These notes discuss the connection between finite projective planes, complete families of orthogonal latin squares and certain resolvable block designs. First we give/recall some definitions:

Complete family of orthogonal latin squares: *A complete family of orthogonal latin squares of order n is a family $L^{(1)}, L^{(2)}, \ldots, L^{(n-1)}$ of $(n-1)$ pairwise orthogonal Latin squares of order n.*

Resolvable $(n^2, n^2+n, n+1, n, 1)$ block design: This is also called a finite *affine plane*. We assume a nontriviality condition, that there are at least 3 points that are not all in the same block/line. *A resolvable $(n^2, n^2+n, n+1, n, 1)$ block design is a family $B(i,j)$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n+1$ of n^2+n size n subsets (called blocks or lines) of an n^2 element set (called varieties) such that each variety appears in $n+1$ blocks and each pair of varieties appears together in a block exactly once. Furthermore the blocks can be partitioned into parallel classes with each parallel class consisting of n disjoint blocks which themselves form a partition of the n^2 varieties.*

The numbers in the notation $(n^2, n^2+n, n+1, n, 1)$ indicate, respectively, the number of varieties, the number of blocks, the number of blocks containing each variety, the size of the blocks and the number of times each variety appears together in a block. We note that it can be shown that the second sentence in the description does not need to be a part of the definition. That is, if the conditions in the first sentence hold then the partition into parallel classes described in the second sentence will necessarily exist.

A more informal but also more intuitive description of these resolvable block designs is as follows: A class of n^2 students is divided into n lab groups on each of $n+1$ days in such a way that each pair of students works together on exactly one day.

Finite projective planes: *A finite projective plane of order n is a collection of n^2+n+1 points and n^2+n+1 'lines' (sets of points or blocks) such that each pair of points appears together on exactly one line and each pair of lines intersects in exactly one point. Each line contains $n+1$ points and each point lies on $n+1$ lines.*

We also assume a nontriviality condition for projective planes that there at least 4 points no three of which are together on a line.

In fact, we can define a finite projective plane to be any collection of points and 'lines' (sets of points) meeting the two conditions: every pair of points lies on exactly one line and every pair of lines intersect in exactly one point, along with a non triviality condition. It can then be shown that there exists an n such that the finite projective plane is as defined above.

We will first by example and then more formally show the following:

There exists:

$$\text{A complete family of orthogonal latin squares of order } n$$
$$\Updownarrow$$
$$\text{A resolvable } (n^2, n^2+n, n+1, n, 1) \text{ block design}$$
$$\Updownarrow$$
$$\text{A finite projective plane of order } n$$

We illustrate the ideas with $n = 3$.

Given the following pair of orthogonal latin squares of order 3 (this is a complete family in this case)

$$L^{(1)} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \quad L^{(2)} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$$

In addition use the row array $L^{(4)}$ and the column array $L^{(3)}$ defined as follows

$$L^{(4)} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix} \quad L^{(3)} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

Now construct a resolvable $(9, 12, 4, 3, 1)$ design using the varieties $(1,1), (1,2), (1,3)$, $(2,1), (2,2), (2,3), (3,1), (3,2), (3,3)$ which will correspond to the locations in the arrays. Define the blocks using the common entries in the $L^{(i)}$. So the blocks $B(1,i), B(2,i), B(3,i)$ correspond to the locations of the $1's$, $2's$ and $3's$ respectively in $L^{(i)}$. So, dropping $\{\}$ for the sets for ease of notation, we get:

$B(1,1) = \{(1,1),(2,3),(3,2)\}$ $B(2,1) = \{(1,2),(2,1),(3,3)\}$ $B(3,1) = \{(1,3),(2,2),(3,1)\}$

$B(1,2) = \{(1,1),(2,2),(3,3)\}$ $B(2,2) = \{(1,2),(2,3),(3,1)\}$ $B(3,2) = \{(1,3),(2,1),(3,2)\}$

$B(1,3) = \{(1,1),(2,1),(3,1)\}$ $B(2,3) = \{(1,2),(2,2),(3,2)\}$ $B(3,3) = \{(1,3),(2,3),(3,3)\}$

$B(1,4) = \{(1,1),(1,2),(1,3)\}$ $B(2,4) = \{(2,1),(2,2),(2,3)\}$ $B(3,4) = \{(3,1),(3,2),(3,3)\}$

To take this resolvable design $(9, 12, 4, 3, 1)$ design and construct a finite projective plane of order 3 we call the varieties points and add 4 new points $1, 2, 3, 4$. One line ('the line at infinity') is $B'(\infty) = \{1, 2, 3, 4\}$. The remaining n^2 lines are based on the $B(i, j)$ of the design by adding $1, 2, 3$ or 4 depending on which latin square the block came from. That is $B'(i, j) = B(i, j) \cup \{j\}$. So, for example, B'(3,2) = {(1,3),(2,1),(3,2),2}. If we think of the resolvable design in terms of lab groups as then we can think of $\{1, 2, 3, 4\}$ as lab aids, one for each day, meeting with each of the groups. We add one more meeting of all the lab aids.

To reverse this process we need to fix notation as ordered pairs first, then the rest is a straightforward reversal of what we just did. Given a finite projective plane of order 4:

$$\begin{array}{cccc} B_1 = \{1,2,3,4\} & & & \\ B_2 = \{1,6,8,13\} & B_3 = \{2,5,9,13\} & B_4 = \{3,5,8,11\} & B_5 = \{4,5,6,7\} \\ B_6 = \{1,7,9,11\} & B_7 = \{2,6,10,11\} & B_8 = \{3,6,9,12\} & B_9 = \{4,8,9,10\} \\ B_{10} = \{1,5,10,12\} & B_{11} = \{2,7,8,12\} & B_{12} = \{3,7,10,13\} & B_{13} = \{4,11,12,13\} \end{array}$$

Pick one block, say B_1 and call this $B'(\infty)$. The other blocks B_2, B_6, B_{10} are distinct from those B_3, B_7, B_{11} containing 2 and similarly for those containing 3 and 4. Identify (in any way) the blocks containing 1 with $B'(i,1)$ and similarly for those containing $2,3,4$. For example $B'(1,1) = B_2, B'(2,1) = B_6, B'(3,1) = B_{10}$ and $B'(1,2) = B_3, B'(2,2) = B_7, B'(3,2) = B_{11}$ and $B'(1,3) = B_4, B'(2,3) = B_8, B'(3,3) = B_{12}$ and $B'(1,4) = B_5, B'(2,4) = B_9, B'(3,4) = B_{13}$. Then relabel the points based on which $B'(r,4)$ and $B'(s,3)$ they are in. So a point is identified with (r,s) if it is in $B(r,4)$ and $B(s,3)$. We can see this via the following array

$$A = \begin{array}{ccc} 5 & 6 & 7 \\ 8 & 9 & 10 \\ 11 & 12 & 13 \end{array}$$

Here we have identified for example 6 with $(1,2)$ and 11 with $(3,1)$. Having done this we delete $\{1,2,3,4\}$ from the remaining blocks to get a resolvable design with blocks $B(i,j)$ and use them to define latin squares. So to get $L^{(1)}$ we use $B(1,1), B(2,1), B(3,1)$ with the entries of $B(i,1)$ giving the locations of the i's in $L^{(1)}$ via the correspondence A above. So we get

$$L^{(1)} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad L^{(2)} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$$

Note for example that $B(2,1) = \{1,7,9,11\}$ and we have 2's in $L^{(1)}$ in the same locations as $7,9,11$ in the array A. Similarly we have $B(1,2) = \{2,5,9,13\}$ and we have 1's in $L^{(2)}$ in the same locations as $5,9,13$ in the array A.

If we are given just the blocks of the resolvable design $B(i,j)$ and not a finite projective plane we can determine the parallel classes, pick two of them to determine rows and columns (like the $B(j,3)$ and $B(i,4)$ above) and define an array like A and proceed in the same fashion.

Formal description of the correspondences between complete families of orthogonal latins squares of order n, resolvable $(n^2, n^2+n, N=1, n, 1)$ designs and finite projective planes of order n.

There exists:

$$\begin{array}{c} \textit{A complete family of orthogonal latin squares of order } n \\ \Updownarrow \\ \textit{A resolvable } (n^2, n^2+n, n+1, n, 1) \textit{ block design} \end{array}$$

In a slight abuse of notation we will use ordered pairs (i,j) to represent elements of the design and to represent locations in the Latin squares. Given a resolvable $(n^2, n^2+n, n+1, n, 1)$ design label the blocks $B(i,j)$ such that the parallel classes are $\{B(1,j), B(2,j), \ldots, B(n,j)\}$ for $j = 1, 2, \ldots, n+1$. We can label the elements with pairs (i,j) as follows: Since $\{B(1,n+1), B(2,n+1), \ldots, B(n,n+1)\}$ form a parallel class they partition the elements and we can assume that $B(i,n+1) = \{(i,1),(i,2),\ldots,(i,n)\}$. Furthermore, since each $B(i,n+1)$ intersects each $B(j,n)$ exactly once we can assume that the labels are such that $B(j,n) = \{(1,j),(2,j),\ldots,(n,j)\}$. Now, for $r = 1, 2, \ldots, n-1$ define Latin squares $L^{(r)}$ by

$$L^{(r)}(i,j) = k \leftrightarrow (i,j) \in B(r,k)$$

This is well defined since each element appears in exactly one block of the r^{th} parallel class. These really are Latin squares since: If $L^{(r)}(i,j) = L^{(r)}(i,j') = k$ then the elements (i,j) and (i,j') appear in blocks $B(i,n+1)$ and $B(r,k)$ contradicting $\lambda = 1$ for the design and if $L^{(r)}(i,j) = L^{(r)}(i',j) = k$ then the elements (i,j) and (i',j) appears in blocks $B(j,n)$ and $B(r,k)$ contradicting $\lambda = 1$ for the design. For orthogonality consider (for $r \neq s$) $L^{(r)}(i,j) = L^{(r)}(i'j') = k$ and $L^{(s)}(i,j) = L^{(s)}(i',j') = k'$. Then the elements (i,j) and (i',j') appear in both $B(r,k)$ and $B(s,k')$. With $r \neq s$ these are distinct blocks and thus $\lambda = 1$ implies that $(i,j) = (i',j')$ and we get orthogonality.

Conversely, given $(n-1)$ orthogonal Latin square of order n we can define blocks $B(i,j)$ for $i = 1, 2, \ldots, n-1$ and $j = 1, 2, \ldots, n$ from the Latin squares as in the displayed equation above and we can define $B(j,n)$ and $B(i,n+1)$ as in the paragraph preceding that equation. The blocks have size n since each element appears n times in a Latin square. For $r = 1, 2, \ldots, n-1$ the blocks $\{B(r,1), B(r,2), \ldots, B(r,n)\}$ form a parallel class since each $L^{(r)}(i,j)$ is uniquely defined. These are parallel classes for $r = n$ and $n+1$ immediately from the definition. The elements (i,j) and (i,j') appear together in $B(i,n+1)$. If they also appear together in $B(r,k)$ we must have $r \leq n-1$ and then we get $L^{(r)}(i,j) = L^{(r)}(i,j') = k$ contradicting distinct row entries for Latins square. The elements (i,j) and (i',j) appear together in $B(j,n)$. If they also appear together in $B(r,k)$ we must have $r \leq n-1$ and then we get $L^{(r)}(i,j) = L^{(r)}(i',j) = k$ contradicting distinct column entries for Latin squares. Finally, consider elements (i,j) and (i',j') with $i \neq i'$ and $j \neq j'$. If these are in block $B(r,k)$ and $B(s,k')$ together then $r, s \leq n-1$. Since we have the parallel classes $r \neq s$. Then we must have $L^{(r)}(i,j) = L^{(r)}(i',j') = k$ and $L^{(s)}(i,j) = L^{(s)}(i',j') = k'$ contradicting orthogonality so each pair appears together at most once. This implies they appear together exactly once since we have $\binom{n^2}{2} n^2(n^2-1)/2$ pairs of elements (pairs of ordered pairs

$(i,j),(i'j'))$ and from the $(n+1)(n) = n^2+n$ blocks of size n we set that there are a total of $(n^2+n)\binom{n}{2} = n^2(n^2-1)/2$ appearances of pairs of elements.

There exists:

$$\begin{array}{c} \textit{A resolvable } (n^2, n^2+n, n+1, n, 1) \textit{ block design} \\ \Updownarrow \\ \textit{A finite projective plane of order } n \end{array}$$

Given a resolvable $(n^2, n^2+n, n+1, n, 1)$ block design we can assume that the elements are labeled with pairs as described in the previous section. we add new elements $\{1, 2, \ldots, n\}$ and define blocks of a projective plane by $B'(i,j) = B(i,j) \cup \{j\}$ along with a new block $B'(\infty) = \{1, 2, \ldots, n\}$. To check that these form a projective plane each pair of original elements appeared together in exactly one block $B(i,j)$ and so they do also $B'(i,j)$. Each new element k is added to the blocks of a parallel class and hence appears exactly once with each original element. The new elements appear together only in the new block $B(\infty)$ and each new element is in this block so each pair appear together exactly once. Each $B'(i,j)$ contains one new element and thus intersects $B(\infty)$ in exactly one element. A pair of blocks $B'(i,j) = B(i,j) \cup \{j\}$ and $B'(i',j) = B(i',j) \cup \{j\}$ intersect only in j since $B(i',j)$ and $B(i,j)$ were in a parallel class and have empty intersection. Consider $B(i,j)$ and $B(i',j')$ for $j \neq j'$. The intersection has size at most one since if it was greater some pair of elements would appear together in more than one block of the design. If the intersection was empty then since the $B(i',j')$ form a parallel class of blocks of size n and since $B(i,j)$ has size n, the n elements of $B(i,j)$ must appear in the $(n-1)$ blocks $B(i'',j')$ for $i'' \neq i'$. By the pigeonhole principle some block $B(i'',j')$ contains two elements of $B(i,j)$, a contradiction. Hence the intersection has size one and thus the intersection of $B(i,j)$ and $B(i',j')$ has size one.

Conversely, given a finite projective plane, we can assume that some line is labeled $B(\infty) = \{1, 2, \ldots, n, n+1\}$. The blocks containing j other than $B(\infty)$ are labeled in some manner $B'(i,j)$ for $i = 1, 2, \ldots, n$. Note that these form a parallel class of the remaining elements as no two lines intersect in more than one point and so the union of the $B'(i,j)$ for a given j contain n^2 elements other than j. Thus each remaining element appears in exactly one $B'(j,n)$ and exactly one $B'(i,n+1)$. Label such an element (i,j). Then let $B(i,j) = B'(i,j) - \{1, 2, \ldots, n, n+1\}$. These form the blocks of a resolvable $(n^2, n^2+n, n+1, n, 1)$ block design. That the size are correct can easily be checked. We have already noted the parallel classes. That each element appear together in exactly one block follows directly from the analogous statement for the projective plane.