An Invitation to Continuous Logic

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- One step more abstract than abstract algebra (the rationals → a field → a model)...
- And yet has found applications in combinatorics, algebra, analysis, and more!
- This is a talk in *applied* model theory—I'll talk about a connection between model theory and functional analysis!

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- Algebraic structures (groups, fields, pre-Hilbert spaces, etc.)

What Can Model Theory Tell Us?

Theorem

Let ϕ be a statement about fields. The following are equivalent:

1. ϕ is true in every algebraically closed field of characteristic zero (one added to itself is never zero).
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Essence of the Proof

"Having zero characteristic" is not expressible as a single first-order statement!

The Ultraproduct!

$$\left\{\overline{\mathbb{F}_p}\right\} \rightsquigarrow \mathbb{C}$$

Fundamental Theorem of the Ultraproduct (Łoś)

For any discrete first-order sentence ϕ,

$$\phi \text{ true in most of the } \left\{\overline{\mathbb{F}_p}\right\} \Leftrightarrow \phi \text{ true in } \mathbb{C}$$

But Not So Good For Complete Metric Structures...

$$\{\mathbb{R}\} \rightsquigarrow {}^*\mathbb{R}$$

Failure of Discrete First-order Logic

${}^*\mathbb{R}$ is not complete!

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- 1968: Ax proves the Ax–Grothendieck Theorem using model theoretic methods.
- 2008: The de facto standard "Model Theory For Metric Structures" (Ben Yaacov, Berenstein, Henson, and Usvyatsov) for continuous logic is published, uniting the two ultraproducts.

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- Complete metric spaces (Hilbert spaces, probability spaces, C*-algebras, etc.)

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And all the results carry over! (Compactness, Löwenheim–Skolem, omitting types, elimination of imaginaries, stability theory, etc.)

Big Question

The Connes Embedding Problem

Big open conjecture for a long time that was recently solved: Can I always embed a certain type of “operator algebra” into a certain ultrapower?

Equivalent to a number of different problems from all different fields across math and computer science:

- C*-algebras
- Quantum information theory
- Quantum complexity theory

And *continuous logic* forms the bridge!