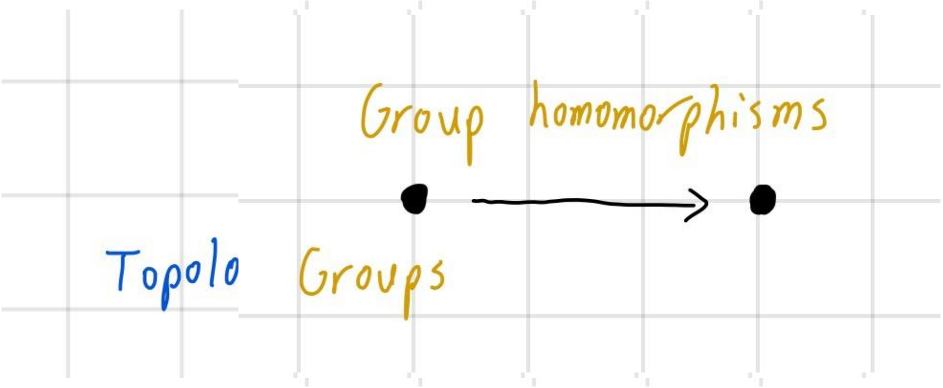
Small Examples in Category Theory

By Eric Yu, Class of 2026

Abstract

Categories are among the most general and far-reaching constructions in mathematics. In this talk, we will motivate and define what a category is and explore some interesting properties of functors, maps between categories, using concrete examples. No prior knowledge of category theory is assumed.

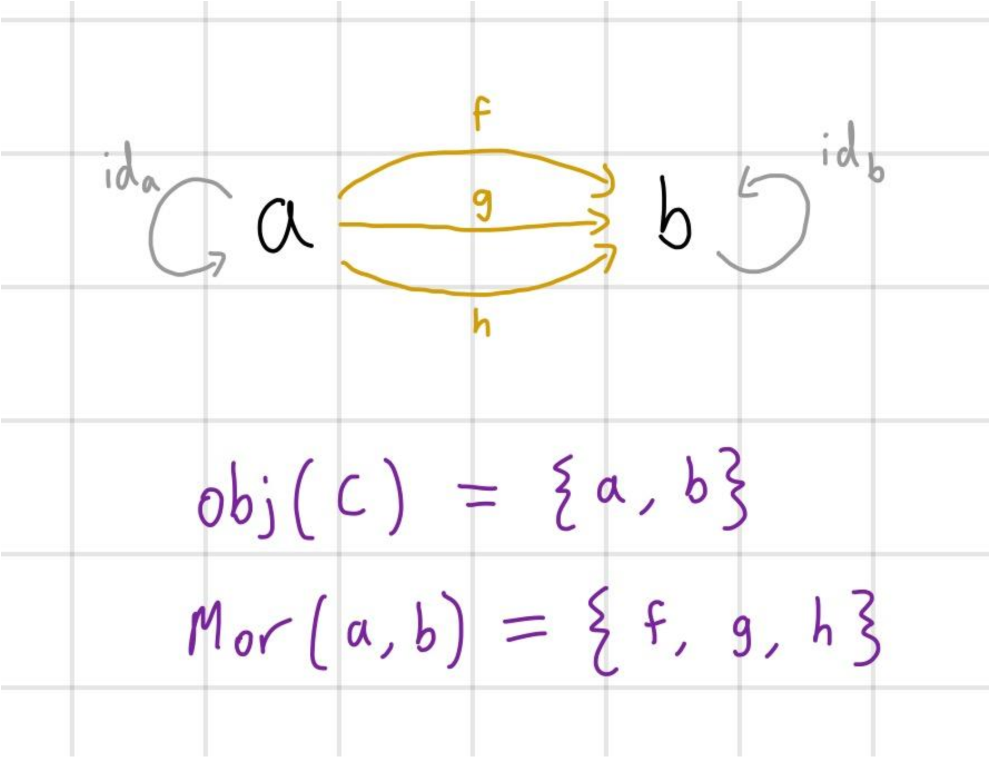
Motivation: Sets, Vect, Top, Grp



What is a Category?

A category C consists of a class obj(C) of objects,

along with a set Mor(a,b) of morphisms between any two objects a,b.



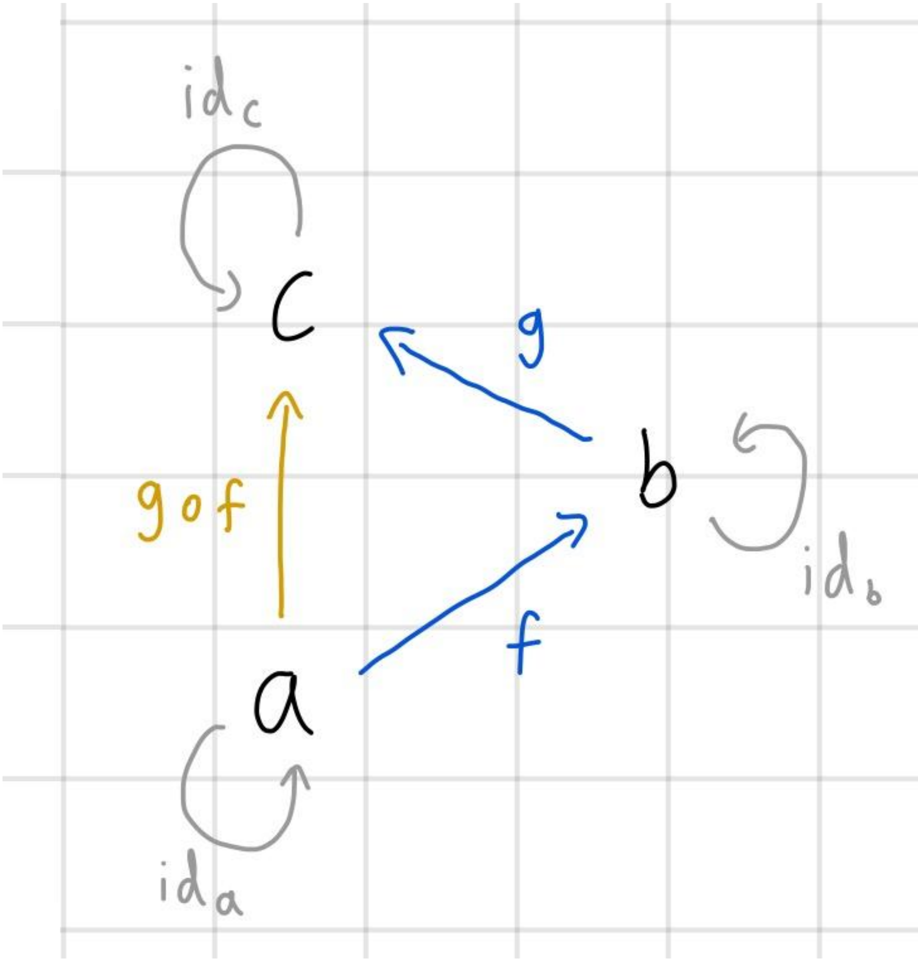
A category must satisfy the following conditions:

What is a Category?

1. Morphisms can be composed.

2. Morphism composition is associative: $h \circ (g \circ f) = (h \circ g) \circ f$.

3. Existence of identity morphism: $id_a \in Mor(a,a)$ for all objects a.



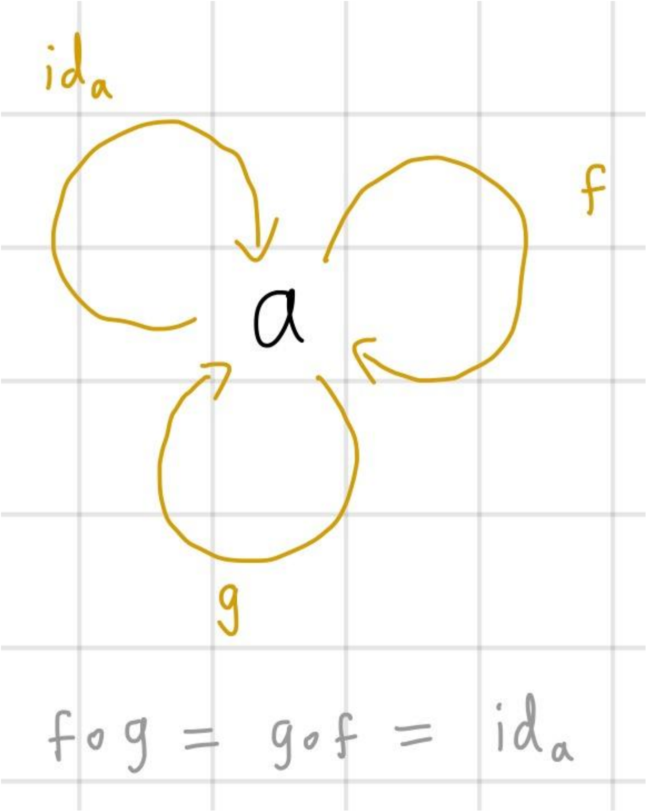
Isomorphisms

Morphisms don't necessarily have inverses. When a morphism has an inverse, it's called an **isomorphism**, and the objects it goes between are said to be **isomorphic**.

Example: two sets are isomorphic if and only if they have the same cardinality.

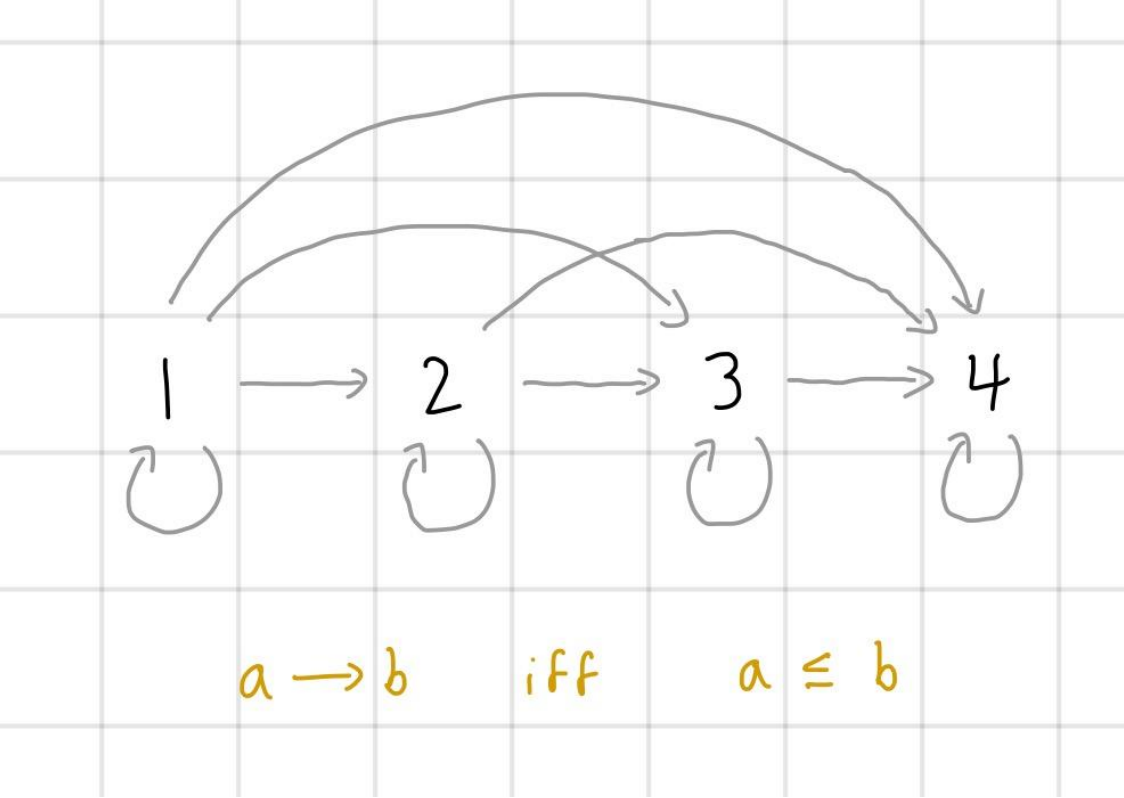
Remark: The identity morphism is an isomorphism. Therefore, every object is isomorphic to itself.

Example: Groupoids with One Object



Z/3Z as a groupoid with one object

Partially Ordered Sets



What is a Functor?

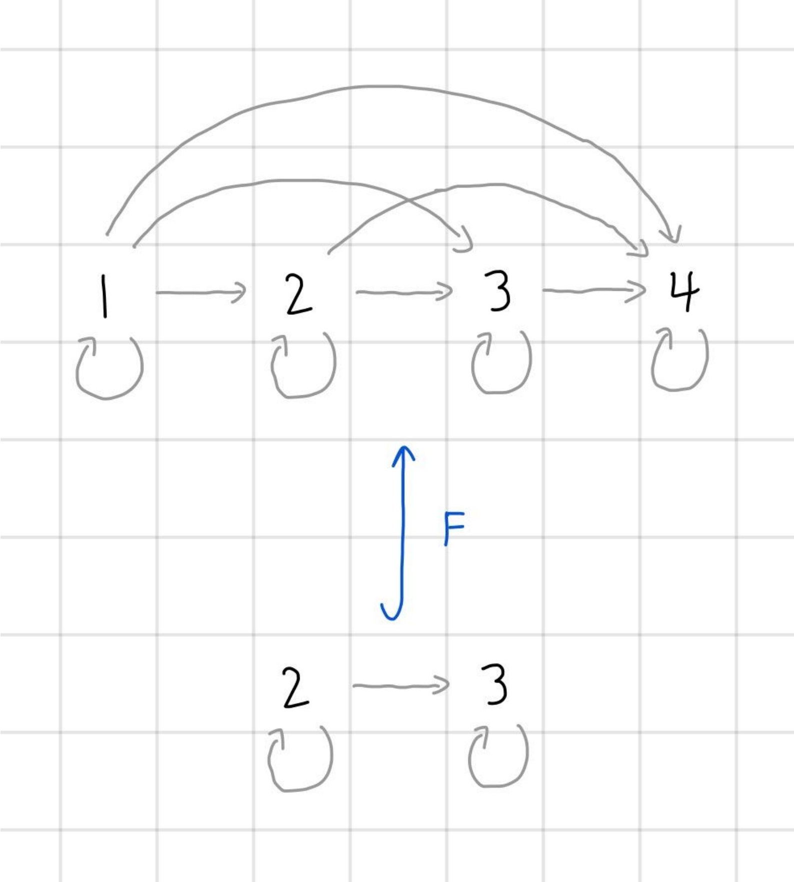
A functor is a map between categories.

More specifically, a (covariant) functor $F: A \to B$ maps objects of A to objects of B and morphisms of A to morphisms of B in a way that respects the following rules:

1. Respects morphism composition: $F(g \circ f) = F(g) \circ F(f)$

2. Maps identities to identities: $F(id_a) = id_{F(a)}$

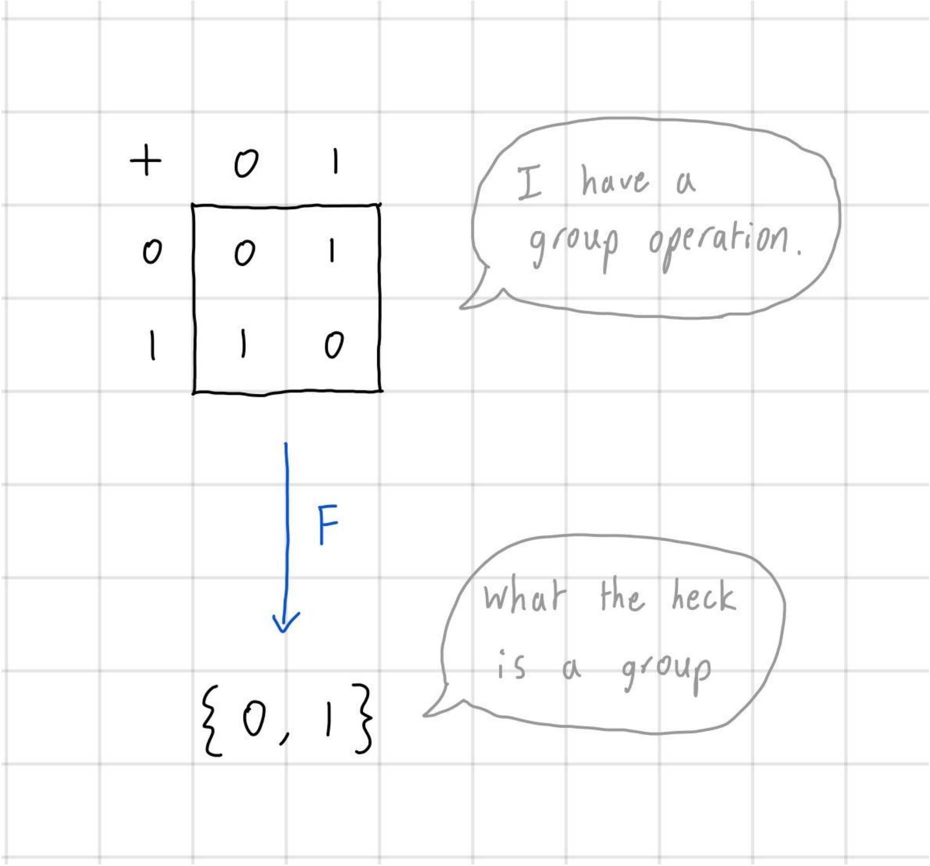
Example: Inclusion Functor



Example: Forgetful Functor from Grp to Sets

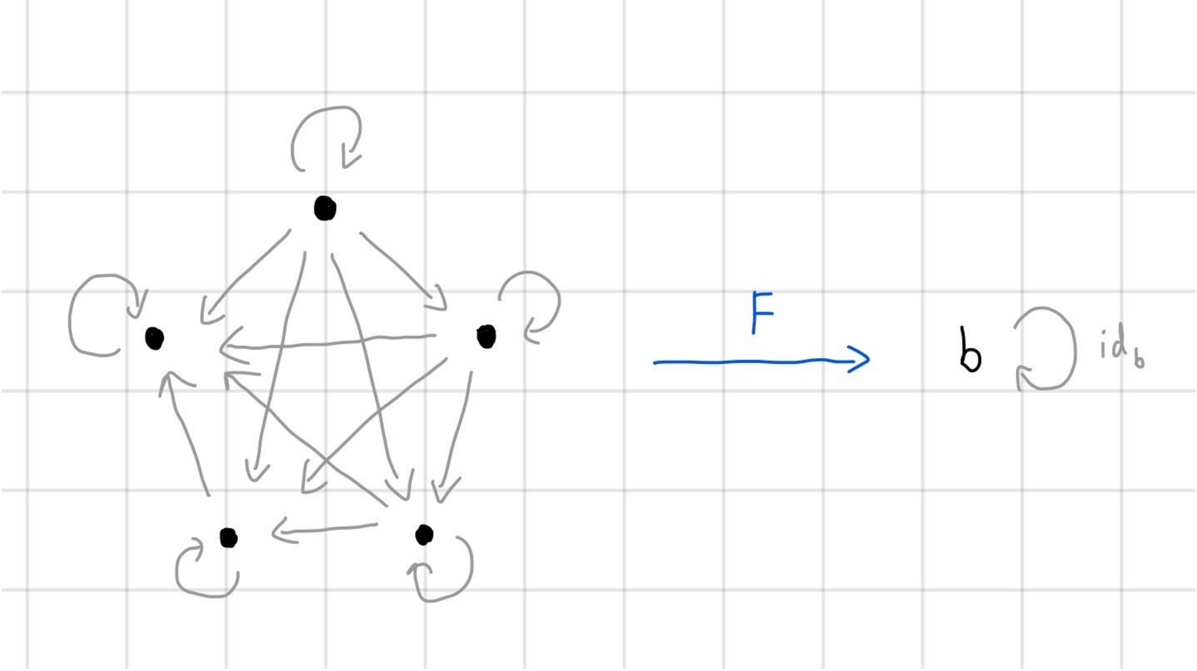
Takes each group to its underlying set.

Takes each group homomorphism to its underlying set function.

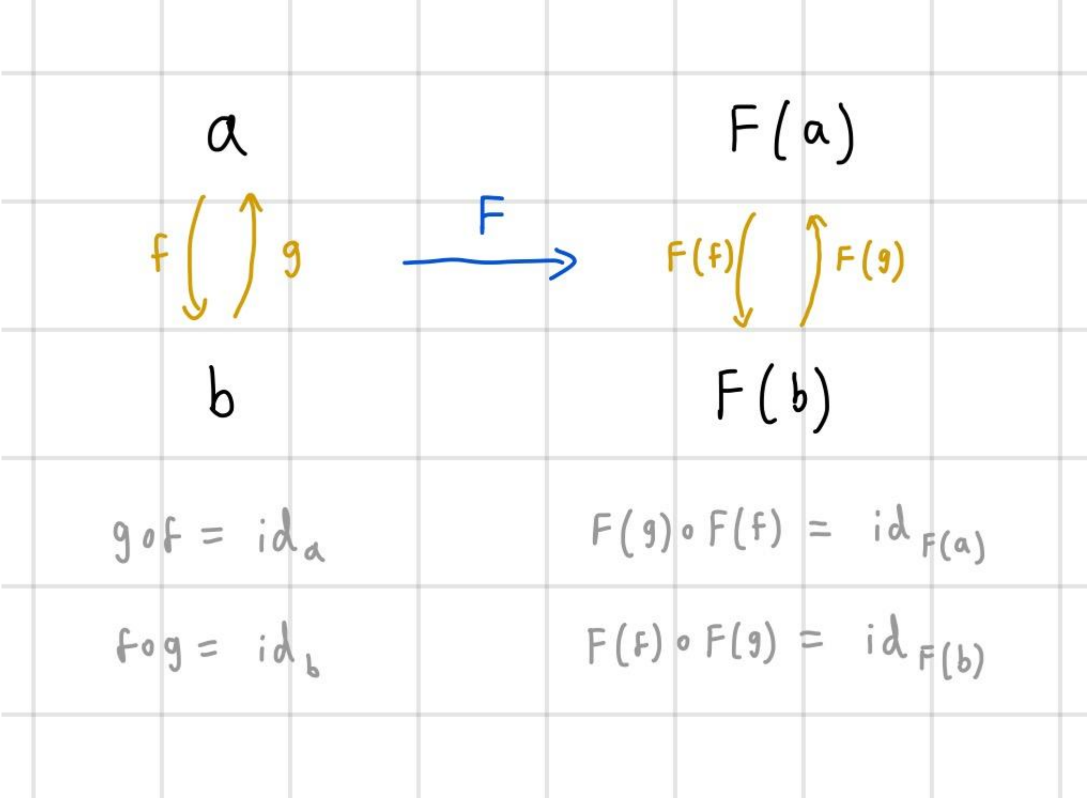


Example: Constant Functor

Maps all objects to a single object, and maps all morphisms to the identity.



Theorem: Functors Map Isomorphic Objects to Isomorphic Objects



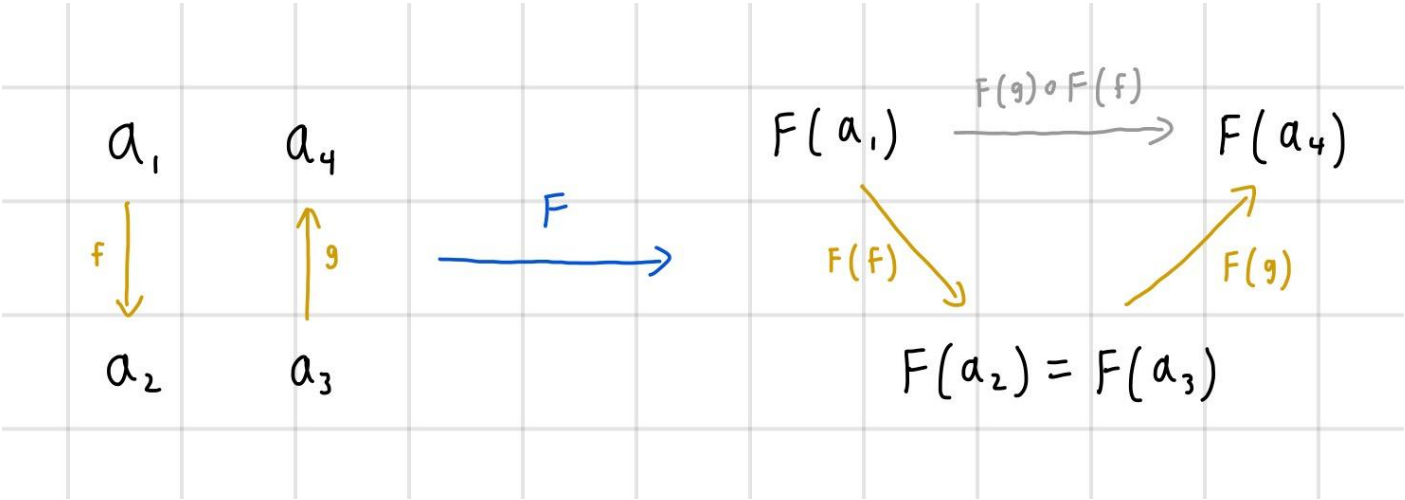
Theorem: Functors Map Isomorphic Objects to Isomorphic Objects

Note that non-isomorphic objects are not necessarily mapped by functors to non-isomorphic objects.

Consider the forgetful functor from Grp to Sets. Z/4Z is not isomorphic to the Klein four group, but as sets they are isomorphic.

If a functor does map non-isomorphic objects to non-isomorphic objects, we say it's *essentially injective*.

Surprising and Somewhat Pathological Counterexample: The image of a category under a functor need not be a category



The image is not closed under morphism composition.

Essentially Injective vs. Conservative

Definition: a functor F is **essentially injective** if

F(a) isomorphic to F(b)

implies

a isomorphic to b.

Definition: a functor F is **conservative** if

F(g) is an isomorphism

implies

g is an isomorphism.

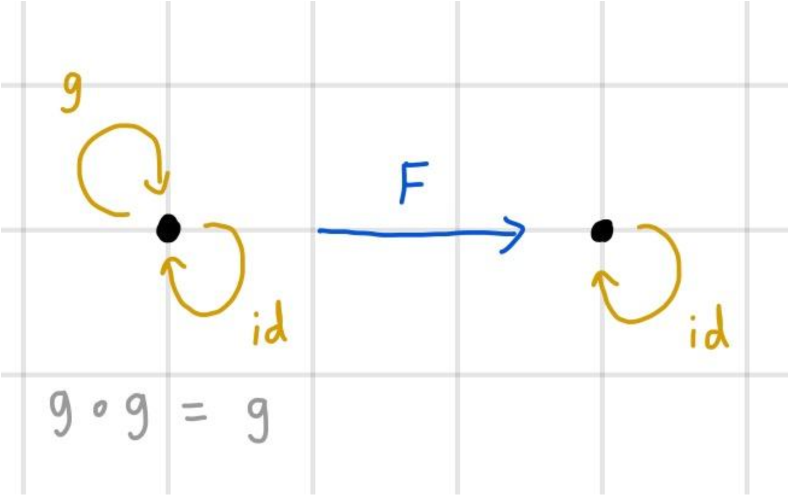
Essentially Injective vs. Conservative

It might be surprising that although the definitions for essentially injective and conservative sound similar, they are not equivalent.

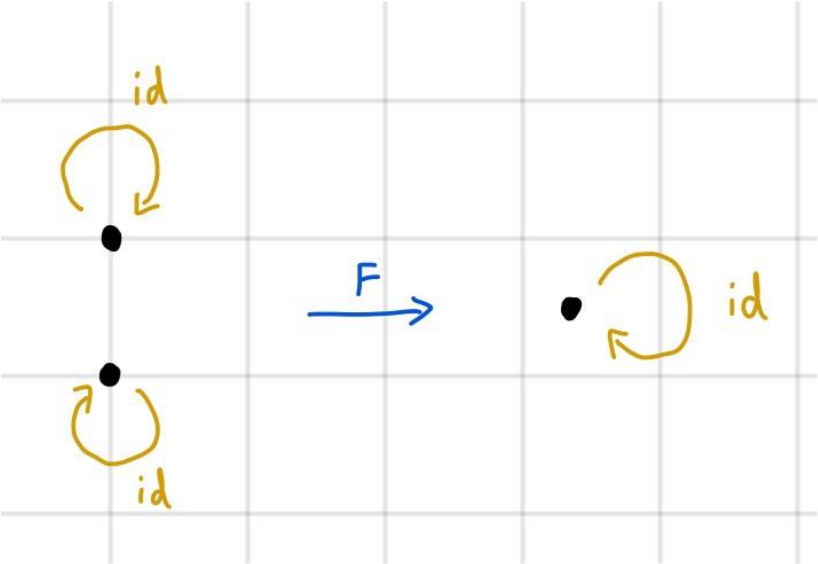
Not only that, but neither of them imply the other.

Essentially Injective vs. Conservative

Essentially injective but not conservative: constant functor on a monoid



Conservative but not essentially injective: constant functor on the disjoint union of two trivial groupoids.



Faithful Functors

Definition: a functor $F: A \rightarrow B$ is **faithful** if $F|_{Mor(a1,a2)}$ is injective for all a1, a2 $\in$ A.

Example: injective group homomorphism between groupoids.

Full Functors

Definition: a functor F: A → B is **full** if $F|_{Mor(a1,a2)}$ is surjective onto Mor(F(a1), F(a2)) for all a1, a2 ∈ A.

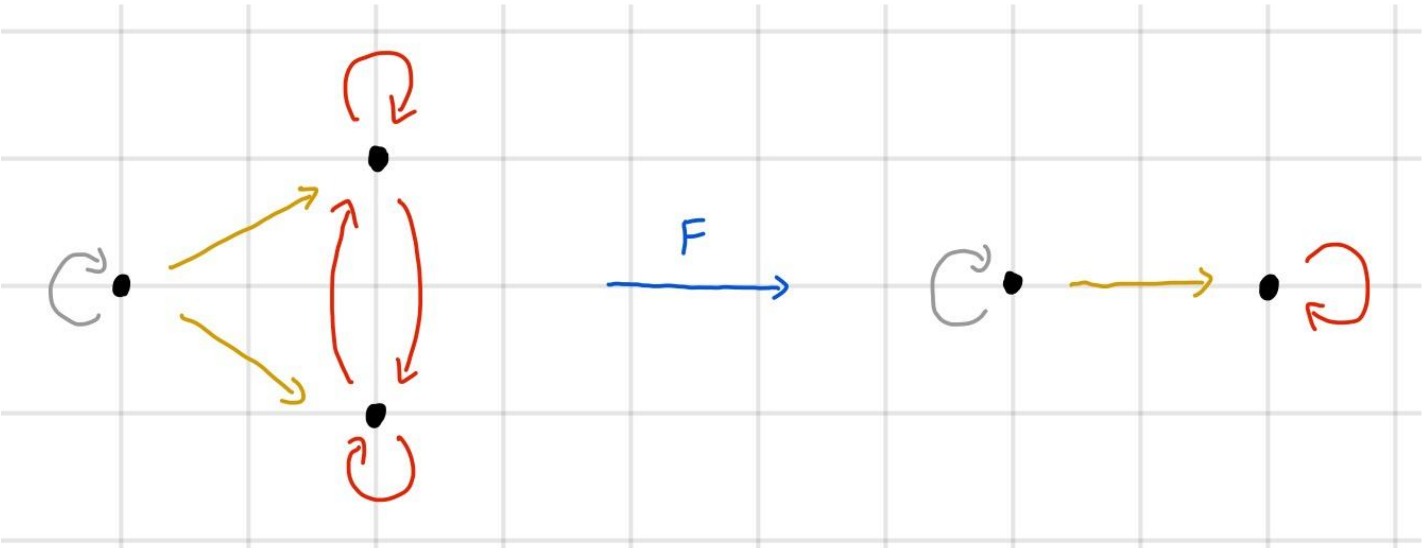
Example: Surjective group homomorphism between groupoids.

Fully Faithful Functors

A functor that is both full and faithful is called **fully faithful**.

Fully Faithful Functors

Example:



You can think of fully faithful functors as just collapsing isomorphic objects.

Properties of Fully Faithful Functors

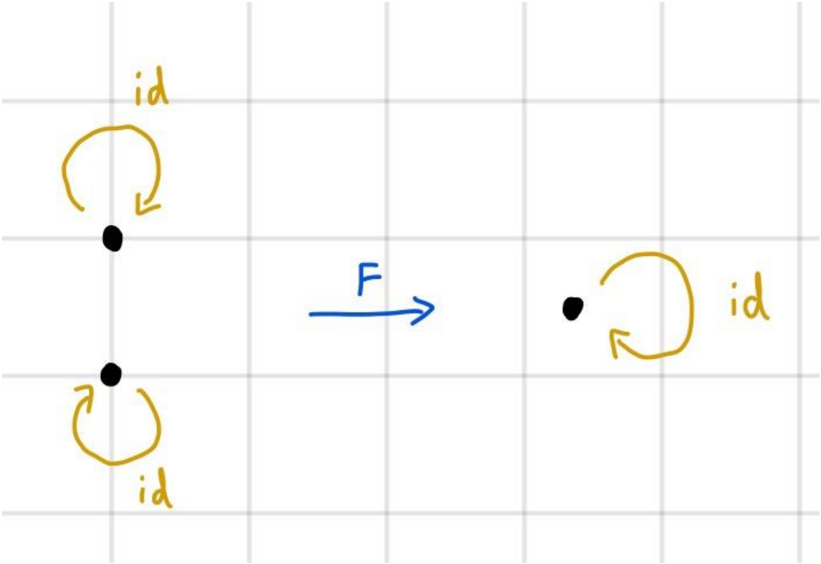
If F is a fully faithful functor, then:

1. F is conservative.

2. F is essentially injective.

3. The image of F is a category (this one only requires fullness).

One Last Thing:



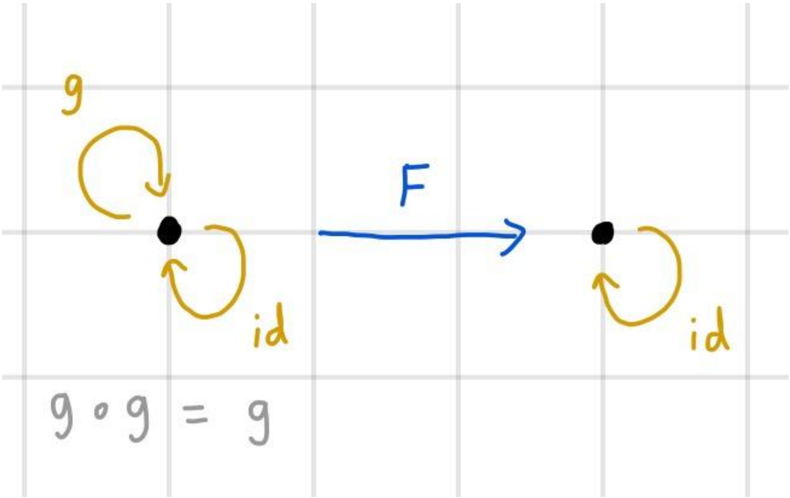
Recall the counterexample we gave of a functor that is conservative but not essentially injective.

This functor is faithful but not full.

You might think that maybe for full functors, conservative does imply essentially injective.

And you would be right.

One Last Thing:



Recall the counterexample we gave of a functor that is essentially injective but not conservative.

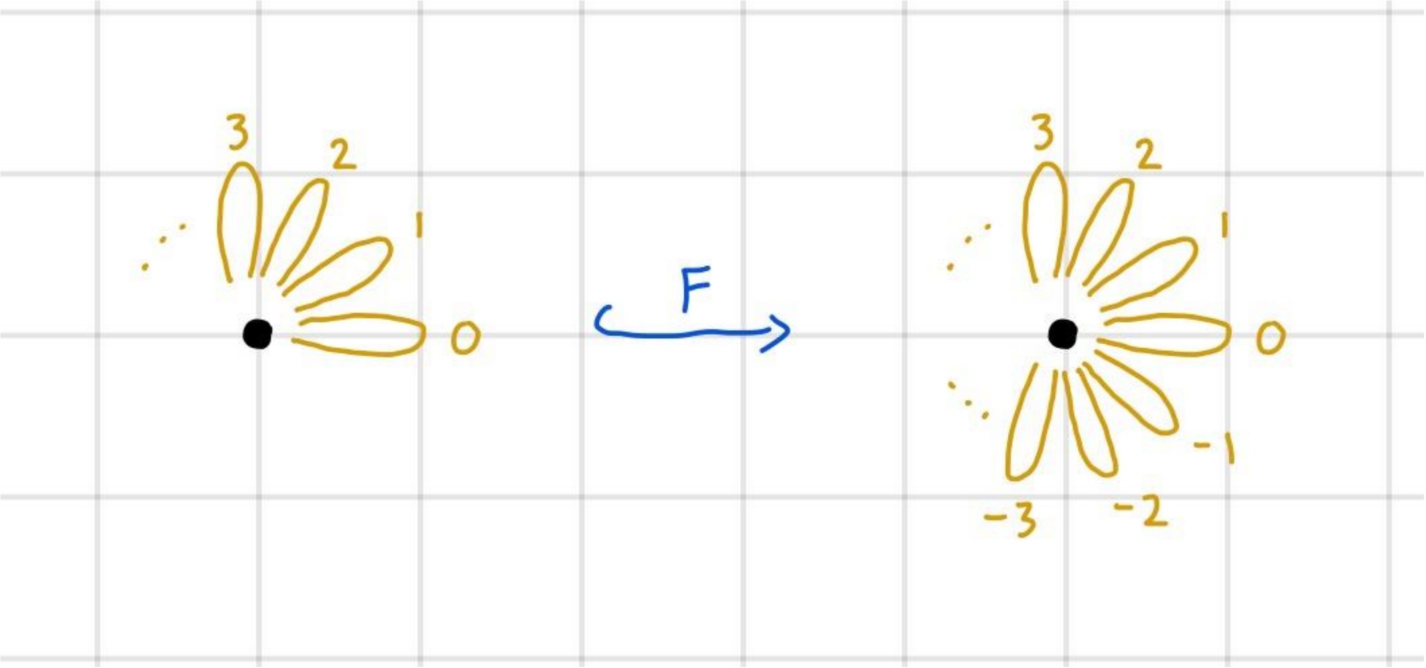
This functor is full but not faithful.

You might think that maybe for faithful functors, essentially injective does imply conservative.

But you would be wrong.

Final Counterexample: Essentially Injective and Faithful, but not Conservative.

The inclusion of a monoid into a groupoid.



Thank You!

Special thanks to Ben.