_{i\text{th trial}} \cdots$$

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- The probability of typing a given text correctly at any trial is greater than zero, say $p = \left(\frac{1}{k}\right)^n$ where k is the number of characters on the typewriter and n is the length of the text.
- Therefore, $P(A_i) = (1-p)^i$ since the trials are independent.

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- Since $\frac{1}{p} < \infty$, the Borel-Cantelli Lemma implies $P(B) = 0$.
- Therefore, the probability that the monkey never types any given text correctly is zero, i.e., the probability that they do type the text is 1.
- We say an event occurs **almost surely** if the probability of that event happening is 1.
- Hence, the monkey will almost surely type the text at some point, proving the Infinite Monkey Theorem.

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- In other words, it would take on average $3.43 \times 10^{183,946}$ trials before the monkey starts typing *Hamlet*.

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- Suppose every atom were a "monkey" typing at 2000 characters per minute, which translates to a billion characters in a year. Chances increase to only 1 in

$$3.43 \times 10^{183,946}/(10^{80} \cdot 10^{20} \cdot 10^{9}) = 3.43 \times 10^{183,819}$$

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- This theorem, while captivating, is more of a thought experiment than a practical reality
- Interesting to explore the limits of probability and the boundaries of possibility.

References

- Durrett, R. (2019). *Probability: Theory and Examples* (5th ed.). Cambridge University Press.

Thank you!