Gaussian Free Field (GFF) and Related Topics

Greg Lawler
University of Chicago

March 16, 2022

Introduction: fractal curves in random geometries

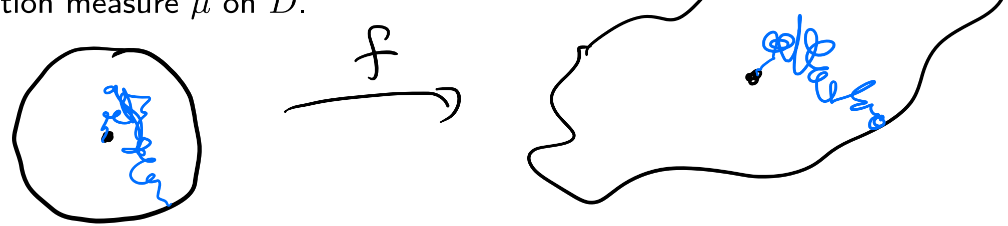
- Let B_t be a standard complex Brownian motion in the unit disk $\mathbb{D}$ stopped at time T when reaches the boundary $\partial\mathbb{D}$. This induces the random measure μ on $\mathbb{D}$,

$$\mu(V) = \int_0^T 1\{B_t \in V\}\, dt.$$

- Suppose $f : \mathbb{D} \to D$ is a conformal transformation. Then $\hat{B}_t = f(B_{\sigma(t)})$ is a Brownian motion in D provided we choose the random time change $\sigma(t)$ such that

$$t - s = \int_{\sigma(s)}^{\sigma(t)} |f'(B_r)|^2\, dr.$$

  This gives an occupation measure $\hat{\mu}$ on D.



- Conformal covariance:

$$\hat{\mu} = |f'|^2\, (f \circ \mu).$$

  The exponent 2 indicates that Brownian motion paths are 2-dimensional paths.

- We could look at the paths on $\mathbb{D}$ but multiply the intrisic metric on the $\mathbb{D}$ by $|f'|$.
- Many models in theoretical physics involve random curves (configurations) and random metrics (geometries). Once often averages (or more generally integrates) over both.
- Sometimes the curve and the metric are independent but the interesting quantities come from the integrated quantities.
- One may hope to understand the random curves by understanding how they look in random geometries. (However, this may not always work!)
- Even if not, the combined random curve/geometry combination can be the underpinning of deep models in theoretical physics (as well as being beautiful mathematics!)

This tutorial

- Write a conformal transformation $f' = e^{u+iv}$. Then u is a "random harmonic function" and $|f'|$ is the "exponential of a random harmonic function".
- We will consider the case where u is a Gaussian free field (GFF).
- The random curves will be Schramm-Loewner evolution (SLE).
- We start by introducing the GFF and the idea of exploration of the field.

Discrete Gaussian Free Field (GFF)

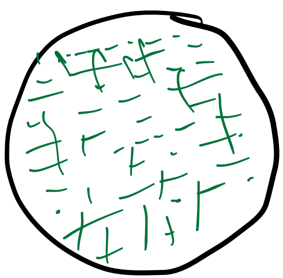
- There is a GFF associated to any transient Markov chain (and even more general than that) but we will restrict ourselves to simple random walk in $\mathbb{Z}^2$ in $A \subsetneq \mathbb{Z}^2$ killed upon leaving the set.
- S_j simple random walk in A. T is the first time k with

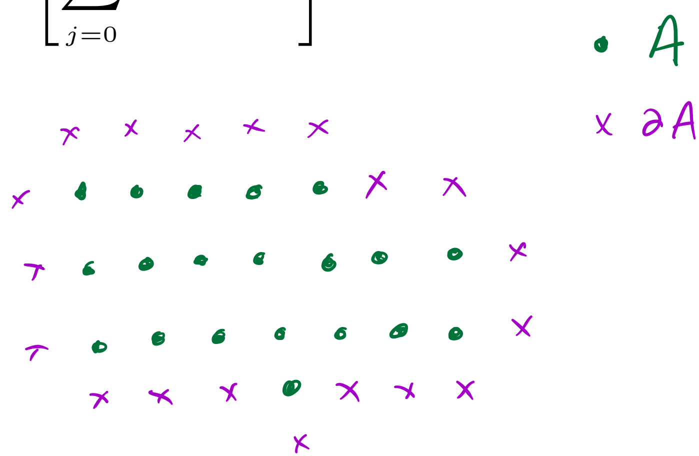
$$S_k \in \partial A := \{w \in \mathbb{Z}^2 : \mathrm{dist}(w, A) = 1\}.$$

- G_A is the Green's function

$$G_A(z, w) = \mathbb{E}^z \left[\sum_{j=0}^{T-1} 1\{S_j = w\} \right].$$

- Let $\mathcal{C}_n = \{z \in \mathbb{Z}^2 : |z| < e^n\}$.



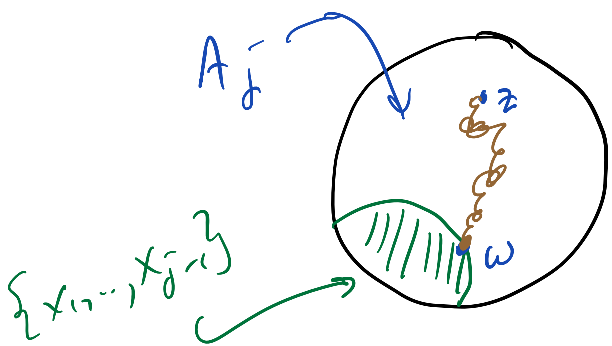


Definition

The Dirichlet Gaussian free field on A is a centered (mean zero) multivariate normal random variable indexed by A, $\mathbf{Z} = \{Z_x : x \in A\}$, with covariance matrix $G_A(z, w)$.

$$\mathbb{E}[Z_z] = 0, \qquad \mathbb{E}[Z_z \, Z_w] = G_A(z, w).$$

- We describe a construction using an ordering of $A = \{x_1, x_2, \ldots\}$ for exploring the field. The construction depends on the ordering but the distribution of the field does not.
- We start with a collection of independent standard normal random variables $\{N_x : x \in A\}$. We view the $\{N_x\}$ as white noise.
- Let $A_j = A \setminus \{x_1, \ldots, x_{j-1}\}$.
- Let $H_{A_j}(z, w)$ be the Poisson kernel, that is, the probability that a random walker starting at z leaves A_j at w. If $z \notin A_j$, then $H_{A_j}(z, z) = 1$.



- For ease write Z_j for Z_{x_j}.
- Define Z_k recursively,

$$Z_1 = \sqrt{G_A(x_1, x_1)}\, N_1,$$

$$Z_k = Z_{x_k} = \sum_{j=1}^{k-1} H_{A_k}(x_k, x_j)\, Z_j + \sqrt{G_{A_k}(x_k, x_k)}\, N_k.$$

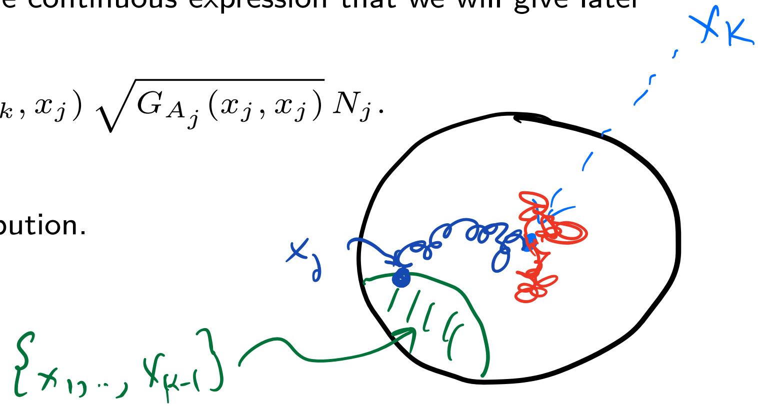
$$\sum_{j=1}^{k-1} H_{A_k}(x_k, x_j)\, Z_j = \text{ Expected value of } Z_k \text{ given } Z_1, \ldots, Z_{k-1}$$

$$\sqrt{G_{A_k}(x_k, x_k)}\, N_k = \text{additional independent randomness}$$

- There is an equivalent form which is more like the continuous expression that we will give later

$$Z_k = \sum_{j=1}^{k} H_{A_j}(x_k, x_j)\, \sqrt{G_{A_j}(x_j, x_j)}\, N_j.$$

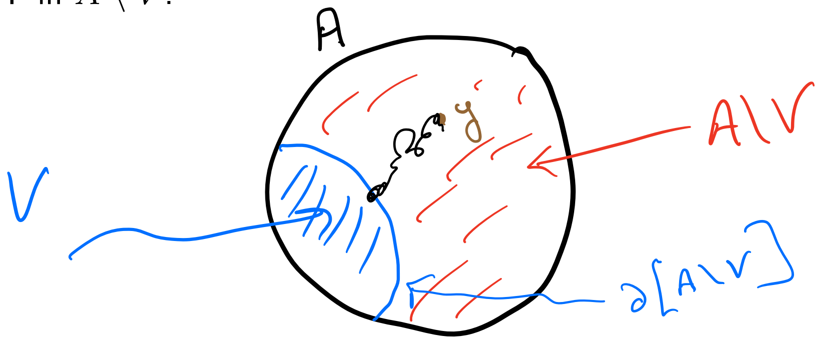
- Exercise to see that these have the correct distribution.



- If $V \subset A$, then the distribution of $\{Z_x : x \in A \setminus V\}$ given $\{Z_x : x \in V\}$ is that of

$$Z_y = v(y) + \tilde{Z}_y$$

where $v(y)$ is the harmonic function with boundary values $\{Z_x : z \in A \setminus V\}$ and $\tilde{Z}_y$ is an independent Dirichlet GFF in $A \setminus V$.



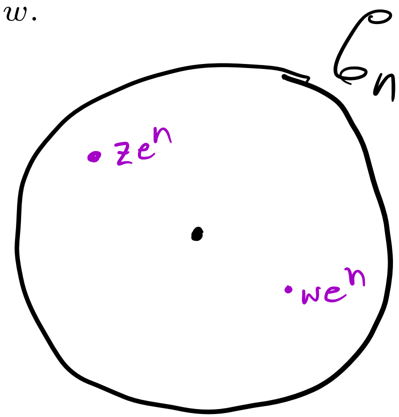
- The field $\{v(y) : y \in A \setminus V\}$ is the harmonic extension of the field on V to $A \setminus V$.
- We sometimes call $\tilde{Z}_y$ the residual field.
- The harmonic function

$$v(y), \qquad y \in A \setminus V$$

depends only on the values $\{Z_x : z \in \partial[A \setminus V]\}$. This is a Markovian field property.

Example: $A = \mathcal{C}_n = \{z \in \mathbb{Z}^2 : |z| < e^n\}$

- $$G_A(0,0) = \frac{2}{\pi}\, n + k_0 + O(e^{-n}),$$
  for a known constant k_0.
- If $z, w \in \mathbb{D}$ with $z \neq w$,
  $$\frac{\pi}{2}\, G_A(ze^n, we^n) = g(z,w) + O(e^{-un}).$$
  where $g(\cdot,\cdot)$ denotes the Brownian motion Green's function in $\mathbb{D}$ normalized so that
  $$g(z,w) = -\log|z-w| + O(1), \quad z \to w.$$
- More precisely, $g(0,z) = -\log|z|$ and g is conformally invariant.
- The continuous Gaussian free field is defined formally by
  $$h(z) = \lim_{n\to\infty} \sqrt{\frac{\pi}{2}}\, Z^{(n)}_{ze^n}$$
  where $Z^{(n)}$ is the GFF on $\mathcal{C}_n$.



(Brownian motion) Green's function in $\mathbb{C} = \mathbb{R}^2$

- $D \subset \mathbb{C}$, domain (open, connected subset) containing 0. We will restrict to $D = \mathbb{C} \setminus (K_1 \cup \cdots \cup K_n)$ where $1 \leq n < \infty$; $K_1, \ldots, K_n$ are the connected components of $\hat{\mathbb{C}} \setminus D$; and each K_j contains more than one point.
- B_t complex Brownian motion,

$$T = T_D = \inf\{t : B_t \notin D\}, \qquad \sigma_s = \inf\left\{t : |B_t| \leq e^{-s}\right\}.$$

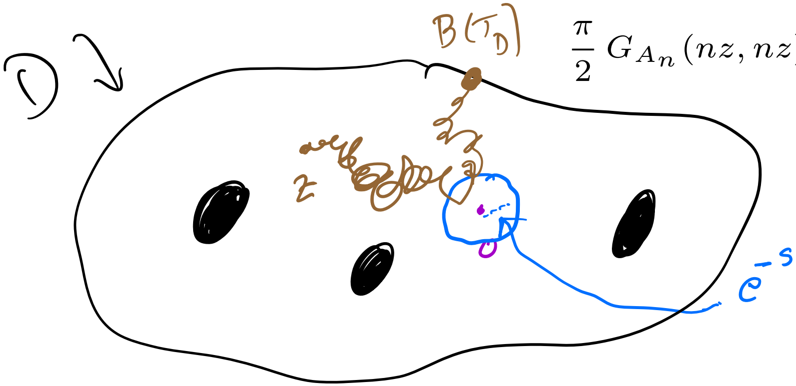
$$G_D(z) := G_D(z, 0) = \lim_{s \to \infty} s\, \mathbb{P}\{\sigma_s < T\}.$$

- More generally,

$$G_D(z, w) = G_{D-w}(z - w, 0).$$

- If $\{A_n\}$ is a sequence of finite subsets of $\mathbb{Z}^2$ with $n^{-1} A_n$ "converging" to D, then

$$\frac{\pi}{2}\, G_{A_n}(nz, nz) \to G_D(z, w).$$



- The conformal radius $r_D(z) = \mathrm{crad}_D(z)$ is defined by

$$G_D(z,w) = -\log|z-w| + \log r_D(z) + o(1), \qquad w \to z.$$

- If $f : D \to f(D)$ is a conformal transformation,

$$G_{f(D)}(f(z), f(w)) = G_D(z,w), \qquad r_{f(D)}(f(z)) = |f'(z)|\, r_D(z).$$

- In particular, if D is simply connected and $f : \mathbb{D} \to D$ is a conformal transformation with $f(0) = z$, then $r_D(z) = |f'(0)|$.

We will now use G for the Brownian motion Green's function on the unit disk $\mathbb{D}$.

Definition

The Gaussian free field (GFF) on $\mathbb{D}$ is a centered Gaussian process $\{h_f\}$ indexed by functions f on $\mathbb{D}$ with covariance function

$$\mathbb{E}[h_f \, h_g] = G(f,g) = \int_{\mathbb{D}} \int_{\mathbb{D}} f(z) \, g(w) \, G(z,w) \, dA(z) dA(w).$$

We write formally

$$h(f) = \int f(z) \, h(z) \, dA(z).$$

- Can define on a countable collection of simple functions (for example step functions with rational values on dyadic squares) and then close in the real Hilbert space with inner product

$$\langle f, g \rangle = \int \int f(z)\, g(w)\, G(z,w)\, dA(z) dA(w).$$

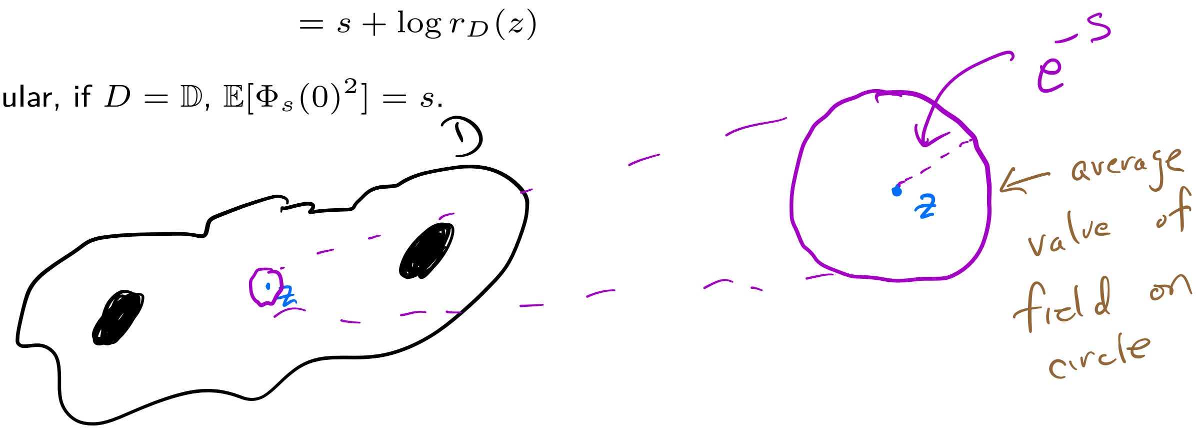
- The closure includes measures ρ that satisfy

$$\int \int G(z,w) \|\rho\|(dz)\, \|\rho\|(dw) < \infty.$$

- This defines random variables in L^2 sense, but it is useful to have a large collection of random variables defined up to one event of probability zero.
- For example, one can define Brownian motion $\{X_t : t \geq 0\}$ as a Gaussian process with covariance $\mathbb{E}[X_s X_t] = s \wedge t$, but one generally chooses a version of this such that with probability one X_t is a continuous function of t.

Let $C_s(z)$ denote the circle of radius e^{-s} centered at z.

- If $z \in D$ with $\mathrm{dist}(z, \partial D) > e^{-s}$ define the circle average of h on the circle of radius e^{-s} about z formally by
$$\Phi_s(z) = \text{“mean value of } h \text{ on } C_s(z)\text{”} = h(\mu),$$
where μ is the uniform probability measure on $C_s(z)$. Using the fact that $G(z,w)$ and $G(z,w) + \log|z-w|$ are harmonic in each variable, we see that
$$\begin{aligned}\mathbb{E}\left[\Phi_s(z)^2\right] &= \text{mean value of } G(\tilde{w}, w) \text{ on } C_s(z) \times C_s(z) \\ &= \text{mean value of } G(z,w) \text{ on } C_s(z) \\ &= s + [\text{mean value of } (G(z,w) + \log|z-w|) \text{ on } C_s(z)] \\ &= s + \log r_D(z)\end{aligned}$$
- In particular, if $D = \mathbb{D}$, $\mathbb{E}[\Phi_s(0)^2] = s$.



- A version of the Dirichlet GFF on $\mathbb{D}$ can be found such that with probability one the circle averages $\Phi_s(z)$ are continuous in s and z. Moreover, sharp estimates can be given for the modulus of continuity.
- Proof idea: define it for $\Phi_s(z)$ over rationals and then proof a result on uniform continuity. Define $\Phi_s(z)$ for all s, z by continuity.
- This allows us to write quantities such as

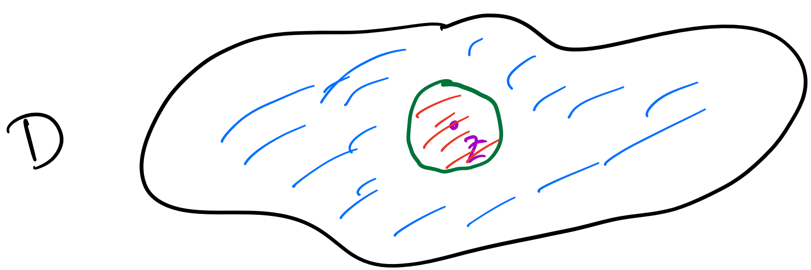
$$\int_0^t \Phi_s(z)\, ds, \qquad \int_V \Phi_s(z)\, dA(z)$$

- We can similarly define the Dirichlet Gaussian field on any domain D with finite Green's function $G_D(z, w)$. In this case,

$$\mathbb{E}\left[\Phi_s(z)^2\right] = s + \log r_D(z).$$

Markovian property

- Let h be a Dirichlet Gaussian free field on a domain D and suppose $z \in D$ with $\mathrm{dist}(z, \partial D) > e^{-s}$.
- Let $\mathcal{D} = \mathcal{D}_s(z)$ be the open disk of radius e^{-s} about z with boundary $C_s(z)$.
- On $\mathcal{D}$ we can write
$$h = v \; + \; \tilde{h}$$
where
$$v(z) = \text{expected value of } h(z) \text{ given } h \text{ on } D \setminus \mathcal{D} = h(\mathrm{hm}_{\mathcal{D},z})$$
$$\tilde{h} = \text{ residual field}$$
where $\mathrm{hm}_{\mathcal{D},z}$ denotes the harmonic measure (hitting measure of boundary by Brownian motion) starting at z and $\tilde{h}$ is an independent Dirichlet GFF on $\mathcal{D}$.
- A "boundary condition" for a GFF on D is a harmonic function on D although the function may not be well defined on the boundary.



- If $t > s$,

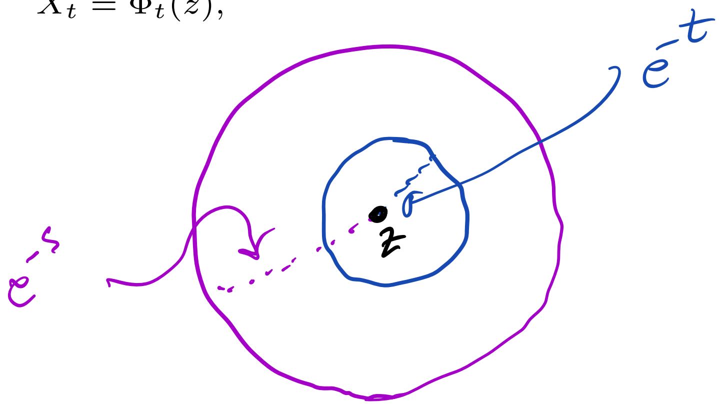
$$\Phi_t(z) = \Phi_s(z) + \tilde{\Phi}_t(z), \quad \tilde{\Phi}_t(z) := \Phi_t(z) - \Phi_s(z).$$

These are independent centered normals and $\mathbb{E}\left[\tilde{\Phi}_t(z)^2\right] = t - s$.

- This uses the fact that the average of $\text{hm}_{\mathcal{D},z}$ where z lies on the circle $C_t(z)$ is the uniform measure on $C_s(z)$.
- Another way of stating this is that if

$$X_t = \Phi_t(z),$$

then X_t is a standard Brownian motion.



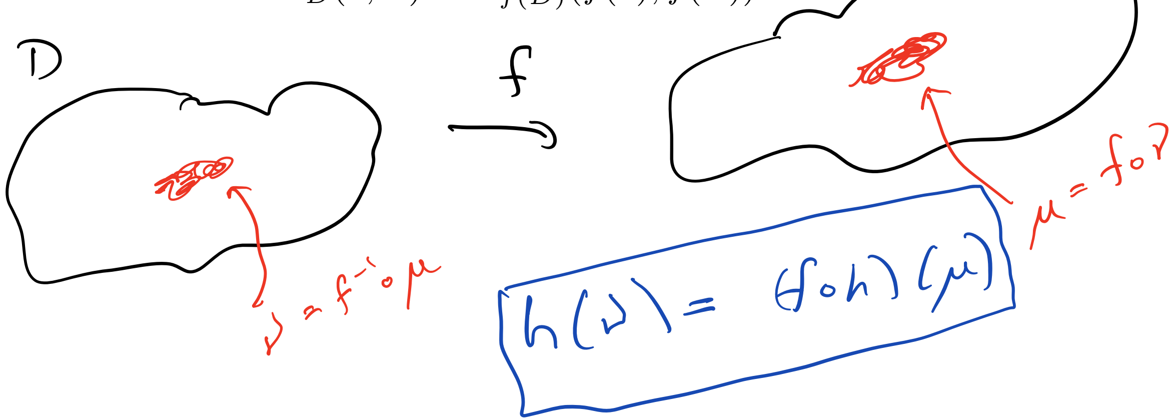
Conformal invariance

- Suppose $f : D \to f(D)$ is a conformal transformation and h is a Dirichlet GFF on D. We will define $f \circ h$ (more precisely written as $h \circ f^{-1}$) to be the field on $f(D)$ given as follows.
- Suppose μ is a measure on $f(D)$. Define ν on D by $\nu(K) = \mu[f(K)]$. Then

$$(f \circ h)(\mu) = h(\nu).$$

- Fact: $f \circ h$ is a Dirichlet Gaussian field on $f(D)$.
- Not difficult to check using the fact that the Green's function is conformally invariant

$$G_D(z, w) = G_{f(D)}(f(z), f(w)).$$

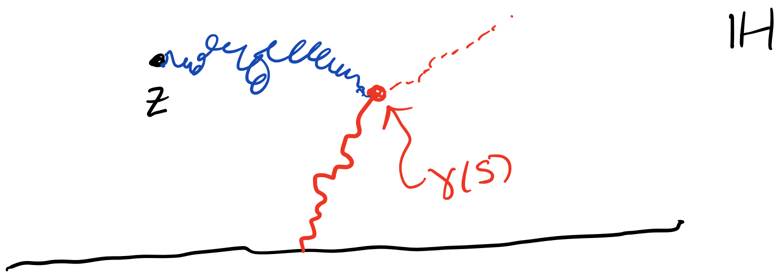


Exploration of the GFF

- We will consider the GFF in the upper half plane $\mathbb{H}$ with Green's function

$$G(z,w) = G_{\mathbb{H}}(z,w) = -\log|z-w| + \log|z-\overline{w}|.$$

- Let $\gamma(t)$ be a simple curve starting at the origin taking values in $\mathbb{H}$.
- We will explore the GFF by traversing the curve γ and seeing the values of the field on γ.



- In analogy with discrete we want to write formally that the expected value of the field at z given the field on $\gamma_t := \gamma(0,t]$ is

$$\int_0^t H_{\mathbb{H}\setminus\gamma_s}(z,\gamma(s)) \text{ (value of } N \text{ at } \gamma(s))\, ds,$$

where N is some "white noise".

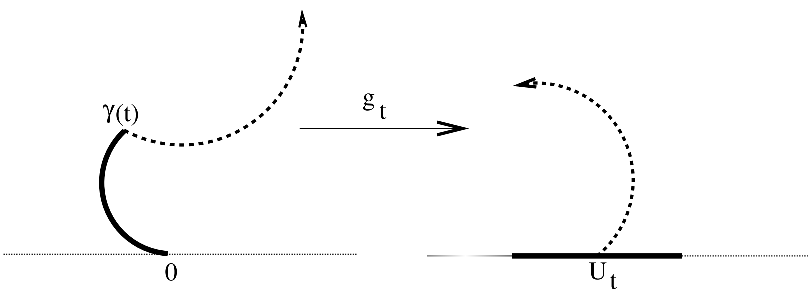
- Make this precise using the Loewner differential equation and (Itô) stochastic integrals

- Suppose $\gamma : [0, \infty) \to \mathbb{H}$ with $\gamma(0) = 0$. We write γ_t for $\gamma[0, t]$.
- Let $a(t) = \text{hcap}[\gamma_t]$ be the half-plane capacity. This can be defined in several ways. For example, if B_s is a complex Brownian motion and $\tau = \tau_t = \min\{s : B_s \in \gamma_t \cup \mathbb{R}\}$, then

$$\text{hcap}[\gamma_t] = \lim_{y \to \infty} y \, \mathbb{E}^{iy} \left[\text{Im}(B_\tau) \right].$$

- $a(t)$ increases with t. We will assume that it is strictly increasing (non-crossing condition). We say that γ is parametrized by capacity if $a(t) = at$ for some $a > 0$; $a = 2$ is most common choice.
- Let D_t be the unbounded component of $\mathbb{H} \setminus \gamma_t$ and let $g_t : D_t \to \mathbb{H}$ be the conformal transformation satisfying $g_t(z) = z + o(1)$ as $z \to \infty$. Then the expansion is given by

$$g_t(z) = z + \frac{a(t)}{z} + O(|z|^{-2}), \qquad z \to \infty.$$



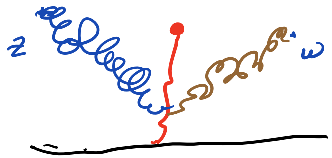
- Loewner differential equation: with capacity parametrization, g_t satisfies

$$\dot{g}_t(z) = \frac{a}{g_t(z) - U_t} = \frac{a}{Z_t(z)} = R_t(z) - i\, H_t(z), \qquad g_0(z) = z. \tag{1}$$

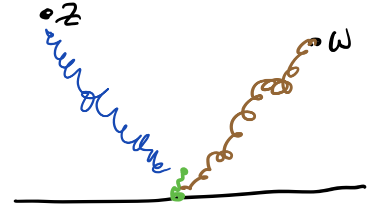
  where $U_t = g_t(\gamma(t))$,

$$Z_t(z) = g_t(z) - U_t, \qquad \frac{1}{Z_t(z)} =: R_t(z) - i\, H_t(z).$$

- We will explore the GFF by traversing the curve γ and seeing the values of the field on γ.



Heuristic picture



- Note that

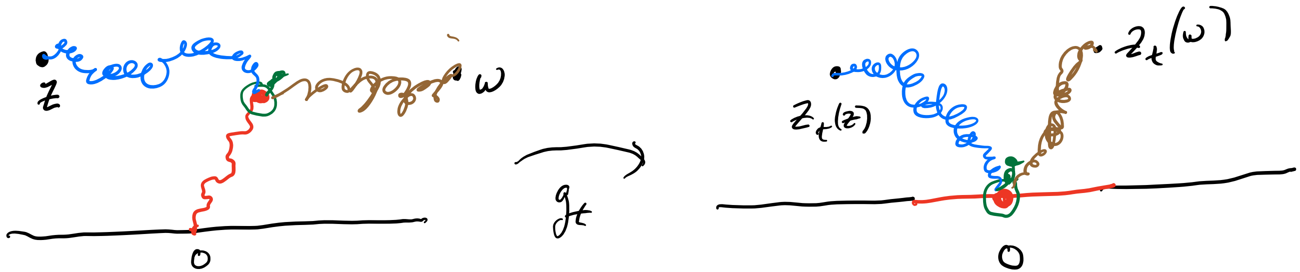
$$H_t(z) = \frac{Y_t(z)}{|Z_t(z)|^2}$$

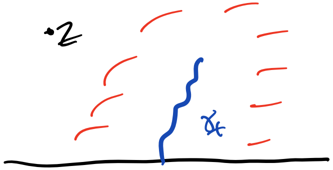
  is (a multiple of) the Poisson kernel in $\mathbb{H}$.

- In a small time interval t, we view the field on γ_t with $\mathrm{hcap}[\gamma_t] = at$. From the perspective of a point away from the origin, this is a Gaussian random variable Y with variance $2at$.
- This contribute a factor of $H_0(z)\, Y$ to the value of field $h(z)$. The remainder of $h(z)$ comes from the values of the field in D_t.
- The Green's function in D_t between z and w is equal to the original Green's function minus

$$2a \int_0^t H_s(z)\, H_s(w)\, ds.$$

  This is the measure of paths from z to w in $\mathbb{H}$ that go through γ_t and hence are not in D_t.





- On D_t we will write the field given its value on γ_t as

$$h(z) = v_t(z) + h_t(z),$$

where v_t is the conditional expectation given the values on γ_t and h_t is an independent Dirichlet GFF on D_t.

- Let W_t be a standard Brownian motion (this is the analogue of the independent normals $\{N_x\}$ from the discrete field)

$$v_t(z) = \int_0^t \sqrt{2a}\, H_s(z)\, dW_s.$$

Compare to discrete: $$Z_k = \sum_{j=1}^{k} H_{A_j}(x_k, x_j)\, \sqrt{G_{A_j}(x_j, x_j)}\, N_j.$$

- Using Loewner equation, we can see that

$$G_{D_t}(z,w) = G(Z_t(z), Z_t(w)) = G(z,w) - 2a \int_0^t H_s(z)\, H_s(w)\, ds.$$

$$\log r_{D_t}(z,z) = \log r_{\mathbb{H}}(z,z) - 2a \int_0^t H_s(z)^2\, ds$$

Theorem

Suppose γ is a noncrossing curve parametrized so that $\mathrm{hcap}[\gamma_t] = at$ and let $D = \mathbb{H} \setminus \gamma$. For $z \in D$, let

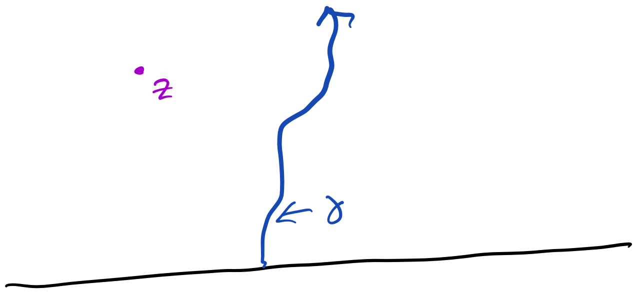
$$v(z) = \sqrt{2a} \int_0^\infty H_s(z)\, dW_s.$$

Let $\tilde{h}$ denote an independent Dirichlet GFF in D. Then

$$v + \tilde{h}$$

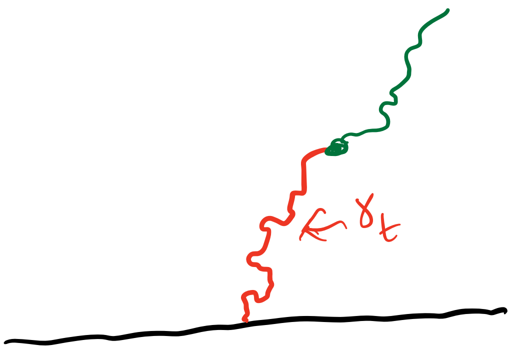
is a Dirichlet GFF in $\mathbb{H}$.

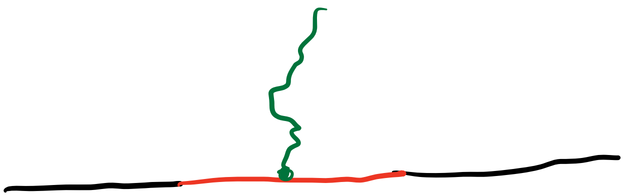
- We can also choose γ to be a random curve independent of W_t
- We can also choose γ that depends on the field — this will lead to an important coupling result.



Schramm-Loewner evolution (SLE_κ)

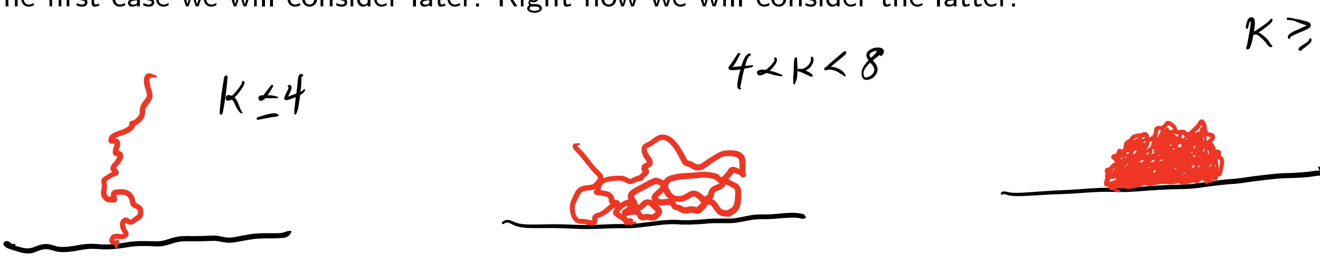
- Probability distribution on simple (or more generally "non-crossing") curves from 0 to ∞ in $\mathbb{H}$ with the conformal Markov property.
- Given an initial segment γ_t, the distribution of the remaining curve after mapping back to $\mathbb{H}$ is the same as the original distribution.
- If we parametrize by half-plane capacity, this determines the distribution up to an additive constant.





Schramm-Loewner evolution (SLE_κ)

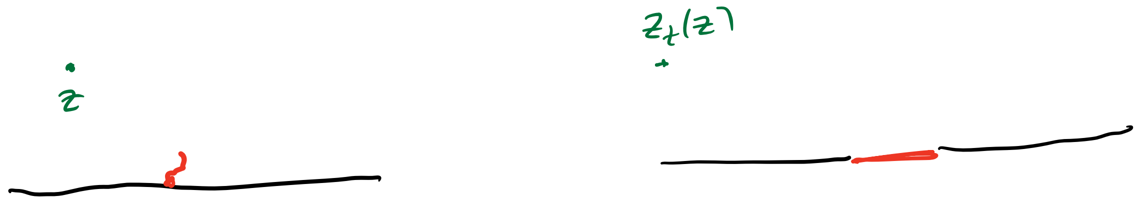
- Solution of Loewner equation with U_t a standard Brownian motion and $a = 2/\kappa$.
- If $Z_t(z) = g_t(z) - U_t$, then
$$dZ_t(z) = \frac{a}{Z_t(z)}\, dt - dU_t.$$
- For $\kappa \le 4$, with probability one we get a simple curve with fractal dimension $d = 1 + \frac{\kappa}{8}$.
- For $4 < \kappa < 8$, the curve has self-intersections but still has dimension $d = 1 + \frac{\kappa}{8}$.
- If $\kappa \ge 8$, the curve is space filling
- While SLE for all values of κ is interesting, we will consider $\kappa \le 4$ primarily.
- We can view field on γ — we can choose W_t to be independent of U_t or we can choose $W_t = U_t$.
- The first case we will consider later. Right now we will consider the latter.



Coupling of SLE with GFF

$$dZ_t(z) = \frac{a}{Z_t(z)}\,dt - dU_t = a\left[R_t(z)\,dt - dU_t + i\,H_t(z)\,dt\right].$$

- We can couple a GFF with an SLE by choosing the same Brownian motion for the field and the curve, that is, $U_t = W_t$.
- Heuristically, we take a small slit with $\mathrm{hcap} = a\epsilon$; view the GFF on this slit, and then use the value of the GFF; on the slit to determine the value of U_ϵ.



- Mathematically, we find the appropriate martingales using the Loewner equation and Itô's formula.
- If $\Theta_t(z) = \arg Z_t(z)$,

$$d\Theta_t(z) = (1-2a)\,R_t(z)\,H_t(z)\,dt + H_t(z)\,dW_t.$$

- In particular, $\Theta_t(z)$ is a martingale for $\kappa = 4$ but not for other values of κ.

- Let $\psi_t(z) = \arg g_t'(z) = \mathrm{Im}[\log g_t'(z)]$ (which is harmonic in z) and

$$M_t(z) = \Theta_t(z) + \frac{2a-1}{2a}\,\psi_t(z)$$

then $M_t(z)$ is a martingale satisfying

$$dM_t(z) = H_t(z)\,dW_t.$$

- More generally, if ρ is a measure, we set

$$M_t(\rho) = \int M_t(z)\,\rho(dz)$$

and see that $M_t(\rho)$ is a martingale satisfying

$$dM_t(\rho) = H_t(\rho)\,dW_t, \qquad \text{where} \qquad H_t(\rho) = \int H_t(z)\,\rho(dz).$$

This gives

$$d\langle M_t(\rho)\rangle_t = \int\int H_t(z)\,H_t(w)\,\rho(dz)\,\rho(dw)\,dt = -(2a)^{-1}\,\partial_t\,G_{D_t}(\rho).$$

Theorem

Let γ be an SLE_κ path in $\mathbb{H}$, $\kappa < 8$. Let $D = \mathbb{H} \setminus \gamma$ and on D define

$$v_\infty(z) = \sqrt{2a}\, M_\infty(z) = \sqrt{2a}\, \left[\Theta_\infty(z) + \frac{2a-1}{2a}\, \psi_\infty(z) \right]$$

(note that $\Theta_\infty(z) \in \{0, \pi\}$) Let $\tilde{h}$ be a Dirichlet GFF on D conditionally independent of γ. Then $h := v_\infty + \tilde{h}$ is a GFF on $\mathbb{H}$ with boundary condition $v_0(z) = \sqrt{2a}\, \arg(z)$.

This is nicest when $\kappa = 4$.

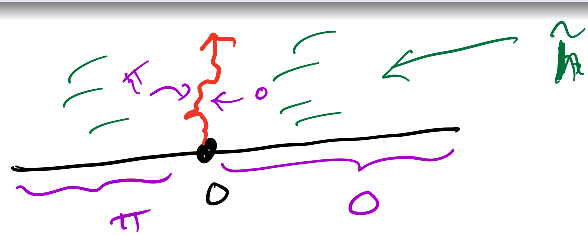
Theorem

Let γ be an SLE_4 path in $\mathbb{H}$. Let $D = \mathbb{H} \setminus \gamma$ and on D define

$$v_\infty(z) = M_\infty(z) = \begin{cases} 0 & \text{right side of } \gamma \\ \pi & \text{left side of } \gamma \end{cases}$$

Let $\tilde{h}$ be a Dirichlet GFF on D conditionally independent of γ. Then $h := v_\infty + \tilde{h}$ is a GFF on $\mathbb{H}$ with boundary condition 0 on $(0, \infty)$ and π on $(-\infty, 0)$.

$\kappa = 4$



Basic idea of proof

- If we can write a random variable Y as
$$Y = \int_0^\infty A_s \, dB_s, \qquad \int_0^\infty A_s^2 \, ds = \sigma^2,$$
where σ^2 is a constant, then $Y \sim N(0,\sigma^2)$. It is not required that each A_s be a constant.
- Similarly, if
$$Y = \int_0^\infty A_s \, dB_s, \qquad \int_0^\infty A_s^2 \, ds \leq \sigma^2,$$
then Y may not be normal but if given Y we add Z, a conditionally independent centered Gaussian with variance
$$\sigma^2 - \int_0^\infty A_s^2 \, ds$$
then $Y + Z \sim N(0,\sigma^2)$.

Full plane GFF

- Gaussian field h defined up to a random constant, that is, if Y is a real-valued random variable that may depend on h, then h and $h+Y$ are the same field.
- Field indexed by signed measures ρ with compact support and $\rho(\mathbb{C})=0$.
- $h(\rho)$ is a centered Gaussian random variable with variance

$$\int\int G(z,w)\,\rho(dz)\,\rho(dw), \qquad G(z,w):=-\log|z-w|.$$

$$(h+Y)(\rho)=h(\rho)+\int Y\,\rho(dz)=h(\rho).$$

- The choice of G is not unique even if we require symmetry and harmonic in each variable. For example, we could choose

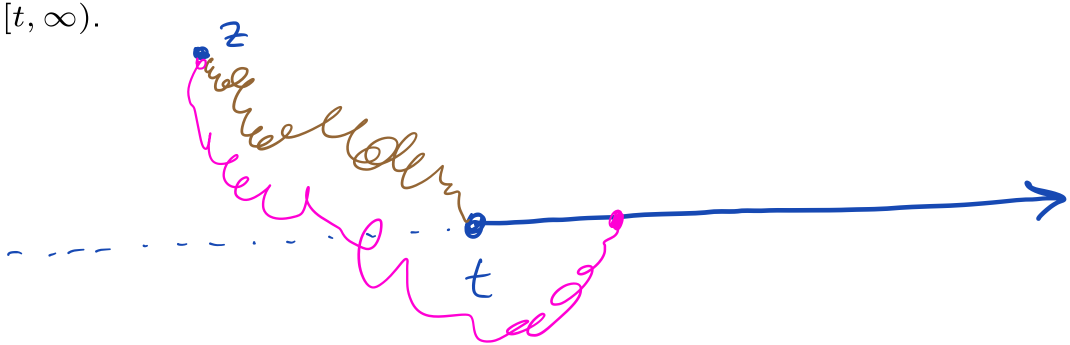
$$G(z,w)=-\log|z-w|+\phi(z)+\phi(w),$$

where ϕ is harmonic since

$$\int\int \phi(z)\,\rho(dz)\,\rho(dw)=\int\phi(z)\,\left[\int\rho(dw)\right]\,\rho(dz)=0.$$

Construction of full plane GFF

- Idea: "get it started at some point (0 or ∞)" and then use Dirichlet GFF to define the rest.
- We will do one way by defining the field on the real line first and then extending. We will explore on the real line from ∞ to $-\infty$.
- Let $D_t = \mathbb{C} \setminus [t, \infty)$ and for $z \in D$, formally let $v_t(z)$ be the expected value of the field at z given the field on $[t, \infty)$.



- While $v_t(z)$ is not well defined, we will be able to define $v_t(\rho)$ if $\rho(\mathbb{C}) = 0$.
- Then for each t we can write

$$h = \tilde{h}_t + v_t$$

where $\tilde{h}_t$ is a Dirichlet GFF on D_t. In particular, $h = \tilde{h} + v_\infty$ where $\tilde{h}$ is a Dirichlet GFF on $\mathbb{C} \setminus \mathbb{R}$.

- Let

$$u(z) = u_0(z) = H_{\mathbb{H}}(\sqrt{z}) = \frac{\sin(\theta_z/2)}{|z|^{1/2}},$$

$$u_t(z) = u(z+t), \qquad u_t(\rho) = \int u_t(z)\, \rho(dz).$$

- We think of $u_t(z)$ as the value of the Poisson kernel $H_{D_{-t}}(z, -t)$ even though t is not a smooth point on ∂D_{-t}.

$$v_t(z) = \int_{-\infty}^{t} u_t(z)\, dW_t, \qquad v_t(\rho) = \int_{-\infty}^{t} u_t(\rho)\, dW_t.$$

- For fixed z, ρ, with $\rho(\mathbb{C}) = 0$, as $t \to \infty$,

$$u_{-t}(z) = O(|t|^{-1/2}), \qquad u_t(z) = O(t^{-3/2}),$$

$$u_{-t}(\rho) = O(|t|^{-3/2}), \qquad u_t(\rho) = O(t^{-3/2})$$

  In particular $v_\infty(\rho)$ is a centered Gaussian random variable with variance

$$\int_{-\infty}^{\infty} u_t(\rho)^2\, dt < \infty$$

- The full plane GFF written as $h = v + \tilde{h}$ contains two independent GFFs on $\mathbb{H}$:

$$h_+(z) = \tilde{h}(z), \qquad h_-(z) = \tilde{h}(\overline{z}).$$

Let

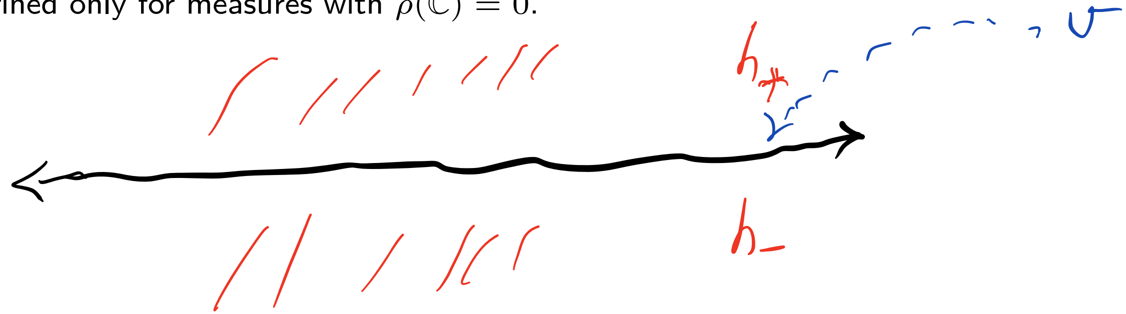
$$\tilde{\Phi}(z) = \frac{h(z) - h(\overline{z})}{\sqrt{2}} = \frac{h_+(z) - h_-(z)}{\sqrt{2}},$$

$$\Phi = \frac{h(z) + h(\overline{z})}{\sqrt{2}} = \frac{h_+ + h_-}{\sqrt{2}} + \sqrt{2}\, v.$$

- $\tilde{\Phi}$ is a Dirichlet GFF on $\mathbb{H}$. We call Φ a Neumann GFF on $\mathbb{H}$. $\tilde{\Phi}(\rho)$ is defined for measures with $\rho(\mathbb{C}) = 0$. We can write Φ as two independent pieces

$$\Phi = \left[\text{expected value given } \sqrt{2}\, h \text{ on } \mathbb{R}\right] + [\text{Dirichlet GFF on } \mathbb{H}]$$

The first is defined only for measures with $\rho(\mathbb{C}) = 0$.



- If ρ is a measure with compact support in $\mathbb{H}$ with $\rho(\mathbb{C}) = 0$, then $\Phi(\rho)$ is a centered Gaussian random variable with variance
$$\int\int \tilde{G}_{\mathbb{H}}(z,w)\,\rho(dz)\,\rho(dw)$$
where
$$\tilde{G}_{\mathbb{H}}(z,w) = -\log|z-w| - \log|z-\overline{w}|.$$
- This can be compared to
$$G_{\mathbb{H}}(z,w) = -\log|z-w| + \log|z-\overline{w}|.$$
- As for the full plane GFF, the Neumann Green's function is not unique. For example, we could choose
$$\tilde{G}_{\mathbb{H}}(z,w) = -\log|z-w| - \log|z-\overline{w}| + \phi(z) + \phi(w)$$
where ϕ is harmonic in $\mathbb{H}$.
- The Neumann GFF is conformally invariant.
- I will discuss this more in my Saturday lecture.

Part II: Random Fractal Measures

- Simply connected domain D and distinct boundary points $z, w \in D$.
- γ: chordal SLE_κ $(\kappa < 8)$ path from z to w in D. We write $d = 1 + \frac{\kappa}{8}$ for the corresponding fractal dimension.
- h Dirichlet GFF in D
- d-dimensional Minkowski content: fractal measure given by γ
- Liouville quantum gravity (LQG): fractal measure with "density" $e^{\sqrt{\kappa}\, h}$ (area measure) or $e^{(\sqrt{\kappa}/2)\, h}$ (length measure).
- Quantum length: A combination of Minkowski content and LQ length measure.

Notes on parameters

- It is standard to use $\gamma \in (0,2)$ for the parameter $\sqrt{\kappa}$. We choose not to do this since we use γ for our SLE_κ paths.
- Other parameters are $Q = Q_\gamma$ and the central charge $\boldsymbol{c}$

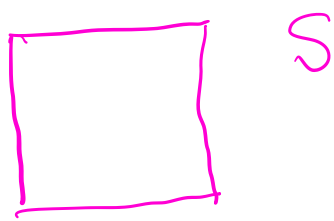
$$Q = Q_\kappa = \frac{\sqrt{\kappa}}{2} + \frac{2}{\sqrt{\kappa}}, \qquad \boldsymbol{c} = \boldsymbol{c}_\kappa = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}.$$

Construction of fractal measures

- If n is an integer, we call S a dyadic square at level n if it is of the form

$$S = \left\{ x + iy : \frac{j}{2^n} \le x < \frac{j+1}{2^n}, \;\; \frac{k}{2^n} \le y < \frac{k}{2^n} \right\}$$

  for some integers j, k, n.
- We choose half-open, half-closed squares so that the squares of a fixed level partition the entire plane.
- Let $\mathcal{Q}_n = \mathcal{Q}_n(D)$ denote the set of dyadic squares S at level n with $\overline{S} \subset D$ and $\mathcal{Q} = \mathcal{Q}(D) = \cup_n \mathcal{Q}_n$.
- A function $\mu : \mathcal{Q} \to [0, \infty)$ is called a (dyadic) field if it is finitely additive, that is, whenever S is written as a finite disjoint union, $S = S_1 \cup \cdots \cup S_k$ with $S_1, \ldots, S_k \in \mathcal{Q}$, then $\mu(S) = \mu(S_1) + \cdots + \mu(S_k)$.
- If μ is a Borel measure on D, then $\{\mu(S) : S \in \mathcal{Q}\}$ is a field. However, fields can be more general than those given by measures. For example, if h is a discrete GFF on D, then $\{h(S) : S \in \mathcal{Q}\}$ is a field.



- If $S \in \mathcal{Q}_n$ and $m \geq n$, let $\partial_m S$ be the union of all $S' \in \mathcal{Q}_m$ with $\mathrm{dist}(S', \partial S) = 0$.
- We define
$$\mu(\partial S) = \lim_{m \to \infty} \mu[\partial_m(S)].$$
The limit exists since $\mu[\partial_m(S)]$ decreases with (large enough) m.
- Let $\mathcal{M} = \mathcal{M}(D)$ denote the set of positive dyadic fields in D such that $\mu(\partial S) = 0$ for every $S \in \mathcal{Q}$.
- Let $\mathcal{Q}^*$ denote the set of finite unions of elements of $\mathcal{Q}$.
- μ extends to an additive function on $\mathcal{Q}^*$ such that $\mu(\partial S) = 0$ for every $S \in \mathcal{Q}^*$.
- A (positive) Borel measure μ on D with $\mu(S) < \infty$ and $\mu(\partial S) = 0$ for all $S \in \mathcal{Q}$ gives an element of $\mathcal{M}$. Indeed, this is a bijection.
- Standard measure theory shows that if $\mu \in \mathcal{M}$, then μ can be extended uniquely to a Borel measure on D.

- To construct μ, we need to show that the limit

$$\mu(S) = \lim_{n \to \infty} \mu_n(S) \tag{2}$$

exists for each $S \in \mathcal{Q}$ and that the limit satisfies $\mu(\partial S) = 0$ for each S.

- Since $\mathcal{Q}$ is countable we need only show an almost sure limit for each S.
- We will write

$$\mu_n(S) = \int_S F_n(z)\, dA(z)$$

where $F_n(z)$ is a random (depending on γ and/or h) measurable function.

- In (2) we have a limit along a sequence. More generally we will consider

$$\mu_t(S) = \int_S F_t(z)\, dA(z), \quad t \in \mathbb{R}$$

and take the limit

$$\mu(S) = \lim_{t \to \infty} \mu_t(S).$$

Liouville Quantum Gravity (Area Measure)

(Roughly speaking, $\mu(dz) = e^{\sqrt{\kappa}\, h(z)}\, dA(z)$)

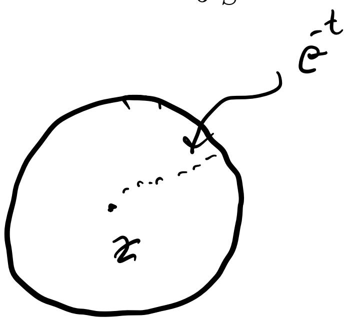
- Fix $0 < \kappa < 4$ and $S \in \mathcal{Q}$
- Recall: if $\mathrm{dist}(S, \partial D) > e^{-t}$ and $z \in S$, $\Phi_t(z)$ is the circle average of h at radius e^{-t} about z.
- $\Phi_t(z)$ is a centered Gaussian random variable with variance $t + \log r_D(z)$.

$$\mathbb{E}\left[\exp\{\sqrt{\kappa}\, \Phi_t(z)\}\right] = R(z)\, e^{\kappa\, t/2}, \qquad R(z) = r_D(z)^{\sqrt{\kappa}/2}.$$

We set

$$F_t(z) = e^{-\kappa\, t/2} \exp\{\sqrt{\kappa}\, \Phi_t(z)\}, \qquad \mathbb{E}[F_t(z)] = R(z).$$

$$\mu_t(S) = \int_S F_t(z)\, dA(z), \qquad \mathbb{E}[\mu_t(S)] = \int_S R(z)\, dA(z).$$



Straightforward calculation for large deviations of normal distributions

- Let N be a standard normal random variable and $\lambda > 0$.

moment generating function $$\mathbb{E}\left[e^{\lambda N}\right] = e^{\lambda^2/2}. \tag{3}$$

- If we tilt the probability measure $\mathbb{P}$ by $e^{-\lambda^2/2}\, e^{\lambda N}$, then in the new probability measure $\mathbb{P}^*$, N has the distribution of a normal random variable with mean λ and variance 1.

$$\mathbb{P}^*\{a \leq N - \lambda \leq b\} = \int_a^b \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx.$$

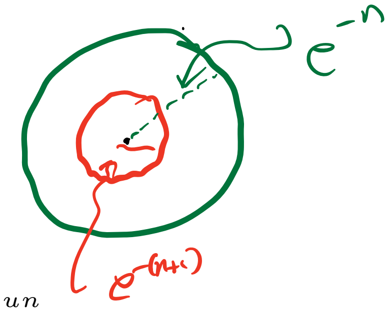
or equivalently,

$$e^{-\lambda^2/2}\, \mathbb{E}\left[e^{\lambda N}; \lambda + a \leq N \leq \lambda + b\right] = \int_a^b \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx.$$

- Roughly speaking, the expectation in (3) is carried on the event that $N = \lambda + O(1)$ and the probability of this event is of order $e^{-\lambda^2/2}$.

Back of the envelope calculation

- The integral
$$\mu_t(S) = \int_S F_t(z)\, dA(z)$$
is carried on disks of radius e^{-t} where the circle average $\Phi_t(z)$ equals about $\sqrt{\kappa}\, t$.
- The probability of such a disk is of order $e^{-(\kappa/2)t}$.
- The number of disks of radius e^{-t} needed to cover S is $O(e^{2t})$. The fractal dimension of the set of points carrying the integral is $2 - \frac{\kappa}{2}$.
- We choose $\kappa < 4$ so that this dimension is positive.



- Fix a dyadic square S and let $Y_t = \mu_t(S)$.
- It suffices to show that there exist c, u such that for $0 \leq t \leq 1$,

$$\|Y_{n+t} - Y_n\|_1 = \mathbb{E}\left[|Y_{n+t} - Y_n|\right] \leq c\, e^{-un}$$

It follows that there exists an L^1 limit Y along every subsequence and

$$\|Y_t - Y_\infty\|_1 \leq c' e^{-ut}, \qquad \mathbb{P}\{|Y_t - Y_\infty| \geq e^{-ut/2}\} \leq c'\, e^{-ut/2}.$$

- It follows that for every arithmetic sequence we get convergence with probability one.
- To get that with probability one

$$\lim_{t \to \infty} Y_t = Y$$

we use these estimates and the estimates on the modulus of continuity of the circle averages.

- Try L^2 limits first — these are easiest for sums or integrals of random variables.
- If L^2 norms do not exists, try truncating the random variables to remove unlikely large values that have a negligible effect on $\mathbb{E}[Y]$ but a nonnegligible effect on $\mathbb{E}[Y^2]$.

- For ease, let $t = 1$ and write

$$\begin{aligned} \Psi_n(z) &= F_{n+1}(z) - F_n(z) \\ &= F_n(z) \left[\exp\left\{ \sqrt{\kappa}\, J_n(z) - \frac{\kappa}{2} \right\} - 1 \right], \end{aligned}$$

  where $J_n(z) = \Phi_{n+1}(z) - \Phi_n(z)$.

- Recall that $J_n(z)$ is a standard normal random variable independent of $\mathcal{F}_n^z$, the σ-algebra associated to the values of the field in $\{\zeta : |z - \zeta| \geq e^{-n}\}$.

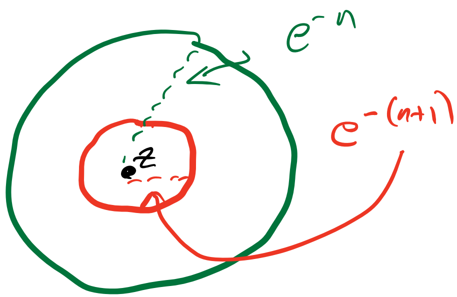
- In particular,

$$\mathbb{E}\left[\Psi_n(z) \mid \mathcal{F}_n^z \right] = \mathbb{E}\left[\Psi_n(z) \right] = 0.$$

- Note that

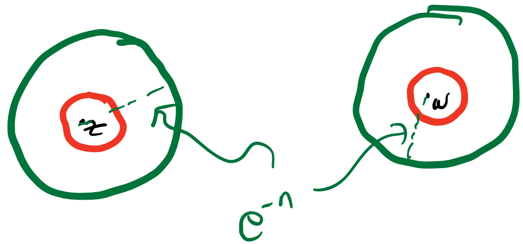
$$Y_{n+1} - Y_n = \int_S \Psi_n(z)\, dA(z),$$

$$\mathbb{E}[(Y_{n+1} - Y_n)^2] = \int_S \int_S \mathbb{E}\left[\Psi_n(z)\, \Psi_n(w) \right]\, dA(z)\, dA(w).$$



- If $|z - w| \geq 2e^{-n}$, then $\Psi_n(w)$ is $\mathcal{F}_n^z$-measurable. Hence

$$\begin{aligned} \mathbb{E}\left[\Psi_n(z)\, \Psi_n(w)\right] &= \mathbb{E}\left[\mathbb{E}\left(\Psi_n(z)\, \Psi_n(w) \mid \mathcal{F}_n^z\right)\right] \\ &= \mathbb{E}\left[\Psi_n(w)\, \mathbb{E}\left(\Psi_n(z) \mid \mathcal{F}_n^z\right)\right] = 0 \end{aligned}$$



- For $|z - w| \leq 2e^{-n}$ we use Cauchy-Schwartz,

$$\mathbb{E}\left[\Psi_n(z)\, \Psi_n(w)\right] \leq \sqrt{\mathbb{E}\left[\Psi_n(z)^2\right] \mathbb{E}\left[\Psi_n(w)^2\right]} \leq \beta_n := \sup_{z \in S} \mathbb{E}[\Psi_n(z)^2].$$

- The measure of the set of (z, w) with $|z - w| \leq 2e^{-n}$ is $O(e^{-2n})$ and hence

$$\mathbb{E}[(Y_{n+1} - Y_n)^2] \leq c\, e^{-2n}\, \beta_n.$$

  On the next slide we will show that $\beta_n \asymp e^{\kappa n}$. Hence we get

$$\mathbb{E}[(Y_{n+1} - Y_n)^2] \leq c\, e^{(\kappa - 2)n}$$

  which gives our estimate for $\kappa < 2$.

$$
\begin{aligned}
\mathbb{E}\left[\Psi_n(z)^2\right] &= \mathbb{E}\left[F_n(z)^2\left(\exp\left\{\sqrt{\kappa}\,J_n(z)-\frac{\kappa}{2}\right\}-1\right)^2\right] \\
&= \mathbb{E}\left[F_n(z)^2\right]\,\mathbb{E}\left[\left(\exp\left\{\sqrt{\kappa}\,J_n(z)-\frac{\kappa}{2}\right\}-1\right)^2\right] \\
&\asymp \mathbb{E}\left[F_n(z)^2\right] \\
&\asymp e^{-\kappa n}\,\mathbb{E}[\exp\{2\sqrt{\kappa}\Phi_n(z)\}] \asymp e^{\kappa n}
\end{aligned}
$$

- The last expectation concentrates on z with $\Phi_n(z) = 2\sqrt{\kappa}n + O(1)$. But our measure should concentrate on z with $\Phi_n(z) = \sqrt{\kappa}n + O(1)$
- We replace $F_n(z)$ with

$$\tilde{F}_n(z) = F_n(z)\,1\{\Phi_n(z) \le (\sqrt{\kappa}+\delta)n\}.$$

$$\tilde{Y}_n(z) = \int \tilde{F}_n(z)\,dA(z).$$

$$
\begin{aligned}
&\mathbb{E}[\exp\{2\sqrt{\kappa}\Phi_n(z)\}; \Phi_n(z) \leq (\sqrt{\kappa}+\delta)n] \\
&\qquad \asymp \quad \exp\{2\sqrt{\kappa}(\sqrt{\kappa}+\delta)n\}\,\mathbb{P}\{\Phi_n(z) \geq (\sqrt{\kappa}+\delta)n\} \\
&\qquad \asymp \quad \exp\left\{n\,\left[2\sqrt{\kappa}(\sqrt{\kappa}+\delta) - \frac{(\sqrt{\kappa}+\delta)^2}{2}\right]\right\} \\
&\qquad \leq \quad \exp\left\{n(\frac{3\kappa}{2} + \delta\sqrt{\kappa})\right\}
\end{aligned}
$$

- We thus get

$$
\begin{aligned}
\mathbb{E}\left[\tilde{\Psi}_n(z)^2\right] &\leq e^{-\kappa n}\,\exp\left\{n\left(\frac{3\kappa}{2} + \delta\sqrt{\kappa}\right)\right\} \\
&= \exp\left\{n\left(\frac{\kappa}{2} + \delta\sqrt{\kappa}\right)\right\}.
\end{aligned}
$$

- For each $\kappa < 4$, we can find $\delta > 0$ such that the last quantity equals $e^{(2-\epsilon)n}$ for some $\epsilon > 0$ giving

$$
\mathbb{E}\left[|\tilde{Y}_{n+1} - \tilde{Y}_n|\right] \leq \sqrt{\mathbb{E}\left[(\tilde{Y}_{n+1} - \tilde{Y}_n)^2\right]} \leq c\,e^{-\epsilon n/2}.
$$

- Straightforward estimate using normal random variables gives

$$\begin{aligned}
&\mathbb{E}[F_n(z); \Phi_n(z) \geq (\sqrt{\kappa}+\delta)n] \\
&\quad \asymp \quad e^{-\kappa n/2}\, \mathbb{E}[\exp\{\sqrt{\kappa}\, \Phi_n(z)\}; \Phi_n \geq (\sqrt{\kappa}+\delta)n] \\
&\quad \leq \quad \exp\left\{-n\left(\frac{\kappa n}{2} + \delta\,(\sqrt{\kappa}+\delta)\right)\right\}\, \mathbb{E}\left[\exp\{(\sqrt{\kappa}+\delta)\,\Phi_n(z)\}\right] \\
&\quad = \quad \exp\left\{-n\left(\frac{\kappa n}{2} + \delta\,(\sqrt{\kappa}+\delta)\right)\right\}\, \exp\left\{\frac{(\sqrt{\kappa}+\delta)^2\, n}{2}\right\} \\
&\quad = \quad e^{-\delta n/2}
\end{aligned}$$

- Therefore,

$$\mathbb{E}\left[Y_n - \tilde{Y}_n\right] = \int \mathbb{E}[F_n(z); \Phi_n(z) \geq (\sqrt{\kappa}+\delta)n]\, dA(z) \leq c\, e^{-\delta n/2}.$$

$$|Y_{n+1} - Y_n| \leq |\tilde{Y}_{n+1} - \tilde{Y}_n| + |Y_n - \tilde{Y}_n| + |Y_{n+1} - \tilde{Y}_{n+1}|.$$

$$\mathbb{E}[|Y_{n+1} - Y_n|] \leq c\, e^{-(\epsilon\wedge\delta)n/2}.$$

Conformal covariance

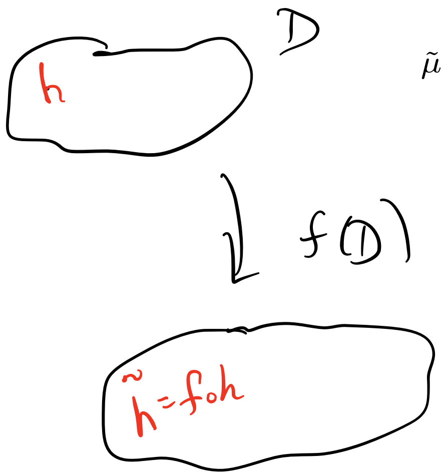
Fact

$f : D \to \tilde{D}$ be a conformal transformation; h a Dirichlet GFF in D and $\tilde{h} = f \circ h$ the corresponding GFF in $\tilde{D}$. Let $\mu, \tilde{\mu}$ be the LQG measures associated to $h, \tilde{h}$. Then,

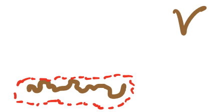
$$\tilde{\mu} = |f'|^{2+\frac{\kappa}{2}} \, (f \circ \mu).$$

- If $z \in D$ with $|f'(z)| = e^s$, we have

$$F_t(z) = e^{-\kappa t/2} \, e^{\Phi_t(z)} \sim e^{-\kappa t/2} \, e^{\tilde{\Phi}_{t-s}(f(z))} \sim |f'(z)|^{-\frac{\kappa}{2}} \, \tilde{F}_{t-s}(f(z)).$$



$$\begin{aligned}
\tilde{\mu}(F(V)) &\sim \int_{f(V)} \tilde{F}_{t-s}(w) \, dA(w) \\
&\sim \int_V \tilde{F}_{t-s}(f(z)) \, |f'(z)|^2 \, dA(z) \\
&\sim \int_V F_t(z) \, |f'(z)|^{2+\frac{\kappa}{2}} \, dA(z)
\end{aligned}$$



Minkowski content

- If $V \subset \mathbb{R}^n$ is compact and $0 < d < n$, then the Minkowski content of V is given by

$$\lim_{\epsilon \to 0} \epsilon^{d-n} \, \mathrm{Vol}_n\{z : \mathrm{dist}(z, V) \leq \epsilon\}$$

  provided that the limit exists.
- This limit often exists for random fractal subsets that exhibit "statistical self-similarity" as we zoom in.
- Similar concepts: local time, occupation measure.
- d-dimensional random fractals tend to have zero d-dimensional Hausdorff measure, Minkowski content is the way to measure the size of the set.
- Generally requires finer analysis of a fractal to show existence of Minkowski content than show Hausdorff dimension.
- One "template" for a proof of existence of Minkowski content for random fractals is similar to the LQG proof. However, it requires fine analysis of the random object.
- The first step is a sharp one-point estimate (Green's function)

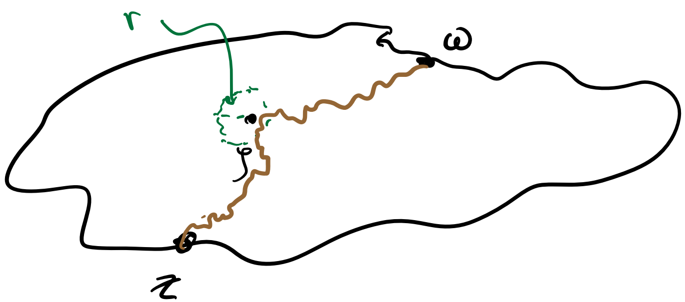
Theorem

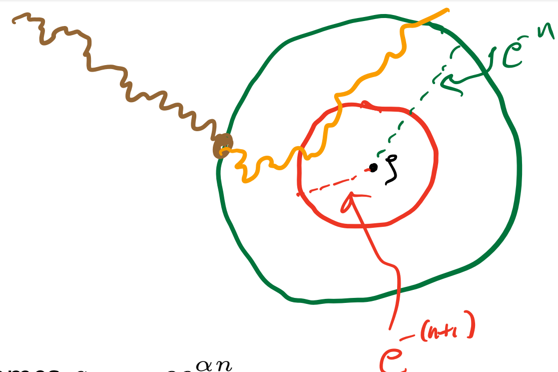
If $0 < \kappa < 8$, there exists $\hat{c} > 0$ and $u > 0$ such that if γ is a chordal SLE_κ path from z to w in simply connected D, then for $r < \mathrm{dist}(\zeta, \gamma)/2$,

$$\mathbb{P}\{\mathrm{dist}(\zeta,\gamma) < r\} = \mathbb{G}_D(\zeta; z, w)\, r^{2-d} \left[1 + O(r^u)\right],$$

where

$$\mathbb{G}_D(\zeta; z, w) = \hat{c}\, r_D(\zeta)^{d-2} \left[\sin \arg_D(\zeta; z, w)\right]^{4a-1}.$$





- General idea to prove that $f(t) \sim c\, t^\alpha$ as $t \to \infty$ for some c.
- First try along a geometric sequence $a_n = f(e^n)$ so that it becomes $a_n \sim c e^{\alpha n}$.
- Try to show that

$$a_{n+1} = a_n\, e^\alpha \left[1 + O(e^{-un})\right].$$

- This implies that there exists c such that

$$a_n = c\, e^{\alpha n} \left[1 + O(e^{-un})\right]$$

  but does not compute the c.

L^2-estimate — second moment bound

Fact

Fix a dyadic square S, There exists c such that if $z, w \in S$, then

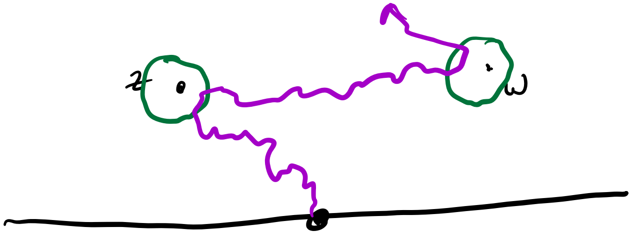
$$\mathbb{P}\{\mathrm{dist}(z,\gamma) < r, \mathrm{dist}(w,\gamma) < r\} \leq c\, r^{2(2-d)}\, |z-w|^{d-2}.$$

This gives

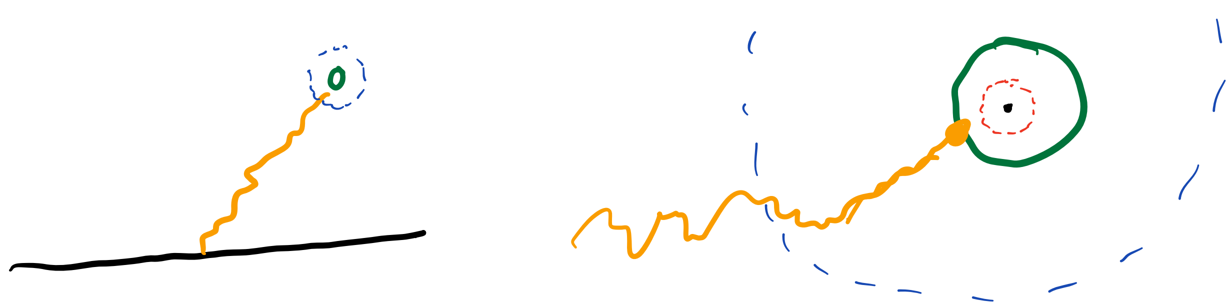
$$\mathbb{E}[F_n(z)\, F_n(w)] \leq |z-w|^{d-2}, \qquad |z-w| \geq e^{-n},$$

$$\mathbb{E}[F_n(z)\, F_n(w)] \leq e^{(2-d)n}\, \mathbb{E}[F_n(z)] \leq c\, e^{(2-d)n}, \quad |z-w| \leq e^{-n}.$$

$$\mathbb{E}\left[\mu_n(S)^2\right] = \int_S \int_S \mathbb{E}[F_n(z)\, F_n(w)]\, dA(z)\, dA(w) \leq C < \infty$$



- For SLE we need to find the probability that $\mathrm{dist}(\gamma, z) < e^{-(n+1)}$ given that $\mathrm{dist}(\gamma, z) < e^{-n}$.
- This probability depends strongly on what the path γ looks like near z but is almost independent of the other parts.
- Find an invariant distribution corresponding to "what SLE paths look like near z" given that they reach the circle of radius e^{-n}.



- In our set-up we get that

$$\begin{aligned}\mathbb{E}[\Psi_n(z)] &= \mathbb{E}[F_{n+1}(z) - F_n(z)] \\ &= \mathbb{E}\left[F_n(z)\,\mathbb{E}\left(e^{2-d}\,1\{\mathrm{dist}(\gamma, z) < e^{-(n+1)}\} - 1 \mid F_n(z)\right)\right] \\ &= O(e^{-un}).\end{aligned}$$

- The random variable

$$\mathbb{E}\left(e^{2-d}\,1\{\mathrm{dist}(\gamma, z) < e^{-(n+1)}\} - 1 \mid F_n(z)\right)$$

depends on the path up to the first time it gets within distance e^{-n} of z.
- It is of order 1 but its expectation with respect to the "invariant distribution" is zero.

- The hardest part of the estimate is the two-point estimate. Roughly speaking, if z, w are not to close, then for n large

$$\mathbb{E}\left(e^{2-d}\, 1\{\mathrm{dist}(\gamma, z) < e^{-(n+1)}\} - 1 \mid F_n(z)\right)$$

and

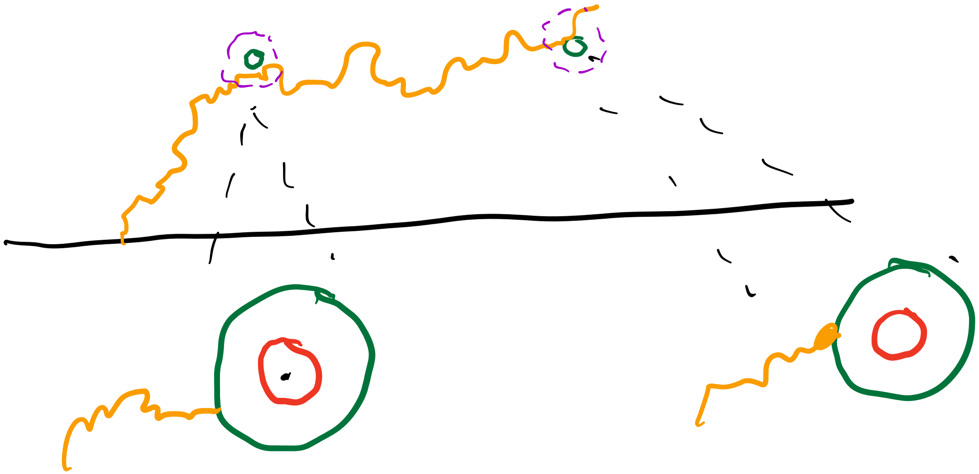
$$\mathbb{E}\left(e^{2-d}\, 1\{\mathrm{dist}(\gamma, w) < e^{-(n+1)}\} - 1 \mid F_n(w)\right)$$

are almost independent random variables.

- This is used to bound

$$\mathbb{E}\left[\Psi_n(z)\, \Psi_n(w)\right].$$

- We will not discuss details — requires careful analysis of SLE path.



Conformal covariance

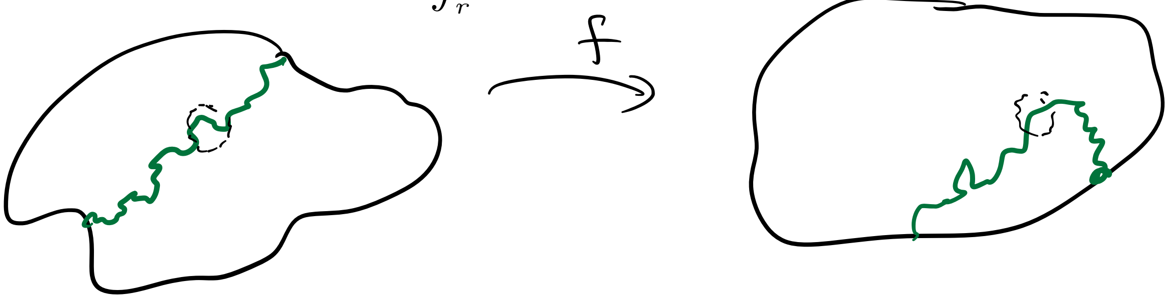
Fact

$f : D \to \tilde{D}$ be a conformal transformation; μ the Minkowski content measure in D and $\tilde{\mu}$ the Minkowski content measure in $\tilde{D}$. Then,

$$\tilde{\mu} = |f'|^d \, (f \circ \mu).$$

- Another way of saying this is that if γ is a path in D with the natural parametrization and $f \circ \gamma$ is the image of γ also with the natural parametrization, then the amount of time for $f \circ \gamma$ to traverse $f(\gamma[r, s])$ is

$$\int_r^s |f'(\gamma(t))|^d \, dt.$$



Dyadic square perspective to fractal measures

- Fix domain D and let $\mathcal{Q}_n = \mathcal{Q}_n(D)$ be the set of (half-open, half-closed) dyadic squares whose closure are contained in D.
- Approximating measures μ_n will be absolutely continuous with respect to Lebesgue measure with density F_n constant on each $S \in \mathcal{Q}_n$.
- Simplest example: $F_n \equiv 1$ and μ is area.
- For $0 < \alpha \leq 2$, define $\mu_n^{(\alpha)}$ by

$$\mu_n^{(\alpha)}(S) = [\mu_n(S)]^{\alpha/2}, \qquad S \in \mathcal{Q}_n.$$

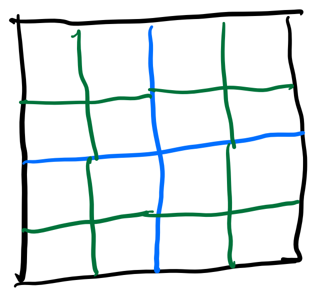
- If $V \subset D$, write

$$\mu_{V,n}^{(\alpha)}(S) = \mu_n^{(\alpha)}(S)\, 1\{S \cap V \neq \emptyset\}, \qquad S \in \mathcal{Q}_n.$$

- If $F_n \equiv 1$ and $\alpha = d$, then for random V we expect

$$\mu_V^{(d)} := \lim_{n\to\infty} \mu_{V,n}^{(d)}$$

  is (a multiple of) Minkowski content measure on V.



Dyadic square perspective of GFF

- $\{\mu_n(S) : S \in \mathcal{Q}_n\}$ is a (finitely or countably infinite) indexed centered Gaussian process with

$$\mathbb{E}\left[\mu_n(S)\,\mu_n(S')\right] = \int_S \int_{S'} G_D(z,w)\,dA(z)\,dA(w).$$

- $\mathbb{E}[\mu_n(S)^2] = 2^{-2n}\ [n\log 2 + O(1)]$.
- Well defined but does not have the Markov property.
- Define

$$\Phi_n(S) = \text{average of } h \text{ on } S = 2^{2n}\,\mu_n(S).$$

- $\{\Phi_n(S) : S \in \mathcal{Q}_n\}$ is also a centered Gaussian field with

$$\mathbb{E}[\Phi_n(S)^2] = n\ \log 2 + O(1), \quad \mathbb{E}\left[e^{\lambda\,\Phi_n(S)}\right] \asymp 2^{\lambda^2 n/2}.$$

- LQG area ($\alpha = 2$) measure with parameter λ (recall that we used $\lambda = \sqrt{\kappa}$ before)

$$F_n(S) = 2^{-\lambda^2 n/2} \, e^{\lambda \Phi_n(S)},$$

$$\mu_n(S) = 2^{-2n} \, F_n(S) = 2^{-(4+\lambda^2)n/2} \, e^{\lambda \Phi_n(S)}.$$

- LQG α-measure with parameter λ: if $S \in \mathcal{Q}_n$,

$$\mu_n^{(\alpha)}(S) = \mu_n(S)^{\alpha/2} = 2^{-(4+\lambda^2)\alpha n/4} \, e^{(\alpha\lambda/2)\, \Phi_n(S)}.$$

- If V is a random subset of fractal dimension d, independent of the field and $S \in \mathcal{Q}_n$,

$$\mu_{V,n}^{(\alpha)}(S) = \mu_n(S)^{\alpha/2} \, 1\{S \cap V \neq \emptyset\}$$

$$= 2^{-(1+\frac{\lambda^2}{4})\alpha n} \, e^{(\alpha\lambda/2)\, \Phi_n(S)} \, 1\{S \cap V \neq \emptyset\}.$$

- Note that

$$\mathbb{E}\left[\mu_{V,n}^{(\alpha)}(S)\right] = 2^{-\beta n}, \qquad \beta = \alpha \left[1 + \frac{\lambda^2}{4}\right] - \frac{\alpha^2 \lambda^2}{8} + 2 - d.$$

KPZ (Knizhnik, Polyakov, Zamolodchikov) Relation

- If S is a fixed dyadic square then it is a union of $O(2^{2n})$ squares in $\mathcal{Q}_n$, and hence if $n \to \infty$,

$$\mathbb{E}[\mu^{(\alpha)}_{V,n}(S)] \asymp 2^{2n}\, 2^{-n\beta}.$$

- The quantum dimension d_q with respect to the parameter λ is the α such that the right-hand side is of order one. This gives the KPZ relation

$$d = d_q \left[1 + \frac{\lambda^2}{4}\right] - \frac{{d_q}^2\,\lambda^2}{8}.$$

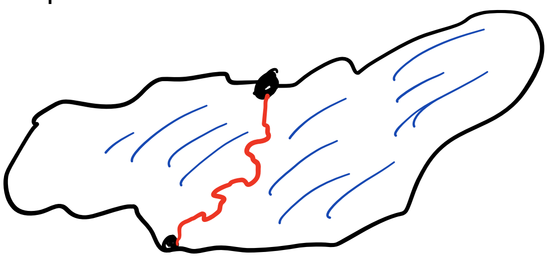
- If $\kappa < 4$ and we choose $\lambda = \sqrt{\kappa}$, then the quantum dimension of an SLE_κ path γ equals one.
- The corresponding measure $\mu^{(1)}_\gamma$ is called the quantum length of the SLE_κ path.

Rigorous construction of quantum length of SLE path

- Take independent GFF and SLE_κ.
- Independent GFF h and SLE_κ path γ in D.

$$F_n(z) = e^{-n(\kappa/8)} \exp\left\{\frac{\sqrt{\kappa}}{2}\,\Phi_n(z)\right\} e^{-n(7\kappa/8)}\, 1\{\mathrm{dist}(\gamma, z) < e^{-n}\}.$$

- Blue measure supported on set of fractal dimension $2 - \frac{\kappa}{8}$. (codimension $\frac{\kappa}{8}$)
- Red measure supported on set of fractal dimension $1 + \frac{\kappa}{8}$. (codimension $\frac{7\kappa}{8}$)
- The limit measure is supported on a set of codimension 1, that is, dimension 1.
- Establishing the limit uses same techniques as earlier this lecture.



Conformal covariance

- Let $f : D \to D'$ be a conformal transformation.
- If h is a GFF in D, let $\tilde{h}$ be the image in D'.
- Let μ be the quantum length in D using h and $\tilde{\mu}$ the quantum length in D' using $\tilde{h}$. Then
$$\tilde{\mu} = |f'|^{1+\frac{\kappa}{4}} \, (f \circ \mu).$$
- We think of
$$|f'|^{1+\frac{\kappa}{4}} = |f'|^{d} \, |f'|^{\frac{\kappa}{8}}$$